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Numerical Generation of Two-Dimensional Orthogonal Curvilinear Coordinates in an Euclidean Space*

Z. U. A. Warsi and J. F. Thompson
Department of Aerospace Engineering
Mississippi State University
Mississippi State, MS 39762

Abstract

In this paper a non-iterative method for the numerical generation of orthogonal curvilinear coordinates for plane annular regions between two arbitrary smooth closed curves has been developed. The basic generating equation is the Gaussian equation for an Euclidean space which has been solved analytically. The method has been applied in many cases and these test results demonstrate that the proposed method can be readily applied to a wide variety of problems. The method can also be used for simply connected regions only by obtaining the solution of the linear equation (19) under the changed boundary conditions. Details on the work reported in this paper are available in Reference [1].

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Fundamental Ideas of the Method

All methods of numerical coordinate generation in a two-dimensional plane and classified under the method of "elliptic equations" (Refs. [2]-[10]), have depended invariably on the solution of Poisson equations for the curvilinear coordinates $\xi(x,y)$ and $\eta(x,y)$:

$$\nabla^2\xi = -\frac{1}{g}(g_{11}\Gamma^1_{22} - 2g_{12}\Gamma^1_{12} + g_{22}\Gamma^1_{11})$$

$$= -\frac{g_{22}}{g} P(\xi,\eta) \qquad (1a)$$

$$\nabla^2\eta = -\frac{1}{g}(g_{11}\Gamma^2_{22} - 2g_{12}\Gamma^2_{12} + g_{22}\Gamma^2_{11})$$

$$= -\frac{g_{11}}{g} Q(\xi,\eta) \qquad (1b)$$

where $P(\xi,\eta)$, $Q(\xi,\eta)$ are arbitrarily specified control functions, the g_{ij} are the fundamental metric coefficients, the Γ^i_{jk} are the Christoffel symbols of the second kind

$$\Gamma^i_{jk} = g^{i\ell}[jk,\ell] \qquad (2)$$

$$[jk,\ell] = \frac{1}{2}\left(\frac{\partial g_{j\ell}}{\partial x^k} + \frac{\partial g_{k\ell}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell}\right)$$

and

$$g = g_{11}g_{22} - (g_{12})^2$$

Implicitly equation (1) implies two things: (i) that the coordinates for the same domain can also be obtained by solving the Laplace equations

$$\nabla^2\xi = 0, \quad \nabla^2\eta = 0 \qquad (3)$$

and (ii) since the Γ^i_{jk} have first partial derivatives of g_{ij} in them, equation (1) can also be interpreted as providing a set of constraints or relations among the g_{ij} and their first partial derivatives.

In this paper we present another method based on elliptic equations and state the problem as follows.

The three functions g_{11}, g_{12}, g_{22} of the curvilinear coordinates ξ,η define an element of length ds in a plane if the Gaussian equation with zero curvature

$$\frac{\partial}{\partial\eta}\left(\frac{\sqrt{g}\,\Gamma^2_{11}}{g_{11}}\right) - \frac{\partial}{\partial\xi}\left(\frac{\sqrt{g}\,\Gamma^2_{12}}{g_{11}}\right) = 0 \tag{4}$$

holds for every point in the plane, and then the Cartesian coordinates are given as

$$x = x(\xi,\eta),\ y = y(\xi,\eta)$$

Equation (4) is identically satisfied by a function $\alpha(\xi,\eta)$ defined as

$$\alpha_\xi = \frac{-\sqrt{g}}{g_{11}}\Gamma^2_{11},\quad \alpha_\eta = \frac{-\sqrt{g}}{g_{11}}\Gamma^2_{12}$$

Specifically, α is the angle of inclination with respect to the x-axis of the tangent to the coordinate line η = const. directed in the sense of increasing values of the parameter ξ. The first partial derivatives of x and y are

$$\left.\begin{aligned} x_\xi &= \sqrt{g_{11}}\cos\alpha,\quad y_\xi = -\sqrt{g_{11}}\sin\alpha \\ x_\eta &= \frac{1}{\sqrt{g_{11}}}(g_{12}\cos\alpha + \sqrt{g}\sin\alpha),\quad y_\eta = -\frac{1}{\sqrt{g_{11}}}(\sqrt{g}\cos\alpha - g_{12}\sin\alpha) \end{aligned}\right\} \tag{5}$$

Then

$$\left.\begin{aligned}
x &= \int [\sqrt{g_{11}} \cos\alpha \, d\xi + \frac{1}{\sqrt{g_{11}}} (g_{12} \cos\alpha + \sqrt{g} \sin\alpha) d\eta] \\
y &= -\int [\sqrt{g_{11}} \sin\alpha \, d\xi - \frac{1}{\sqrt{g_{11}}} (\sqrt{g} \cos\alpha - g_{12} \sin\alpha) d\eta] \\
\alpha &= -\int \frac{\sqrt{g}}{g_{11}} (\Gamma_{11}^{2} \, d\xi + \Gamma_{12}^{2} \, d\eta)
\end{aligned}\right\} \qquad (6)$$

The inverse relations of (5) are

$$\left.\begin{aligned}
\xi_x &= (\sqrt{g} \cos\alpha - g_{12} \sin\alpha)/\sqrt{g g_{11}} \\
\xi_y &= -(g_{12} \cos\alpha + \sqrt{g} \sin\alpha)/\sqrt{g g_{11}} \\
\eta_x &= \sqrt{g_{11}/g} \sin\alpha \\
\eta_y &= \sqrt{g_{11}/g} \cos\alpha
\end{aligned}\right\} \qquad (7)$$

