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Strong Turbulence by Renormalization

I. KINETIC THEORY OF TURBULENCE

The hydrodynamical equations of turbulent motions are inhomogeneous and nonlinear in their inertia and force terms, and will generate a hierarchy. Dr. C. M. Tchen has developed a kinetic method to transform the hydrodynamic equations into a master equation governing the velocity distribution $f(t, \underset{\sim}{x}, \underset{\sim}{v})$, as a function of the time t, the position $\underset{\sim}{x}$, and the velocity $\underset{\sim}{v}$ as an independent variable. The master equation presents the advantage of being homogeneous and having fewer nonlinear terms, and is therefore simpler for the investigation of closure.

After the closure by means of a cascade-scaling procedure, the kinetic equation is derived and possesses a memory which represents the non-Markovian character of turbulence. The said kinetic equation is transformed back to the hydrodynamical form to yield an energy balance in the cascade form, i.e. with a mode-transfer that is explicit. Note that the underlying mechanism is a "collisionless damping", similar to the "nonlinear Landau damping" in plasma turbulence.

The theory analyzes the eddy transports along a gradient (i.e. normal transports), counter the gradient or without a gradient (i.e. anomalous transports).

The kinetic theory is described in PAPER #1 for the incompressible turbulence, in PAPER #3 for the compressible turbulence, and in PAPER #2 and #5 for the plasma turbulence.

The normal and anomalous transports are described in PAPER #4.

II. APPLICATIONS

Theory of turbulence developed above by Dr. Tchen served as a statistical foundation to the following problems. The application to sound generation is given in PAPER 6.

The application to the propagation of light in a non-frozen turbulence is given in PAPER #7. In this connection two questions are of particular importance. The first deals with the remote sensing of the cross-wind by using two laser beams. The theory of remote sensing in non-frozen turbule derives a formula that can measure both the wind velocity and the eddy viscosity, while the formula based upon the hypothesis of frozen turbulence (Taylor's hypothesis), as well known in the literature, can only measure the wind velocity, but not the characteristics of turbulence. The second question deals with the frequency spectrum of phase fluctuations. The theory based upon the assumption of frozen turbulence (i.e. with an unperturbed path of light) gives a spectrum of -8/3 power, as well known in the literature. Our new result as obtained by treating the closure without the hypothesis of frozen turbulence (i.e. with a perturbed path of light) yields a spectrum of -2 power law, which is in a better agreement with available experimental data.

Other applications to an atmospheric medium with particulates are also given in PAPER #7.

III. PUBLICATIONS OF RESEARCH RESULTS UNDER THE NASA SPONSORSHIP

(AUTHOR: C. M. TCHEN)

1. "Kinetic approach of incompressible turbulence", C. R. Acad. Sci. Paris, 287B, 175 (1978).

2. "Kinetic model of closure by relaxation in a strong plasma turbulence", C. R. Acad. Sci, Paris, 286 A, 605 (1978).

3. "Kinetic approach to compressible turbulence", Proc. Int'l Symposium of Rarefied Gas Dynamics, Cannes, published by the Commissariat a l'Energie Atomique, Paris 1979. Edited by R. Campargue, pp 105-114.

4. "Normal and anomalous turbulent transports in a stratified medium", C. R. Acad. Sci. Paris, 290 B, 167 (1980).

5. "Turbulence in a low beta plasma", Plasma Phys. 22, 817 (1980).

6. "Spectral structure of noise generated by turbulence. Manuscript 1980.

7. "Propagation of light through random refractive index fluctuations in non-frozen turbulence". Manuscript 1981.

MÉCANIQUE DES FLUIDES. — *Modèle cinétique de turbulence dans un fluide incompressible*. Note (*) de **Chan M. Tchen**, transmise par M. Marcel Barrère.

Nous développons un traitement statistique pour l'étude de la turbulence dans un fluide incompressible obéissant aux équations de Navier-Stokes, en considérant la pression comme un potentiel d'interaction entre les éléments de fluide. Un procédé d'échelles divise une fluctuation en trois rangs représentant les trois processus de transport : évolution macroscopique, propriété de transport et relaxation. Nous obtenons une fermeture par relaxation, et déduisons une équation cinétique pour la fluctuation de rang macroscopique de la fonction de distribution. La solution donne la fonction de transfert et la viscosité tourbillonnaire. Comme application dans la zone d'inertie du spectre d'énergie, nous trouvons la loi de Kolmogoroff et son coefficient numérique.

We develop a statistical approach for investigating incompressible Navier-Stokes turbulence by considering pressure as a potential of interaction between fluid elements. A scaling procedure divides a fluctuation into three ranks, representing three transport processes: macroscopic evolution, transport property and relaxation. We obtain a closure by relaxation, and derive a kinetic equation for the distribution function of macroscopic rank. The solution yields a transfer function and an eddy viscosity. Application to inertia subrange finds the Kolmogoroff law with numerical coefficient.

I. Introduction. — La représentation cinétique de la turbulence dans un fluide incompressible est basée sur l'hypothèse suivant laquelle la pression agit comme un potentiel d'interaction entre deux éléments de fluide, par analogie avec le potentiel entre deux particules dans la théorie cinétique d'un système discret. Cette représentation conduit à une hiérarchie de fonctions de distribution à plusieurs points de l'espace, semblable à la hiérarchie de BBGK (Bogoliubov, Born, Green, Kirkwood) à plusieurs particules [(1), (4)].

Dans la théorie de Bogoliubov, la fermeture de la hiérarchie consiste à dégénérer une distribution d'ordre supérieur en distributions d'ordres inférieurs dont la justification peut être donnée par un développement en puissances d'un petit paramètre. Le succès de la théorie cinétique de la turbulence dans un système discret est, en grande partie, dû à l'existence de ce paramètre qui n'existe pas dans notre système continu. Pour cette raison, nous développons une théorie de fluctuations, par un procédé d'échelle des fluctuations de fonctions de distribution dont nous cherchons la fermeture par un processus de relaxation asymptotique.

Comme la fonction de distribution normalisée à l'unité est équivalente à la densité de probabilité, ce modèle cinétique coïncide avec le modèle probabiliste.

II. Représentation cinétique. — Les équations de Navier-Stokes et de continuité, dans un fluide incompressible avec vitesse $\mathbf{u}$, pression p, densité ρ et viscosité cinématique ν, peuvent être considérées comme des moments de l'équation microscopique :

$$(1)\quad (\partial_t + \mathrm{L})\,\mathrm{N}(t,\,\mathrm{x},\,\mathrm{v}) = 0 \qquad \text{avec} \qquad \mathrm{L} \equiv \mathrm{v}\cdot\nabla + \mathrm{E}\cdot\partial_v - \nu\nabla^2$$

La fonction de distribution $\mathrm{N}(t,\,\mathrm{x},\,\mathrm{v}) \equiv \delta[\mathbf{v} - \mathbf{u}(t,\,\mathrm{x})]$ satisfait à la condition de normalisation

$$(2)\quad \int_{-\infty}^{\infty} d\mathrm{v}\,\mathrm{N}(t,\,\mathrm{x},\,\mathrm{v}) = 1.$$

Comme le champ est défini par

$$(3)\quad \mathrm{E}(t,\,\mathrm{x}) \equiv \rho^{-1}\,\nabla p(t,\,\mathrm{x}) = -g(\mathrm{x}/\mathrm{x}',\,\mathrm{v}')\{\mathrm{N}(t,\,\mathrm{x}',\,\mathrm{v}')\},$$

on peut écrire (1) sous la forme

$$(4)\quad (\partial_t + \mathrm{v}\cdot\nabla - \nu\nabla^2)\,\mathrm{N}(t,\,\mathrm{x},\,\mathrm{v}) = \partial_v g(\mathrm{x}/\mathrm{x}',\,\mathrm{v}')\{\mathrm{N}(t,\,\mathrm{x},\,\mathrm{v})\,\mathrm{N}(t,\,\mathrm{x}',\,\mathrm{v}')\},$$

où

$$g(\mathrm{x}/\mathrm{x}', \mathrm{v}')\{\ldots\} \equiv \frac{1}{4\pi}\iint_{-\infty}^{\infty} d\,\mathrm{x}'\,d\mathrm{v}'\,\nabla|\mathrm{x}-\mathrm{x}'|^{-1}\,\nabla'\nabla' : \mathrm{v}'\mathrm{v}'\{\ldots\}$$

est un opérateur intégral, et $\partial_t \equiv \partial/\partial_t$, $\nabla \equiv \partial/\partial \mathrm{x}$, $\nabla' \equiv \partial/\partial \mathrm{x}'$, $\partial_v \equiv \partial/\partial \mathrm{v}$.

Dans un mouvement turbulent, la distribution $\mathrm{N} = \bar{f} + \tilde{f}$ se décompose en une distribution moyenne $\bar{f}$ et une fluctuation $\tilde{f}$, où $\bar{f}$ peut s'appeler densité de probabilité, au titre de la normalisation à l'unité d'après (2).

En prenant la moyenne de (4), au moyen de l'opérateur $\bar{\mathrm{A}}$, et en déduisant la fluctuation au moyen de l'opérateur $\tilde{\mathrm{A}} \equiv 1 - \bar{\mathrm{A}}$, nous obtenons :

(5 *a*) $(\partial_t + \bar{\mathrm{A}}\mathrm{L})\bar{f} = -\bar{\mathrm{A}}\,\partial_v \cdot (\tilde{\mathrm{E}}\tilde{f})$,

(5 *b*) $(\partial_t + \tilde{\mathrm{A}}\mathrm{L})\tilde{f} = -\tilde{\mathrm{E}} \cdot \partial_v \bar{f}$.

On dispose alors de deux méthodes : la première méthode consiste à traiter la distribution moyenne en écrivant (5 *a*) sous la forme

(6) $(\partial_t + \bar{\mathrm{A}}\mathrm{L})\bar{f} = \partial_v g(\mathrm{x}/\mathrm{x}', \mathrm{v}')\{\langle \tilde{f}(t, \mathrm{x}, \mathrm{v})\tilde{f}(t, \mathrm{x}', \mathrm{v}')\rangle\}$,

qui engendre une hiérarchie. Cette forme a été aussi déduite par Monin ([1]) et Lundgren ([2]) par d'autres moyens; la seconde méthode que nous envisageons ici consiste à traiter les fluctuations par un procédé d'échelles, en écrivant :

(7) $\tilde{f} = f^{(0)} + f'$, $\quad f' = f^{(1)} + f''$.

Les trois rangs $f^{(0)}$, f' et f'' représentent les trois processus de transport en plusieurs cascades : évolution macroscopique, propriété de transport et relaxation, ayant leurs durées de corrélation

(8) $\tau_c^{(0)} > \tau_c' > \tau_c''$,

d'après la discrimination par les opérateurs $\mathrm{A}^{(0)}$, A', A''. Les échelles instantanées des rangs peuvent se dépasser, mais leurs échelles statistiques (8) sont distinctes, de telle façon que les intensités

$$\langle \mathrm{E}^{(0)2}\rangle = 2\int_0^k dk'\,\mathrm{S}(k'), \qquad \langle \mathrm{u}^{(0)2}\rangle = 2\int_0^k dk'\,\mathrm{F}(k')$$

sont réparties dans un domaine $(0, k)$ des spectres $\mathrm{S}(k')$ et $\mathrm{F}(k')$.

Par l'intermédiaire des opérateurs $\mathrm{A}^{(0)}$, A', A'' appliqués à (5 *b*), nous obtenons les équations des rangs, et en particulier, par l'élimination du rang microscopique caractérisé par A', nous obtenons l'équation cinétique suivante :

(9 *a*) $(\partial_t + \mathrm{A}^{(0)}\mathrm{L})f^{(0)} = -\mathrm{E}^{(0)} \cdot \partial_v \bar{f} + \mathrm{A}^{(0)}\,\partial_v \cdot \mathscr{D}' \cdot \{\partial_v f^{(0)}\}$,

avec

(9 *b*) $\mathscr{D}' = \int_0^{t\to\infty} d\tau \langle \mathrm{E}'(t, \mathrm{x})\,\mathrm{U}(t, t-\tau)\,\mathrm{E}'(t-\tau)\rangle$.

Cette équation cinétique gouverne la fluctuation $f^{(0)}$, et diffère de l'équation cinétique suivante, déduite de la même manière

$$(10)\quad (\partial_t+\overline{\mathrm{A}}\mathrm{L})\bar{f}=\partial_v\cdot\tilde{\mathscr{D}}\cdot\{\partial_v\bar{f}\}\qquad\text{avec}\quad \tilde{\mathscr{D}}'=\int_0^{t\to\infty}d\tau\langle\tilde{\mathrm{E}}(t,\mathbf{x})\,\mathrm{U}(t,t-\tau)\,\tilde{\mathrm{E}}(t-\tau)\rangle.$$

Cette équation peut être considérée comme une forme fermée de la hiérarchie (6). Une équation cinétique pour la distribution $\bar{f}_2(t,\mathbf{x},\mathbf{v},\mathbf{x}',\mathbf{v}')$ à deux points peut aussi se déduire d'une manière analogue. Elle pourrait servir à développer une fonction de corrélation et par conséquent, une fonction spectrale. L'équation cinétique de $f^{(0)}$ en (9 *a*) a l'avantage de se dispenser de f_2 et de permettre, directement, la détermination de la fonction spectrale.

III. Turbulence isotrope et homogène. — Une corrélation lagrangienne $\langle\mathrm{E}'(t,\mathbf{x})\,\mathrm{U}(t,t-\tau)\,\mathrm{E}'(t-\tau)\rangle$ apparaît en (9 *b*). Son calcul suit une trajectoire qui est perturbée par la turbulence de $\mathrm{E}(t,\mathbf{x})$. Dans $\mathbf{l}=\bar{\mathbf{l}}+\mathbf{l}^{(0)}+\mathbf{l}^{(1)}+\mathbf{l}''$, nous distinguons le parcours au rang submicroscopique $\mathbf{l}''(\tau)$ qui traduit une diffusion des trajectoires, fournit une relaxation ainsi que les autres parcours qui correspondent à la convection. Par l'isolement des rangs E' et $\mathbf{l}''(\tau)$, l'expression (9 *b*) peut être découplée et s'écrire sous la forme scalaire qui est la trace du tenseur isotrope

$$(11)\quad \mathrm{D}'=2\int_0^{t\to\infty}d\tau\iint_{-\infty}^{\infty}d\omega'\,d\mathbf{k}'\,\mathrm{S}(\omega',\mathbf{k}')$$
$$\times\langle\exp-i\,\mathbf{k}\cdot\mathbf{l}''(\tau)\rangle\langle\exp\,[-\nu k'^2\tau+i(\omega'-\mathbf{k}'\cdot\mathbf{v})\tau-i\,\mathbf{k}'\cdot(\bar{\mathbf{l}}+\mathbf{l}^{(0)}+\mathbf{l}^{(1)})]\rangle,$$

la moyenne se calcule par une probabilité de transition obéissant à une équation de Fokker-Planck, et $\mathrm{S}(\omega',\mathbf{k}')$ est la fonction spectrale dans l'espace de $(\omega',\mathbf{k}')$. Une turbulence faible permet de négliger la relaxation, tandis qu'une turbulence forte permet de négliger la convection. Dans le dernier cas, nous réduisons (11) à

$$(12)\quad \mathrm{D}'=\frac{2}{3}\Gamma\left(\frac{4}{3}\right)\int_k^{\infty}dk'\,\mathrm{S}(k')/\omega'_{\mathrm{D}}(k'),\qquad \omega'_{\mathrm{D}}(k')=\left[\frac{1}{3}k'^2\,\mathrm{D}'(k')\right]^{1/3}.$$

La résolution de l'équation cinétique (9 *a*) et la méthode des moments donnent alors le bilan d'énergie

$$(13)\quad \frac{1}{2}\partial_t\langle\mathbf{u}^{(0)2}\rangle=-\mathrm{T}^{(0)}\equiv-\mathrm{K}'\,\mathrm{R}^{(0)}.$$

La fonction de transfert $\mathrm{T}^{(0)}$ se compose d'un produit de viscosité tourbillonnaire

$$(14)\quad \mathrm{K}'=\frac{2}{3}\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)\int_k^{\infty}dk'\,\mathrm{S}(k')/\omega'^3_{\mathrm{D}}$$

et d'une fonction de vorticité

$$(15)\quad \mathrm{R}^{(0)}=2\int_0^k dk'\,k'^2\,\mathrm{F}(k').$$

Si nous remplaçons (13) par le bilan d'énergie qui égale $\mathrm{T}^{(0)}$ au taux de dissipation ε dans la zone d'inertie, nous trouvons la fonction spectrale

$$(16)\quad \mathrm{F}=\mathrm{A}\,\varepsilon^{2/3}\,k^{-5/3},$$

conforme à la loi de Kolmogoroff ([5]). Le coefficient numérique A peut être calculé par une équation qui relie **E** à **u** en partant de (3) et également $S(k)$ à $F(k)$. Le calcul donne ainsi la valeur du coefficient $A = 1,6$, pour $k/k_v = 100$, où k_v^{-1} est l'échelle interne de Kolmogoroff gouvernant le diamètre des plus petits tourbillons avant leur dissipation par viscosité.

Ce travail est fait sous l'auspice de National Aeronautics and Space Administration.

(*) Séance du 11 septembre 1978.

([1]) A. S. Monin, *Appl. Math. Mech.*, 31, 1967, p. 1057.
([2]) T. S. Lundgren, *Phys. Fluids*, 10, 1967, p. 969.
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([4]) V. M. Ievlev, *Dokl. Akad. Nauk S.S.S.R.*, 208, 1973, p. 1044; *Sov. Phys. Dokl.*, 18, 1973, p. 117.
([5]) A. N. Kolmogoroff, *C. R. Acad. Sc. U.R.S.S.*, 30, 1941, p. 301.

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PHYSIQUE MATHÉMATIQUE. — *Fermeture par relaxation d'un modèle cinétique de plasma en turbulence forte.* Note (*) de **Chan Mou Tchen**, présentée par M. André Lichnerowicz.

La décomposition en cascade répétée permet de dériver une équation cinétique de Fokker-Planck pour un plasma turbulent. Le calcul du propagateur et la solution de l'équation cinétique déterminent les coefficients de transports (diffusivité et viscosité tourbillonnaires), moyennant une fermeture basée sur un processus de relaxation responsable de l'approche à l'équilibre. Ensuite la loi spectrale k^{-3} est prévue, et gouverne le couplage des fluctuations de vitesse et de champ électrostatique.

A scaling procedure by repeated-cascade derives a kinetic equation in the form of Fokker-Planck equation for turbulent plasmas. The calculation of the propagator and the solution of the kinetic equation determine transport coefficients (eddy diffusivity and eddy viscosity), with a closure based upon a relaxation process that is responsible for the approach to equilibrium. A spectral law k^{-3} is found to govern the coupling between velocity and electrostatic field fluctuations, and is in agreement with experiments and numerical modeling. The same kinetic method may serve in determining the k^{-3} spectrum for velocity fluctuation in compressible turbulence and the Kolmogoroff law $k^{-5/3}$ in incompressible turbulence.

I. Cascade répétée. — La fonction de distribution $f(t, \mathbf{x}, \mathbf{v})$ de vitesse $\mathbf{v}$ dans un plasma sans collision est gouvernée par l'équation de Vlasov :

$$(1) \qquad (\partial_t + \mathbf{v}\cdot\nabla + \mathbf{E}\cdot\partial) f = 0, \qquad \nabla \equiv \partial/\partial\mathbf{x}, \qquad \partial \equiv \partial/\partial\mathbf{v}.$$

Le champ self-consistant E qui sera précisé plus tard, est supposé fluctuer sans valeur moyenne.

On décompose la fonction de distribution en

$$(2) \qquad f(t, \mathbf{x}, \mathbf{v}) = \bar{f}(t, \mathbf{v}) + \tilde{f}(t, \mathbf{x}, \mathbf{v}),$$

où $\bar{f}$ est la moyenne et $\tilde{f}$ est une fluctuation, et le champ fluctuant en un macro-champ $E^{(0)}$, un micro-champ E', et un champ submicroscopique E'', comme suit :

$$(3) \qquad E = E^{(0)} + E', \qquad E' = E^{(1)} + E'', \qquad (E^{(0)} + E^{(1)} = E_1).$$

La discrimination de ces composantes ou rangs se fait par les moyennes d'ensemble, représentées par les opérateurs filtres $A^{(0)}$, A', A'', qui s'applique aussi bien à $\tilde{f}(t, \mathbf{x}, \mathbf{v})$.

L'application des opérateurs filtres $A^{(0)}$, A' à (1) conduit au système suivant :

$$(4) \qquad (\partial_t + A^{(0)} L) f^{(0)} = -E^{(0)}\cdot\partial\bar{f} - \partial\cdot\langle E' f'\rangle, \qquad (\partial_t + A' L) f' = -E'\cdot\partial(\bar{f} + f^{(0)}),$$

où $\tilde{f} = f^{(0)} + f'$, et l'élimination de f' nous donne l'équation cinétique sous la forme de Fokker-Planck :

$$(5) \qquad (\partial_t + A^{(0)} L) f^{(0)} = -E^{(0)}\cdot\partial\bar{f} + A^{(0)}\partial\cdot D'\cdot\partial f^{(0)}, \quad L \equiv \mathbf{v}\cdot\nabla + E\cdot\partial,$$

avec une diffusivité

$$(6) \qquad D' = \int_0^t d\tau\, \langle E'(t, \mathbf{x})\, U(t, t-\tau)\, E'(t-\tau)\rangle, \qquad U = \text{propagateur}.$$

La décomposition en cascade répétée permet de définir une fonction spectrale $S(k)$ d'énergie électrostatique, telle que

$$(7) \qquad \langle E^{(0)2}\rangle = 2\int_0^k dk'\, S(k'),$$

appartient aux nombres d'onde plus petits que k, tandis que la diffusivité $\mathbf{D}'$ est effectuée par le micro-champ E' aux nombres d'onde plus grands que k. Finalement une relaxation contrôle le temps nécessaire aux propriétés de transport pour s'approcher à l'équilibre; elle régit les nombres d'onde plus grands que k' et par conséquence sera gouvernée par un champ submicroscopique E''. Les nombres d'onde k et k' sont pris comme variables indépendantes. Les rangs $\mathrm{E}^{(0)}$, E', E'' sont autorisés à outrepasser leurs échelles instantanées, mais leurs durées de corrélation $\tau_c^{(0)} > \tau_c' > \tau_c''$ doivent être nettement séparées comme échelles statistiques. Une fonction spectrale $\mathrm{F}(k)$ pour la fluctuation de vitesse $\mathbf{u}$ peut être définie d'une manière analogue. Le coefficient χ en (7) provient de la transformation de Fourier bornée.

Il faut remarquer que les décompositions (2) et (3) sont usuelles en turbulence hydrodynamique : (2) s'appelle la décomposition de Reynolds, et (3) s'appelle la décomposition en cascade répétée qui est une généralisation de la cascade de Heisenberg ([1]). Les trois rangs caractérisent les trois processus dans l'ordre d'échelles décroissantes ou dans l'ordre croissant au titre aléatoire : l'évolution macroscopique, le développement des propriétés de transport, et la relaxation, forment un couplage qui rejoint la hiérarchie cinétique de B.B.G.K. ([2]).

