Effect of Grid System on Finite Element Calculation

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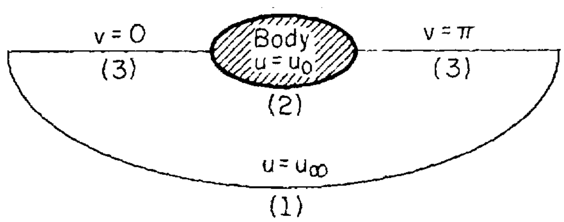
We have made detailed parametric studies of the effect of grid system on finite element calculation for potential flows. These studies have led to the formulation of a design criteria for optimum mesh system and the development of two methods to generate the optimum mesh system. The guidelines for optimum mesh system are:

1. The mesh structure should be regular.
2. The element should be as regular and equilateral as possible.
3. The distribution of size of element should be consistent with that of flow variables to insure maximum uniformity in error distribution.
4. For non-Dirichlet boundary conditions, smaller boundary elements or higher-order interpolation functions should be used.
5. The mesh should accommodate the boundary geometry as accurately as possible.

We shall present in this paper the results of our parametric studies.



(u,v): Elliptic-Cylindrical Coordinate System
(Subscript ∞ Denotes Free Stream Condition)

		Problem I	Problem II	Problem III
Type of Boundary Conditions		Dirichlet	Neumann	Mixed
Variable		Stream Function	Velocity Potential	Velocity Potential
Boundary Conditions	at (1)	$\Psi = \Psi_\infty$	$\Phi = \Phi_\infty$	$\Phi = \Phi_\infty$
	at (2)	$\Psi = 0$	$\frac{\partial \Phi}{\partial u} = 0$	$\frac{\partial \Phi}{\partial u} = 0$
	at (3)	$\Psi = 0$	$\frac{\partial \Phi}{\partial v} = 0$	$\Phi + A\left(\frac{\partial \Phi}{\partial v}\right) = B$

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Fig. 1. We choose three potential flow problems around an elliptic cylinder as the test problems to evaluate and to compare computational errors. In these problems, the computational domain is transformed into a rectangular domain by using the elliptic-cylindrical coordinate system (u,v). This corresponds to an isoparametric element in the physical plane where element boundaries are curved isoparametric lines.

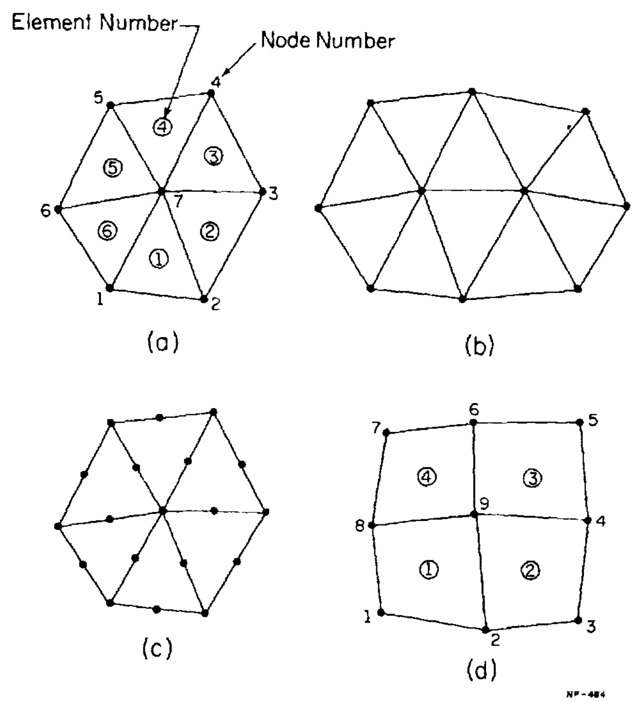


Fig. 2. Numerical solutions are obtained for the test problems using a sector method. A sector is defined by a combination of elements surrounding a node or nodes. It becomes the finite cut-off zone of influence of the interior node or nodes. The solution procedure is to construct the sector matrix for each sector and to iterate by sweeping all the sectors. This method provides a way to avoid the tedious data handling in constructing the system stiffness matrix and facilitates the treatment of boundary conditions.