For the case of orthogonal coordinates, the coefficient $g_{12} = 0$, i.e.,

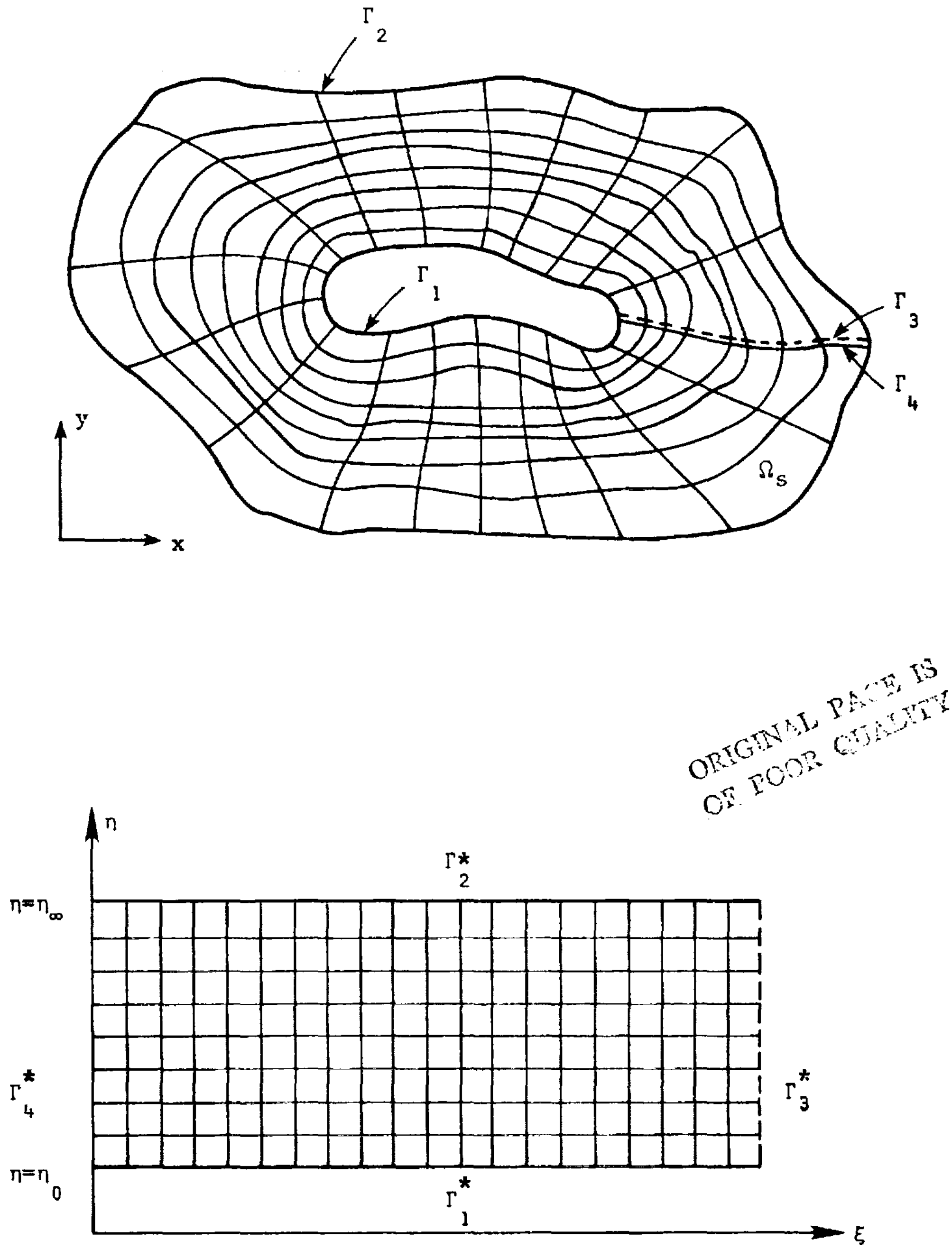
$$g_{12} = x_\xi x_\eta + y_\xi y_\eta = 0 \qquad (8)$$

which is satisfied by the equations

$$\left.\begin{aligned}
x_\eta &= -F y_\xi \\
y_\eta &= F x_\xi
\end{aligned}\right\} \qquad (9)$$

where $F > 0$ is a continuous function of g_{11} and g_{22} [11] .

Referring to Figure 1, let the boundary Γ_2 of a bounded region in an Euclidean two-dimensional space be a simple curve $x = x_\infty(\xi)$, $y = y_\infty(\xi)$, with a uniformaly turning tangent. In the region Ω, let



Transformed Plane
(Natural Coordinates)

Figure 1.- Physical and transformed planes.

Ω_s be an annular subregion bounded by the inner boundary Γ_1 and the outer boundary Γ_2. The region Ω_s is to be mapped onto a rectangular region R in the $\xi\eta$-plane by a transformation so as to have

$$\left.\begin{aligned} x &= x(\xi,\eta) \\ y &= y(\xi,\eta) \end{aligned}\right\} \qquad \eta_\beta \le \eta \le \eta_\infty$$

where η_β and η_∞ are the actual parametric values associated with the boundaries Γ_1 and Γ_2, respectively, and x,y are periodic in the ξ-argument.

Substituting $g_{12} = 0$ in the fundamental equation (4) we get

$$\frac{\partial}{\partial\xi}\left[\frac{1}{Fg_{11}}\frac{\partial}{\partial\xi}(F^2 g_{11})\right] + \frac{\partial}{\partial\eta}\left[\frac{1}{Fg_{11}}\frac{\partial g_{11}}{\partial\eta}\right] = 0 \tag{10}$$

where

$$\left.\begin{aligned} g_{22} &= F^2 g_{11} \\ g &= (Fg_{11})^2 \end{aligned}\right\} \tag{11}$$

Before we solve the problem of orthogonal coordinate generation based on the elliptic equation (10), we digress and state the following results: Following Potter and Tuttle [6] we assume that the ξ-curves in the physical xy-plane are free from sources and sinks. This condition establishes a unique correspondence between the ξ-points on each pair of η = const. lines. In the absence of sources and sinks, we have

$$\operatorname{div}[\operatorname{grad} \psi(\eta)] = 0 \tag{12}$$

where $\psi(\eta)$ is an arbitrary differentiable function of η and grad $\psi(\eta)$ is oriented along the normal to the curve η = const. Carrying out the differential operation in (12) and using the expressions

$$|\operatorname{grad} \eta| = 1/\sqrt{g_{22}}$$

$$\nabla^2\eta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial\eta} \left(\sqrt{\frac{g_{11}}{g_{22}}}\right)$$

in (12), we obtain

$$\frac{\partial}{\partial\eta} \left(\ln \sqrt{g_{11}/g_{22}}\right) = - \frac{d^2\psi}{d\eta^2} / \frac{d\psi}{d\eta}$$

Writing $\frac{d\psi}{d\eta} = 1/\nu(\eta)$ and denoting the arbitrary function of ξ due to integration by $\ln\mu(\xi)$, we obtain the result

$$\sqrt{g_{11}/g_{22}} = \mu(\xi)\nu(\eta)$$

$$= 1/F \qquad (13)$$

This result shows that for the case of orthogonal coordinates the ratio g_{11}/g_{22} is a product of the positive functions $\mu(\xi)$ and $\nu(\eta)$. The result in (13) also provides the condition for the two distinct families of orthogonal curves

$$\xi = \text{const.}, \quad \eta = \text{const.}$$

to divide the physical plane in infinitensimal squares. (See Cohen [12]).

We now introduce new coordinates $\xi'(\xi)$ and $\eta'(\eta)$ as

$$\xi' = \int\mu(\xi)d\xi, \quad \eta' = \int \frac{d\eta}{\nu(\eta)} \qquad (14)$$

Thus

$$g'_{11} = g_{11}/\mu^2, \quad g'_{22} = g_{22}\nu^2$$

so that

$$g'_{22} = g'_{11}$$

Defining

$$P' = \ln g'_{11}$$

it can be shown that

$$\frac{\partial^2 P'}{\partial \xi'^2} + \frac{\partial^2 P'}{\partial \eta'^2} = \frac{\nu(\eta)}{\mu(\xi)} \{\frac{\partial}{\partial \xi} [\frac{1}{Fg_{11}} \frac{\partial}{\partial \xi} (F^2 g_{11})] + \frac{\partial}{\partial \eta} [\frac{1}{Fg_{11}} \frac{\partial g_{11}}{\partial \eta}]\} \tag{15}$$

Using (15) in (10), we get a much simpler equation

$$\frac{\partial^2 P'}{\partial \xi'^2} + \frac{\partial^2 P'}{\partial \eta'^2} = 0 \tag{16}$$

Another important result can be obtained based on (13). Using the orthogonality condition $g_{12} = 0$ in (7), we have

$$\eta_y = \xi_x/F, \; \eta_x = - \xi_y/F$$

so that

$$\frac{\partial}{\partial \xi} (\xi_x/F) + \frac{\partial}{\partial y} (\xi_y/F) = 0 \tag{17}$$

Carrying out the transformation (14) in (17), we get

$$\nabla^2 \xi' = 0 \tag{18}$$

Equation (18) provides the uniqueness condition for the solution of equation (16).