II. Fermeture. — D'après la théorie de diffusion, la diffusivité devient asymptotique, i.e. $\mathbf{D}'(t > \tau_c') = \mathbf{D}'(t \to \infty)$, quand $t > \tau_c'$, ce qui est vrai puisque t est l'échelle d'évolution de $f^{(0)}$ en (5). Par le même raisonnement une diffusivité qui possède un temps disponible τ plus petit doit choisir le rang $\mathbf{D}''(\tau)$ pour devenir asymptotique, i.e.

$$\mathbf{D}''(\tau > \tau_c'') = \mathbf{D}''(\tau \to \infty),$$

correspondant au rang $\mathbf{l}''(\tau)$ du parcours $\mathbf{l}(\tau) = \mathbf{l}_1(\tau) + \mathbf{l}''(\tau)$.

Le calcul du propagateur U transforme la diffusivité (7) sous la forme de Fourier :

$$\mathbf{D}' = \int_0^{t \to \infty} d\tau \iint_{-\infty}^{\infty} d\omega' \, d\mathbf{k}' \chi \langle \mathbf{E}'(\omega', \mathbf{k}') \, \mathbf{E}'(-\omega', -\mathbf{k}') \rangle \times \langle \exp - i \, \mathbf{k}' \cdot \bar{\mathbf{l}}''(\tau) \rangle \exp i [\omega' \tau - \mathbf{k}' \cdot (\mathbf{v}\tau + \mathbf{l}_1)], \tag{8}$$

qui contient une convection libre $\exp i[\omega'\tau - \mathbf{k}' \cdot (\mathbf{v}\tau + \mathbf{l}_1)]$ et une relaxation $\langle \exp - i \, \mathbf{k}' \cdot \mathbf{l}''(\tau) \rangle$ qui peut se découpler du champ E' à cause de leur différence de rangs. L'effet de convection libre se manifeste en turbulence faible, comme il est d'usage dans la théorie quasilinéaire, et l'effet de relaxation domine en turbulence forte, ce que nous traitons ici. La moyenne de la fonction exponentielle est calculée par une probabilité de transition gouvernée par une autre équation de Fokker-Planck semblable à (5). Le calcul transforme (8) en une équation intégrale dont la solution donne la trace du tenseur isotrope suivante :

$$\mathrm{D}' = \left[\frac{8}{9} \Gamma\left(\frac{4}{3}\right) \int_k^{\infty} dk' \, \mathrm{S}(k') \Big/ \left(\frac{1}{3} k'^2\right)^{1/3} \right]^{3/4}. \tag{9}$$

III. Structure spectrale. — Considérons un plasma quasi-neutre, composé d'ions de masse M, au sein d'électrons chauds de température T_- ayant une densité n qui est reliée au champ $\mathbf{E} = -\nabla\psi$, suivant la loi d'équilibre $n = n_0 \exp(\psi/c_0^2)$, où $c_0 = (\varkappa \mathrm{T}_-/\mathrm{M})^{1/2}$ est la vitesse thermique.

On peut définir une fonction d'enstrophie pour le potentiel par l'expression $\mathrm{J}^{(0)} = \mathrm{M} \, n_0 \, c_0^{-2} \langle (\nabla \psi^{(0)})^2 \rangle$, qui se réduit à J quand k s'accroît indéfiniment, et une fonction d'enstrophie analogue $R^{(0)}$ pour la vitesse. Les équations d'évolution d'énergie cinétique et potentielle peuvent se déduire comme moments de l'équation (5) de Fokker-Planck gouvernant la distribution $f^{(0)}$, d'où l'on établit les équations du bilan d'énergies. En omettant les détails, on obtient dans le cas de stationarité statistique et dans la zone d'échange :

$$(10) \qquad C^{(0)} - K' R^{(0)} = \mathrm{Cte}, \qquad C^{(0)} - \mathrm{Pr}\, K' J^{(0)} = \mathrm{Cte},$$

où la viscosité tourbillonnaire est :

$$(11) \qquad K' = \frac{2}{3}\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)\int_k^\infty dk'\, S(k') \Big/ \frac{1}{3} k'^2 D'(k'),$$

et la fonction d'échange est : $C^{(0)} = n_0 M D^{(0)}$.

L'énergie cinétique est excitée par les fluctuations électrostatiques occasionnant un échange $C^{(0)}$ entre le spectre d'énergie cinétique et le spectre d'énergie potentielle. Les deux énergies poursuivent leurs dissipations tourbillonnaires par leurs transferts $K' R^{(0)}$ et $\mathrm{Pr}\, K' J^{(0)}$ dus au terme d'interaction non linéaire des ondes à travers leurs spectres individuels. Le transfert dans le spectre d'énergie potentielle s'effectue avec une propriété de transport modifiée par un nombre de Prandtl turbulent Pr.

Il apparaît donc que le spectre d'énergie cinétique se trouve dans le régime de développement par le gain d'échange où $R^{(0)} \cong 0$, tandis que le spectre d'énergie potentielle tend vers son régime terminal par les deux pertes qu'elle subit, où $J^{(0)} \cong J$ à grand k. Le flux d'énergie, décrit par la forme différentielle de (10), peut s'écrire, par approximation :

$$(12) \qquad \dot{C}^{(0)} - K' \dot{R}^{(0)} = 0, \qquad -\dot{C}^{(0)} - \mathrm{Pr}\, \dot{K}' J = 0,$$

où $(\dot{}) = d/dk$. L'enstrophie J définit un puits dans le processus d'échange et de dissipations tourbillonnaires des deux spectres, et remplace le taux de dissipation dans la théorie de turbulence hydrodynamique [3]. Des équations (12), on tire les solutions

$$(13) \qquad F(k) = \mathrm{Pr}\, J k^{-3}, \qquad \bar{S}(k) = \mathrm{Cte}\,(J/n_0 M)^2 k^{-3}.$$

La loi k^{-3} est bien reconnue dans les mesures [4] et calculs numériques [5]. Dans la comparaison avec les calculs numériques, il faut remarquer que la loi $E_k \sim k^{-1}$ obtenue numériquement comme composante de série de Fourier conforme à la prédiction (13).

Notre méthode cinétique peut servir à déterminer la loi spectrale k^{-3} et la loi de Kolmogoroff $k^{-5/3}$ dans les cas de turbulence compressible et incompressible, respectivement.

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A KINETIC APPROACH TO COMPRESSIBLE TURBULENCE

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We develop a kinetic theory of strong turbulence, by decomposing fluctuation of distribution function into scaled components, or ranks, to represent three transport processes: macroscopic evolution, transport property and relaxation. Closure is obtained by formulating a relaxation which determines the approach of eddy diffusivity to equilibrium. We solve the kinetic equation and derive transport functions of turbulence in adiabatic gas: two transfer functions describe turbulent mode transfer, and a coupling function describes the build-up of kinetic energy at the expense of internal energy. Their balance in the inertial subrange of homogeneous and isotropic turbulence in an adiabatic gas yields a k^{-3} law for kinetic energy and internal energy. Their intensities predict a k^{-2} law, in agreement with numerical computations from one dimensional model of compressible turbulence.

I. INTRODUCTION

By considering pressure as responsible for the interaction between fluid elements, Monin[1], Lundgren[2], Ulinich, Lyubimov[3], and Ievlev[4,5] have described a kinetic treatment of turbulence. The kinetic hierarchy generated is similar to the BBGK hierarchy in plasmas, and has to be closed by degenerating a high order distribution function into lower order ones.

The success in this direction will depend on finding a small parameter, as in the case of plasmas. However, this parameter is not present in neutral turbulence. In order to avoid this difficulty, we develop a fluctuation theory, by scaling a fluctuation into components of various degree of random character, or ranks, to represent macroscopic evolution of distribution fluctuation, eddy diffusivity from microscopic fluctuation, and eddy relaxation from submicroscopic fluctuation. The relaxation takes the form of two streamings (free and dynamical) and submicroscopic diffusion which is made asymptotic for the approach of eddy diffusivity to equilibrium. Such an asymptotic diffusion renders the relaxation frequency deterministic and degenerates a high order correlation into lower order ones, thus achieving a closure.

This closure produces a kinetic equation with an explicit diffusivity. By solving the kinetic equation, we can calculate the transport functions consisting of a coupling function and two transfer functions, governing the energy balance for homogeneous and isotropic turbulence in an adiabatic gas. Eddy viscosity is found. We derive spectral distributions of kinetic energy and internal energy, from velocity and density fluctuations, respectively. The k^{-2} law of these energies agree with a one-dimensional numerical modeling developed by Tokunaga.[6]

II. FLUCTUATION THEORY OF TRANSPORT

A. Interaction Between Fluid Elements

We introduce a distribution function

$$f(t,\underset{\sim}{x},\underset{\sim}{v}) = \rho(t,\underset{\sim}{x})\ \delta[\underset{\sim}{v}-\underset{\sim}{u}(t,\underset{\sim}{x})] \tag{1}$$

of velocity $\underset{\sim}{v}$ at time t and position $\underset{\sim}{x}$. It gives density and momentum by

$$\int d\underset{\sim}{v}\ f = \rho\ , \quad \int d\underset{\sim}{v}\ \underset{\sim}{v}\ f = \rho\underset{\sim}{u}\ . \tag{2}$$

A self-consistent field

$$\underset{\sim}{E} \equiv -\frac{1}{\rho}\underset{\sim}{\nabla} p = -\frac{1}{4\pi\rho}\underset{\sim}{\nabla}\iint d\underset{\sim}{x}'d\underset{\sim}{v}'\varphi(|\underset{\sim}{x}-\underset{\sim}{x}'|,\underset{\sim}{x}',\underset{\sim}{v}')f(t,\underset{\sim}{x}',\underset{\sim}{v}') \tag{3}$$

from pressure gradient $\underset{\sim}{\nabla} p$ can be written. Here the operator

$$\varphi(|\underset{\sim}{x}-\underset{\sim}{x}'|,\underset{\sim}{x}',\underset{\sim}{v}') = |\underset{\sim}{x}-\underset{\sim}{x}'|^{-1}(\partial_t+\underset{\sim}{v}'\cdot\underset{\sim}{\nabla}')\underset{\sim}{v}'\cdot\underset{\sim}{\nabla}' \tag{4}$$

represents the interaction between two fluid elements in compressible fluid. It will reduce to $|\underset{\sim}{x}-\underset{\sim}{x}'|^{-1}(\underset{\sim}{v}'\cdot\underset{\sim}{\nabla}')^2$ in incompressible fluid, and to the Coulomb potential $|\underset{\sim}{x}-\underset{\sim}{x}'|^{-1}$ in a plasma. It is easy to verify that the Liouville equation

$$(\partial_t+L)f = 0, \qquad \text{with} \quad -L = \underset{\sim}{v}\cdot\underset{\sim}{\nabla}+\underset{\sim}{E}\cdot\underset{\sim}{\partial} \tag{5}$$

will reproduce the equations of continuity and momentum by means of moments. Our transport theory, which will derive eddy transport functions and coefficients, can neglect viscous effects. Here $\partial_t \equiv \partial/\partial t$, $\underset{\sim}{\nabla}\equiv\partial/\partial\underset{\sim}{x}$, $\underset{\sim}{\partial}\equiv\partial/\partial\underset{\sim}{v}$.

B. Scaling

We can write

$$\underset{\sim}{E} = \overline{\underset{\sim}{E}} + \widetilde{\underset{\sim}{E}}, \quad \widetilde{\underset{\sim}{E}} = \underset{\sim}{E}^{(o)} + \underset{\sim}{E}', \quad \underset{\sim}{E}' = \underset{\sim}{E}^{(1)} + \underset{\sim}{E}''. \tag{6}$$

The components $\underset{\sim}{E}^{(o)}$, $\underset{\sim}{E}'$ and $\underset{\sim}{E}''$ partake in macroscopic evolution, eddy diffusivity, and relaxation, respectively.

The various components, or ranks, may overlap in instantaneous scales, but have statistically distinct correlation times, or scales, in the decreasing order

$$\tau_c^{(o)} > \tau_c' > \tau_c'' \quad , \tag{7}$$

signifying their increasing degree of random character, and, consequently, the quasi-stationarity of one macroscopic rank with respect to the next random rank.

The ensemble averages $\bar{A}\langle\cdot\cdot\rangle$, $A_o\langle\cdot\cdot\rangle$, $A_1\langle\cdot\cdot\rangle$ of increasing finer scales help in discerning the ranks $\bar{\underset{\sim}{E}}$, $\underset{\sim}{E}^{(o)}$, $\underset{\sim}{E}'$, $\underset{\sim}{E}''$, by means of filter operators $\tilde{A} = 1-\bar{A}$, $A^{(o)} = A_o-\bar{A}$, $A' = 1-A_o$, $A'' = 1-A_1$.

C. Kinetic Equation

By the application of $A^{(o)}$, A', we transform Eq. (5) into

$$(\partial_t + A^{(o)}L)f^{(o)} = -\underset{\sim}{E}^{(o)}\cdot\underset{\sim}{\partial}\bar{f} - A^{(o)}\{\underset{\sim}{E}'\cdot\underset{\sim}{\partial}f'\}$$
$$(\partial_t + L)f' = -\underset{\sim}{E}'\cdot\underset{\sim}{\partial}(\bar{f}+f^{(o)}) + A_o\langle Lf'\rangle, \qquad (8)$$

and by the elimination of f', we derive a kinetic equation

$$(\partial_t + A^{(o)}L)f^{(o)} = -\underset{\sim}{E}^{(o)}\cdot\underset{\sim}{\partial}\bar{f} + A^{(o)}D'\{\partial^2 f^{(o)}\}_{t-\tau}, \qquad (9)$$

with a diffusivty

$$\underset{\approx}{D}' = \int_0^t d\tau\, A_1\langle \underset{\sim}{E}'(t)A'U(t,t-\tau)\underset{\sim}{E}'(t-\tau)\rangle \qquad (10)$$

which may also serve as an integral operator on functions within brackets $\{\cdots\}$. The propagator U indicates a Lagrangian representation.

III. CLOSURE BY RELAXATION

We can write Eq. (10) in Fourier form:

$$\underset{\approx}{D}' = \int_0^t d\tau \iint d\omega' d\underset{\sim}{k}'\, \chi\, A_o\langle \underset{\sim}{E}'(-\omega',-\underset{\sim}{k}')\underset{\sim}{E}'(\omega',\underset{\sim}{k}')A_1\hat{h}\rangle, \qquad (11)$$

where $\hat{h}$ is a kernel, and χ is a factor of truncation in the Fourier transformation of a function which is quasi-stationary within a certain time interval or space interval, or both.

We have the following kernels:

$$A_1\hat{h} = A_1 h h_v, \quad A_1 h = A_1 h'' h_1,$$
$$h_v = \exp i(\omega' - \underset{\sim}{k}'\cdot\underset{\sim}{v}), \quad h = \exp[-\underset{\sim}{k}'\cdot\underset{\sim}{\ell}(\tau)], A_1 h'' = A_1 e^{-i\underset{\sim}{k}'\cdot\underset{\sim}{\ell}''} \qquad (12)$$

The total kernel h consists of an Eulerian kernel h_v representing a streaming of velocity $\underset{\sim}{v}$, and of a Lagrangian kernel h from the perturbed path $\ell = \ell_1 + \ell''$. In its turn, the Lagrangian kernel h is separated into h_1 and A_1h'', representing a dynamical streaming of path ℓ_1 and a relaxation by diffusion of the random path ℓ'', respectively. We shall rely upon this micro-diffusion for obtaining the approach of $\underset{\approx}{D}'$ to equilibrium and closure.

From the inequalities

$$\tau_c'' < \tau < \tau_c' < t \ , \tag{13}$$

the upper limit t can be replace by ∞ in Eq. (11). For the same reason, while τ is larger enough for $\underset{\sim}{E}''$ and A_1h'' to reach asymptotic diffusion, i.e. $t \to \infty$, it is allowing $\underset{\sim}{E}_1$ and h_1 to afford a dynamical streaming only.

Since

$$A_1\hat{h} \cong h_v, \text{ for weak turbulence} \tag{14a}$$

$$\cong A_1h'', \text{ for strong turbulence,} \tag{14b}$$

an interpolation formula

$$A_1\hat{h} \cong A_1h''h_v \tag{14c}$$

can be written by neglecting dynamic streaming.

The relaxation A_1h'' can be calculated, by means of a Fokker-Planck equation for the transition probability $p(\tau, \underset{\sim}{\ell}'')$, with a diffusion consistent to the same process as governed by Eq. (11). We find:

$$A_1h'' = \exp(-\omega_D''^3\tau^3), \text{ with } \omega_D''^3 = \frac{1}{3}k'^2D''. \tag{15}$$

The relaxation frequency ω_D'' and the asymptotic diffusivity

$$\begin{aligned} D'' &= \mathrm{tr}\int_0^\infty d\tau' \iint d\omega'' d\underset{\sim}{k}'' \chi A_1\langle \underset{\sim}{E}''(-\omega'', -\underset{\sim}{k}'')\underset{\sim}{E}''(\omega'', \underset{\sim}{k}'')\rangle A_2h''' \\ &= \mathrm{tr}\int_0^\infty d\tau' \int d\underset{\sim}{k}'' \chi A_1\langle \underset{\sim}{E}''(-\underset{\sim}{k}'')\underset{\sim}{E}''(\underset{\sim}{k}'')\rangle \exp(-\omega_D''^{\prime 3}\tau^3) \end{aligned} \tag{16}$$

are found independent of $\underset{\sim}{v}$ in strong turbulence.

By introducing a spectral function $F_E(k')$, defined by the trace

$$\mathrm{tr}\int d\underset{\sim}{k}'' \chi A_1\langle \underset{\sim}{E}''(-\underset{\sim}{k}'')\underset{\sim}{E}''(\underset{\sim}{k}'')\rangle = \frac{2}{3}\int_{k'}^\infty dk'' \, F_E(k'') \ , \tag{17}$$

and by performing the τ'-integration, we reduce Eq. (16) to

$$\begin{aligned} D'' &= \frac{2}{3}\Gamma\left(\frac{4}{3}\right)\int_{k'}^\infty dk'' \, F_E(k'')/\omega_D''' \\ &= \left[\frac{8}{9}\Gamma\left(\frac{4}{3}\right)\int_{k'}^\infty dk'' \, F_E(k'')/(\tfrac{1}{3}k''^2)^{1/3}\right]^{3/4}. \end{aligned} \tag{18}$$

Note that

$$\begin{aligned} D'' &\sim |\underset{\sim}{E}''|^{3/2}, \text{ in strong turbulence} \\ &\sim |\underset{\sim}{E}''|^2, \text{ in weak turbulence .} \end{aligned} \tag{19}$$

IV. PHYSICAL BASIS OF BALANCE BETWEEN KINETIC ENERGY AND INTERNAL ENERGY

Introduce an auxiliary variable

$$w = w_o(\rho/\rho_o)^{(\gamma-1)/2}, \quad \text{with} \quad w_o = [2p_o/(\gamma-1)\ \rho_o]^{\frac{1}{2}}, \qquad (20)$$

having the dimension of velocity, in an aidabatic gas, with an equation of state

$$p = p_o(\rho/\rho_o)^{\gamma}$$

$$= \frac{\gamma-1}{2}\rho w^2 . \qquad (21)$$

The equation of continuity

$$(\partial_t + \underset{\sim}{u}\cdot\underset{\sim}{\nabla})\rho = -\rho\underset{\sim}{\nabla}\cdot\underset{\sim}{u} \qquad (22)$$

can be written as

$$(\partial_t + \underset{\sim}{u}\cdot\underset{\sim}{\nabla})w = -\frac{\gamma-1}{2} w\,\underset{\sim}{\nabla}\cdot\underset{\sim}{u}, \qquad (23)$$

to form the equation of energy

$$\tfrac{1}{2}\,\partial_t \rho w^2 + \nabla_j(u_j \tfrac{1}{2}\rho w^2) = -\underset{\sim}{\nabla}\cdot(\underset{\sim}{u}p) - \rho\,\underset{\sim}{u}\cdot\underset{\sim}{E} . \qquad (24)$$

We can verify that

$$\tfrac{1}{2}w^2 = c_v T \qquad (25a)$$

is the internal energy, and that

$$\tfrac{1}{2}\gamma w^2 = c_p T \equiv \psi \qquad (25b)$$

is the enthalpy playing the role of an effective potential, such that

$$\underset{\sim}{E} = -\frac{1}{\rho}\underset{\sim}{\nabla} p = -\underset{\sim}{\nabla}\psi . \qquad (26)$$

Hence Eq. (24) will be called equation of internal energy.

The two forms taken by Eqs. (23) and (24) are analogous to the equations of momentum

$$(\partial_t + \underset{\sim}{u}\cdot\underset{\sim}{\nabla})\underset{\sim}{u} = \underset{\sim}{E} \qquad (27)$$

and of kinetic energy

$$\tfrac{1}{2}\partial_t \rho u^2 + \nabla_j(u_j \tfrac{1}{2}u^2) = \rho\,\underset{\sim}{u}\cdot\underset{\sim}{E} . \qquad (28)$$

In homogeneous turbulence, the two energy equations reduce to

$$\partial_t \mathcal{E}_w = -C \qquad (29a)$$

$$\partial_t \mathcal{E}_u = C , \qquad (29b)$$

where

$$C = \langle \rho \underset{\sim}{u} \cdot \underset{\sim}{E} \rangle \tag{29c}$$

is the coupling function which describes the build-up of kinetic energy $\mathcal{E}_u \equiv \frac{1}{2}\langle \rho \; u^2 \rangle$ at the expense of internal energy $\mathcal{E}_w \equiv \frac{1}{2}\langle \rho \; w^2 \rangle$. Note that

$$\tfrac{1}{2} \partial_t \rho \, w^2 \equiv \psi \partial_t \rho \quad . \tag{29d}$$

The energy balance described above is written in the non-scaled from, i.e. k = ∞ . The corresponding scaled form, i.e. k = finite, must retain internal transfers along individual spectra, as follows:

$$\partial_t \; \mathcal{E}_u^{(o)} = C^{(o)} - T_u^{(o)} \tag{30a}$$

$$\partial_t \; \mathcal{E}_w^{(o)} = - C^{(o)} - T_w^{(o)} \; . \tag{30b}$$

The transfer functions $T_u^{(o)}$, $T_w^{(o)}$ govern the internal mode-transfers. We have the conditions:

$$C^{(o)} = C, \quad T_u^{(o)} = 0, \quad T_w^{(o)} = 0, \quad \text{at } k = \infty \; , \tag{31}$$

so that the system of Eqs. (30) reverts to Eqs. (29) at k = ∞, as expected.

We shall write the kinetic equation (9) in the form:

$$(\partial_t + L) f^{(o)} = \bar{J}^{(o)} \equiv J_E^{(o)} + J_D^{(o)} \; , \tag{32}$$

where

$$J_E^{(o)} \equiv - \underset{\sim}{E}^{(o)} \cdot \underset{\sim}{\partial} \bar{f} \quad , \qquad J_D^{(o)} = D' \{ \partial^2 f^{(o)} \}_{t-\tau} \; , \tag{33}$$

so that the solution

$$f^{(o)}(t, \underset{\sim}{x}, \underset{\sim}{v}) = A^{(o)} \int_0^t d\tau' \iint d\omega' d\underset{\sim}{k}' \; J^{(o)}(\omega', \underset{\sim}{k}', \underset{\sim}{v}) A_o \hat{h} e^{-i(\omega' t - \underset{\sim}{k}' \cdot \underset{\sim}{x})}$$

$$= f_E^{(o)} + f_D^{(o)} \tag{34}$$

consists of two parts, as contributed by $J_E^{(o)}$ and $J_D^{(o)}$, respectively.