Types of Sectors shown are:

(a) Six Triangular Elements, One Interior Node.
(b) Ten Triangular Elements, Two Interior Nodes.
(c) Six Triangular Elements, Seven Interior Nodes.
(d) Four Quadrilateral Elements, One Interior Node.

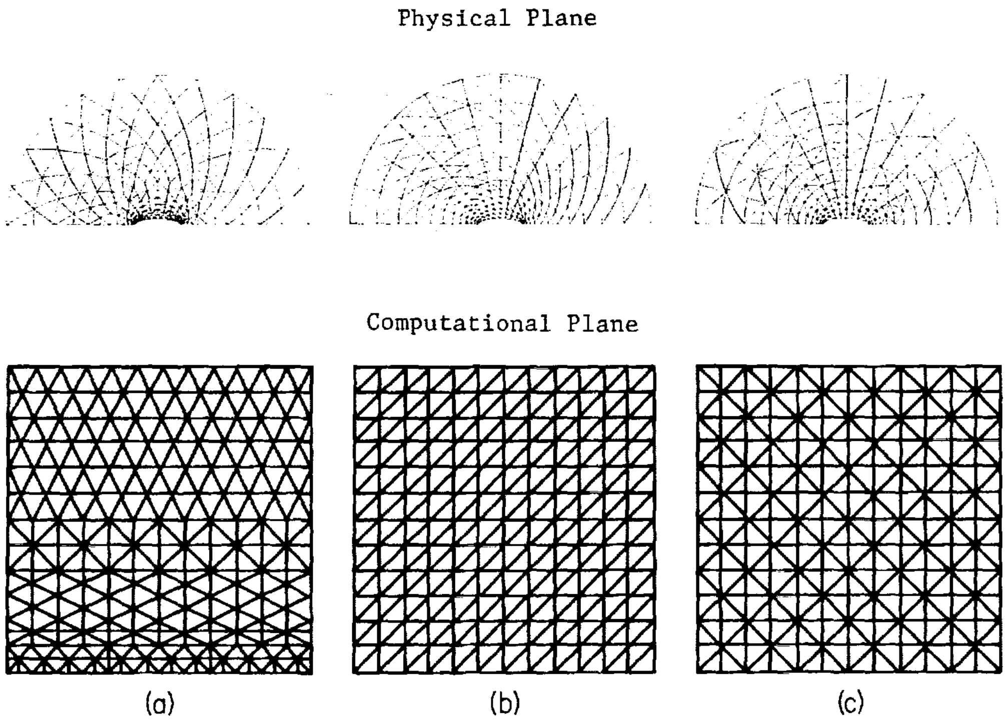
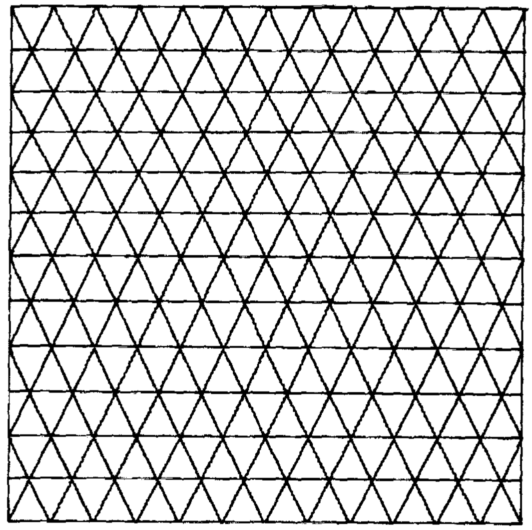
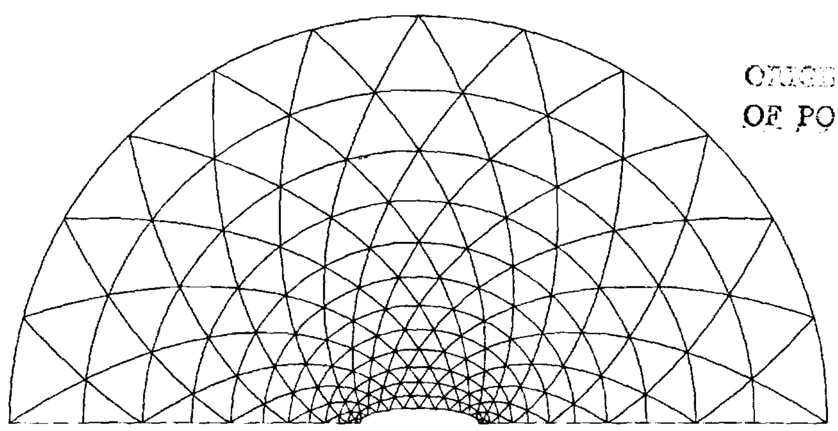