Based on the preceeding analysis we can state that if an exact analytic solution of equation (10) can be obtained for F = 1, i.e., $g_{22} = g_{11}$, then the solution for any other coordinate system $\bar{\xi}$ and $\bar{\eta}$, where $\xi = \phi\ (\bar{\xi})$ and $\eta = f(\bar{\eta})$, can simply be obtained by the substitution of ϕ and f in place ξ and η respectively. With this scheme in mind, we solve the equation

$$\frac{\partial^2 P}{\partial \xi^2} + \frac{\partial^2 P}{\partial \eta^2} = 0 \tag{19}$$

where

$$g_{22} = g_{11}$$

$$P = \ln g_{11} \tag{20}$$

under the boundary conditions

$$\left.\begin{aligned} P &= P_\beta(\xi) \text{ at } \eta = 0^\dagger \\ &= P_\infty(\xi) \text{ at } \eta = \eta_\infty \end{aligned}\right\} \quad 0 \le \xi \le 2\pi \tag{21}$$

where the subscripts β and ∞ denote the inner and outer boundaries, respectively. The periodicity requirement is that

$$P(\xi,\eta) = P(\xi+2\pi,\eta) \tag{22}$$

Further, the ξ-coordinate must be such that the equation

$$\nabla^2\xi = 0 \tag{23}$$

is always satisfied.

A general analytic solution of equation (19) under the boundary conditions (21) and the periodicity condition (22) is

$$\begin{aligned} P(\xi,\eta) = a_o + \eta\overline{K} + \sum_{n=1}^{\infty} \sinh n(\eta_\infty-\eta) \\ (a_n \cos n\xi + b_n \sin n\xi)/\sinh n\eta_\infty \\ + \sum_{n=1}^{\infty} \sinh n\eta(c_n \cos n\xi + d_n \sin n\xi)/\sinh n\eta_\infty \end{aligned} \tag{24}$$

where

$$\overline{K} = (c_o-a_o)/\eta_\infty \tag{25}$$

and

$$a_o = \frac{1}{2\pi}\int_o^{2\pi} P_\beta(\xi)d\xi, \; c_o = \frac{1}{2\pi}\int_o^{2\pi} P_\beta(\xi)d\xi$$

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†There is no loss of generality in setting the parametric value $\eta_\beta = 0$. The value η_∞ must be interpreted as the difference between the actual values of η at the outer and inner boundaries.

$$a_n = \frac{1}{\pi}\int_o^{2\pi} P_\beta(\xi)\cos n\xi\, d\xi,\quad b_n = \frac{1}{\pi}\int_o^{2\pi} P_\beta(\xi)\sin n\xi\, d\xi$$
$$c_n = \frac{1}{\pi}\int_o^{2\pi} P_\infty(\xi)\cos n\xi\, d\xi,\quad d_n = \frac{1}{\pi}\int_o^{2\pi} P_\infty(\xi)\sin n\xi\, d\xi \qquad (26)$$

For orthogonal Coordinates

$$\alpha_\xi = \frac{1}{2\sqrt{g}}\frac{\partial g_{11}}{\partial\eta},\quad \alpha_\eta = -\frac{1}{2\sqrt{g}}\frac{\partial g_{22}}{\partial\xi}$$

therefore for $g_{22} = g_{11}$

$$\alpha_\xi = \frac{1}{2}\frac{\partial P}{\partial\eta},\quad \alpha_\eta = -\frac{1}{2}\frac{\partial P}{\partial\xi}$$

and consequently

$$\alpha(\xi,\eta) = \alpha(\xi,o) + \sum_{n=1}^{\infty}\frac{\cosh n(\eta_\infty-\eta)}{2\sinh n\,\eta_\infty}(b_n\cos n\xi - a_n\sin n\xi)$$
$$+ \sum_{n=1}^{\infty}\frac{\cosh n\eta}{2\sinh n\,\eta_\infty}(c_n\sin n\xi - d_n\cos n\xi)$$
$$- \sum_{n=1}^{\infty}\frac{\cosh n\,\eta_\infty}{2\sinh n\,\nu_\infty}(b_n\cos n\xi - a_n\sin n\xi)$$
$$- \sum_{n=1}^{\infty}\frac{1}{2\sinh n\,\eta_\infty}(c_n\sin n\xi - d_n\cos n\xi)$$

Having determined g_{11} and α, we can find the Cartesian coordinates

$$x(\xi,\eta) = x(\xi,o) + \int_o^\eta \sqrt{g_{22}}\sin\alpha\, d\eta$$
$$y(\xi,\eta) = y(\xi,o) + \int_o^\eta \sqrt{g_{22}}\cos\alpha\, d\eta \qquad (28)$$

The preceding solution is for the case when $g_{22} = g_{11}$, i.e., $F = 1$. However, as stated earlier, the solution for any other coordinate system $\bar{\xi},\bar{\eta}$ in which $\bar{g}_{22} \neq \bar{g}_{11}$ can be obtained by replacing ξ and η in (24), (27) and (28) by $\phi(\bar{\xi})$ and $f(\bar{\eta})$, respectively. This feature can be used to redistribute the coordinate lines in the desired regions. Since the functions ϕ and f are at our disposal, they play the same role as P and Q in equation (1). Further, since the Fourier coefficients (26) are invariant to a coordinate transformation $\xi = \phi(\bar{\xi})$, where $\phi(\bar{\xi}_o) = 0$, $\phi(\bar{\xi}_m) = 2\pi$ and $\bar{\xi}_o$ corresponds to $\xi = 0$, $\bar{\xi}_m$ corresponds to $\xi = 2\pi$, these coefficients need not be recalculated.

The procedure of transformation from ξ,η to $\bar{\xi},\bar{\eta}$ is as follows.