The presence of h_v and of the τ'-integration facilitate the transformation of $- i\omega$ into

$$m(\tau') = 2 \; \delta(\tau') - i\underset{\sim}{k}' \cdot \underset{\sim}{v} \tag{35}$$

from time differentiation, obtaining

$$\partial_t f^{(o)}(t,\underset{\sim}{x},\underset{\sim}{v})=A^{(o)}\int_0^{t\to\infty} d\tau' \iint d\omega' d\underset{\sim}{k}' \, J_D^{(o)}(\omega',\underset{\sim}{k}',\underset{\sim}{v}) \times m(\tau')A_o\hat{h}e^{-i(\omega' t-\underset{\sim}{k}'\cdot\underset{\sim}{x})} \tag{36}$$

In terms of distribution functions, the transport functions in Eqs. (29) - (31) can now be written as

$$T_u^{(o)} = -\int d\underset{\sim}{v}\, v_i \bar{A} \left\langle u_i^{(o)} \partial_t f_D^{(o)} \right\rangle \tag{37a}$$

$$T_w^{(o)} = -\int d\underset{\sim}{v}\, \bar{A} \left\langle \psi^{(o)} \partial_t f_D^{(o)} \right\rangle \tag{37b}$$

$$C^{(o)} = \bar{A} \left\langle \underset{\sim}{E}^{(o)} \cdot \int d\underset{\sim}{v}\, \underset{\sim}{v}\, f_E^{(o)} \right\rangle . \tag{37c}$$

V. DERIVATION OF TRANSPORT FUNCTIONS AND TRANSPORT COEFFICIENTS

The procedure of calculations of transport functions requires a closure. This is made by the use of the interpolation formula

$$A_o\hat{h} = A_o h' h_v \tag{38}$$

on an analogous basis as Eq. (14c). The partial integrations with respect to $\underset{\sim}{v}$ and the integration with respect to time are made. The details of calculations will be omitted.

For strong turbulence, we find the following transport functions:

$$T_u^{(o)} = \eta' R_u^{(o)} \tag{39a}$$

$$T_w^{(o)} = Pr^{-1} \eta' R_w^{(o)} \tag{39b}$$

$$C^{(o)} = 3\bar{\rho}\, D^{(o)} , \tag{39c}$$

the transport coefficients:

$$\eta' = \frac{8}{9}\left[\Gamma\left(\tfrac{4}{3}\right)\right]^2 \int_k^{\infty} dk'\, F_E(k')/\omega_D''^3 \tag{40a}$$

$$D^{(o)} = \frac{2}{3}\Gamma\left(\tfrac{4}{3}\right) \int_0^k dk'\, F_E(k')/\omega_D' , \tag{40b}$$

the vorticity functions:

$$R_u^{(o)} = 2\int_0^k dk' \; k'^2 F_u(k') \tag{41a}$$

$$R_w^{(o)} = 2\int_0^k dk' \; k'^2 F_w(k'), \quad R_w = R_w^{(o)}\Big|_{k=\infty}, \tag{41b}$$

the spectral functions:

$$\int_0^k dk' \; F_u(k') = \bar{A} \left\langle u_i^{(o)} \int d\underset{\sim}{v} \; v_i f^{(o)} \right\rangle \tag{42a}$$

$$\int_0^{k=\infty} dk' \; F_w(k') = \tfrac{1}{2} \bar{A} \langle \rho \, w^2 \rangle \tag{42b}$$

and the turbulent Prandtl number

$$Pr = 16/3\gamma \; . \tag{43}$$

VI. ENERGY BALANCE AND SPECTRAL STRUCTURE

In statistical equilibrium, the inertia subrange in compressible turbulence is governed by the following coupled system

$$C^{(o)} - T_u^{(o)} = \underset{-}{C} - \varepsilon \tag{44a}$$

$$-C^{(o)} - T_w^{(o)} = -C \; , \tag{44b}$$

for kinetic energy and internal energy. The system is a generalization of the energy balance

$$T_u^{(o)} = \varepsilon$$

in incompressible turbulence, for which $C^{(o)}$ and C vanish, where ε is the rate of energy dissipation.

The system describes a build-up of kinetic energy at the expense of internal energy for an amount $C^{(o)}$, to be followed by mode-transfers $T_u^{(o)}$ and $T_w^{(o)}$ in respective spectra. In differential form, we have:

$$\dot{C}^{(o)} - \eta' \dot{R}_u^{(o)} - \dot{\eta}' R_u^{(o)} = 0 \tag{45a}$$

$$-\dot{C}^{(o)} - Pr^{-1} \eta' \dot{R}_w^{(o)} - Pr^{-1} \dot{\eta}' R_w^{(o)} = 0. \tag{45b}$$

By this coupling, the two spectra in the inertia subrange are in two different regimes: the F_u spectrum is in the regime of production where its vorticity function $R_u^{(o)}$ is negligible, while the F_w spectrum, after having suffered two losses, falls in the regime of eddy dissipation where its vorticity function becomes asymptotic: $R_w^{(o)} \simeq R_w$. Under these circumstances, the energy balance reduces to:

$$\dot{C}^{(o)} - \eta' \dot{R}^{(o)} = 0 \qquad (46a)$$

$$-\dot{C}^{(o)} - Pr^{-1} \dot{\eta}' R_w = 0. \qquad (46b)$$

The upper dot denotes a differentiation with respect to k.

With the aid of Eqs. (39) and (40), we can solve the system of Eqs. (46). The solutions can be expressed in terms of $Pr^{-1}R_w$ or in terms of the constant relaxation frequency

$$\omega_D' = a_\omega R_w / Pr \bar{\rho} \;, \quad a_\omega = \tfrac{2}{3}\left[\Gamma(\tfrac{4}{3})\right]^{\frac{1}{2}} = 0.63, \qquad (47)$$

as a parameter.

We find the spectral functions

$$\begin{aligned} F_u &= a_u \bar{\rho}\, \omega_D'^2 k^{-3}, \quad a_u = a_\omega^{-2} = 2.52 \\ F_E &= a_E\, \omega_D'^4 k^{-3}, \quad a_E = 9/\Gamma(4/3) = 10.08, \end{aligned} \qquad (48)$$

the intensities

$$\begin{aligned} \mathcal{E}_u' &= \tfrac{1}{2} a_u \bar{\rho}\, \omega_D'^2 k^{-2} \\ \langle \underset{\sim}{E}'^2 \rangle &= a_E\, \omega_D'^4 k^{-2}, \end{aligned} \qquad (49)$$

and the transport coefficients

$$\begin{aligned} \eta' &= a_\eta\, \omega_D' k^{-2}, \quad a_\eta = 4\Gamma(4/3) = 3.57 \\ D' &= 3\, \omega_D'^3 k^{-2}. \end{aligned} \qquad (50)$$

We can also calculate the intensity of enthalpy fluctuation

$$\langle \psi'^2 \rangle = a_E\, \omega_D'^4 k^{-4} \qquad (51)$$

from Eqs. (26) and (48), and the internal energy

$$\tfrac{1}{2}\langle w'^2 \rangle = a_w\, \omega_D'^2 k^{-2}, \quad a_w = \gamma^{-1} a_E^{\frac{1}{2}} = 3.17\,\gamma^{-1} \qquad (52)$$

from Eq. (25b).

The k^{-2} law for kinetic and internal energies in Eqs. (49) and (52) are in agreement with results of numerical computations for one-dimensional model by Tokunaga.[6]

VII. CONCLUSIONS

Instead of closing a kinetic hierarchy of mean distributions, we develop a fluctuation theory, and close a hierarchy of correlations. The closure is facilitated by a scaling procedure into ranks. When the microscopic diffusion of path becomes asymptotic, a relaxation is obtained, providing equilibrium to diffusivity and rendering it deterministic, and in so doing, it closes the hierarchy. The kinetic equation of macroscopic distribution is then solved for deriving transport functions and transport coefficients. The balance between kinetic energy and internal energy derives the spectra. In a customary kinetic treatment without scaling, one would require a doublet distribution function to calculate a velocity correlation, and a Fourier transformation to derive the spectrum. The present method by scaling fluctuations, a singlet distribution function suffices for the same purpose.

With the transport coefficients determined in this manner, the method could also lend to analyse the distribution of velocities around a mean value and the deviation from normality. This problem will be left for a later opportunity.

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TURBULENCE. — *Transports turbulents normaux et anormaux dans un milieu stratifié.* Note (*) de **Tchen Chan-Mou**, présentée par Alexandre Favre.

L'étude est consacrée à la dérivation analytique des flux de transport dans un milieu stratifié. La mémoire, le couplage entre les fluctuations de vitesse et de température et les propriétés tensorielles apportent aux transports anormaux un caractère contre-gradient ou sans gradient, outre le transport normal du type Boussinesq suivant un gradient.

This Note presents an analytical derivation of the fluxes of transport in a stratified medium. The memory, the coupling between velocity and temperature fluctuations and the tensorial properties of transport give rise to an anomalous counter-gradient or without gradient transport, in addition to the normal transport of the Boussinesq type along a gradient.

I. Introduction. — Les transports anormaux de quantité de mouvement ([1] à [4]) et de chaleur qui ne suivent pas la relation de Boussinesq ont été mis en évidence dans des écoulements turbulents en laboratoire [5], dans l'atmosphère ([6], [7]) et en astrophysique [8]. On peut expliquer ce phénomène par un effet de mémoire, un couplage entre les fluctuations de vitesse et de température, et les propriétés tensorielles du transport, que nous examinons séparément dans la suite.

II. Mémoire et couplage. — On distingue les valeurs moyennes $\bar{\mathbf{u}}$, $\bar{\mathrm{T}}$, de vitesse et de température, et leur fluctuations turbulentes $\mathbf{u}$, θ. Introduisons une vitesse auxiliaire $w \equiv c\theta/\mathrm{T}_0$, un champ $\mathrm{E} \equiv -\nabla(p/\bar{\rho})$, deux gradients $\gamma \equiv \nabla \bar{w} + (g/c)\hat{\mathbf{z}}$, $\lambda \equiv \nabla \bar{w} + \lambda_0 \hat{\mathbf{z}}$, et deux opérateurs différentiels L_w et L_u tels que $\mathrm{L}_w\, w \equiv \tilde{\mathrm{A}}(\mathbf{u}+\mathbf{u}).\nabla w$, $\mathrm{L}_u \mathbf{u} \equiv \mathrm{L}_w \mathbf{u} - \mathrm{E}$, où $c \equiv (\mathrm{RT}_0)^{1/2}$ est la vitesse de propagation, R est la constante des gaz parfaits, T_0^{-1} correspond au coefficient d'expansion thermique, g est l'accélération de la pesanteur; λ_0 est un taux adiabatique, $\hat{\mathbf{z}} = (0, 0, 1)$ est le vecteur unité, $\bar{\rho}$ est la masse volumique moyenne, p est la fluctuation de pression, et $\tilde{\mathrm{A}}$ est l'opérateur de fluctuation. Les équations fondamentales de quantité de mouvement et de chaleur peuvent s'écrire sous la forme suivante :

$$(1) \qquad (\partial_t + \mathrm{L}_u)\mathbf{u} = -\mathbf{u}.\nabla\bar{\mathbf{u}} + w\,\gamma, \qquad (\partial_t + \mathrm{L}_w) w = -\mathbf{u}.\lambda.$$

Par intégration de (1), nous obtenons les vitesses $\mathbf{u}$, w, au moyen desquelles nous calculons les flux

$$(2) \qquad \begin{cases} \langle u_j u_i \rangle = -\sum_s (\mathrm{K}_u)_{js} \{\nabla_s \bar{u}_i\} + (\mathrm{M}_u)_j \{\gamma_i\}; \\ \langle u_j w \rangle = -\sum_s (\mathrm{K}_w)_{js} \{\lambda_s\}; \qquad j \neq i. \end{cases}$$

Les coefficients de transports

$$(3) \qquad \begin{cases} (\mathrm{K}_{u,\,w})_{js} = \displaystyle\int_0^t d\tau \langle u_j(t, \mathrm{x})\, \mathrm{U}_{u,\,w}(t, t-\tau)\, u_s(t-\tau) \rangle; \\ (\mathrm{M}_u)_j = \displaystyle\int_0^t d\tau \langle u_j(t, \mathrm{x})\, \mathrm{U}_u(\mathrm{t}, \mathrm{t}-\tau)\, w(t-\tau) \rangle \end{cases}$$

agissent comme opérateurs. L'intégration se fait suivant les trajectoires qui sont perturbés par les champs turbulents, comme l'indiquent les propagateurs $\mathrm{U}_{u,\,w}$ qui sont les inverses de $\mathrm{L}_{u,\,w}$. Ainsi la forme lagrangienne, qui porte sur les gradients en (2), représente un effet de mémoire. Cet effet a servi de base pour expliquer le décalage [9] entre flux et gradient. Le couplage entre vitesse et température et les propriétés tensorielles donneront d'autres transports anormaux.

III. Propriétés tensorielles de transport. — Les coefficients de transport (3) sont exprimés avec $\langle u_j u_s \rangle$ ou $\langle u_i w \rangle$, pour un temps fini qui est le temps d'approche à l'équilibre, appelé temps de relaxation ω_u^{-1} ou ω_w^{-1}, en introduisant les notations

$$Q_{js}(\omega, \mathbf{k}) \equiv \chi \langle u_j(\omega, \mathbf{k}) u_s(-\omega, -\mathbf{k}) \rangle, \qquad R_j \equiv \chi \langle u_j(\omega, \mathbf{k}) w(-\omega, -\mathbf{k}) \rangle,$$

où le facteur χ provient de la transformation de Fourier bornée. Nous pouvons exprimer (3) sous la forme de Fourier [10] :

$$(4) \qquad (K_{u,w})_{js} = \iint_{-\infty}^{\infty} d\omega\, d\mathbf{k}\, \omega_{u,w}^{-1} Q_{js}(\omega, \mathbf{k}), \qquad (M_u)_j = \iint_{-\infty}^{\infty} d\omega\, d\mathbf{k}\, \omega_u^{-1} R_j(\omega, \mathbf{k}),$$

et ainsi transformer (2) en

$$(5) \qquad \begin{cases} Q_{ji}(\omega, \mathbf{k}) = -\sum_s \omega_u^{-1} Q_{js}(\omega, \mathbf{k}) \nabla_s \bar{u}_i + \omega_u^{-1} R_j(\omega, \mathbf{k}) \gamma_i, & j \neq i, \\ R_j(\omega, \mathbf{k}) = -\sum_s \omega_w^{-1} Q_{js}(\omega, \mathbf{k}) \lambda_s. \end{cases}$$

Ce système peut se résoudre en fonction de la composante diagonale Q_{jj} pour donner

$$(6) \qquad Q_{ji} = -Q_{jj} S_{jir}, \qquad R_j = Q_{jj} \omega_w^{-1} \sigma_{jir}(\lambda)$$

et par conséquent pour obtenir :

$$(7) \qquad (K_{u,w})_{ji} = -(K_{u,w})_{jj} \{ S_{jir} \}, \qquad (M_u)_j = -(K_u)_{jj} \{ \omega_w^{-1} \sigma_{jir}(\lambda) \},$$

avec $j \neq i \neq r$. Nous remarquons que les coefficients de transport $(K_{u,w})_{jj}$ opèrent sur

$$(8a) \qquad S_{jir} \equiv (\Gamma_{rr} \Gamma_{ji} - \Gamma_{jr} \Gamma_{ri}) / (\Gamma_{ii} \Gamma_{rr} - \Gamma_{ri} \Gamma_{ir})$$

et

$$(8b) \qquad \begin{cases} \sigma_{jir}(\lambda) \equiv \lambda_j - S_{jir} \lambda_i - S_{jri} \lambda_r, \\ \sigma_{jir}(\nabla \bar{u}_i) = (\nabla_j - S_{jir} \nabla_i - S_{jri} \nabla_r) \bar{u}_i, \end{cases}$$

avec

$$(8c) \qquad \Gamma_{si} \equiv \delta_{si} + \nabla_s \bar{u}_i / \omega_u + \lambda_s \gamma_i / \omega_u \omega_w.$$

Une théorie d'échelle pour la fermeture par relaxation [10] permet de déterminer la structure des fréquences de relaxation ω_u, ω_w. Ici il suffit de considérer (8 *c*) comme un gradient non dimensionnel donné, sans entrer dans la structure analytique des fréquences de relaxation.

IV. Transports normaux et anormaux. — Par la substitution de (7) en (2), nous aboutissons aux flux

$$(9) \qquad \begin{cases} \langle u_j u_i \rangle = -(K_u)_{jj} \{ \sigma_{jir}(\nabla \bar{u}_i) + \sigma_{jir}(\lambda) \gamma_i / \omega_w \}; \\ \langle u_j w \rangle = -(K_w)_{jj} \{ \sigma_{jir}(\lambda) \}. \end{cases}$$

Rappelons que $(K_{u,w})_{jj}$ sont les coefficients de transport sous forme de Fourier (4) et opèrent sur les gradients composés (8 *b*), pour donner un transport normal du type Boussinesq avec mémoire et les transports anormaux. Le transport de quantité de mouvement est plus compliqué que celui de la chaleur, et peut devenir plus facilement anormal. Si de tels transports anormaux prédominent, les flux peuvent changer de signe, de sorte que la viscosité effective peut formellement devenir négative [8].

Nous illustrons ces effets par un exemple, où la présence des gradients non nuls $\nabla_3 \bar{u}_1$, $\nabla_1 \bar{w}$, et $\lambda_3 \equiv \nabla_3 \bar{w} + \lambda_0$ réduisent (9) au système suivant :

$$\begin{cases} \langle u_3 u_1 \rangle = -(\mathrm{K}_u)_{33} \{ \Delta_1^{-1} (\nabla_3 \bar{u}_1 + a_w \lambda_3) \}; \\ \langle u_3 w \rangle = -(\mathrm{K}_w)_{33} \{ \Delta_1^{-1} (\lambda_3 - a_u \nabla_3 \bar{u}_1) \}, \end{cases} \tag{10}$$

avec $a_{u,\,w} \equiv \nabla_1 \bar{w} / \omega_{u,\,w}$; $\Delta_1 = 1 + a_u a_w$. Les flux verticaux (10) à contre-gradient résultent d'un gradient horizontal $\nabla_1 \bar{w}$ qui apparaît dans $a_{u,\,w}$.

Dans une stratification uniquement verticale avec gradients $\nabla_3 \bar{u}_1$ et $\nabla_3 \bar{w}$, les transports deviennent

$$\langle u_3 u_1 \rangle = -(\mathrm{K}_u)_{33} \{ \nabla_3 \bar{u}_1 \}; \qquad \langle u_3 w \rangle = -(\mathrm{K}_w)_{33} \{ \nabla_3 \bar{w} + \lambda_0 \}, \tag{11}$$

appartenant au type de Boussinesq avec mémoire. L'absence de tous gradients $\nabla \mathbf{u}$, $\nabla \bar{w}$, sauf le taux adiabatique, peut encore donner lieu à un flux de chaleur

$$\langle u_3 w \rangle = -(\mathrm{K}_w)_{33} \lambda_0. \tag{12}$$

Si les gradients varient très lentement, le système (11) se réduit aux transports normaux sans mémoire

$$\langle u_3 u_1 \rangle = -(\mathrm{K}_u)_{33} \nabla_3 \bar{u}_1, \qquad \langle u_3 w \rangle = -(\mathrm{K}_w)_{33} (\nabla_3 \bar{w} + \lambda_0), \tag{13}$$

qui sert à donner la justification de la théorie de similitude de Monin et Obukhoff [11]. Cependant cette théorie de similitude ne traduit pas la dépendance quantitative des coefficients de transport à la longueur de stabilité. Notre théorie d'échelle [10] permet de déterminer la structure des coefficients de transport et de la relaxation pour analyser cette dépendance.

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Résumé d'un texte qui sera conservé 5 ans par les Archives de l'Académie et dont copie peut être obtenue.

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STRONG TURBULENCE IN LOW-β PLASMAS

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Abstract—An investigation of the spectral structure of turbulence in a plasma confined by a strong homogeneous magnetic field was made by means of a fluid description. The turbulent spectrum is divided into subranges. Mean gradients of velocity and density excite turbulent motions, and govern the production subrange. The spectra of velocity and potential fluctuations interact in the coupling subrange, and the energy is transferred along the spectrum in the inertia subrange. Applying the method of cascade decomposition, the spectral laws k^{-3}, k^{-3}, k^{-2} are obtained for the velocity fluctuations, and k^{-3}, k^{-5}, $k^{-3/2}$ for the potential fluctuations in the production, coupling and inertia subranges, respectively. The coefficient of Bohm diffusion is reproduced, and its role in electrostatic coupling is derived. Comparison is made with measured power laws reported in the literature, from Q-devices, hot-cathode reflex arc, Stellarator, Zeta discharge, ionospheric plasmas, and auroral plasma turbulence.

1. INTRODUCTION

COLLECTIVE anomalous properties of turbulent plasmas are found important in characterising plasma motions in fusion devices, in astrophysics and in ionospheric heating. Observations have been reported on the spectral structure of turbulence for density, electric potential and electrostatic field fluctuations, (e.g. BOL, 1964; CHEN 1965; D'ANGELO and ENRIQUES, 1966; BERKL and GRIEGER, 1967; REECE ROTH 1971; ROBINSON and RUSBRIDGE, 1971; LEONARD and LINNERUD, 1972; GURNETT and FRANK, 1977). It is our purpose to present a hydrodynamic formulation of the theory of turbulence for comparison with these observations. A kinetic formulation was given previously by TCHEN (1978*a*, *b*).

Theories of weak plasma turbulence have been frequently reported in the literature (KADOMTSEV, 1965). Theories of strong turbulence (DUPREE, 1966; WEINSTOCK, 1969) are limited to weak couplings. For strong couplings, a procedure was developed for smoothing the fluctuations by a repeated-cascade as an advanced Reynolds' decomposition (TCHEN, 1969). This procedure was applied to the Navier-Stokes model for a two-dimensional magnetized plasma (TCHEN, 1976). Here we shall generalize the problem to three dimensions and deal with the spectral structure of the fluctuating components in the plane perpendicular to the magnetic field. Further details of the analysis presented in the following sections are available in the form of a report (TCHEN *et al.*, 1977).

2. MODEL EQUATIONS AND CASCADE DECOMPOSITIONS

The motion of ions, of density n, velocity $\mathbf{v}$, temperature T_i, mass M, charge e, under an external magnetic field $\mathbf{B}=(0,0,B)$ and a self-consistent field

$$\mathbf{E}=-\nabla\phi, \tag{1}$$

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is governed by the following equations of conservation of mass and momentum:

$$\frac{\partial n}{\partial t}+\nabla\cdot(n\mathbf{v})=0, \tag{2a}$$

$$nM\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)-en(-\nabla\phi+\mathbf{v}\times\mathbf{B})=-\kappa T_i\nabla n. \tag{2b}$$

The hot electrons of temperature T_e can maintain quasi-neutrality by flowing along the magnetic line of force, according to the equation

$$\kappa T_e\,\nabla n = ne\,\nabla\phi \tag{3}$$

for a Boltzmann distribution. By introducing

$$\psi=\alpha\ln(n/n_0) \tag{4}$$

with a normalizing constant n_0 and

$$\alpha=\sqrt{\kappa(T_e+T_i)/M} \tag{5}$$

we can rewrite (2a) and (2b) in the form:

$$\frac{\partial \psi}{\partial t}+\mathbf{v}\cdot\nabla\psi=-\alpha\nabla\cdot\mathbf{v}, \tag{6a}$$

$$\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}=-\alpha\nabla\psi+\frac{e}{M}\mathbf{v}\times\mathbf{B}. \tag{6b}$$

The system of conservation equations (2) and its reduced form (6) will serve as a basis for formulating a turbulent transport theory, as being separate from collisions or molecular dissipations. Therefore these dissipative effects are not written in (2) and (6), but will, however, be restituted as sinks in energy balance later on.