Fig. 3. Three different grid systems for an elliptic boundary. The system (a) has a larger number of nodes and better resolution near the body, however, the structure of the elements is irregular. Table 1 shows the maximum percent error obtained for Problem 1 for the case of Dirichlet boundary conditions. The grid system (a) has much larger error despite the fact that it has more mesh points near the body. This larger error comes from unfair treatment of the influence of neighboring points. The unfair treatment results not only from the irregular shapes of the elements but also from the use of several types of sectors, i.e., sectors consisting of different number of elements. The error increases as more types of sectors are used. The fact that the error in grid system (c) is greater than that in grid system (b) is a further indication of this effect. Only one type of sector, which consists of six elements is used in grid system (b), while two types of sectors, one with eight elements and the other with four, are used in grid system (c). Note that five different types of sectors are used in grid system (a).

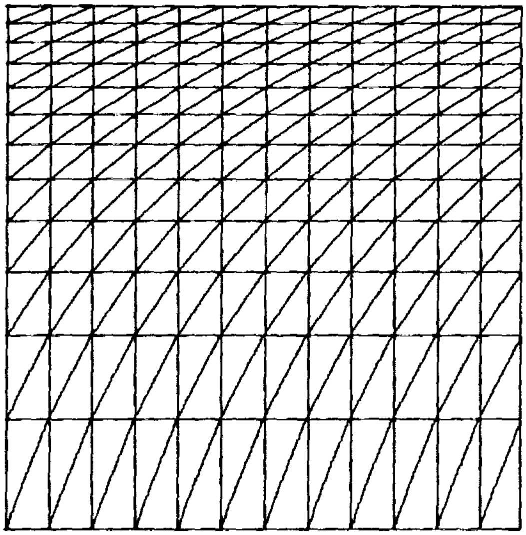


(a) Computational Plane

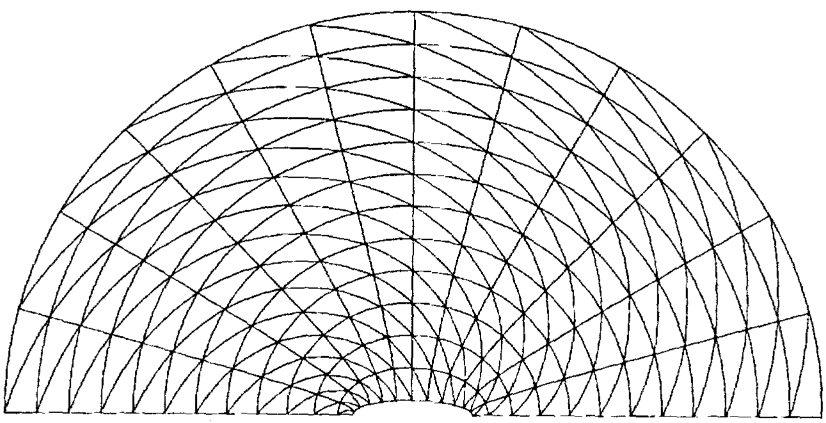


(b) Physical Plane

Fig. 4. The mesh structure of Fig. 3(b) is used to study the effect of element shapes. The shapes considered are equilateral or isosceles triangles, as shown here, in addition to the right-angled triangles, as shown in Fig. 3(b). The maximum errors at both the body surface and the outer boundary are tabulated. The evaluation of the effect of the element shape on the computational errors is based on the comparison of these two errors. For case (1) with right-angled triangles, the error at the body surface is much greater; therefore, the error due to element shape dominates. For case (2) with isoceles triangles, the outer boundary error dominates. For case (3) with equilateral triangles, the two errors are nearly equal. In fact, the error distribution is almost uniform. Such a uniformity in error distribution is important for any flow field computation.