On transformation from (ξ,η) to $(\bar{\xi},\bar{\eta})$, the covariant metric coefficients transform through the equation

$$\bar{g}_{ij} = g_{k\ell} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^\ell}{\partial \bar{x}^j}$$

so that, on using the relations $g_{22} = g_{11}$ and $g_{12} = 0$, we have

$$\left.\begin{aligned} \bar{g}_{11} &= \left[\left(\frac{\partial \xi}{\partial \bar{\xi}}\right)^2 + \left(\frac{\partial \eta}{\partial \bar{\xi}}\right)^2\right] g_{11} \\ \bar{g}_{22} &= \left[\left(\frac{\partial \xi}{\partial \bar{\eta}}\right)^2 + \left(\frac{\partial \eta}{\partial \bar{\eta}}\right)^2\right] g_{11} \end{aligned}\right\} \tag{29}$$

We now introduce the transformation

$$\left.\begin{aligned} \xi &= \phi(\bar{\xi}) \\ \eta &= f(\bar{\eta}) \end{aligned}\right\} \tag{30}$$

where the functions ϕ and f are continuously differentiable and satisfy the conditions

$$\phi(\bar{\xi}_o) = 0, \; f(\bar{\eta}_\beta) = 0$$

where $\xi = 0$ corresponds to $\bar{\xi} = \bar{\xi}_o$ and $\eta = 0$ corresponds to $\bar{\eta} = \bar{\eta}_\beta$. Defining

$$\lambda = \frac{d\phi}{d\bar{\xi}}, \; \theta = \frac{df}{d\bar{\eta}}$$

we obtain from (30)

$$\bar{g}_{22}(\bar{\xi},\bar{\eta}) = \frac{\theta^2}{\lambda^2}\ \bar{g}_{11}(\bar{\xi},\bar{\eta})$$

Comparing with (13), we find $\mu = \lambda,\ \nu = \frac{1}{\theta}$. (31)

The salient feature of the preceding analysis is that the solution for the case $g_{22} = g_{11}$ can be used to obtain the solution for the case $g_{22} \neq g_{11}$ by coordinate transformation.

Before solving any specific problem, it is important first to establish an orthogonal correspondence between unique points of the inner and outer boundary curves. This condition is satisfied if we choose the ξ-curves satisfying the equation

$$\nabla^2\xi = 0 \tag{32}$$

The inner and outer boundary curves are available either in tabular or functional form as

$$y_\beta = y(x_\beta),\ y_\infty = y(x_\infty) \tag{33}$$

For equation (32) to be satisfied, we can take ξ as the angle traced out in a clockwise sense by the common radius of the concentric circles in a conformal representation of the inner and outer boundary curves. If a and A, respectively, are the radii of the inner and outer circles in the transformed conformal plane, then

$$\left.\begin{aligned} a &= \frac{1}{2\pi}\int_0^{2\pi} [x_\beta(\xi)\ \cos\xi - y_\beta(\xi)\ \sin\xi]d\xi \\ A &= \frac{1}{2\pi}\int_0^{2\pi} [x_\infty(\xi)\ \cos\xi - y_\infty(\xi)\ \sin\xi]d\xi \end{aligned}\right\} \tag{34}$$

As is well known, the preceding scheme is an iterative numerical scheme. In lieu of this, we have developed a method which is fast and direct, and is equivalent to satisfying equation (32).

We circumscribe circles around the inner and outer boundary curves. Two cases arise depending on whether the circles are concentric or nonconcentric.

<u>Case I</u>: If the circumscribed circles are concentric (Fig. 2(a)), then we select those values of the ordinates which correspond to the abscissae

$$x_\beta = r_s \cos \xi, \; x_\infty = r_L \cos \xi \tag{35}$$

where r_s and r_L are the radii of the circumscribed circles in the physical plane.

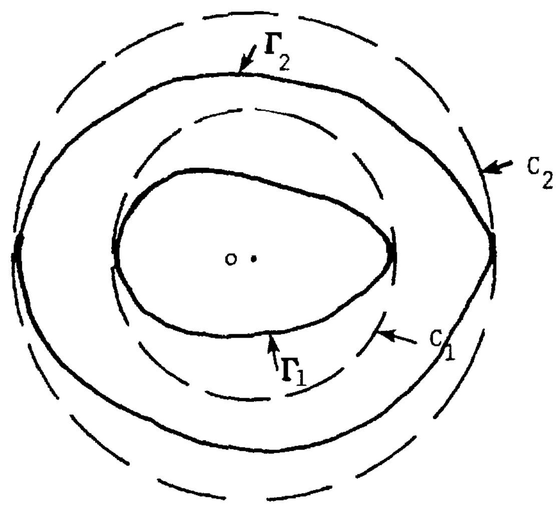
<u>Case II</u>: If the circumscribed circles are nonconcentric (Fig. 2(b)), then we first use the formula for the conformal transformation of non-concentric to concentric circles, Kober [13], and choose the ordinates corresponding to abscissae given by the formula

$$\begin{aligned} x(\xi) = [&(1 - c\gamma \cos \xi) \{x_L(1 - c\gamma \cos \xi) + c\gamma y_L \sin \xi \\ &+ r_L(c \cos \psi - \gamma \cos(\xi-\psi))\} \\ &- c\gamma \sin \xi\{y_L(1 - c\gamma \cos \xi) - c\gamma x_L \sin \xi \\ &- r_L(c \sin \psi + \gamma \sin(\xi-\psi))\}] \\ &/(1 - 2c\gamma \cos \xi + c^2\gamma^2) \end{aligned} \tag{36}$$

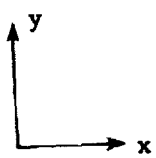
where

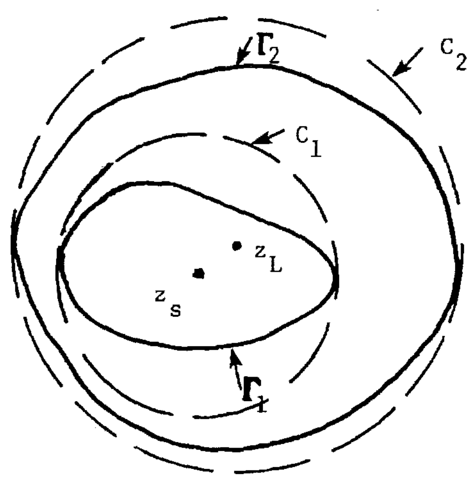
r_L, r_s = radii of outer and inner circumscribed circles

(x_L, y_L) and (x_s, y_s) = coordinates of the centers



(a) Concentric circumscribed circles C_1 and C_2 of radii r_s and r_L, respectively, with center at the origin.





(b) Nonconcentric circumscribed circles C_1 and C_2 of radii r_s and r_L and centers at z_s and z_L, respectively.