If the plasma motion is strictly two-dimensional in the plane perpendicular to the magnetic field, we can reduce the system (6) into an equation for $\nabla\times\mathbf{v}$ of the Navier-Stokes type without pressure. Such an equation can serve as a simple model of turbulence. However, if the above condition is not fulfilled, the complete system (6) has to be considered. This latter problem will be treated here.

We decompose the velocity $\mathbf{v}$ into a mean velocity $\bar{\mathbf{v}}$ and a turbulent fluctuation $\mathbf{u}$, as

$$\mathbf{v}=\bar{\mathbf{v}}+\mathbf{u}, \tag{7a}$$

and subsequently we redecompose the fluctuation into a macroscopic component $\mathbf{u}^{(0)}$ and a random fluctuation $\mathbf{u}'$, as:

$$\mathbf{u}=\mathbf{u}^{(0)}+\mathbf{u}'. \tag{7b}$$

The decomposition (7a) is known as Reynolds' decomposition in turbulence, and the decomposition (7b) is an extension. The macroscopic component $\mathbf{u}^{(0)}$ and the random fluctuation $\mathbf{u}'$ can be screened by means of a distribution function, so that, after an ensemble average,

$$\tfrac{1}{2}\langle(u_i^{(0)})^2\rangle=\int_0^k \mathrm{d}k'F(k') \tag{8a}$$

and

$$\tfrac{1}{2}\langle(u_1')^2\rangle = \int_k^\infty dk' F(k'), \tag{8b}$$

represent two consecutive portions of a spectral function $F(k')$ for velocity fluctuations separated by a wavenumber k considered as an independent variable ranging from 0 to ∞. Before this ensemble average, $\mathbf{u}^{(0)}$ and $\mathbf{u}'$ may have overlapping length scales, but should predominantly lie in separate spectral portions. The same decomposition applies to ψ, so that

$$\psi = \bar{\psi} + \psi^{(0)} + \psi'. \tag{9}$$

and

$$\tfrac{1}{2}\langle(\psi^{(0)})^2\rangle = \int_0^k dk' G(k'), \qquad \tfrac{1}{2}\langle(\psi')^2\rangle = \int_k^\infty dk' G(k') \tag{10}$$

are two consecutive portions of the spectral function $G(k')$ for ψ-fluctuations. Similarily, for $\mathbf{E}$-fluctuations, we have:

$$\tfrac{1}{2}(e/M)^2\langle(E_i^{(0)})^2\rangle = \int_0^k dk' G_E(k'), \qquad \tfrac{1}{2}(e/M)^2\langle(E_i')^2\rangle = \int_k^\infty dk' G_E(k'). \tag{11}$$

Decomposing the system (6) into macroscopic and random components, the following equations are obtained in the form of a cascade for components in the plane perpendicular to the magnetic field:

$$D_t u_i^{(0)} - \omega_c \varepsilon_{ij} u_j^{(0)} = -u_j^{(0)} \partial_j \bar{v}_i + \frac{e}{M} E_i^{(0)} - \langle u_j' \partial_j u_i' \rangle \tag{12a}$$

$$D_t \psi^{(0)} = -u_j^{(0)} \partial_j \bar{\psi} - \alpha \partial_j u_j^{(0)} - \langle u_j' \partial_j \psi' \rangle \tag{12b}$$

$$(\partial_t + v_j \partial_j) u_i' - \omega_c \varepsilon_{ij} u_j' = \frac{e}{M} E_i' - u_s' \partial_s (\bar{v}_i + u_i^{(0)}) \tag{13a}$$

$$(\partial_t + v_j \partial_j) \psi' = -\alpha \partial_j u_j' - u_s' \partial_s (\bar{\psi} + \psi^{(0)}) \tag{13b}$$

with

$$D_t = \partial_t + (\bar{\mathbf{v}} + \mathbf{u}^{(0)}) \cdot \nabla, \qquad \partial_t = \frac{\partial}{\partial t}$$

$$\{\varepsilon_{ij}\} = \begin{Bmatrix} 0 & 1 \\ -1 & 0 \end{Bmatrix}$$

and

$$\frac{e}{M} \mathbf{E} = -\alpha \nabla \psi. \tag{14}$$

The fluxes

$$\langle u_j' u_i' \rangle, \quad \langle u_j' \psi' \rangle \tag{15}$$

are Reynolds' stresses accounting for the statistical effects of random fluctuations upon the evolution of macroscopic components $u^{(0)}$ and $\psi^{(0)}$ in (12). In turn, the random fluctuations are produced by the gradients of macroscopic variables, as in

(13). The discrimination between the two systems (12) and (13) is obtained by a scaled ensemble average. As the fluxes (15) are of macroscopic rank, they are omitted in (13), governing fluctuations of random rank.

3. TRANSPORT COEFFICIENTS

The system (13) can be integrated formally to give the random components:

$$u_i' = -[\partial_s(\bar{v}_i + u_i^{(0)})]\int_0^t \mathrm{d}\tau U_\omega(t, t-\tau)u_s'(t-\tau) + \frac{e}{M}\int_0^t \mathrm{d}\tau U_\omega(t, t-\tau)E_i'(t-\tau) \quad (16a)$$

$$\psi' = -[\partial_s(\bar{\psi} + \psi^{(0)})]\int_0^t \mathrm{d}\tau U(t, t-\tau)u_s'(t-\tau) - \alpha\partial_i\int_0^t \mathrm{d}\tau U(t, t-\tau)u_i'(t-\tau) \quad (16b)$$

which serve to form the gradients of Reynolds' stresses:

$$\langle u_j'\partial_j u_i'\rangle \simeq -K_{js}'\partial_j\partial_s(\bar{v}_i + u_i^{(0)}) \quad (17a)$$

$$\langle u_j'\partial_j \psi'\rangle \simeq -\lambda_{js}'\partial_j\partial_s(\bar{\psi} + \psi^{(0)}) \quad (17b)$$

and the cross-correlation

$$\frac{e}{M}\langle E_j'u_s'\rangle \equiv D_{js}'. \quad (17c)$$

The eddy viscosity in x-space, the diffusivity in x-space, and the diffusivity in volocity space, which are assumed homogeneous, are:

$$K_{js}' = \int_0^t \mathrm{d}\tau\langle u_j'(t, \mathbf{x})U_\omega(t, t-\tau)u_s'(t-\tau)\rangle \quad (18a)$$

$$\lambda_{js}' = \int_0^t \mathrm{d}\tau\langle u_j'(t, \mathbf{x})U(t, t-\tau)u_s'(t-\tau)\rangle \quad (18b)$$

$$D_{js}' = \left(\frac{e}{M}\right)^2\int_0^t \mathrm{d}\tau\langle E_j'(t, \mathbf{x})U_\omega(t, t-\tau)E_s'(t-\tau)\rangle, \quad (18c)$$

respectively. The fluxes $\langle u_j'u_i'\rangle$ and $\langle u_j'\psi'\rangle$ in (17a) and (17b) are calculated approximately by only retaining the gradient of the macroscopic quantities transported, i.e. $\partial_s(\bar{v}_i + u_i^{(0)})$ and $\partial_s(\bar{\psi} + \psi^{(0)})$, while the cross correlation (17c) is calculated approximately as a transport of non-gradient type. The propagator $U_\omega(t, t-\tau)$ implies a Lagrangian representation of the correlation function following a trajectory that is perturbed by **u** in a magnetic field. For a strong magnetic field, we can write

$$U_\omega(t, t-\tau) \simeq \exp[-(\omega_k + i\omega_c)\tau], \quad (19)$$

where $\omega_c = eB/M$ is the cyclotron frequency, and ω_k is a relaxation frequency governing the approach to equilibrium of the transport property, and it is approximately

$$\omega_k \simeq \omega_c. \quad (20)$$

Thus we can write (18a) and (18c) in the form of traces as

$$K' \simeq a_k\langle u_i'(t, \mathbf{x})]^2\rangle\omega_c^{-1} \quad (21)$$

and

$$D' \simeq a_D\left(\frac{e}{M}\right)^2\langle E_i'(t, \mathbf{x})]^2\rangle\omega_c^{-1}, \quad (22a)$$

respectively. Using the definition (14), we can rewrite (22a) as

$$D' = \lambda_c \langle (\nabla\psi')^2 \rangle \tag{22b}$$

where

$$\lambda_c = a_D \alpha^2 \omega_c^{-1}, \tag{23}$$

is found as the Bohm diffusion. The transport coefficients $K^{(0)}$, $\lambda^{(0)}$ and $D^{(0)}$, which are defined from macroscopic fluctuations $u^{(0)}$ and $E^{(0)}$, can be written similarly. The numerical coefficients a_K and a_D are not determined. It should be remarked that most analytical theories of turbulence have not reached the stage where numerical coefficients can be determined. The present analytical description of plasma turbulence, as different from similarity theories which are based upon dimensional arguments, aims at a description of turbulent transport processes and the derivation of the spectral structure of the transport coefficients.

4. ENERGY BALANCE

Multiplying (12a) and (12b) by $u_i^{(0)}$ and $\psi^{(0)}$ and averaging, we find the following equations of energy balance in collisionless form:

$$\langle u_i^{(0)} D_t u_i^{(0)} \rangle = P_u^{(0)} + D^{(0)} - T_u^{(0)} \tag{24a}$$

$$\langle \psi^{(0)} D_t \psi^{(0)} \rangle = P_\psi^{(0)} - D^{(0)} - T_\psi^{(0)}. \tag{24b}$$

These equations consist of the production functions

$$P_u^{(0)} = K^{(0)} \Gamma_u^{\,2}, \tag{25a}$$

$$P_\psi^{(0)} = \lambda^{(0)} \Gamma_\psi^{\,2}, \tag{25b}$$

a coupling function $D^{(0)}$, and the transfer functions

$$T_u^{(0)} = K' R^{(0)}, \tag{26a}$$

$$T_\psi^{(0)} = \lambda' J^{(0)}. \tag{26b}$$

The gradient and vorticity functions are:

$$\Gamma_u^{\,2} \equiv (\partial_j \bar{v}_i)^2, \qquad \Gamma_\psi^{\,2} \equiv (\partial_j \bar{\psi})^2, \tag{27a}$$

$$R^{(0)} \equiv \langle (\partial_j u_i^{(0)})^2 \rangle = 2\int_0^k \mathrm{d}k' k'^2 F(k') \tag{27b}$$

$$J^{(0)} \equiv \langle (\partial_j \psi^{(0)})^2 \rangle = 2\int_0^k \mathrm{d}k' k'^2 G(k') \tag{27c}$$

respectively, with asymptotic values:

$$R \equiv \langle (\partial_j u_i)^2 \rangle = 2\int^{k=\infty} \mathrm{d}k' k'^2 F(k') \tag{27d}$$

$$J \equiv \langle (\partial_j \psi)^2 \rangle = 2\int_0^{k=\infty} \mathrm{d}k' k'^2 G(k'). \tag{27e}$$

The transfer functions (26), which arises from nonlinear mode-couplings, represent an energy cascade, as was postulated by KOLMOGOROFF (1941) and

HEISENBERG (1948). Transfer functions in the form (26) has a kinetic basis (TCHEN, 1978a, b). They do not exist in theories of weak turbulence.

All the transport functions are considered homogeneous in the locally homogeneous plasma.

5. SPECTRAL STRUCTURE

A. *Classification of spectral subranges*

We shall distinguish between universal and non-universal ranges in a spectrum and shall be exclusively concerned with the universal range. The non-universal range depends on the conditions for the particular experiments.

The universal range can be subdivided into production, coupling, and inertia subranges. The individual subranges are investigated separately.

B. *The production subrange*

Mean gradients of velocity and potential will feed energy into the fluctuations for further transfer across the individual spectra. In order to describe this transport explicitly, we write the energy equations in the following differential form, where the upper dot represents differentiation with respect to k;

$$\Gamma_u^2 \dot{K}^{(0)} - K'\dot{R}^{(0)} - \dot{K}'R^{(0)} = 0 \tag{28a}$$

$$\Gamma_\psi^2 \dot{\lambda}^{(0)} - \lambda'\dot{J}^{(0)} - \dot{\lambda}'J^{(0)} = 0 \tag{28b}$$

obtained by retaining the production and transfer functions in (24).

As the production occurs at low wavenumbers, we may neglect the last terms in (28a) and (28b) on account of small $R^{(0)}$ and $J^{(0)}$. The simplified equations then become

$$\Gamma_u^2 \dot{K}^{(0)} = K'\dot{R}^{(0)}, \tag{29a}$$

$$\Gamma_\psi^2 \dot{\lambda}^{(0)} = \lambda'\dot{J}^{(9)}, \tag{29b}$$

yielding the solutions

$$F(k) = \text{const}\, \Gamma_u^2 k^{-3}, \tag{30}$$

$$G(K) = \text{const}\, \Gamma_\psi^2 k^{-3}. \tag{31}$$

C. *The coupling subrange*

Density fluctuations which appear as potential energy can excite the kinetic energy and give rise to a transfer down the spectrum. This circumstance defines the coupling subrange. The governing transport functions in this subrange are thus the coupling function and the transfer function. From (24), we have, in differential form

$$\dot{D}^{(0)} - \dot{T}_u^{(0)} = 0 \tag{32a}$$

$$-\dot{D}^{(0)} - \dot{T}_\psi^{(0)} = 0 \tag{32b}$$

or, with the substitution of (22), (25) and (26),

$$\lambda_c \dot{J}^{(0)} - \dot{K}'R^{(0)} - K'\dot{R}^{(0)} = 0, \tag{33a}$$

$$-\lambda_c \dot{J}^{(0)} - \dot{\lambda}'J^{(0)} - \lambda'\dot{J}^{(0)} = 0. \tag{33b}$$

Because the ψ spectrum is dissipated at a rate controlled by the Bohm diffusion, the following approximations can be introduced

$$R^{(0)} \simeq 0, \quad J^{(0)} \simeq J \quad \text{and} \quad \lambda' \ll \lambda_c,$$

reducing (33) to

$$\lambda_c \dot{J}^{(0)} - K'\dot{R}^{(0)} = 0 \tag{34a}$$

$$-\lambda_c \dot{J}^{(0)} - \dot{\lambda}' J = 0 \tag{34b}$$

or, after addition,

$$-K'\dot{R}^{(0)} - \dot{\lambda}' J = 0. \tag{35}$$

Substitution of (21) into (35) gives

$$-\frac{\langle u'^2\rangle \dot{R}^{(0)}}{\dot{\lambda}'} = \omega_0{}^3, \tag{36}$$

with

$$\omega_0{}^3 \equiv \omega_c J. \tag{37}$$

As the left-hand side of equation (36) is a function of F and k alone, we obtain the solution

$$F(k) = \text{const}\ \omega_0{}^2 k^{-3}. \tag{38}$$

On the other hand, equation (34a) gives, by means of the definition (21),

$$2k^2 G(k) = \text{const}\,\frac{\langle u'^2\rangle}{\lambda_c \omega_c}\,\dot{R}^{(0)}. \tag{39}$$

Substituting the solution (38) for $F(k)$, we find from equation (39):

$$G(k) = \text{const}\ (\omega_0{}^4/\lambda_c\omega_c)k^{-5} \tag{40}$$

corresponding to a spectrum for the electrostatic field fluctuations

$$G_E(k) = \text{const}\ (\alpha^2\omega_0{}^4/\lambda_c\omega_c)k^{-3} \tag{41}$$

using relation (14).

We can conclude that the ψ-spectrum (40) falls off much faster than the u-spectrum (38) in the present subrange, on account of the effective dissipation by the Bohm diffusion in the ψ-spectrum alone. The velocity and the electric field spectra share a k^{-3} law, as evidenced by available data.

D. *Inertia subrange*

The inertia subrange is characterized by a constant transfer of energy across the spectrum, i.e.

$$T_u^{(0)} = \varepsilon_u \tag{42a}$$

$$T_\psi^{(0)} = \varepsilon_\psi \tag{42b}$$

or, with the definitions (26)

$$K'R^{(0)} = \varepsilon_0 \tag{43a}$$

$$\lambda' J^{(0)} = \varepsilon_\psi. \tag{43b}$$

Here ε_u and ε_ψ are the rates of energy dissipation, or sinks. They serve as parameters governing inertia and dissipation subranges.

The rate of energy dissipation ε_ψ can be identified as

$$\varepsilon_\psi \equiv D = \lambda_c J \tag{44}$$

from (22b) and (27c). Note that ε_ψ is not due to collisional losses but to a collisionless Bohm diffusion. This is to be expected, since the equation of continuity (2a) is conserved. On the other hand, the equation for the momentum (2b) may have a collision term giving rise to ε_u. The particular form of the collision term (kinematic viscosity by self-collisions, or damping by collisions between species) is not of interest here.

The energy balance derived in (43) has a form similar to that postulated in the theories of KOLMOGOROFF (1941) and HEISENBERG (1948), using dimensional considerations. Our transport theory, which derives from (43) and the transfer functions, is based upon a kinetic foundation (TCHEN, 1978a, b). Since the structure of our eddy transport coefficients differ from that assumed by HEISENBERG (1948), we expect different solutions of the equations of energy balance (43).

To obtain the solutions, we write (21) in spectral form

$$K' = \frac{\text{const}}{\omega_c}\int_k^\infty \mathrm{d}k' F(k')$$

and rewrite (43a) as

$$\frac{\text{const}}{\omega_c}\int_k^\infty \mathrm{d}k' F(k') 2\int_0^k \mathrm{d}k'' k''^2 F(k'') = \varepsilon_u \tag{45}$$

or

$$\int_0^k \mathrm{d}k' k'^2 F(k') = \text{const}\ \varepsilon_u \omega_c \frac{1}{\displaystyle\int_k^\infty \mathrm{d}k' F(k')}. \tag{46}$$

After differentiation with respect to k, we obtain

$$k^2 F(k) = \text{const}\ \varepsilon_u \omega_c \frac{F(k)}{\left[\displaystyle\int_k^\infty \mathrm{d}k' F(k')\right]^2} \tag{47}$$

or

$$\int_k^\infty \mathrm{d}k' F(k') = \text{const}\ (\varepsilon_u \omega_c)^{1/2} k^{-1}. \tag{48}$$

Differentiation of (48) finally yields

$$F(k) = \text{const}\ (\varepsilon_u \omega_c)^{1/2} k^{-2}. \tag{49}$$

On substituting (49) into (43b), we obtain

$$G(k) = \text{const}\ \varepsilon_\psi (\varepsilon_u \omega_c)^{-1/4} k^{-3/2}. \tag{50}$$

As the ψ spectrum is quickly dissipated by Bohm diffusion, as mentioned in Subsection 5C, we may expect that the inertia subrange (50) is not fully developed under certain circumstances.

6. COMPARISON WITH EXPERIMENTS

We shall compare our theoretical predictions with data on turbulent spectra reported in the literature. In most experiments, the spectra are measured as a function of frequency rather than wavenumber. This introduces the problem of Eulerian-Lagrangian transformation. If the streaming velocity predominates, we obtain a linear relation between the spectral functions in ω and k spaces, according to Taylor's hypothesis, or frozen turbulence, which is very widely accepted in the interpretation of turbulence measurements in plasmas and fluids (e.g. KOFOED-HANSEN and WANDEL, 1967). The streaming velocity consists effectively of the mean velocity $\bar{\mathbf{v}}$ and the velocity representing that portion of the spectrum of larger scale than the subrange concerned. Consequently, the Taylor hypothesis is valid for plasmas with a strong drift $\bar{\mathbf{v}}$, or, in its absence, for subranges (coupling and inertia subranges) of large wavenumbers. The hypothesis may not be valid for the production subrange (of small k) without a sufficient drift $\bar{\mathbf{v}}$. Under circumstances where the Taylor hypothesis is not valid, a nonlinear dispersion will prevail and this gives the frequency spectrum as a Fourier transform of the Lagrangian correlation in (18c):

$$G_E(\omega) = \frac{1}{\pi}\left(\frac{e}{M}\right)^2 \int_0^\infty \mathrm{d}\tau e^{-i\omega\tau}\langle E_i'(t, x)U_\omega(t, t-\tau)E_i'(t-\tau)\rangle. \tag{51}$$

For $\omega = 0$, this spectrum becomes linearly proportional to diffusivity:

$$G_E(\omega = 0) = \frac{1}{\pi} D^{(0)}(k = \infty). \tag{52}$$

We note that $G_E(\omega = 0) > 0$, while $G_E(k = 0) = 0$. This feature is observed in all experimental data.

An analytical study of the nonlinear dispersion relation requires the calculation of the propagator U_ω and of the Lagrangian correlation (51). Although the propagator has been investigated (TCHEN, 1978a, b), the calculation of the Langrangian correlation has not yet been made. However, a comparison between the theoretical prediction of the spectrum in k-space and the available data in ω-space can be justifiably made applying the Taylor hypothesis for the coupling subrange, with spectra

$$F(k) \sim k^{-3}, \qquad G_E(k) \sim k^{-3}, \qquad G(k) \sim k^{-5}, \tag{53}$$

according to (38), (41) and (40), respectively. The predictions are found in reasonably good accord with experiments.

In a strong streaming which is the necessary condition for the Taylor hypothesis, it is not clear what correction the non-uniformity of the stream may bring to the conversion between ω and k representations. It may conjectured that a weak non-uniformity may not be important in the conversion of small scale turbulence.

It is to be noted that, in the lack of a nonlinear dispersion relation, the Taylor hypothesis of frozen turbulence is widely accepted, not only in comparison between theory and experiment relating to ω and k-spaces, but also in turbulent propagation of electromagnetic waves, in remote optical sensing of wind velocity, and in laser doppler measurement of fluid velocity.

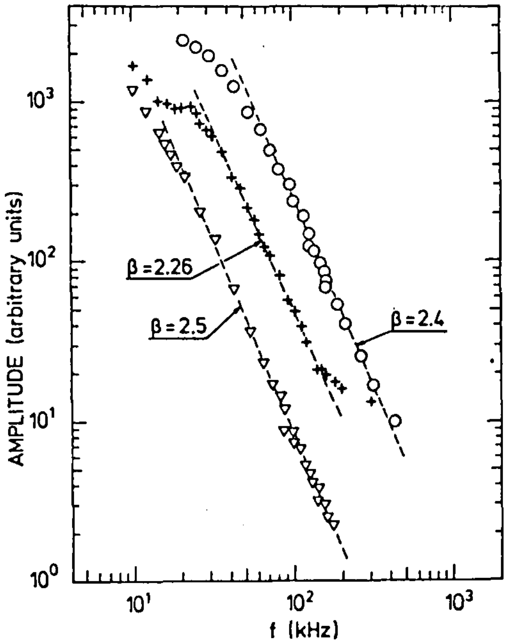

FIG. 1.—Turbulent spectra obtained in Q-devices. All curves show the amplitude of the fluctuations as a function of frequency. The curves reproduced from D'ANGELO and ENRIQUES (1971) are marked ∇ and O; those of BERKL and GRIEGER (1966) by +. The measurements marked ∇ and O refer to fluctuations in floating potential, while the experiment denoted + measured fluctuations in the ion saturation current to a Langmuir probe. The spectral index for the amplitude variations is denoted β, i.e. $A(f) \sim f^{-\beta}$. The amplitude is obtained by taking the rms output from a constant bandwidth filter.