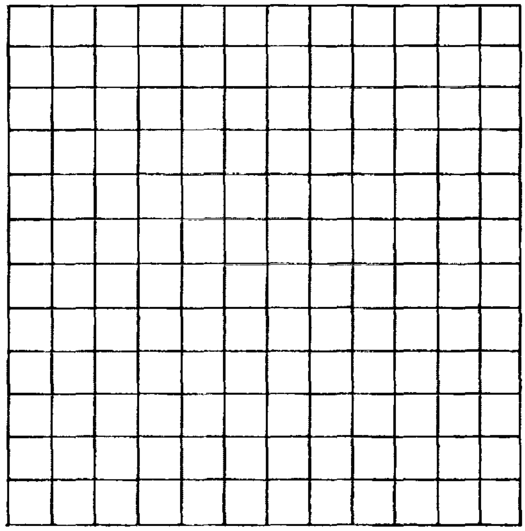


(a) Computational Plane

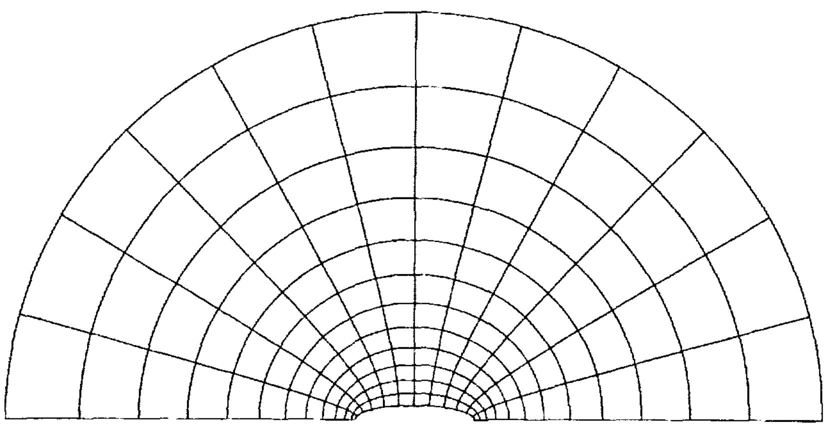


(b) Physical Plane

Fig. 5. The effect of element size distribution was studied by comparing the error for two grid systems shown in Fig. 3(b) and Fig. 5 respectively. These two systems have the same structure; however, the distribution of nodes in the system shown in Fig. 5 is not as uniform. The comparison of errors is given in Table 3. The error for the system of Fig. 5 is greater because the distribution of nodes deviates significantly from the change of field variables.



(a) Computational Plane



(b) Physical Plane

Fig. 6. The error of the system with the triangular element is compared with that with the quadrilateral elements. The interpolation functions in both cases are of second order in the field variables, but differ in their derivatives. The triangular element has a first order accuracy while the quadrilateral element has a second order accuracy. The results are summarized in Table 4. Even though the difference in error in the stream function between the two cases is small, the difference in errors in the velocities is appreciable. In comparing the errors in velocities, it may be more informative to examine the maximum deviations from the exact solutions. This maximum deviation is found to be of $O[10^{-4}]$ per unit free stream velocity for the quadrilateral element and $O[10^{-2}]$ for the triangular element.