Figure 2.- Circumscribed circles.

$$d^2 = (x_s - x_L)^2 + (y_s - y_L)^2$$

$$\psi = \pi - \tan^{-1}\left(\frac{y_s - y_L}{x_s - x_L}\right)$$

$$c = [(d^2 + r_L^2 - r_s^2) + \{(d^2 + r_L^2 - r_s^2)^2 - 4d^2 r_L^2\}^{1/2}]/2r_L d$$

$$\gamma = 1 \text{ for the outer boundary}$$

$$\gamma = \frac{r_L}{r_s}\left|\frac{d-t}{t}\right| \text{ for the inner boundary}$$

$$t = cr_L$$

Having determined the appropriate sets $(x_\beta(\xi), y_\beta(\xi))$ and $(x_\infty(\xi), y_\infty(\xi))$, we use (34) to obtain the values of a and A. The parametric difference η_∞ is connected in some manner with the "modulus" of the domain which, however, by itself is a separate problem (see Burbea [14] and Gaier [15]). In this work we have defined η_∞ based on the knowledge of a and A as discussed above by the formula

$$\eta_\infty = \ln\left(\frac{A}{a}\right) \tag{37}$$

For Figures 3 to 8, we have used the following functional forms of ϕ and f:

$$\phi(\bar{\xi}) = \frac{2\pi(\bar{\xi} - \bar{\xi}_o)}{\bar{\xi}_m - \bar{\xi}_o}$$

$$f(\bar{\eta}) = \frac{\eta_\infty(\bar{\eta} - \bar{\eta}_\beta)}{\bar{\eta}_\infty - \bar{\eta}_\beta} \quad \frac{K^{(\bar{\eta} - \bar{\eta}_\beta)}}{K^{(\bar{\eta}_\infty - \bar{\eta}_\beta)}}$$

so that

$$\lambda = \frac{2\pi}{\overline{\xi}_m - \overline{\xi}_o}$$

$$\theta = \frac{\eta_\infty}{\overline{\eta}_\infty - \overline{\eta}_\beta} [1 + (\overline{\eta} - \overline{\eta}_\beta) \ln K] \frac{K^{(\overline{\eta} - \overline{\eta}_\beta)}}{K^{(\overline{\eta}_\infty - \overline{\eta}_\beta)}}$$

where $K > 1$ is an arbitrary constant, and $\overline{\xi} = \overline{\xi}_m$, $\overline{\eta} = \overline{\eta}_\infty$ correspond, respectively, to $\xi = 2\pi$ and $\eta = \eta_\infty$. We treat $\overline{\xi}$ and $\overline{\eta}$ as integers so that $\overline{\xi}_o = 1$, $\overline{\xi}_m = \text{IMAX}$, $\overline{\eta}_\beta = 1$, and $\overline{\eta}_\infty = \text{JMAX}$. Since η_∞ is known from (37), hence by specifying the numerical values to K and JMAX we can create the desired mesh control in the direction of η. The value of K between 1.05 and 1.1 is quite sufficient [16] to have a fine grid near the inner boundary.

The number of terms to be retained in the series (24) is usually small for convex inner and outer boundaries, though we have retained (IMAX-1)/2 number of coefficients in each computation. This number is the optimum number of terms in a discrete Fourier series [17] having IMAX number of points in one period.

Figure 3 shows the classic case of confocal ellipses with coordinate contraction in η. The value of K is 1.05. The orthogonal correspondence between ξ-points of the inner and outer boundary has been established by using Case I, Eq. (35).

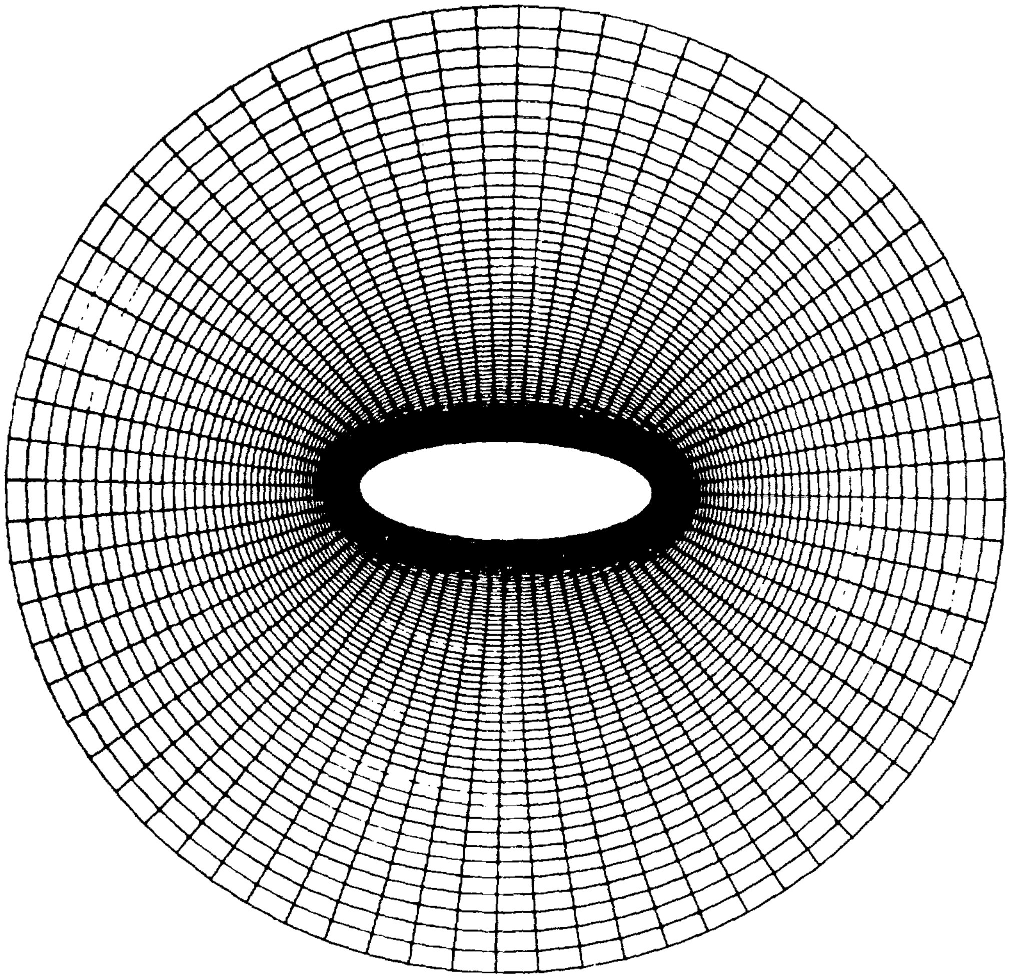


Figure 3.- Confocal ellipses. Semimajor axes 1.48, 5.0, and semiminor axes 0.5, 4.802, respectively. Only 38 η = Const. lines shown for detail.

Figure 4 presents orthogonal coordinates for a blunt body with elliptical outer boundary. Here K = 1.01. For orthogonal correspondence between ξ-points, Eq. (35) has been used.