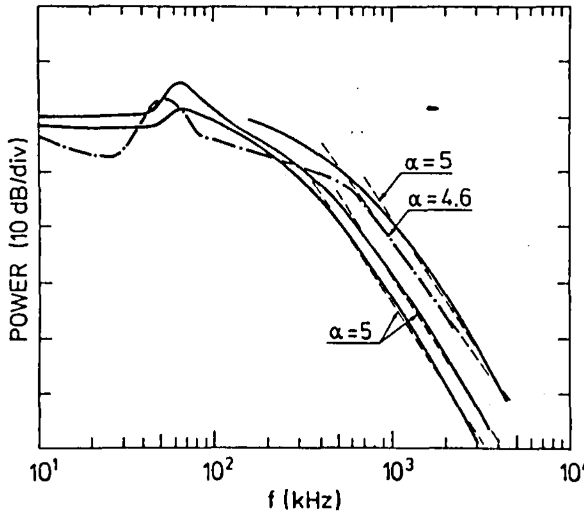

FIG. 2.—Turbulent power spectra obtained in a hot-cathode reflex arc ((–·–·–) by CHEN (1965) and in the Etude stellarator (——) by BOL (1964). The measurements refer to density fluctuations. The spectral index for the power spectrum is denoted α, i.e., $P(f) \sim f^{-\alpha}$.

The inertia subrange may not be fully developed in experiments for the reason mentioned in Subsection 5C, and therefore the spectra (49) and (50) will not be compared with experiments. On the other hand, strong mean gradients are unlikely to be found in plasma devices, except e.g. in the ionosphere. Therefore the spectral predictions (30) and (31) in the production subrange may not be easily found in data from plasma devices.

With the reservations stipulated above we compare our theoretical predictions with experiments. Figure 1 shows spectra obtained in conventional Q-devices reproduced from D'ANGELO and ENRIQUES (1966) BERKL and GRIEGER (1967). The reported spectra show the amplitude variation as a function of frequency. We note that the measurements of BERKL and GRIEGER (1967) have a large $\mathbf{E}\times\mathbf{B}$ rotation of the entire plasma column, rendering Taylor's hypothesis effective.

Figure 2 shows turbulent spectra for the density fluctuations obtained in a hot-cathode reflex arc (CHEN, 1965) and the Etude stellarator (BOL, 1964). Spectral measurements of the fluctuating electric field in the Zeta discharge (ROBINSON and RUSBRIDGE, 1971) are shown in Fig. 3.

Figure 4 shows a spectral analysis of turbulent density fluctuations in barium plasmas released in the upper atmosphere (LEONARD and LINNERUD, 1972). Because the data were reduced by photographic technique, it was possible to present the spectrum as a function of wavenumber in this case.

Finally, Fig. 5 shows satellite data for the fluctuating electric field on high-latitude auroral field lines (GURNETT and FRANK, 1977).

In view of the difficulties associated with frozen turbulence mentioned above, the lack of supporting experimental data for determining parameters, and because diverse conditions of turbulence can produce a k^{-3} law (e.g. in neutral compressible turbulence, see TCHEN (1979)) we must caution any overemphasis of the agreement between theoretical predictions and experimental spectra.

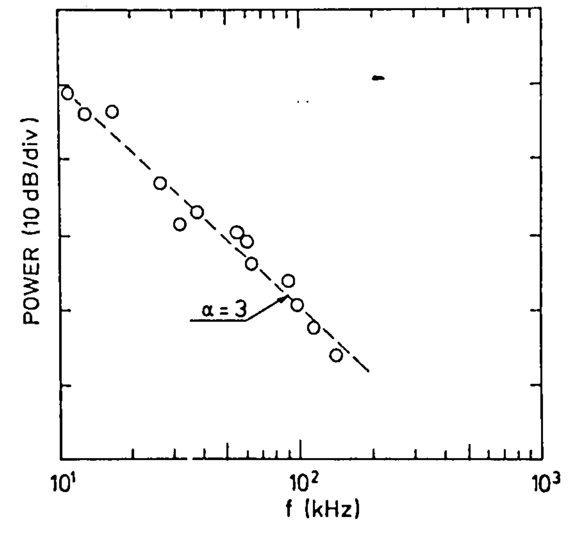

FIG. 3.—Power spectrum for the fluctuating electric field in the Zeta discharge (ROBINSON and RUSHBRIDGE, 1971).

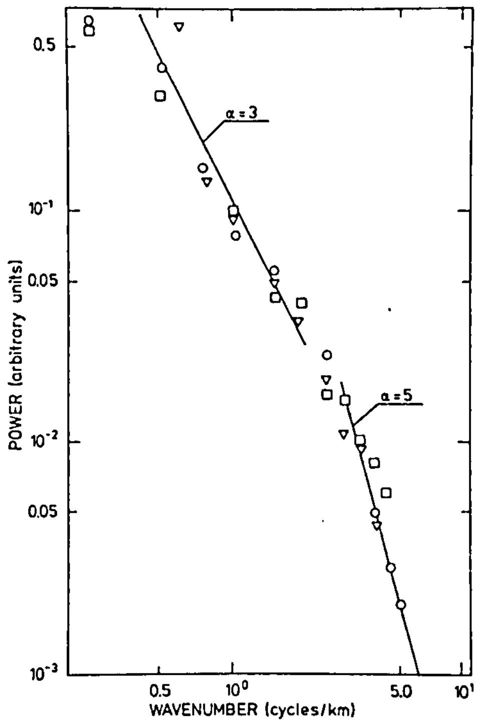

FIG. 4.—Turbulent density fluctuations in a barium plasma artificially released in the upper atmosphere (LEONARD and LINNERUD, 1972). The spectrum is shown as a function of wavenumber.

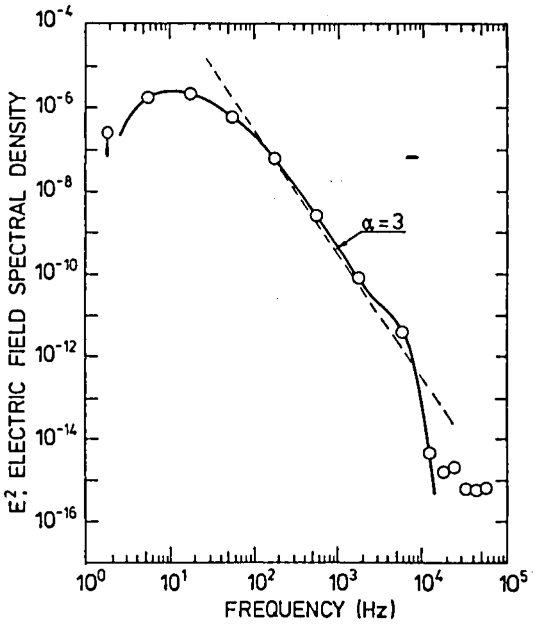

FIG. 5.—Power spectrum for the fluctuating electric field detected by the Hawkeye satellite on high-latitude auroral field lines (GURNETT and FRANK, 1977).

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Spectral Structure of Noise Generated by Turbulence*

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ABSTRACT

From the Lighthill equation for the generation of sound by turbulence, we investigate the spectral structure of sound pressure fluctuations in a turbulent velocity field that possesses Kolmogoroff's spectrum $F(k) \sim k^{-5/3}$. The pressure spectrum is found to be $\Pi(k) \sim k^{-7/3}$, where k is wavenumber. For the transformation of k-spectrum into ω-spectrum, a nonlinear dispersion relation for non-frozen turbulence is derived. The structure of $\Pi(k)$ contains a contribution from large scale motions which are linearly dispersed by free streaming (i.e. frozen turbulence) and a contribution from small-scale motions which are nonlinearly dispersed by diffusion process. The theory of transformation yields $\Omega(\omega) \sim \omega^{-5/3}$. The result compares favorably with measurements.

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I. INTRODUCTION

The theoretical foundation of turbulent jet noise has been laid in the work of Lighthill [1]. Concerning the turbulent structure of noise, a large body of experimental measurements of two point pressure correlation has been carried out by Maestrello [2]. An analysis of the correlation has been developed by Ribner [3]. The theory used an empirical Gaussian correlation function for the turbulent velocity fluctuations and found a good agreement with experiments. Because of the difficulties of correlation measurements at small distances between two points, spectral measurements in frequency have been performed [4,5].

From the Kolmogoroff law $k^{-5/3}$ in the turbulent velocity field, dimensional reasonings can predict a spectrum $k^{-7/3}$ for the micro-fluctuatio of sound pressure, where k is wavenumber. The dimensional theory does not provide detailed dynamics of sound emission.

Since theory deals in k-space and measurements are in ω-space, a comparison requires a space-time transformation, i.e. a dispersion relation. A linear relation based upon frozen turbulence is well known [6], but is not valid for small scale turbulence. We shall attempt to resolve these difficulties here.

II. INHOMOGENEOUS WAVE EQUATION

In incompressible, homogeneous and isotropic turbulence, the pressure fluctuation $p(t,\underset{\sim}{x})$ is related to velocity fluctuation $\underset{\sim}{u}(t,\underset{\sim}{x})$ by the Poisson equation

$$-\nabla^2 p = \rho_0 r , \tag{1}$$

through a source

$$r \equiv \underset{\sim}{\nabla}\underset{\sim}{\nabla} : \underset{\sim}{u}\underset{\sim}{u} . \tag{2}$$

Here ρ_0 is the constant density in the turbulent medium. Equation (1) can be generalized to

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \rho_0 r \tag{3}$$

for describing the generation of noise of infinitesimal amplitude by turbulence [1]. The linearized equation (3) has a constant speed of propagation c and a turbulent driving source r . The problem of acoustic turbulence which is fully nonlinear is treated separately [7] . Beside these two extremes of linear sound emission, or self-noise, and of acoustic turbulence, there exist other models, including sound scattering. A derivation of these models is given in APPENDIX, including a rederivation of (3).

The solution of the inhomogeneous wave equation (3) is

$$p(t,\underset{\sim}{x}) = \frac{\rho_0}{4\pi}\int d\underset{\sim}{x}' \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\, r\left(t - \frac{|\underset{\sim}{x}-\underset{\sim}{x}'|}{c}, \underset{\sim}{x}'\right), \tag{4a}$$

or, in Fourier form,

$$p(\omega,\underset{\sim}{k}) = \rho_0\, k^{-2}\Delta^{\frac{1}{2}}(\omega/kc)\; r(\omega,\underset{\sim}{k}), \tag{4b}$$

where

$$\Delta(\omega/kc) \equiv \left[1 - (\omega/kc)^2\right]^{-2}. \tag{5}$$

Here and in the following, the limits of integration are understood to extend to the whole available domain, unless otherwise indicated.

The relation (4b) serves as a fundamental basis for analysing the spectrum of sound pressure fluctuations, as emitted in an isotropic and homogeneous turbulent field, whose velocity fluctuation is known to possess the Kolmogoroff spectrum.

III. SPECTRAL STRUCTURE IN FREQUENCY AND WAVENUMBER SPACES

A. Source

In this section, we investigate the . . .

(2) in terms of the spectrum of the turbulent velocity field. To this end, we decompose the velocity $\underset{\sim}{u}$ into a constant mean velocity $\bar{\underset{\sim}{u}}$, a macroscopic fluctuation $\underset{\sim}{u}^{(o)}$ and a microscopic fluctuation $\underset{\sim}{u}'$, as

$$\underset{\sim}{u} = \bar{\underset{\sim}{u}} + \underset{\sim}{u}^{(o)} + \underset{\sim}{u}' \quad . \tag{6}$$

The operators $A^{(o)}$, A' select the ranks $\underset{\sim}{u}^{(o)}$, $\underset{\sim}{u}'$, and are

$$A^{(o)} = 1 - \bar{A} - A'$$

$$A' = 1 - A_o$$

to be regulated by ensemble averages $\bar{A}\langle\cdots\rangle$, $A_o\langle\cdots\rangle$ of increasingly finer scales, as denoted by the ranks in the operators $\bar{A}$, A_o .

We can write the macroscopic source

$$\begin{aligned} r^{(o)} &\equiv A^{(o)} r \\ &= \underset{\sim}{\nabla}\underset{\sim}{\nabla} : A^{(o)}\left[\underset{\sim}{u}^{(o)}\underset{\sim}{u}^{(o)} + A_1\langle \underset{\sim}{u}'\underset{\sim}{u}'\rangle\right] , \end{aligned} \tag{7}$$

the condition of incompressibility

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{u}^{(o)} = 0 , \tag{8}$$

and the stress

$$A_1\langle \underset{\sim}{u}'\underset{\sim}{u}'\rangle = -\eta' \underset{\sim}{\nabla}\underset{\sim}{u}^{(o)} , \tag{9}$$

where η' is an eddy viscosity which is deterministic and stationary in $t, \underset{\sim}{x}$, to show that

$$\underset{\sim}{\nabla}\underset{\sim}{\nabla} : \left(\bar{\underset{\sim}{u}}\underset{\sim}{u}^{(o)} + \underset{\sim}{u}^{(o)}\bar{\underset{\sim}{u}}\right) = 0, \quad \text{and} \quad \underset{\sim}{\nabla}\underset{\sim}{\nabla} : A_1\langle \underset{\sim}{u}'\underset{\sim}{u}'\rangle = 0,$$

and to reduce (7) to

$$r^{(o)} = \underset{\sim}{\nabla}\underset{\sim}{\nabla} : A^{(o)} \underset{\sim}{u}^{(o)} \underset{\sim}{u}^{(o)} \quad . \tag{10a}$$

In wavenumber space, this is

$$r^{(o)}(\underset{\sim}{k}) = -\int d\underset{\sim}{k}' (k_i - k_i') k_j' A^{(o)} u_i^{(o)}(\underset{\sim}{k}') u_j^{(o)}(\underset{\sim}{k} - \underset{\sim}{k}') . \tag{10b}$$

We can calculate

$$\bar{A}\langle|n^{(0)}(\underset{\sim}{k})|^2\rangle = \bar{A}\iint d\underset{\sim}{k}' d\underset{\sim}{k}''(k_i - k_i')k_j'(-k_\ell - k_\ell'')k_m''\langle u_i^{(0)}(\underset{\sim}{k}')u_j^{(0)}(\underset{\sim}{k}-\underset{\sim}{k}')u_\ell^{(0)}(\underset{\sim}{k}'')u_m^{(0)}(-\underset{\sim}{k}-\underset{\sim}{k}'')\rangle$$
$$= -\bar{A}\iint d\underset{\sim}{k}' d\underset{\sim}{k}''(k_i - k_i')k_j'(-k_\ell + k_\ell'')k_m''\langle u_i^{(0)}(\underset{\sim}{k}')u_j^{(0)}(\underset{\sim}{k}-\underset{\sim}{k}')u_\ell^{(0)}(-\underset{\sim}{k}'')u_m^{(0)}(-\underset{\sim}{k}+\underset{\sim}{k}'')\rangle .$$

(11)

Simplifications will be made by using the following properties:

(i) isotropy

$$\bar{A}\langle u_i^{(0)}(\underset{\sim}{k})\, u_j^{(0)}(-\underset{\sim}{k})\rangle = \tfrac{1}{3}\bar{A}\langle u_i^{(0)}(\underset{\sim}{k})\, u_j^{(0)}(-\underset{\sim}{k})\rangle\, \delta_{ij} \,, \tag{12a}$$

(ii) homogeneity

$$\bar{A}\langle \underset{\sim}{u}^{(0)}(\underset{\sim}{k}')\, \underset{\sim}{u}^{(0)}(\underset{\sim}{k}'')\rangle = \chi_1 \bar{A}\langle \underset{\sim}{u}^{(0)}(\underset{\sim}{k}')\, \underset{\sim}{u}^{(0)}(-\underset{\sim}{k}')\rangle\, \delta(\underset{\sim}{k}'+\underset{\sim}{k}'') . \tag{12b}$$

(iii) incompressibility of turbulence

$$\underset{\sim}{k} \cdot \underset{\sim}{u}^{(0)}(\underset{\sim}{k}) = 0 \tag{12c}$$

(iv) quasi-normality

$$\langle u_i^{(0)} u_j^{(0)} u_\ell^{(0)} u_m^{(0)}\rangle = \langle u_i^{(0)} u_j^{(0)}\rangle\langle u_\ell^{(0)} u_m^{(0)}\rangle + \langle u_i^{(0)} u_\ell^{(0)}\rangle\langle u_j^{(0)} u_m^{(0)}\rangle + \langle u_i^{(0)} u_m^{(0)}\rangle\langle u_j^{(0)} u_\ell^{(0)}\rangle . \tag{12d}$$

Here

$$\chi_1 \doteq (\pi/X)^3 \tag{13a}$$

is a factor of truncation of Fourier transformation of a function which is stationary within an interval of space 2X. When the Fourier transformati is extended also to time within an interval 2T, the factor concerned is

$$\chi_2 \doteq \frac{\pi}{T}\left(\frac{\pi}{X}\right)^3 \tag{13b}$$

instead of (13a).

We introduce an energy spectrum F(k), such that

$$\bar{A}\,\langle \underset{\sim}{u}^{(0)2}\rangle = \int d\underset{\sim}{k}'\,\chi_1\,\bar{A}\,\langle|\underset{\sim}{u}^{(0)}(\underset{\sim}{k}')|^2\rangle$$
$$= 2\int_0^k dk'\,F(k')\,, \tag{14a}$$

with

$$F(k) = 2\pi k^2\chi_1\,\bar{A}\,\langle|\underset{\sim}{u}^{(0)}(\underset{\sim}{k})|^2\rangle = A\,\varepsilon^{2/3}k^{-5/3} \tag{14b}$$

in the inertia subrange of isotropic turbulence, and A = 1.6. It follows an enstrophy function

$$R^{(0)} = \int d\underset{\sim}{k}'\,k'^2\chi_1\,\bar{A}\,\langle|\underset{\sim}{u}^{(0)}(\underset{\sim}{k}')|^2\rangle$$
$$= 2\int_0^k dk'\,k'^2\,F(k')\,. \tag{15}$$

Similarly, we introduce a spectrum S(k) for the fluctuations of the source

$$\bar{A}\,\langle r^{(0)2}\rangle = \int d\underset{\sim}{k}'\,\chi_1\,\bar{A}\,\langle|r^{(0)}(\underset{\sim}{k}')|^2\rangle$$
$$= 2\int_0^k dk'\,S(k')\,, \tag{16a}$$

and a pressure spectrum $\Pi(k)$, such that

$$\bar{A}\,\langle p^{(0)2}\rangle = \int d\underset{\sim}{k}'\,\bar{A}\,\langle|p^{(0)}(\underset{\sim}{k}')|^2\rangle$$
$$= 2\int_0^k dk'\,\Pi(k')\quad. \tag{16b}$$

The quadruple correlation in (11) can be reduced by the use of (12a) - (12d). It is shown that the first product

$$\langle u_i^{(0)}(\underset{\sim}{k}')\,u_j^{(0)}(\underset{\sim}{k}-\underset{\sim}{k}')\rangle\,\langle u_\ell^{(0)}(-\underset{\sim}{k}'')\,u_m^{(0)}(-\underset{\sim}{k}+\underset{\sim}{k}'')\rangle$$

in (12d) gives no contribution, and the remaining products yield

$$\chi_1 \bar{A}\langle|r^{(0)}(\underset{\sim}{k})|^2\rangle = \frac{2}{9}\int d\underset{\sim}{k}'\, k'^2 \chi_1 \bar{A}\langle|\underset{\sim}{u}^{(0)}(\underset{\sim}{k}')|^2\rangle(\underset{\sim}{k}-\underset{\sim}{k}')^2 \chi_1 \bar{A}\langle|\underset{\sim}{u}^{(0)}(\underset{\sim}{k}-\underset{\sim}{k}')|^2\rangle . \tag{17}$$

In the notation of (15) and (16a), we can rewrite (17) in the form:

$$\begin{aligned} 2\int_0^k dk''S(k'') &\equiv \int d\underset{\sim}{k}''\chi \bar{A}\langle|r^{(0)}(\underset{\sim}{k}'')|^2\rangle \\ &= \frac{2}{9}\int dk'\, k'^2 \chi_1 \bar{A}\langle|u^{(0)}(k')|^2\rangle \int dk''(k''-k')^2 \chi_1 \bar{A}\langle|u^{(0)}(k''-k')|^2\rangle \\ &= \frac{2}{9} R^{(0)2} , \end{aligned} \tag{18}$$

giving, by differentiation,

$$S(k) = \frac{4}{9} R^{(0)} k^2 F(k) . \tag{19}$$

This relation is equivalent to

$$\chi_1 \bar{A}\langle|r^{(0)}(\underset{\sim}{k})|^2\rangle = \frac{4}{9} R^{(0)} k^2 \chi_1 \bar{A}\langle|u^{(0)}(\underset{\sim}{k})|^2\rangle , \tag{20}$$

or

$$\chi_2 \bar{A}\langle|r^{(0)}(\omega,\underset{\sim}{k})|^2\rangle = \frac{4}{9} R^{(0)} k^2 \chi_2 \bar{A}\langle|\underset{\sim}{u}^{(0)}(\omega,\underset{\sim}{k})|^2\rangle . \tag{21}$$

B. Pressure

From (4b) and (21), we derive the spectral structure of pressure

$$\begin{aligned} \chi_2 \bar{A}\langle|p^{(0)}(\omega,\underset{\sim}{k})|^2\rangle &= \rho_0^2 k^{-4}\Delta(\omega/kc)\, \chi_2 \bar{A}\langle|r^{(0)}(\omega,\underset{\sim}{k})|^2\rangle \\ &= \frac{4}{9}\rho_0^2 R^{(0)} k^{-2}\Delta(\omega/kc)\, \chi_2 \bar{A}\langle|\underset{\sim}{u}^{(0)}(\omega,\underset{\sim}{k})|^2\rangle , \end{aligned} \tag{22a}$$

and its intensity

$$\bar{A}\langle p^{(0)2}\rangle \equiv \iint d\omega' d\underset{\sim}{k}'\, \chi_2\, \bar{A}\left\langle |p^{(0)}(\omega',\underset{\sim}{k}')|^2\right\rangle$$

$$= \frac{4}{9}\rho_0^2 \iint d\omega' d\underset{\sim}{k}'\, R^{(0)}(k')\, k'^{-2}\Delta(\omega'/k')\, \chi_2 \bar{A}\left\langle |\underset{\sim}{u}^{(0)}(\omega',\underset{\sim}{k}')|^2\right\rangle. \tag{22b}$$

IV. DISPERSION RELATION

A. Eulerian and Lagrangian Representations

We shall investigate the dispersion relation, i.e. a relation between frequency and wavenumber in turbulence, by considering the following transform

$$\bar{A}\left\langle \underset{\sim}{u}^{(0)}(t)\; U(t,t-\tau)\underset{\sim}{u}^{(0)}(t-\tau)\right\rangle = \iint d\omega' d\underset{\sim}{k}'\, \chi_2\, \bar{A}\left\langle |\underset{\sim}{u}^{(0)}(\omega',\underset{\sim}{k}')|^2\right\rangle \hat{h} \tag{23}$$

between the Lagrangian correlation at two times $t-\tau$ and t on the left hand side and the Eulerian spectrum $\chi_2\bar{A}\left\langle |\underset{\sim}{u}^{(0)}(\omega',\underset{\sim}{k}')|^2\right\rangle$, through a kernel on the right hand side. The propagator $U(t,t-\tau)$ gives a Lagrangian form to the velocity $\underset{\sim}{u}^{(0)}(t-\tau)$ which follows a perturbed trajectory. In Fourier space, the propagator becomes a Lagrangian kernel

$$\hat{h} \cong A_1\, h'\, h_u \quad , \tag{24a}$$

consisting of: (a) an Eulerian kernel

$$h_u = e^{-i\underset{\sim}{k}'\cdot\underset{\sim}{\bar{u}}} \quad , \tag{24b}$$

which represents a free streaming (deterministic) of a frozen turbulent pattern by a mean velocity $\underset{\sim}{\bar{u}}$, and (b) a relaxation kernel

$$A_1 h' = e^{-\omega_D''^3\tau^3}, \tag{24c}$$

which represents a diffusive perturbation of the trajectory at a relaxation frequency

$$\omega_D'' = \left(\frac{1}{3} k'^2 D''\right)^{\frac{1}{3}} \tag{25a}$$

dependent on the diffusivity D". In the inertia subrange of strong turbulence, this diffusivity is determined to be

$$D'' = c_D \, \varepsilon \quad , \tag{25b}$$

where ε is the rate of energy dissipation, and $c_D \cong 6.29$ is a numerican coefficient. In weak turbulence, the kernel (24c) should be replaced by another dynamical kernel. This dynamical kernel is negligible in the present strong turbulence. The calculations of the propagator and the kernels f m the basis of closure by relaxation. For specific details, see Ref. 8.