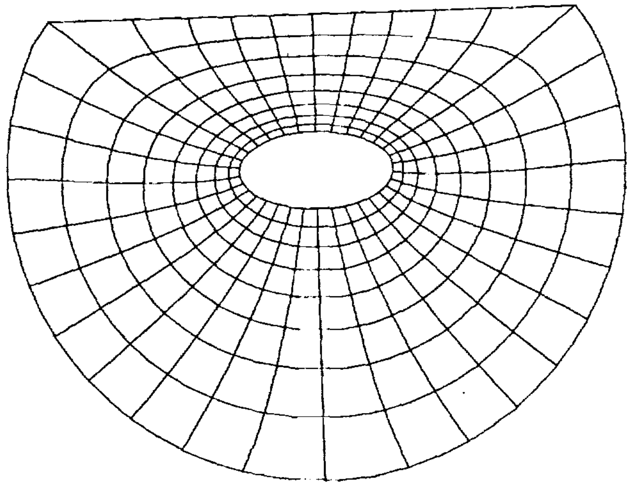


Fig. 7. Optimum mesh system - submerged elliptical cylinder near a free surface.

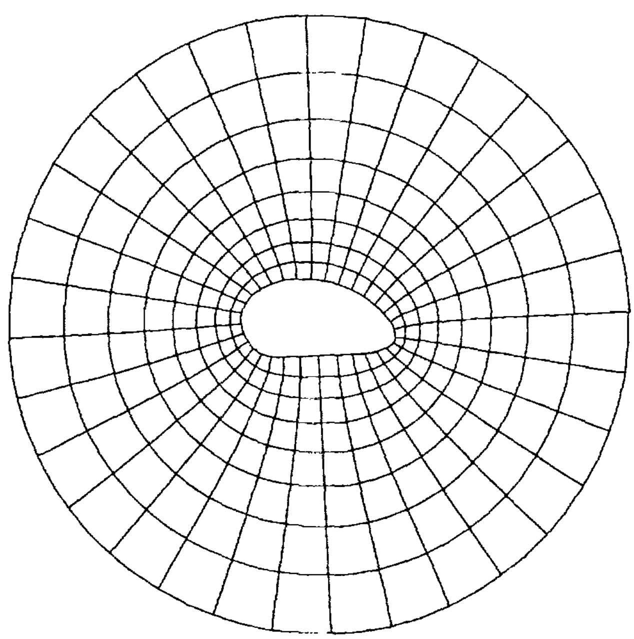


Fig. 8. Optimum mesh system - cylinder of irregular shape.

Two methods of numerical transformation into a set of orthogonal coordinates have been developed to generate an optimum mesh system which meets the guidelines listed above. Figs. 7 and 8 show examples of mesh systems generated.

Table 1. Effect of Mesh Structure

Problem : Problem I, Fig. 1

Element Type : Triangular Element

Mesh System : Fig. 3

Outer Boundary : $u_{out} = u_o + 0.75\pi$

Mesh System	(a)	(b)	(c)
% Error	29.4	0.979	1.64

Table 2. Effect of Element Shapes

Problem : Problem I, Fig. 1

Element Type : Triangular Element

Case	(a)	(b)	(c)
Mesh System	Fig. 3(b)	Fig. 4	Fig. 4
Element Shape	Right-angled Triangles	Isosceles Triangles	Equilateral Triangles
% Error near body	0.979	0.254	0.172
% Error at Outer Boundary	0.363	0.363	0.174
u_{out}	$u_o+0.75\pi$	$u_o+0.75\pi$	$u_o+\pi$

Table 3. Effect of Element Size Distribution

Problem : Problem I, Fig. 1

Element Type : Triangular Element

Number of Nodes : 13 x 13

Outer Boundary : $u_{out} = u_o + 0.75\pi$

Mesh System	Fig. 3(b)	Fig. 5
% Error	0.979	6.652

Table 4. Effect of Element Type and Interpolation Functions

Problem : Problem I, Fig. 1

Number of Nodes : 16 x 16

Outer Boundary : $u_{out} = u_o + \pi$

Element Type		Triangular	Quadrilateral
Mesh System		Fig. 3(b)	Fig. 6
% Error	Stream Function	0.856	0.710
	u-Velocity	34.95	1.435
	v-Velocity	33.98	1.096
Maximum Deviation	u-Velocity	0.010	0.0004
	v-Velocity	0.068	0.0004