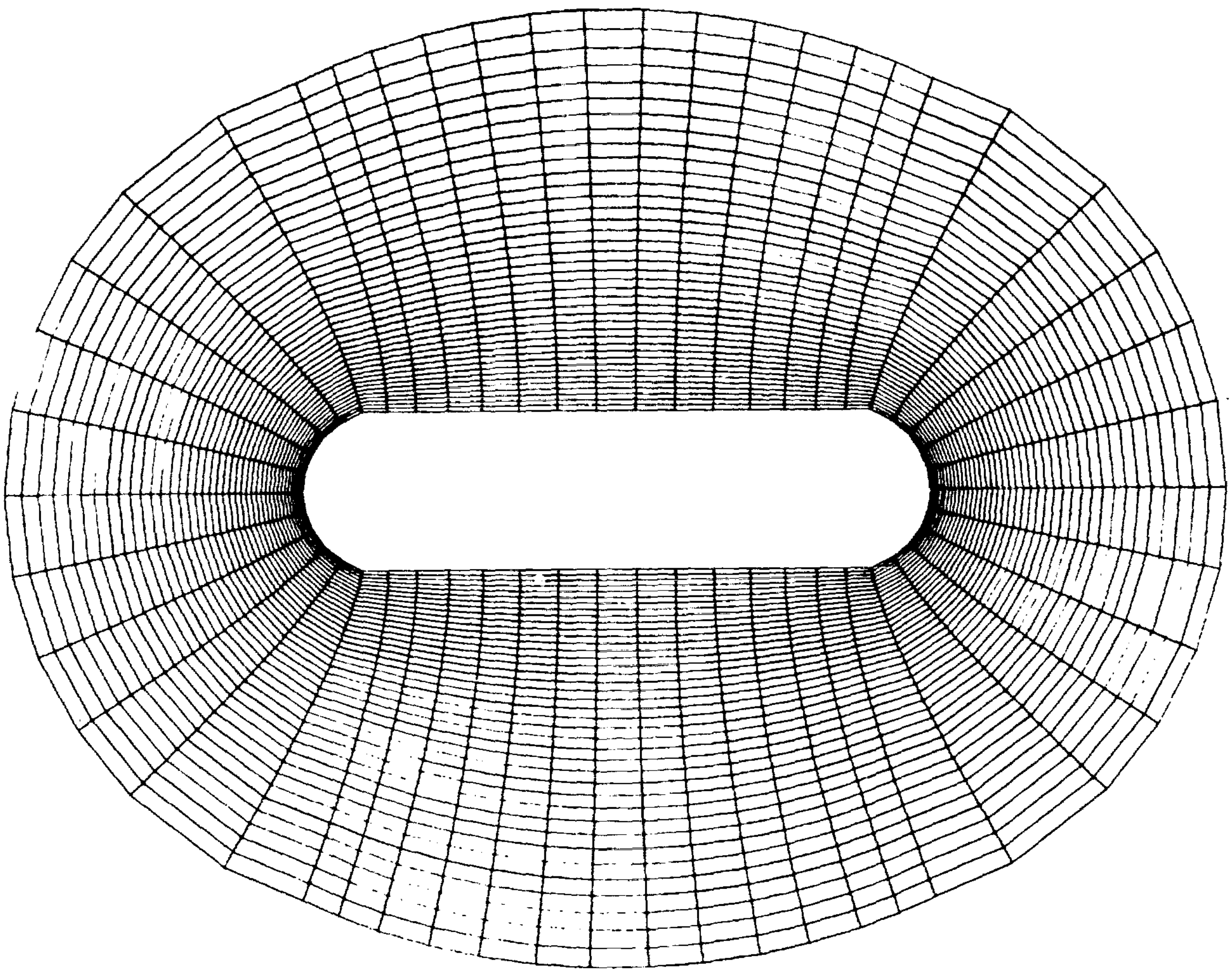


Figure 4.- A blunt body section with elliptical outer boundary.

Orthogonal coordinates for nonconcentric circles are presented in Figure 5. Here K = 1.01. For orthogonal correspondence between ξ-points between the inner and outer boundary, Eq. (36) has been used. Data shown on the figure.

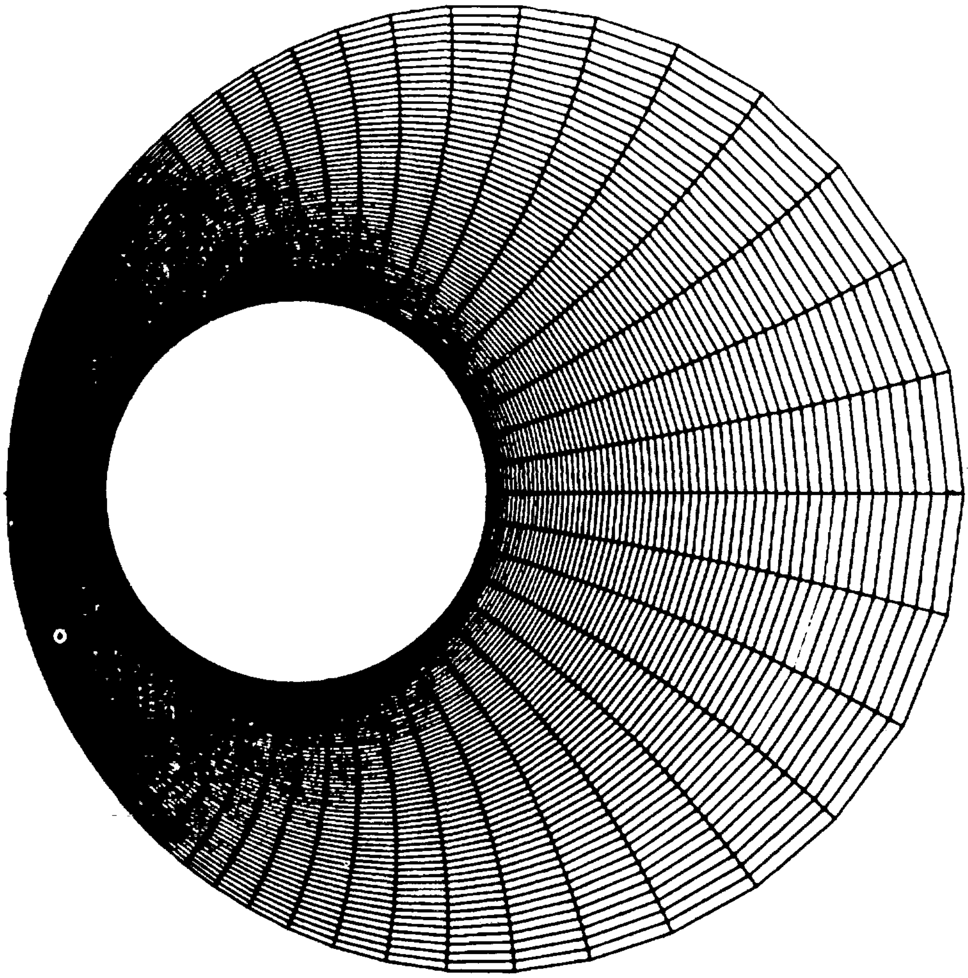


Figure 5.- Nonconcentric circles: $r_S = 1$, $r_L = 2.5$, $z_S = (0,0)$, $z_L = (1,0)$.

Orthogonal coordinates for a Joukowsky's airfoil with slightly rounded trailing edge are shown in Figure 6. Eq. (35) is used for orthogonal correspondence. Here K = 1.02.