B. Diffusion Time

The Lagrangian correlation, (23) defines a frequency spectrum

$$F(\omega) = \frac{1}{\pi}\int_0^\infty d\tau \; e^{i\omega\tau} \chi_2 \; \bar{A}\langle \underset{\sim}{u}^{(0)}(t) \cdot U(t,t-\tau) \; \underset{\sim}{u}^{(0)}(t-\tau)\rangle \tag{26a}$$

which becomes

$$F(\omega) = \iint d\omega' d\underset{\sim}{k}' \; \chi_2 \; \bar{A}\langle |\underset{\sim}{u}^{(0)}(\omega', \underset{\sim}{k}')|^2 \rangle \, \tau_k(\omega, \underset{\sim}{k}') \tag{26b}$$

with the use of (24a). Here the dispersion time is defined by the Fourier transform of $\hat{h}(\tau)$, as

$$\begin{aligned} \tau_k(\omega, \underset{\sim}{k}) &= \text{real} \, \frac{1}{\pi}\int_0^\infty d\tau \; \hat{h}(\tau) \; e^{i\omega\tau} \\ &= \text{real} \, \frac{1}{\pi}\int_0^\infty d\tau \; \exp\left\{-\left[\omega_D''(k)\tau\right]^3 + i\left(\omega - \underset{\sim}{k}\cdot\underset{\sim}{\bar{u}}\right)\tau\right\} \end{aligned} \tag{27}$$

by (24), and satisfies the normalization condition

$$\int d\omega\, \tau_k(\omega, \underset{\sim}{k}) = 1 \quad . \tag{28}$$

It may not be too wrong to replace $(\omega_D''\tau)^3$ by $(\omega_D''\tau)^2$ and to estimate (27) by means of the following approximate form:

$$\tau_k(\omega,k) \cong \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau \;\; \exp\left[-(\omega_D''\tau)^2 + i(\omega - \underset{\sim}{k}\cdot\underset{\sim}{\bar{u}})\tau\right]$$

$$= \frac{1}{\sqrt{4\pi\omega_D''^2}} \exp\left[-\frac{(\omega - \underset{\sim}{k}\cdot\underset{\sim}{\bar{u}})^2}{4\omega_D''^2}\right] . \tag{29}$$

This function may be considered as a distribution of k which peaks at a value of k corresponding to

$$k\cdot\bar{u} = \omega , \quad \text{for small } k, \tag{30a}$$

and

$$\omega_D''(k) = a_\omega \omega , \text{for large } k, \tag{30b}$$

with $a_\omega = \sqrt{3/8}$. A more precise evaluation, based upon (28) instead of (29), will give the same results (30) except with a slightly modified numerical value.

C. Transformation of a Spectrum in ω,k-Space into a Spectrum in k-Space

An intensity

$$\bar{A}\,\langle \underset{\sim}{u}^{(0)2}\rangle = \iiint d\omega\, d\omega'\, d\underset{\sim}{k}'\;\; \chi_2 \bar{A}\,\langle |\underset{\sim}{u}^{(0)}(\omega', \underset{\sim}{k}')|^2\rangle\;\; \tau_k(\omega, \underset{\sim}{k}') \tag{31}$$

is obtained, by integrating (27) with respect to ω from $-\infty$ to ∞. If we perform the integrations with respect to $d\omega'$ and $d\omega$ using (28), we reduce (31) to

$$\bar{A}\langle \underset{\sim}{u}^{(0)2}\rangle = \iint d\omega\, d\underset{\sim}{k}'\ \chi_1\ \bar{A}\langle |u^{(0)}(\underset{\sim}{k}')|^2\rangle\ \tau_k(\omega,\underset{\sim}{k}') \tag{32a}$$

and

$$\bar{A}\langle u^{(0)2}\rangle = \iint d\omega' d\underset{\sim}{k}'\ \chi_2\ \bar{A}\langle |\underset{\sim}{u}^{(0)}(\omega',\underset{\sim}{k}')|^2\rangle, \tag{32b}$$

respectively. A comparison of the right hand sides finds the desired dispersion relation:

$$\chi_2\ \bar{A}\langle |\underset{\sim}{u}^{(0)}(\omega,\underset{\sim}{k})|^2\rangle = \chi_1\ \bar{A}\langle |\underset{\sim}{u}^{(0)}(\underset{\sim}{k})|^2\rangle\ \tau_k(\omega,\underset{\sim}{k}) \tag{33}$$

which serves to transform a function in $\omega,\underset{\sim}{k}$-space into a function in k-space, or reciprocally.

This dispersion relation is now used to transform (22b) into

$$\bar{A}\langle p^{(0)2}\rangle = \frac{4}{9}\rho_0^2 \iint d\omega' d\underset{\sim}{k}'\ R^{(0)}(k')\, k'^{-2}\, \Delta(\omega'/k'c)\,\chi_1 \bar{A}\langle |\underset{\sim}{u}^{(0)}(\underset{\sim}{k}')|^2\rangle\, \tau_k(\omega',\underset{\sim}{k}'). \tag{34}$$

V. SPECTRAL STRUCTURE OF PRESSURE FLUCTUATIONS

A. Frozen Turbulence

In frozen turbulence, the free streaming at velocity $\bar{\underset{\sim}{u}}$ predominates, i.e.

$$|\omega' - \underset{\sim}{k}'\cdot\bar{\underset{\sim}{u}}| \gg \omega_D'' \ ,$$

reducing (29) to

$$\tau_k(\omega',\underset{\sim}{k}') \simeq \delta(\omega' - \underset{\sim}{k}'\cdot\bar{\underset{\sim}{u}}),$$

and (34) to

$$\bar{A}\langle p^{(o)2}\rangle = \frac{8}{9}\rho_o^2\int_0^k dk'\, k'^{-2}\mathcal{R}^{(o)}(k')\, 2\pi k'^2 \chi_1 \bar{A}\langle|\underset{\sim}{u}^{(o)}(\underset{\sim}{k}')|^2\rangle \sigma_1(M)\,, \tag{35}$$

with

$$\begin{aligned}\sigma_1(M) &= \int d\omega' \int_{-1}^{1} d\mu\ \Delta(\omega'/k'c)\ \delta(\omega' - k'\bar{u}\mu) \\ &= \frac{1}{2}\int_{-1}^{1} d\mu\ \Delta(\mu M) \\ &= \frac{1}{2M}\left(\frac{M}{1-M^2} + \frac{1}{2}\ln\frac{1+M}{1-M}\right)\end{aligned} \tag{36}$$

to represent the effect of Mach number $M = \bar{u}/c$ in a free stream of velocity $\bar{u}$. The function (36) is found to be independent of k'.

Note that the available domain of $\underset{\sim}{k}'$-integration corresponds to the limits $(0,k)$ in the k'-integration, because the integrations refer to the truncated functions $\bar{A}\langle p^{(o)2}\rangle$, $R^{(o)}(k')$ and $\bar{A}\langle|\underset{\sim}{u}^{(o)}(\underset{\sim}{k}')|^2\rangle$. Thus, by means of spherical coordinates, we have written

$$\int d\underset{\sim}{k}' \ldots = \int_0^k dk'\, 2\pi k'^2 \int_{-1}^{1} d\mu \ldots$$

in (35), where $\mu \equiv \underset{\sim}{k}'\cdot\underset{\sim}{u}/k'u$ is the co-latitude.

In the inertia subrange (14b), we can calculate (15), and by substitution, reduce (34) to

$$\begin{aligned}\bar{A}\langle p'^2\rangle &= \frac{8}{9}\rho_o^2\sigma_1(M)\int_k^\infty dk'\, k'^{-2}\mathcal{R}^{(o)}(k')\, F(k') \\ &= \frac{4}{3}\rho_o^2 A^2 \sigma_1(M)\, \varepsilon^{4/3}\int_k^\infty dk'\, k'^{-7/3}\end{aligned} \tag{37}$$

or

$$\Pi(k) = \frac{2}{3}\rho_o^2 A^2 \sigma_1(M)\, \varepsilon^{4/3} k^{-7/3}, \tag{38}$$

in terms of $\Pi(k)$ from (16b).

B. Fluid Pressure in Incompressible Turbulence, i.e. $c=\infty$, or $M=0$

With $c=\infty$, we have $\Delta(\omega'/k/c)=1$, and reduce (34) to

$$\bar{A}\langle p^{(0)2}\rangle = \frac{4}{9}\rho_0^2\int d\underset{\sim}{k}'\, R^{(0)}(k')\, k'^{-2}\chi_1 \bar{A}\langle |\underset{\sim}{u}^{(0)}(\underset{\sim}{k}')|^2\rangle, \tag{39}$$

after integration with respect to $d\omega'$ using (28). Further calculations which need not be elaborated give the same result as (38), with

$$\sigma_1(M=0) = 1 . \tag{40}$$

C. ω-Spectrum in the Absence of Free Stream $\underset{\sim}{\bar{u}}$

The dispersion time (29), with $\underset{\sim}{\bar{u}}=0$, which peaks at (30), can be used to calculate (34). We obtain the following intensity

$$\bar{A}\langle p'^2\rangle = \frac{4}{9} b_1 \rho_0^2 \varepsilon^2 \int_{\omega}^{\infty} d\omega'\, \omega'^{-3}\Delta[M_\varepsilon(\omega')] \tag{41}$$

and the spectrum

$$\Pi(\omega) = \frac{2}{9} b_1 \rho_0^2 \varepsilon^2 \omega^{-3}\Delta[M_\varepsilon(\omega)] , \tag{42}$$

where Δ is defined by (5). Since the Mach number

$$M_\varepsilon(\omega) \equiv b_2\varepsilon / c^2\omega$$

may be considered as negligibly small, we can reduce (42) to

$$\Pi(\omega) = \frac{4}{9} b_1 \rho_0^2 \varepsilon^2 \omega^{-3} . \tag{43}$$

The calculations which lead to (41) involve the use of (30), a change of variables from k' into $\omega_D''(k')$ according to (25), and an integration with respect to ω_D''. The details are omitted. The numerical coefficients b_1, b_2 depend on a_ω as :

$$b_1 = 3^{4/3} A^2 a_D^{2/3} a_\omega^{-3}$$
$$b_2 = \frac{1}{3} a_D a_\omega^{-3} \tag{44}$$

The power law ω^{-3}, as predicted by (43), can find agreement with experiments of Gorshkov [4] .

D. ω -Spectrum in Non-frozen Turbulence with Free Stream $\bar{u}$

The spectrum, as determined by (34), is governed by a product of two factors

$$k'^{-2} \Delta(\omega'/k'c)\, \chi_1 \bar{A} \langle | \underset{\sim}{u}^{(0)} (\underset{\sim}{k}') |^2 \rangle \quad \text{and} \quad R^{(0)}(k') \quad , \tag{45}$$

and is weighed by a dispersion time $\tau_k(\omega', \underset{\sim}{k}')$. From (27), (29) and (30), this dispersion time is seen to be bi-modal: When the first factor predominates while the second factor is insignificant at small k', τ_k has the property (30a); on the other hand, when the second factor predominat[es] while the first factor falls off at large k', τ_k has the property (30b). Thu[s] the integration with respect to $d\underset{\sim}{k}'$ in (34), which is weighed by τ_k in the manner described above, amounts to

$$\int d\underset{\sim}{k}'\, R^{(0)}(k')\, k'^{-2} \Delta(\omega'/k'c)\, \chi_1 \bar{A} \langle |\underset{\sim}{u}^{(0)}(\underset{\sim}{k}')|^2 \rangle\, \tau_k(\omega', \underset{\sim}{k}')$$
$$\simeq R^{(0)}(k') \Big|_{\omega_D''(k') = a_\omega \omega'} \int d\underset{\sim}{k}'\, k'^{-2} \Delta(\omega'/k'c)\, \chi_1 \bar{A} \langle |u^{(0)}(k')|^2 \rangle\, \delta(\omega' - \underset{\sim}{k}' \cdot \underset{\sim}{\bar{u}}) . \tag{47}$$

The enstrophy function in (47) is

$$R^{(0)}(k') \Big|_{\omega_D''(k') = a_\omega \omega'} = a_R\, \omega'^2 , \tag{48a}$$

with

$$a_R = \frac{3}{2} A \left(\frac{1}{3} a_D\right)^{-2/3} a_\omega^{\,2} \quad , \tag{48b}$$

and the remaining integral performed in spherical coordinates is

$$A\varepsilon^{2/3} \int_{-1}^{1} d\mu \, (\bar{u}\mu)^{11/3} \Delta(M\mu) \int d\underset{\sim}{k}' \, \delta(\omega' - k'\bar{u}\mu)$$

$$= 2A \, \varepsilon^{2/3} c^{8/3} \sigma_2(M) \, \omega'^{-11/3} \, , \tag{49}$$

with

$$\sigma_2(M) = \frac{1}{M} \int_0^M dx \;\; x^{8/3} \Delta(x) \, . \tag{50}$$

By collecting (47) and (48a) and substituting into (47) and (34), we find an intensity

$$\bar{A}\langle p'^2 \rangle = \frac{8}{9} a_R A \rho_0^2 \varepsilon^{2/3} c^{8/3} \sigma_2(M) \int_k^\infty d\omega' \, \omega'^{-5/3} , \tag{51}$$

and a spectrum

$$\Pi(\omega) = \frac{4}{9} a_R A \rho_0^2 \varepsilon^{2/3} c^{8/3} \sigma_2(M) \;\; \omega^{-5/3} \quad . \tag{52}$$

The power law -5/3, as predicted by (52), agrees with experiments [5].

APPENDIX: EMISSION AND SCATTERING OF SOUND BY TURBULENCE

The propagation of sound, as represented by the variables ρ^L = density, p^L=pressure and $\underset{\sim}{u}^L$= velocity, can be obtained by an operator $\mathcal{L}$ which selects longitudinal mode, as may be driven by a turbulent source. The transversal modes are then selected by a complementary operator $1-\mathcal{L}=\hat{\mathcal{L}}$. The equations of continuity and momentum of the sum of longitudinal and transversal modes are:

$$\partial_t \rho + \nabla_j(\rho u_j) = 0 , \tag{A1}$$

$$\partial_t(\rho u_i) + \nabla_j(\rho u_i u_j) = -\nabla_i p , \tag{A2}$$

the effects of viscosity being neglected.

Upon differentiating (A1) and (A2) with respect to t and $\underset{\sim}{x}$, and by subsequently taking the difference, we obtain

$$\partial_t^2 \rho - \nabla^2 p = \nabla_i \nabla_j \rho u_i u_j . \tag{A3}$$

By applying the operators $\mathcal{L}$ and $\hat{\mathcal{L}}$, we obtain:

$$\partial_t^2 \rho^L - \nabla^2 p^L = \mathcal{L}\, \nabla_{ij}^2 (\rho_0 + \rho^L)(u_i^L u_j^L + u_i^T u_j^T + u_i^L u_j^T + u_i^T u_j^L) , \tag{A4a}$$

$$-\nabla^2 p^T = \hat{\mathcal{L}}\, \nabla_{ij}^2 (\rho_0 + \rho^L)(u_i^L u_j^L + u_i^T u_j^T + u_i^L u_j^T + u_i^T u_j^L) . \tag{A4b}$$

Note that the density in the transverse mode is simply ρ_0 = constant. It is obvious that the sum of (A4a) and (A4b) recovers (A3).

We assume that turbulence is completely transversal on account of its incompressibility, and is not much modified by the propagation of longitudinal modes, and that the sound, which is represented by longitudinal modes, has a linear propagation. Then we reduce (A4a) to

$$\partial_t^2 \rho^L - \nabla^2 p^L \simeq \hat{\mathcal{L}} \nabla^2_{ij} (\rho_0 + \rho^L)\left(u_i^T u_j^T + u_i^L u_j^T + u_i^T u_j^L\right) . \tag{A5a}$$

$$-\nabla^2 p^T \simeq \hat{\mathcal{L}} \nabla^2_{ij} \rho_0 u_i^T u_j^T . \tag{A5b}$$

The term $\rho_0 \nabla^2_{ij} u^T_i u^T_j = r$ represents a turbulent source for the emission of sound, and the terms linear in ρ^L and u^L represent scattering.

By neglecting scatterings, and linearizing the longitudinal modes, we obtain the Lighthill equation in the form

$$\partial_t^2 \rho^L - \nabla^2 p^L \cong r ,$$

or

$$\left(c^2 \partial_t^2 - \nabla^2\right) p \cong r , \tag{A6}$$

if the equation of state

$$dp/d\rho = c^2$$

with the speed of sound c is used.

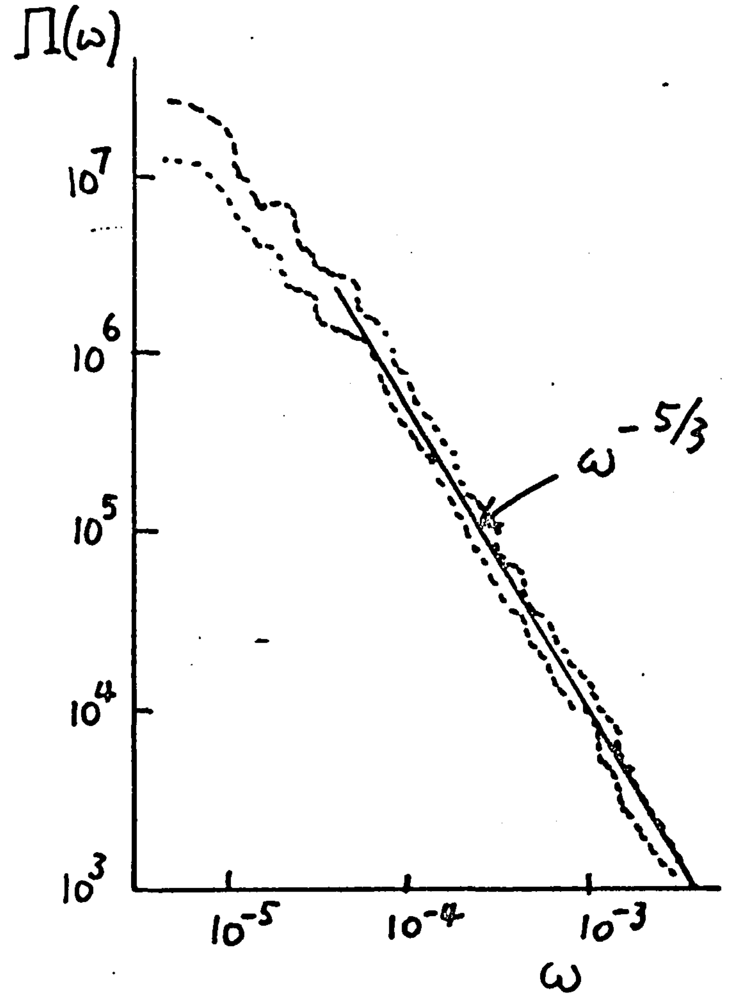

Fig. 1.. Spectrum of pressure fluctuations [5].

Propagation of Light Through Random Refractive Index Fluctuations in Non-Frozen Turbulence

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ABSTRACT

A nonlinear dispersion relation for the space-time transformation in light propagation through a turbulent medium is developed. The derivation is based upon a nonlinear propagator, calculating the perturbed trajectory along which an optical field is transported by turbulent motions. The nonlinear theory of dispersion relation is developed without the hypothesis of frozen turbulence. The frequency spectrum thus found agrees better with experiments than the one based on the hypothesis of frozen turbulence. The remote sensing by two laser beams can measure simultaneously the mean cross-wind and the eddy diffusivity in this method. The dispersion relation without the hypothesis of frozen turbulence finds applications to other remote sensing techniques, e.g. remote sensing by two receivers, and by multi-wavelengths with or without absorption by aerosols, and methods of reduction of turbulence effects in remote sensing.

I. INTRODUCTION

Recent progress of laser technology creates a great interest in optical turbulence, dealing with the coupling between laser irradiation and turbulence in the atmosphere. We shall discuss the following:

(a) laser light propagation in the turbulent atmosphere,

(b) remote sensing of the turbulent characteristics and the structure of the atmosphere,

(c) image retrieval,

(d) propagation through aerosols.

Since the wave field is coupled to the turbulent fluctuations of the refractive index, a parallel effort for investigating the statistics of wave field and atmospheric turbulence becomes necessary.

In the statistics of light waves, the knowledge of turbulence and of the turbulent characteristics of the atmosphere has been meager and limited to the assumptions of validity of Kolmogoroff's spectrum and of Taylor's hypothesis,[1] although the turbulent motions in the atmosphere are of large scales and will clearly deviate from these conditions. Even in small scale turbulence, the mean wind may be weak or there may exist other circumstances where these approximations are not valid. These limitations hamper a correct interpretation of optical data for retrieving the atmospheric characteristics.

The reasons why these assumptions have been made are evident. The assumption of a Kolmogoroff spectrum simplifies the statistics of wave propagation to the problem of a stochastic process with one single parameter. The assumption of Taylor's hypothesis or of frozen turbulence relieves us from the difficulty of a space-time transformation that is generally valid for frozen and non-frozen turbulence conditions. The importance of the general space-time transformation has been pointed out,[2] and there have been suggestions of substituting the standard deviation of wind velocity fluctuations for the mean wind velocity, if the latter is too weak or absent.[3]

The linear theory of light propagation produces a light scintillation σ^2 linearly proportional to the structure coefficient C_n^2 of the refractive index fluctuation n_1, as an index of the intensity of turbulence in the inertia subrange. Experiments show that this linear relation ceases to exist when $\sigma^2 > 0.3$, a limit easily surpassed at a large optical path-length L in space probing. Under this circumstance, a nonlinear theory of light propagation is necessary.

On the one hand, a nonlinearity can appear when the light field E is sufficiently intense as to heat the turbulent medium, so that

$$n_1 \sim |E|^2 .$$

Then the equation of propagation of E-field will contain the nonlinear term

$$|E|^2 E$$

whose role is to produce a self-focusing of laser beam and a thermal blooming. On the other hand, the propagation of a weak E-field in a refractive medium is governed by an equation of propagation linear in n_1 and E. But the presence of the coupling term

$$n_1 E$$

requires a nonlinear statistical treatment for closing the statistical hierarchy entailing moments of ever increasing high orders, even when the statistical characteristics of are given. The literature treats this coupling under the name of "saturation."[4] Experimental devices using differential transmitter and receiver apertures can suppress the sensitivity of large-scale turbulence and thereby correct the effect of saturation.[5] Nevertheless, the optical effects from large eddies and the resulting saturation remain to be an important task of measurements for the purpose of understanding the large-scale structure and characteristics of the atmosphere.

In the following, we shall investigate the space-time transformation by considering the nonlinear perturbation of the trajectory along which a scalar quantity is transported by turbulence. This nonlinear dispersion relation plays an important role in the statistics of optical field and remote sensing. The new parameters which appear in the formula of transformation provides us with a means of remote sensing not only the mean wind, but also the turbulent characteristics. Illustrations are given for: occultation, multi-wavelength laser with and without absorption, cross-wind measurements, and frequency distributions. Comparisons with measurements are made, and show the importance of the nonlinear dispersion. Large eddies having a spectrum different from the Kolmogoroff spectrum are also considered. The space-time transformation which enters in the problem of propagations will also play an important role in "image retrieval."