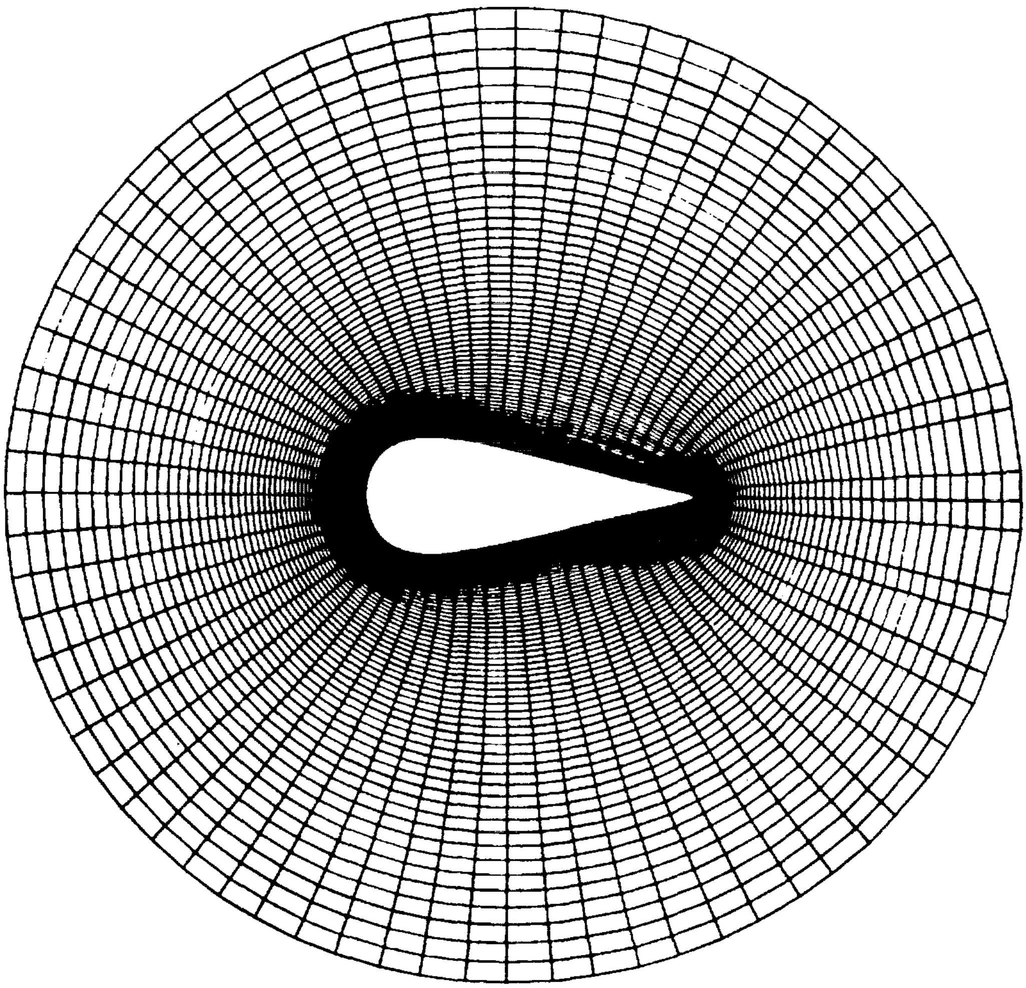


Figure 6.- Joukowsky's airfoil with slightly rounded trailing edge.

Figure 7 presents orthogonal coordinates for nonconcentric ellipses. Centers of the inner and outer ellipses are at (0,0) and (1,0), respectively. Here K = 1.01. For orthogonal correspondence Eq. (36) has been used.

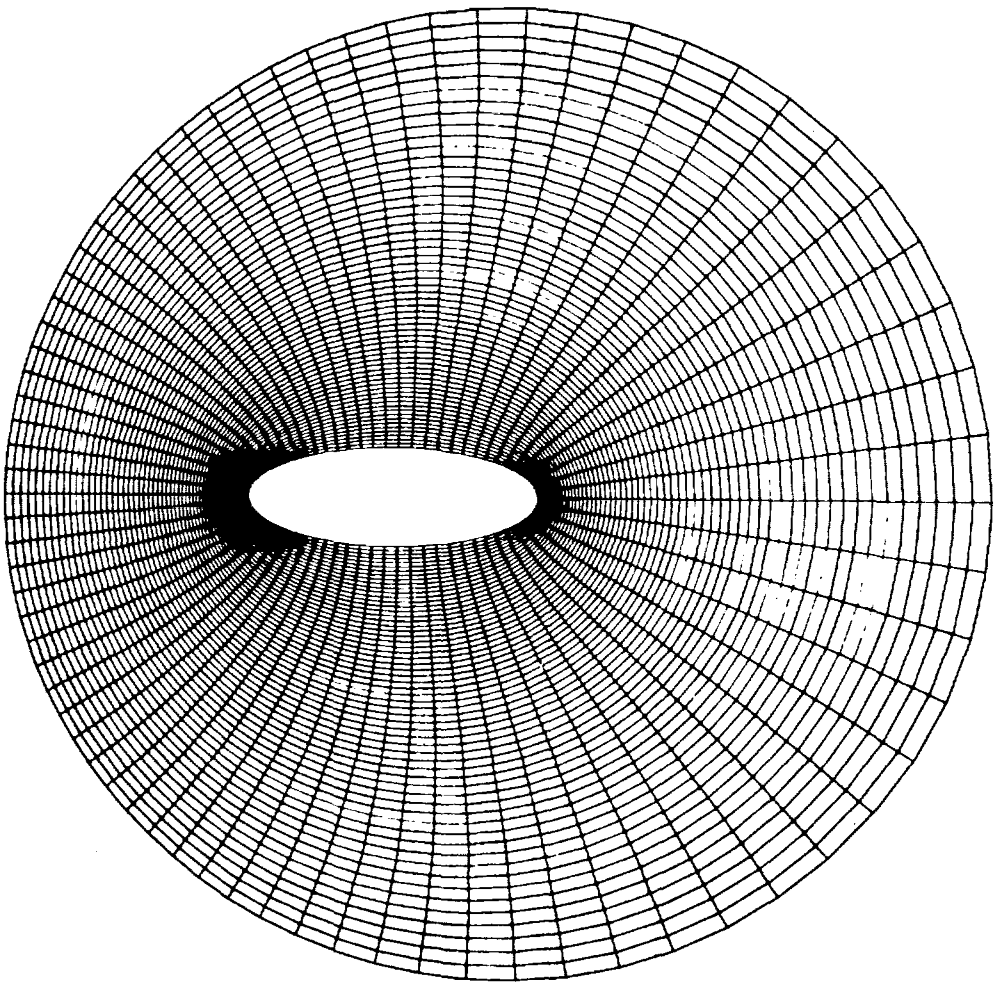


Figure 7.- Nonconcentric ellipses. Size data same as in Figure 3. $z_S = (0,0)$, $z_L = (1,0)$.

Orthogonal coordinates for an arbitrarily deformed upper part of Figure 4 are shown in Figure 8. The placement of outer boundary is limited to avoid intersecting normals (Eiseman [18]). This figure shows that we need some attraction near those sections of the outer boundary which face the concave side.

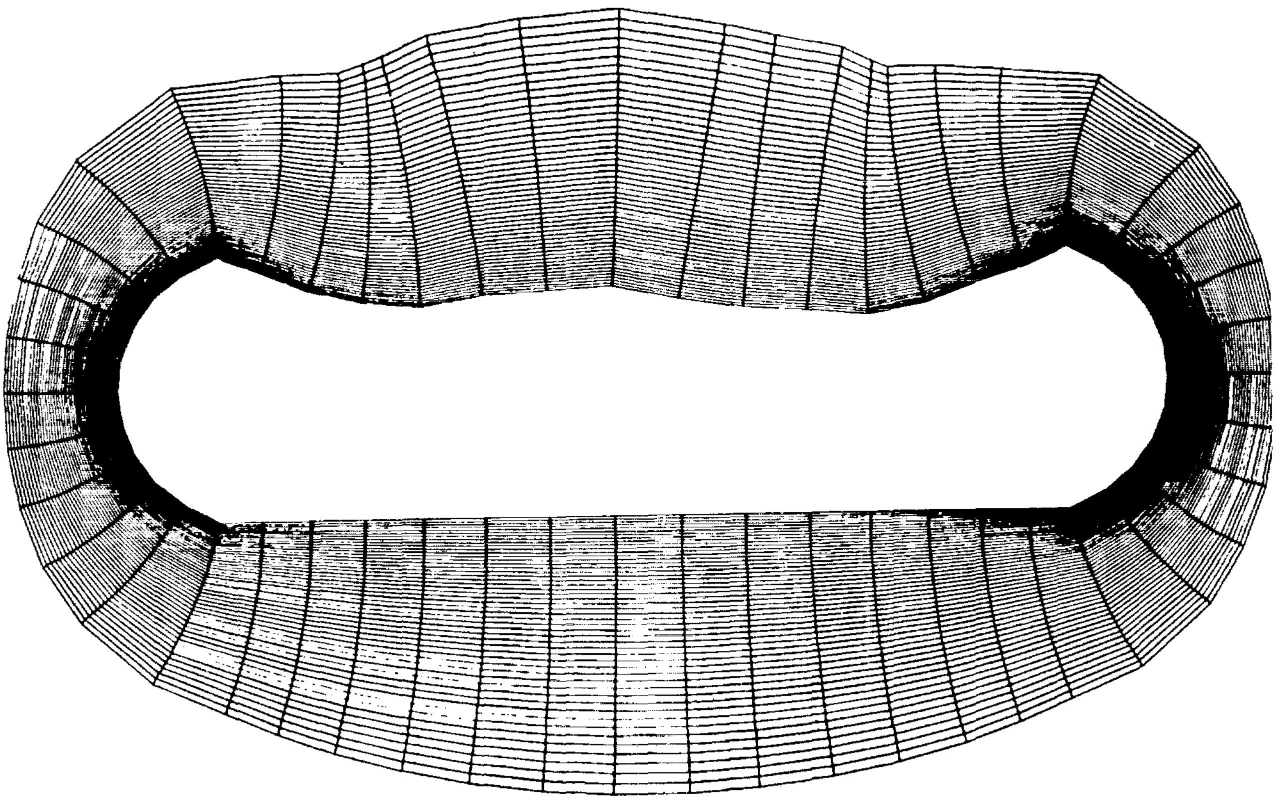


Figure 8.- Generated coordinates for body having convex, concave and straight portions. Placement of outer boundary is decided by the radius of the osculating circles of the concave portions.

Summary of Numerical Experimentation

In the course of this investigation a number of cases of inner and outer boundary shapes and orientations have been tested through the developed computer program. The main conclusions are listed below:

(i) The method works very effectively for smooth and convex boundaries of any shape and orientation.

(ii) For concave boundaries a method similar to that of Eiseman has to be used in the placement of the outer boundary to avoid intersecting normals. Another remedy would be to introduce some type of attraction near the outer boundary facing the concave side of the inner boundary.

(iii) Sharp turns and corners are not admissible and have to be rounded to avoid singularities in the metric data.

References

1. Z. U. A. Warsi, R. A. Weed, and J. F. Thompson, "Numerical Generation of Two-Dimensional Orthogonal Coordinate in an Euclidean Space", Engineering and Experimental Research Station, Mississippi State University, Rep. No. MSSU-EIRS-ASE-80-3, June 1980.

2. A. J. Winslow, "Numerical Solution of the Quasi-Linear Equation in a Non-Uniform Triangular Mesh", Journal of Computational Physics, 2, 149 (1966).

3. W. D. Barfield, "An Optimal Mesh Generator for Lagrangian Hydrodynamic Calculations in Two Space Dimensions", Journal of Computational Physics, 6, 417 (1970).

4. W. H. Chu, "Development of a General Finite Difference Approximation for a General Domain, Part I: Machine Transformation", Journal of Computational Physics, 8, 392 (1971).

5. A. A. Amsden and C. W. Hirt, "A Simple Scheme for Generating General Curvilinear Grids", Journal of Computational Physics, 11, 348 (1973).

6. D. E. Potter and G. H. Tuttle, "The Construction of Discrete Orthogonal Coordinates", Journal of Computational Physics, 13, 483 (1973).

7. J. F. Thompson, F. C. Thames, and C. W. Mastin, "Automatic Numerical Generation of Body-Fitted Curvilinear Coordinate System for Field Containing any Number of Arbitrary Two-Dimensional Bodies", Journal of Computational Physics, 15, 299 (1974).

8. S. B. Pope, "The Calculation of Turbulent Recirculating Flows in General Orthogonal Coordinates", Journal of Computational Physics, 26, 197 (1978).

9. J. F. Middlecoff and P. D. Thomas, "Direct Control of the Grid Point Distribution in Meshes Generated by Elliptic Equations", AIAA Computational Fluid Dynamics Conference, Paper No. 79-1462 (1979).

10. R. L. Sorenson and J. L. Steger, "Simplified Clustering of Nonorthogonal Grids Generated by Elliptic Partial Differential Equations", NASA-TM-73252, August 1977.

11. G. Starius, "Constructing Orthogonal Curvilinear Meshes by Solving Initial-Value Problem", Numerische Mathematik, 28, 25 (1977).

12. A. Cohen, "An Introduction to the Lie Theory of One-Parameter Groups", G. E. Stechert and Co., New York (1931), p. 2-5.

13. H. Kober, "Dictionary of Conformal Representations", Dover Publications, Inc. (1952).

14. J. Burbea, "A Numerical Determination of the Modulus of Doubly Connected Domains by Using the Bergman Curvature", Mathematics of Computation, 25, 743 (1971).

15. D. Gaier, "Determination of Conformal Modulus of Ring Domains and Quadrilaterals", In-Lecture Notes in Mathematics, No. 399, Springer-Verlag, Berlin (1974).

16. Z. U. A. Warsi and J. F. Thompson, "Machine Solutions of Partial Differential Equations in the Numerically Generated Coordinate Systems", Engineering and Experimental Research Stateion, Mississippi State University, Rep. No. MSSU-EIRS-ASE-77-1, August (1976).

17. F. B. Hildebrand, "Introduction to Numerical Analysis", McGraw-Hill Book Company, Inc., (1956).

18. P. R. Eiseman, "A Coordinate System for a Viscous Transonic Cascade Analysis", Journal of Computational Physics, 26, 307 (1978).