2. PROPAGATION OF WEAK LASER IRRADIATION

2.1. Wave Equation

The propagation of light is governed by the following Maxwell equations:

$$
\begin{aligned}
\nabla \times \tilde{E} &= -\frac{1}{c}\frac{\partial \tilde{H}}{\partial t} \\
\nabla \times \tilde{H} &= \frac{1}{c}\frac{\partial \tilde{D}}{\partial t} \\
\nabla \cdot \tilde{D} &= 0, \qquad \tilde{D} = \varepsilon \tilde{E} \\
\nabla \cdot \tilde{H} &= 0,
\end{aligned}
\tag{2-1}
$$

where $\tilde{E}(t, x)$ is the electric field, $\hat{H}(t, x)$ is the magnetic field, ε is the dielectric coefficient, and c is the speed of light.

By eliminating $\tilde{H}$, noting that

$$\nabla \times \nabla \times \tilde{E} = -\nabla^2 \tilde{E} + \nabla \nabla \cdot \tilde{E} \tag{2-2a}$$

$$= -\nabla^2 \tilde{E} - \nabla(\tilde{E} \cdot \nabla \ln \varepsilon), \tag{2-2b}$$

and

$$\frac{\partial^2}{\partial t^2}(\varepsilon \tilde{E}) \cong \varepsilon \frac{\partial^2 \tilde{E}}{\partial t^2}, \tag{2-3}$$

and assuming that the atmospheric inhomogeneities vary slowly with time, i.e.

$$\varepsilon(t,\underset{\sim}{x}) \cong \varepsilon(\underset{\sim}{x}) \quad ,$$

we obtain

$$\nabla^2 \tilde{\underset{\sim}{E}} + \underset{\sim}{\nabla}\left(\tilde{\underset{\sim}{E}} \cdot \nabla \ln \varepsilon\right) - \frac{\varepsilon}{c^2} \frac{\partial^2 \tilde{\underset{\sim}{E}}}{\partial t^2} = 0 , \tag{2-4}$$

or

$$\nabla^2 \tilde{\underset{\sim}{E}} + \nabla\left(\tilde{\underset{\sim}{E}} \cdot \underset{\sim}{\nabla} \ln \varepsilon\right) + \frac{k_o^2}{\varepsilon_o} \varepsilon \tilde{\underset{\sim}{E}} = 0 , \tag{2-5}$$

when we let $\tilde{\underset{\sim}{E}} = \underset{\sim}{E} e^{i\omega t}$. Here

$$k_o = \frac{\omega}{c} \sqrt{\varepsilon_o} \tag{2-6}$$

is the optical wavenumber, ε_o is the unperturbed dielectric coefficient, and the perturbation is

$$\varepsilon_1 = \varepsilon - \varepsilon_o \quad . \tag{2-7}$$

It can be assumed that the scale of atmospheric turbulence is much larger than the optical wave length, so that the depolarization as represented by $\underset{\sim}{E} \cdot \underset{\sim}{\nabla} \ln \varepsilon$ in (2b), or in (2a), can be neglected, obtaining the wave equation for the scalar variable E in the form:

$$\nabla^2 E + \left(k_o^2/\varepsilon_o\right) \varepsilon E = 0 . \tag{2-8}$$

For the dielectric coefficient, we can write

$$\varepsilon = n^2, \qquad \varepsilon = \varepsilon_0 + \varepsilon_1 , \qquad n = n_0 + n_1 , \tag{2-9}$$

giving

$$\begin{aligned} \varepsilon_0 &= n_0^2 + \langle n_1^2 \rangle \\ \varepsilon_1 &= 2 n_0 n_1 + \left(n_1^2 - \langle n_1^2 \rangle \right) , \end{aligned} \tag{2-10}$$

where n is the refractive index, with an unperturbed value n_0 and a perturbation n_1 . If the effect of n_1 on the propagation of E is weak, and so is the heating of the atmosphere by the laser, we can neglect the quadratic term, and write

$$\varepsilon_0 \cong n_0^2 , \qquad \varepsilon_1 \cong 2 n_0 n_1 , \qquad \mu = n_1 / n_0 , \tag{2-11}$$

reducing (8) to

$$\nabla^2 E + k_0^2 (1 + 2\mu) E = 0 . \tag{2-12}$$

For the sake of abbreviation and without loss of generality, we shall write (12) as

$$\nabla^2 E + k_0^2 (1 + 2 n_1) E = 0 , \tag{2-13}$$

with $\varepsilon_0 = 1$ and $\mu = n_1$.

2.2. Rytov's Method of Solution

The problem of single scattering can be best explained from the wave equation (13) as a point of departure. The Rytov method introduces a function

$$\psi = \psi_0 + \psi_1 \tag{2-14}$$

with an unperturbed component ψ_0 and a perturbation ψ_1, such that

$$E(\underset{\sim}{x}) \equiv e^{\psi(\underset{\sim}{x})} = E_0(\underset{\sim}{x})\, e^{\psi_1(\underset{\sim}{x})} \tag{2-15}$$

This transforms (13) into the following linearized system:

$$\nabla^2 \psi_0 + (\nabla \psi_0)^2 + k_0^2 = 0, \tag{2-16a}$$

$$\nabla^2 \psi_1 + 2\, \underset{\sim}{\nabla} \psi_0 \cdot \underset{\sim}{\nabla} \psi_1 + 2 k_0^2 n_1 = 0. \tag{2-16b}$$

Another change of variable

$$\psi_1 = w\, e^{-\psi_0} \tag{2-17}$$

transforms (16b) into the differential equation

$$\nabla^2 w + k_0^2 w + 2 k_0^2 n_1 E_0 = 0 \tag{2-18}$$

which gives the solution:

$$w = \frac{2k_0^2}{4\pi} \int d\underset{\sim}{x}' \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} e^{ik_0|\underset{\sim}{x}-\underset{\sim}{x}'|} n_1(\underset{\sim}{x}') E_0(\underset{\sim}{x}') , \qquad (2\text{-}19)$$

or

$$\psi_1 = \frac{2k_0^2}{4\pi E_0} \int d\underset{\sim}{x}' \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} e^{ik_0|\underset{\sim}{x}-\underset{\sim}{x}'|} n_1(\underset{\sim}{x}') E_0(\underset{\sim}{x}') . \qquad (2\text{-}20)$$

The integration extends to the whole available domain.

For the sake of simplification of writing, we introduce a Green's function

$$G(|\underset{\sim}{x}|) = \frac{1}{4\pi|\underset{\sim}{x}|} e^{ik_0|\underset{\sim}{x}|} \qquad (2\text{-}21)$$

and a kernel

$$h(\underset{\sim}{x}, \underset{\sim}{x}') = 2k_0^2 \, G(|\underset{\sim}{x}-\underset{\sim}{x}'|) \, E_0(\underset{\sim}{x}')/E_0(\underset{\sim}{x}) , \qquad (2\text{-}22)$$

simplifying (20) to the form:

$$\psi_1(\underset{\sim}{x}) = \int d\underset{\sim}{x}' \, h(\underset{\sim}{x}, \underset{\sim}{x}') \, n_1(\underset{\sim}{x}') . \qquad (2\text{-}23)$$

If the scale of turbulence in the x direction is much larger than the scales in the $\underset{\sim}{x}_\perp$ -directions, the contribution to ψ_1 in (23) comes mostly from small values of $|\underset{\sim}{x}_\perp - \underset{\sim}{x}_\perp'|$. This permits one to approximate $|\underset{\sim}{x}|$ by

$$|\underset{\sim}{x}| \cong x - \frac{1}{2} \frac{x_\perp^2}{x} , \qquad (2\text{-}24)$$

or to approximate (21) by

$$G(|\underset{\sim}{x}-\underset{\sim}{x}'|) \cong \frac{1}{4\pi(x-x')} \exp ik_0\left[x-x'+\frac{1}{2}\frac{(\underset{\sim}{x}_\perp-\underset{\sim}{x}_\perp')^2}{x-x'}\right] \tag{2-25}$$

and (22) by

$$h(|\underset{\sim}{x}-\underset{\sim}{x}'|) \cong 2k_0^2 G(|\underset{\sim}{x}-\underset{\sim}{x}'|)\, E_0(\underset{\sim}{x}')/E_0(\underset{\sim}{x}) , \tag{2-26}$$

transforming the solution (23) into

$$\psi_1(x,y,z) = \int_0^x dx' \iint dy'dz'\, h(\underset{\sim}{x},\underset{\sim}{x}')\, n_1(\underset{\sim}{x}') . \tag{2-27}$$

We can write

$$\psi = \chi + iS \tag{2-28}$$

as decomposed into an unperturbed value

$$\psi_0 = \chi_0 + iS_0 \tag{2-29}$$

and a perturbation

$$\psi_1 = \chi_1 + iS_1 , \tag{2-30}$$

from

$$E_0 = A_0 e^{iS_0} , \quad E_1 = A_1 e^{iS_1} ,$$

with

$$A_0 \equiv e^{\chi_0} , \qquad A_1 \equiv e^{\chi_1} .$$

Here χ_1 is called log-amplitude fluctuation, and S_1 is called phase fluctuation.

Upon identifying (27) with (30), we find

$$\chi_1(\underset{\sim}{x}) = \int_0^x dx' \iint dy'dz' \, h_r(\underset{\sim}{x}, \underset{\sim}{x}') \, n_1(\underset{\sim}{x}') \tag{2-31a}$$

$$S_1(x) = \int_0^x dx' \iint dy'dz' \, h_i(\underset{\sim}{x}, \underset{\sim}{x}') \, n_1(\underset{\sim}{x}') , \tag{2-31b}$$

where

$$h = h_r + i h_i . \tag{2-32}$$

2.3. Solution for Plane Wave

We consider a plane incident wave

$$E_0(\underset{\sim}{x}) = e^{i k_0 x} , \tag{2-33}$$

transforming (27) and (26) into

$$\psi_1(x,y,z) = \int_0^x dx' \iint dy'dz' \, h(x-x'; \underset{\sim}{x}_\perp - \underset{\sim}{x}'_\perp) \, n_1(x', \underset{\sim}{x}'_\perp) \tag{2-34}$$

$$\begin{aligned} h(x-x'; \underset{\sim}{x}_\perp - \underset{\sim}{x}'_\perp) &= 2 k_0^2 \, G(|\underset{\sim}{x} - \underset{\sim}{x}'|) \, e^{-ik_0(x-x')} \\ &= \frac{k_0^2}{2\pi} \, \frac{1}{x-x'} \exp \frac{i k_0}{2(x-x')} (\underset{\sim}{x}_\perp - \underset{\sim}{x}'_\perp)^2 . \end{aligned} \tag{2-35}$$

The convolution form, as present in (34), suggests a Fourier transform in the transverse direction. Thus by writing

$$n_1(x, \underset{\sim}{x}_\perp) = \int d\underset{\sim}{x}_\perp \, e^{i \underset{\sim}{k} \cdot \underset{\sim}{x}_\perp} \, n_1(x, \underset{\sim}{k}) \, , \tag{2-36}$$

with

$$\underset{\sim}{x}_\perp \equiv (y, z) \, , \quad \underset{\sim}{k} \equiv (k_y \, , k_z) \, , \tag{2-37}$$

we transform (34) into

$$\begin{aligned} \psi_1(x, \underset{\sim}{k}) &= (2\pi)^2 \int_0^x dx' \, h(x-x'; \underset{\sim}{k}) \, n_1(x'; \underset{\sim}{k}) \\ &= ik_0 \int_0^x dx' \, n_1(x'; \underset{\sim}{k}) \exp\left[-\frac{ik^2}{2k_0}(x-x')\right] , \end{aligned} \tag{2-38}$$

with

$$(2\pi)^2 h(x-x'; \underset{\sim}{k}) = ik_0 \exp\left[-\frac{ik^2}{2k_0}(x-x')\right] \, . \tag{2-39}$$

Since h is complex, by (32), we find

$$\begin{aligned} \chi_1(x, \underset{\sim}{k}) &= (2\pi)^2 \int_0^x dx' \, h_r (x-x'; \underset{\sim}{k}) \, n_1(x'; \underset{\sim}{k}) \\ S_1(x, \underset{\sim}{k}) &= (2\pi)^2 \int_0^x dx' \, h_i (x-x'; \underset{\sim}{k}) \, n_1(x'; \underset{\sim}{k}) , \end{aligned} \tag{2-40}$$

with

$$
\begin{aligned}
(2\pi)^2 h_r(x-x';\underset{\sim}{k}) &= k_0 \sin\left[\frac{k^2}{2k_0}(x-x')\right] \equiv (2\pi)^{-1} m_\chi(x';\underset{\sim}{k}) \\
(2\pi)^2 h_i(x-x';\underset{\sim}{k}) &= k_0 \cos\left[\frac{k^2}{2k_0}(x-x')\right] \equiv (2\pi)^{-1} m_S(x';\underset{\sim}{k}) \quad .
\end{aligned}
\tag{2-41}
$$

3. STATISTICS OF FIELD FLUCTUATIONS: CORRELATION AND SCINTILLATION

3.1. Plane Wave

The correlation function of χ-fluctuations is defined by the mean product

$$B_\chi(\underset{\sim}{\rho}) = \langle \chi(\underset{\sim}{x_\perp})\, \chi(\underset{\sim}{x_\perp} + \underset{\sim}{\rho})\rangle \tag{3-1}$$

of the χ-fluctuations at the two points

$$\underset{\sim}{x_\perp}\,, \qquad \underset{\sim}{x_\perp} + \underset{\sim}{\rho}$$

on the plane transversal to the direction x of propagation. Here and in the following we omit the subscript "$_1$" in χ_1 and n_1.

By the Fourier formula, we can write

$$B_\chi(\underset{\sim}{\rho}) = \int d\underset{\sim}{k}\; e^{-i\underset{\sim}{k}\cdot\underset{\sim}{\rho}}\; \Phi_\chi(\underset{\sim}{k})\,, \tag{3-2}$$

with the aid of a spectral function

$$\Phi_\chi(\underset{\sim}{k}) = \left(\frac{\pi}{X}\right)^2 \langle \chi(\underset{\sim}{k})\, \chi(-\underset{\sim}{k})\rangle\,. \tag{3-3}$$

Here $d\underset{\sim}{k} \equiv dk_y\, dk_z$ is the infinitesimal surface element, and $2X$ is the interval of Fourier truncation.

For the calculation of (3), we write

$$\chi(L,\underset{\sim}{k}) = \int_0^L dx'\, (2\pi)^2 h_r(L-x';\underset{\sim}{k})\; n(x';\underset{\sim}{k})\,, \tag{3-4}$$

at $x = L$, from (2-40). We can formulate

$$\langle \chi(L,\underset{\sim}{k})\,\chi(L,-\underset{\sim}{k})\rangle = \int_0^L dx' \int_0^L dx''\,(2\pi)^4 h_r(L-x';\underset{\sim}{k})\,h_r(L-x'';\underset{\sim}{k}) \times \langle n(x';\underset{\sim}{k})\,n(x'';-\underset{\sim}{k})\rangle . \tag{3-5}$$

For large L as compared to the scale l of turbulence in the x -direction, we can replace the integrations with respect to x', x'' by integrations with respect to

$$x_0 = \tfrac{1}{2}(x' + x'') , \quad x_1 = x' - x''$$

as

$$\int_0^L dx' \int_0^L dx'' \ldots \cong \int_0^L dx_0 \int_{-\infty}^{\infty} dx_1 \ \cdots .$$

The integration with respect to x_1 takes the limits $(-\infty, \infty)$ approximately. The kernel h varies slowly with x' and x'' , so that we can approximate

$$h_r(L-x';\underset{\sim}{k}) \cong h_r(L-x'';\underset{\sim}{k}) . \tag{3-6}$$

Hence we can reduce (5) to

$$\langle \chi(L,\underset{\sim}{k})\chi(L,-\underset{\sim}{k})\rangle = \int_0^L dx' \left[(2\pi)^2 h_r(L-x';\underset{\sim}{k})\right]^2 \times \int_{-\infty}^{\infty} dx_1 \langle n(x';\underset{\sim}{k})\, n(x'+x_1,-\underset{\sim}{k})\rangle . \tag{3-7}$$

The last integral is

$$\left(\frac{\pi}{\chi}\right)^2 \int_{-\infty}^{\infty} dx_1 \left\langle n(x',\underset{\sim}{k})\, n(x'+x_1,-\underset{\sim}{k})\right\rangle = 2\pi\, \Phi_n(\underset{\sim}{k}) , \tag{3-8}$$

and can be expressed in terms of the spectral function $\Phi_n(\underset{\sim}{k})$ of n-fluctuations, so that (3-7) becomes

$$\Phi_\chi(\underset{\sim}{k}) \equiv (\pi/\chi)^2 \left\langle \chi_1(L,\underset{\sim}{k})\chi_1(L,-\underset{\sim}{k})\right\rangle = 2\pi \int_0^L dx'\, \Phi_n(\underset{\sim}{k}) \left[(2\pi)^2 h_\chi(L-x';\underset{\sim}{k})\right]^2 . \tag{3-9}$$

It follows, from (2):

$$B_\chi(L,\underset{\sim}{\rho}) = \int_0^L dx' \int d\underset{\sim}{k}\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{\rho}}\, 2\pi\, \Phi_n(\underset{\sim}{k}) \left[(2\pi)^2 h_\chi(L-x';\underset{\sim}{k})\right]^2 . \tag{3-10}$$

The integration with respect to $\underset{\sim}{k}$ can be written in polar coordinate, as

$$B_\chi(L,\underset{\sim}{\rho}) = \int_0^L dx' \int_0^\infty dk\, k \int_0^{2\pi} d\phi\, e^{-ik\rho\cos\phi}\, 2\pi\, \Phi_n(k) \left[(2\pi)^2 h_\chi(L-x';k)\right]^2$$

$$= (2\pi)^2 \int_0^L dx' \int_0^\infty dk\, k\, J_0(k\rho)\, \Phi_n(k) \left[(2\pi)^2 h_\chi(L-x';k)\right]^2$$

$$= \int_0^L dx' \int_0^\infty dk\, k\, J_0(k\rho)\, \Phi_n(k)\, m_\chi^2(x';k) , \tag{3-11a}$$

or as

$$B_\chi(L,\rho) = 2\pi^2 k_0^2 L \int_0^\infty dk\; k\; J_0(k\rho)\; \Phi_n(k)\; f_\chi(k), \tag{3-11b}$$

with the use of a filter function

$$f_\chi(k) = \frac{2}{L}\int_0^L dx'\, \sin^2\left[\frac{k^2}{2k_0}(L-x')\right]$$

$$= 1 - \frac{\sin \eta^2}{\eta^2}, \tag{3-12a}$$

$$\cong \begin{cases} \frac{1}{3!}\eta^4, & \text{for } \eta < 1 \qquad (3\text{-}12b) \\ 1, & \text{for } \eta > 1 . \qquad (3\text{-}12c) \end{cases}$$

Here

$$\eta = k/k_F \quad \text{and} \quad k_F = (k_0/L)^{\frac{1}{2}} . \tag{3-13}$$

We note that

$$2\pi/k_F = (2\pi\lambda L)^{\frac{1}{2}} \tag{3-14}$$

represents the Fresnel zone size, with $\lambda = 2\pi/k_0$.

By definition, the structure function is

$$D_\chi(L,\rho) = \langle[\chi(L, x_\perp+\rho) - \chi(L, x_\perp)]^2\rangle$$

$$= 2\left[\langle[\chi(L,x_\perp)]^2\rangle - B_\chi(L,\rho)\right] . \tag{3-15}$$

Since B_χ is given by (11b), and $\langle\chi^2\rangle$ is given by

$$\langle\chi^2\rangle = B_\chi(L, \underset{\sim}{\rho}=0)$$

$$= 2\pi^2 k_0^2 L \int_0^\infty dk\ k\ \Phi_n(k)\, f_\chi(k), \tag{3-16}$$

we can rewrite (15) as

$$D_\chi(L, \underset{\sim}{\rho}) = 4\pi^2 k_0^2 L \int_0^\infty dk\ k\ \Phi_n(k)\left[1 - J_0(k\rho)\right] f_\chi(k). \tag{3-17}$$

The statistical characteristics of S-fluctuations are obtained by considering h_i instead of h_r. Analogous calculations yield the following formulas:

$$B_S(L, \underset{\sim}{\rho}) = 2\pi^2 k_0^2 L \int_0^\infty dk\ k\ J_0(k\rho)\, \Phi_n(k)\, f_S(k), \tag{3-18}$$

$$\langle S^2\rangle = 2\pi^2 k_0^2 L \int_0^\infty dk\ k\ \Phi_n(k)\, f_S(k), \tag{3-19}$$

$$D_S(L, \underset{\sim}{\rho}) = 4\pi^2 k_0^2 L \int_0^\infty dk\ k\ \Phi_n(k)\left[1 - J_0(k\rho)\right] f_S(k), \tag{3-20}$$

with a filter function

$$f_S(k) = 1 + \frac{\sin \eta^2}{\eta^2} \tag{3-21a}$$

$$\cong \begin{cases} 2, & \text{for } \eta < 1 \\ 1, & \text{for } \eta > 1. \end{cases} \tag{3-21b}$$

The most usual spectrum is the Kolmogoroff spectrum

$$\Phi_n(k) = 0.033\, C_n^2\, k^{-11/3} . \tag{3-22}$$

It is valid in the inertia subrange

$$K_o < k < k_\nu \quad . \tag{3-23}$$

Because of the presence of the filter function $f_\chi(k)$, the contribution of the product $f_\chi(k)\,\Phi_n(k)$ to the integral $(11b)$ will depend on the value of k_F relative to K_o and k_ν . In this connection, we shall consider the following three regions:

(a) geometric optical region $k_\nu < k_F$ (3-24a)

(b) diffraction region $K_o < k_F < k_\nu$ (3-24b)

(c) large scale turbulence region $k_F < K_o$. (3-24c)

The three regions are illustrated in Fig. 1a-c.

In the region (a), the tail of the Kolmogoroff spectrum near the cutoff at

$$k_c \cong k_\nu$$

contributes most significantly to $\langle\chi^2\rangle$. By (12b), we have

$$\langle\chi^2\rangle \cong \frac{0.033}{3!} C_n^2\, 2\pi^2 k_o^2 L \int_0^{k_c} dk\; k\; k^{-11/3} (k/k_F)^4$$

$$= a_1\, C_n^2 L^3 k_c^{7/3} , \tag{3-25}$$

with

$$a_1 = \frac{3}{7}\,\frac{0.033}{3!}\,2\pi^2$$

In the region (b), the Kolmogoroff law contributes

$$\langle \chi^2 \rangle = 0.033\, C_n^2\, 2\pi^2 k_0^2 L \int_0^\infty dk\, k^{-8/3} f_\chi(k)\,, \tag{3-26}$$

by (16) and (22). Since the integral is

$$\int_0^\infty dk\, k^{-8/3} f_\chi(k) = k_F^{-5/3} \int_0^\infty d\eta\, \eta^{-8/3} f_\chi(\eta)\,, \tag{3-27}$$

we find

$$\langle \chi^2 \rangle = a_2\, C_n^2\, k_0^{7/6} L^{11/6}\,, \quad a_2 = 0.307\;. \tag{3-28}$$

In the region (c), the contribution of the Kolmogoroff law (22), that is restricted to the inertia subrange (23), has the form

$$\langle \chi^2 \rangle = 2\pi^2 k_0^2 L \int_{K_0}^{k_\nu} dk\; k\; \Phi_n(k)\, f_\chi(k)\,, \tag{3-29}$$

or

$$\langle \chi^2 \rangle \cong 2\pi^2 k_0^2 L \int_{K_0}^{k_\nu} dk\; k\; \Phi_n(k)$$

$$\cong A\, C_n^2\, k_0^2 L\,, \tag{3-30}$$

with

$$f_\chi \cong 1$$

$$A \cong \frac{6}{5} \times 0.033\,\pi^2 K_0^{-5/3} .$$

The variance (30) has a milder dependence on L than (25) and (28).

Often the turbulent medium has a spectrum not merely restricted to the inertia subrange (23), but possesses in addition a spectrum

$$\Phi_n = C^2 k^{-3} \qquad (3\text{-}31)$$

in the subrange

$$k < K_0$$

of larger-scale fluctuations concomitant with (12b). This spectrum will contribute a variance

$$\langle \chi^2 \rangle \cong \frac{2\pi^2}{3!} k_0^2 L C^2 \int_0^{K_0} dk\ k^{-2} (k/k_F)^4$$

$$= B\, C^2 (k_0 L)^{3/2} , \qquad (3\text{-}32)$$

with

$$B = \frac{2}{3} \frac{1}{3!} \pi^2 K_0^3$$

The parameter C^2 depends on C_n and $\nabla \bar{n}$. The spectrum (31) takes its origin from an inhomogeneous $\bar{n}(\underset{\sim}{x})$.[6]

3.2. Spherical Wave

The spherical wave

$$E_0(\underset{\sim}{x}) = \frac{1}{4\pi|\underset{\sim}{x}|} e^{ik_0|\underset{\sim}{x}|}$$

$$\cong \frac{1}{4\pi x} \exp ik_0 \left(x + \frac{x_\perp^2}{2x} \right) \tag{3-33}$$

has a form similar to the Green function $G(x)$ in (2-21) and (2-25).

Upon substituting (33) and (2-25) into (2-26), we get

$$h(x,x') = \frac{k_0^2}{2\pi} \frac{1}{\gamma(x-x')} \exp i \frac{k_0}{2} \frac{|\underset{\sim}{x}'_\perp - \gamma \underset{\sim}{x}_\perp|^2}{\gamma(x-x')}, \quad \gamma = x'/x . \tag{3-34}$$

By comparing (2-34) and (34), we note that (2-34) for the plane wave can be converted into (34) for the spherical wave by multiplying all variables except ρ' in (2-34) by γ . In this manner, (2-41) is converted into

$$(2\pi)^2 h_r(L-x';\underset{\sim}{k}) = k_0 \sin \frac{k^2}{2k_0}\gamma(L-x') \equiv (2\pi)^{-1} m_\chi(x';k)$$

$$(2\pi)^2 h_i(L-x';\underset{\sim}{k}) = k_0 \cos \frac{k^2}{2k_0}\gamma(L-x') \equiv (2\pi)^{-1} m_S(x';k), \tag{3-35}$$

and (3-11a) is converted into

$$B_\chi(L,\rho) = \int_0^L dx' \int_0^\infty dk\, k\, J_0(\gamma k \rho)\, \Phi_n(k)\, m_\chi^2(x';k) \,. \tag{3-36}$$

By letting $\rho = 0$ in (36), we obtain the variance

$$\langle \chi^2 \rangle = \int_0^L dx' \int_0^\infty dk\, k\, \Phi_n(k)\, m_\chi^2(x';k). \tag{3-37}$$

When the Kolmogoroff spectrum is applied, we find

$$\langle \chi^2 \rangle = 0.033\,(2\pi)^2 k_0^2 \int_0^L dx'\, C_n^2(x') \int_0^\infty dk\, k^{-8/3} \sin^2\left[\frac{k^2}{2k_0}\gamma(L-x')\right] . \tag{3-38}$$

The last integral is

$$\left[\frac{1}{2k_0}\,\frac{x'(L-x')}{L}\right]^{5/6} \int_0^\infty d\eta\, \eta^{-8/3} \sin^2 \eta^2 , \tag{3-39}$$

so that (38) becomes

$$\langle \chi^2 \rangle = \alpha_1 k_0^{7/6} \int_0^L dx'\, C_n^2(x') \left[\frac{x'(L-x')}{L}\right]^{5/6} , \quad \alpha_1 = 0.563. \tag{3-40}$$

If $C_n^2 =$ constant, (40) reduces to

$$\langle \chi^2 \rangle = a_3\, C_n^2\, k_0^{7/6} L^{11/6} , \quad a_3 = 0.124 , \tag{3-41}$$

having a form similar to the plane wave case (28) with a different numerical coefficient.

The formulas (40) and (41) are valid for the diffraction region (24b).

4. REMOTE SENSING OF A PLANETARY ATMOSPHERE BY THE METHOD OF SCINTILLATION

Consider a planet of radius a . The light propagates from a transmitter T (a space satellite), placed at $x'=0$, and is measured by a detector D placed at $x'=L$. See Fig. 2. The point Q on the line of sight is situated at $x'=L_1$ and is at the shortest distance y from the center P of the planet. A variable point M on the line of sight is situated at x' .

We can assume

$$L \ll y \quad , \tag{4-1}$$

and write

$$\begin{aligned} r &= \left[y^2 + (x'-L_1)^2 \right]^{\frac{1}{2}} \\ &\cong y + \frac{1}{2y}(x'-L_1)^2 \quad . \end{aligned} \tag{4-2}$$

The remote sensing by the light propagation from a space probe can measure the structure of the atmospheric environment. Applications can be found in weather prediction, climate modification, pollution studies, storm and clear air turbulence warnings.

We shall discuss the following case of spherical wave propagation.

If C_n^2 depends on the height $r-a$ from the surface of the planet, as

$$C_n^2 = C_{no}^2 \, e^{-(r-a)/H} \tag{4-3}$$

where H is constant, we can substitute (2) and obtain

$$C_n^2(x') = C_{no}^2 \, e^{-\left[(y-a)/H + k_H^2 (x'-L_1)^2\right]}, \quad k_H \equiv (2Hy)^{-\frac{1}{2}}. \tag{4-4}$$

A further substitution of (4) into (3-40) finds

$$\langle \chi^2 \rangle = \alpha_1 k_o^{7/6} C_{no}^2 \, e^{-(y-a)/H} \times \int_0^L dx' \left[\frac{x'(L-x')}{L}\right]^{5/6} \exp\left[-k_H^2 (x'-L_1)^2\right]. \tag{4-5}$$

In view of large k_H, we can approximate the integral by

$$\left[\frac{x'(L-x')}{L}\right]^{5/6}_{x'=L_1} \int_0^{L\to\infty} dx' \exp\left[-k_H^2 (x-L_1)^2\right] = \frac{\sqrt{\pi}}{k_H} \left(L_1 L_2 / L\right)^{5/6}, \tag{4-6}$$

and obtain

$$\langle \chi^2 \rangle = a_4 \sqrt{\pi}\, k_H^{-1} k_0^{7/6} C_{no}^{\,2}\, e^{-(y-a)/H} \left(L_1 L_2 / L\right)^{5/6}, \quad a_4 = 0.563 \, . \tag{4-7}$$

This result is valid in the diffraction region. We find a reciprocity in L_1 and L_2 .

5. STATISTICS OF FIELD FLUCTUATIONS: FREQUENCY SPECTRUM IN A CROSS-WIND

5.1. Frequency Spectrum

We consider a plane wave propagating along the x-direction. See Fig. 3. Two receivers are located at $\underset{\sim}{x}_\perp$ and $\underset{\sim}{x}_\perp + \underset{\sim}{\rho}$ in a cross-wind of velocity $\hat{\underset{\sim}{u}} = (0, \hat{u}_y, \hat{u}_z)$, and measure

$$\chi(t, \underset{\sim}{x}_\perp) \quad \text{and} \quad \chi(t+\tau, \underset{\sim}{x}_\perp + \underset{\sim}{\rho}), \tag{5-1}$$

to form the correlation function

$$B_\chi(\tau, \underset{\sim}{\rho}) \equiv \langle \chi(t, \underset{\sim}{x}_\perp)\, \chi(t+\tau, \underset{\sim}{x}_\perp + \underset{\sim}{\rho}) \rangle, \tag{5-2}$$

with fixed $\underset{\sim}{\rho}$ and variable τ.

We consider a wind of velocity

$$\hat{\underset{\sim}{u}} = \bar{\underset{\sim}{u}} + \underset{\sim}{u},$$

consisting of a constant average velocity $\bar{\underset{\sim}{u}}$ and a fluctuation $\underset{\sim}{u}$.

Since the pattern $\chi(t+\tau, x_\perp)$ at time $t+\tau$ and position $\underset{\sim}{x}_\perp$ must have come from time t and position

$$\underset{\sim}{x}_\perp - \bar{\underset{\sim}{u}}\tau - \underset{\sim}{\ell}(\tau), \tag{5-3}$$

we can write

$$\chi(t+\tau, \underset{\sim}{x}_\perp) = U(t+\tau, t)\chi(t) \tag{5-4}$$

in the notation of a propagator $U(t+\tau, t)$, or evolution operator, which indicates that the transport of χ from t to $t+\tau$ is made along the trajectory perturbed by turbulence. Thus (2) becomes

$$\begin{aligned} B_\chi(\tau, \underset{\sim}{\rho}) &= \left\langle \chi(t, \underset{\sim}{x}_\perp)\, U(t+\tau, t)\chi(\underset{\sim}{x}_\perp(t) + \underset{\sim}{\rho}) \right\rangle \\ &= \left\langle \chi(t, \underset{\sim}{x}_\perp)\, \chi\left[t, \underset{\sim}{x}_\perp + \underset{\sim}{\rho} - \underset{\sim}{\bar{u}}\tau - \underset{\sim}{\ell}(\tau)\right] \right\rangle \\ &= 2\pi^2 k_0 L \int_0^\infty dk\, k\, J_0(k|\underset{\sim}{\rho} - \underset{\sim}{\bar{u}}\tau|)\zeta(k,\tau) f_\chi(k) \Phi_n(k) . \end{aligned} \tag{5-5}$$

The derivation follows that of (3-11b), with an additional factor $\zeta(k,\tau)$ representing the effect of $\underset{\sim}{\ell}(\tau)$ in the perturbed trajectory. The determination of $\zeta(k,\tau)$ will be given in Sec. 7. Note that the case with

$$\zeta(k,\tau) = 1 , \quad \text{or} \quad \underset{\sim}{\ell}(\tau) = 0 \tag{5-6}$$

is equivalent to frozen turbulence.

By letting $\rho = 0$ in (5), we find the temporal correlation, as follows:

$$B_\chi(\tau) = B_\chi(\tau, \underset{\sim}{\rho} = 0)$$

$$= 2\pi^2 k_o^2 L \int_0^\infty dk\, k\, J_o(k\bar{u}\tau)\, \zeta(k,\tau)\, f_\chi(k)\, \Phi_n(k) .$$

(5-7)

In wave propagation, a frequency spectrum is defined as*

$$W_\chi(\omega) = 2\int_{-\infty}^{\infty} d\tau\; e^{-i\omega\tau} B_\chi(\tau) . \tag{5-8}$$

and becomes

$$W_\chi(\omega) = 8\pi^2 k_o^2 L \int_0^\infty dk\; k\; f_\chi(k)\; \Phi_n(k) \int_0^\infty d\tau\, e^{-i\omega\tau} J_o(k\bar{u}\tau)\, \zeta(k,\tau) ,$$

(5-9a)

or

$$W_\chi(\omega) = 8\pi^2 k_o^2 L \int_0^\infty dk\; k\; f_\chi(k)\, \Phi_n(k)\; M(\omega, k) , \tag{5-9b}$$

---

* Note that this definition differs from the usual one :

$$B_\chi(\omega) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau} B_\chi(\tau) = 4\pi\, W_\chi(\omega) .$$

with

$$M(\omega,k) = \int_0^\infty d\tau\, e^{-i\omega\tau} J_0(k\bar{u}\tau)\, \zeta(k,\tau)\,. \tag{5-10}$$

In particular, if the hypothesis (6) of frozen turbulence is valid, we can simplify (10) to

$$M(\omega,k) = \int_0^\infty d\tau\, e^{-i\omega\tau} J_0(k\bar{u}\tau) = \begin{cases} \left[(k\bar{u})^2 - \omega^2\right]^{-\frac{1}{2}}, & \text{for } k\bar{u} > \omega \\ 0, & \text{for } k\bar{u} < \omega, \end{cases} \tag{5-11}$$

and reduce (9b) to the well-known formula of frequency spectrum in frozen turbulence:

$$W_\chi(\omega) = 8\pi^2 k_0^2 L \int_{\omega/\bar{u}}^\infty dk\, k \left[(k\bar{u})^2 - \omega^2\right]^{-\frac{1}{2}} f_\chi(k)\, \Phi_n(k)\quad . \tag{5-12}$$

5.2. Remote Sensing of Cross-Wind

From phase fluctuation S, we can obtain a correlation function $B_S(\tau, \underset{\sim}{\rho})$ in a formula analogous to (5), if f_χ is replaced by f_S. The structure function

$$D_S(\tau,\rho) = 4\pi^2 k_0^2 L \int_0^\infty dk\, k \left[1 - J_0(k|\underset{\sim}{\rho} - \underset{\sim}{\bar{u}}\tau|)\right] \zeta(\tau,k)\, f_S(k)\, \Phi_n(k) \tag{5-13}$$

takes the form analogous to (3-20).

The time derivative is

$$\left.\frac{\partial D_s(\tau,\rho)}{\partial \tau}\right|_{\tau=0} = -4\pi^2 k_o L \int_0^\infty dk\; k\; \Phi_n(k)\, f_s(k)$$

$$\times \left\{ k\bar{u}\, J_1(k\rho) - [1 - J_0(k\rho)] \left.\frac{\partial \zeta}{\partial \tau}\right|_{\tau=0} \right\} , \qquad \text{(5-14a)}$$

In the hypothesis of frozen turbulence $(\zeta = 1)$, (14a) degenerates to the well-known formula

$$\left.\frac{\partial D_s(\tau,\rho)}{\partial \tau}\right|_{\tau=0} = -4\pi^2 k_o^2 L \bar{u} \int_0^\infty dk\; k^2 \Phi_n(k)\, f_s(k)\; J_1(k\rho) \qquad \text{(5-14b)}$$

for the determination of the mean cross wind $\bar{u}$. On the other hand, in the present formulation where this hypothesis is not necessary, the general formula (14) can determine both the mean wind $\bar{u}$ and the turbulent characteristics as represented by ζ . As one important characteristic property, the eddy viscosity can be measured directly, without going through the intermediary of eddy stress and velocity gradient.

6. REMOTE SENSING BY FREQUENCY SPECTRUM FROM TWO WAVES OF DIFFERENT OPERATING FREQUENCIES

6.1. Without Absorption

We consider now the propagation of two waves of different operating frequencies

$$k_{01} = \omega_1/c \,, \qquad k_{02} = \omega_2/c \,, \tag{6-1}$$

with $\varepsilon_0 = 1$, and measure the log-amplitude fluctuations

$$\chi(k_{01}, t_1, \underset{\sim}{x}) \,, \quad \chi(k_{02}, t_2, \underset{\sim}{x}) \tag{6-2}$$

and their correlation function:

$$B_\chi(k_{01}, t_1, \underset{\sim}{x}; k_{02}, t_2, \underset{\sim}{x}) = \langle \chi(k_{01}, t_1, \underset{\sim}{x}) \, \chi(k_{02}, t_2, \underset{\sim}{x}) \rangle$$

$$= B_\chi(k_{01}, k_{02}, \underset{\sim}{x}, \tau) \,, \quad \tau \equiv t_1 - t_2 \,. \tag{6-3}$$

According to (5-8), the frequency spectrum is

$$W_\chi(k_{01}, k_{02}, \underset{\sim}{x}, \omega) = 2 \int_{-\infty}^{\infty} d\tau \, e^{-i\omega\tau} B_\chi(k_{01}, k_{02}, \underset{\sim}{x}, \tau) \,. \tag{6-4}$$

Calculations find

$$B_{\chi 12} \equiv B_\chi(k_{01}, k_{02}, \underset{\sim}{x}, \tau) = \int_0^L dx' \int_0^\infty dk \, k \, J_0(\gamma k \bar{u} \tau) \, \Phi_n(k) \, m_{\chi_{12}}(x', k') \, \zeta(\omega, k)$$

(6-5)

$$W_{\chi 12}(\omega) \equiv W_{\chi}(k_{01}, k_{02}, \underset{\sim}{x}, \omega) = 2\int_0^L dx' \int_{\omega/\bar{u}}^{\infty} dk\ k\left[(\gamma k\bar{u})^2 - \omega^2\right]^{-\frac{1}{2}} \Phi_n(k)\ m_{\chi 12}(x';k)\,\zeta(\omega,k) \tag{6-6}$$

where

$$m_{\chi 12} = m_{\chi 1}\, m_{\chi 2}\ , \qquad m_{\chi 1} = m_{\chi}\Big|_{k_0 = k_{01}}\ , \quad m_{\chi 2} = m_{\chi}\Big|_{k_0 = k_{02}} \tag{6-7}$$

are defined by (2-41), and

$$\gamma = x'/L, \qquad \text{for a spherical wave}$$
$$= 1, \qquad \text{for a plane wave.}$$

If we write

$$W_{\chi} = C_{\chi} - i\,Q_{\chi} \tag{6-8}$$

into two parts C_{χ} and Q_{χ}, called cospectrum and quadrature spectrum, respectively, a coherence coh_{χ} can be defined by

$$coh_{\chi}(k_{01}, k_{02}, \omega) = \frac{C_{\chi}^2(k_{01}, k_{02}, \omega) + Q_{\chi}^2(k_{01}, k_{02}, \omega)}{W_{\chi}(k_{01}, k_{01}, \omega)\, W_{\chi}(k_{02}, k_{02}, \omega)}\ . \tag{6-9}$$

This coherence has been measured, see Fig. 3.[3]

6.2. With Absorption

Kjelaas et al.[7] made an experimental investigation on laser absorption at two wave lengths, and showed that performance degradation due to scintillation can be reduced by exploiting the coherence between two fluctuations. The analysis is based on the propagation of two waves, described in Subsec. 6.1, but extended to include the absorption.

The laser emits radiation alternatively at two optical wave numbers k_{01} and k_{02} with absorption at k_{01} only, giving the intensities at the receiver

$$I_1 = I_{01}\, e^{-(\alpha C + \sigma_1)L}, \quad \text{at } k_{01} \tag{6-9a}$$

$$I_2 = I_{02}\, e^{-\sigma_2 L}, \quad \text{at } k_{02} \tag{6-9b}$$

in a laminar medium, where α is the absorption coefficient of the pollutant gas of concentration C, while σ_1, σ_2 are losses due to aerosols and other gases, and can be assumed of having a negligible difference

$$\sigma_1 - \sigma_2 \cong 0 .$$

It follows from (9a) and (9b):

$$C = \frac{1}{\alpha L}\left(\ln \frac{I_2}{I_{20}} - \ln \frac{I_1}{I_{10}}\right)$$

$$= \frac{2}{\alpha L}(\chi_2 - \chi_1) , \tag{6-10}$$

or

$$c(t) = \frac{2}{\alpha L}\left[\chi_2(t+\Delta) - \chi_1(t)\right] , \tag{6-11}$$

since χ_2 has no absorption. The expressions (10) and (11) are valid for perturbed concentration as well as for perturbation. In the following, we shall consider c, χ_1, χ_2 as perturbations or turbulent fluctuations.

We can form the correlation function of (11) and obtain

$$\begin{aligned} B_c(\tau) &\equiv \langle c(t)\, c(t+\tau)\rangle \\ &= (2/\alpha L)^2 \left\langle \left[\chi_2(t+\Delta) - \chi_1(t)\right]\left[\chi_2(t+\tau+\Delta) - \chi_1(t+\tau)\right]\right\rangle \\ &= (2/\alpha L)^2 \left[B_{\chi 22}(\tau) + B_{\chi 11}(\tau) - B_{\chi 12}(\tau+\Delta) - B_{\chi 12}(\tau-\Delta)\right], \end{aligned} \tag{6-12}$$

where

$$B_{\chi ij}(\tau) \equiv \langle \chi_i(0)\, \chi_j(\tau)\rangle \tag{6-13}$$

is the correlation function of χ, with $i, j = 1, 2$.

A frequency spectrum in the form

$$W(\omega) = 2\int_{-\infty}^{\infty} d\tau\; e^{-i\omega\tau}\, B(\tau) , \tag{6-14}$$

or

$$e^{i\omega\Delta} W(\omega) = 2\int_{-\infty}^{\infty} d\tau \, e^{-i\omega\tau} B(\tau+\Delta) , \qquad (6\text{-}15)$$

can be written in analogy to (5-8). By means of these definitions, we can transform (12) into

$$W_c(\omega) = (2/\kappa L)^2 \left[W_{\chi_{22}}(\omega) + W_{\chi_{11}}(\omega) - 2 W_{\chi_{12}}(\omega) \cos \omega\tau \right] . \qquad (6\text{-}16)$$

The result is shown in Fig. 4. The spectrum decreases as $\Delta \to 0$ and λ_2/λ_1 increases to approach unity.

It remains the problem of derivation of the spectral structure of c-fluctuations from the spectral structure of refractive index fluctuations. The problem involves again the transformation between the frequency and the wave number spaces in a non-frozen turbulence.

7. SPACE-TIME TRANSFORMATION IN NON-FROZEN TURBULENCE

7.1. Transformation of Eulerian Correlations in Time and Space

The analysis of frequency spectrum has usually been made using the hypothesis of frozen turbulence, in which the pattern of refractive index is assumed frozen during its transport by a constant velocity $\bar{\underset{\sim}{u}}$. In the following we shall analyze the frequency spectrum without this hypothesis. To this end, define an Eulerian spatial correlation

$$B_\chi(\underset{\sim}{\rho}) = \langle \chi(t, \underset{\sim}{x}_\perp)\, \chi(t, \underset{\sim}{x}_\perp + \underset{\sim}{\rho}) \rangle \tag{7-1a}$$

and an Eulerian temporal correlation

$$B_\chi(\tau) = \langle \chi(t, \underset{\sim}{x}_\perp)\, \chi(t+\tau, \underset{\sim}{x}_\perp) \rangle \, . \tag{7-1b}$$

If the dynamics of $\chi(t, x_\perp)$ is governed by

$$(\partial_t + L)\chi = 0 \, , \tag{7-2}$$

with a differential operator

$$L = \hat{\underset{\sim}{u}} \cdot \underset{\sim}{\nabla} \tag{7-3}$$

of transport by a wind velocity $\hat{\underset{\sim}{u}} = \bar{\underset{\sim}{u}} + \underset{\sim}{u}$, consisting of a constant mean velocity $\bar{\underset{\sim}{u}}$ and a fluctuation $\underset{\sim}{u}$, we can conceive that the turbulent pattern, as described by

$$\chi(t+\tau, \underset{\sim}{x}_\perp) \tag{7-4}$$

and occupying the position $\underset{\sim}{x}_\perp$ at time $t+\tau$, must have originated from the position $\underset{\sim}{x}_\perp - \hat{\underset{\sim}{\ell}}(\tau)$ at time t in a pattern described by

$$\chi[t, \underset{\sim}{x}_\perp - \hat{\underset{\sim}{\ell}}(\tau)] \equiv U(t+\tau, t)\, \chi(t). \tag{7-5}$$

The propagator $U(t+\tau,t)$ is an evolution operator that represents the transition from t to $t+\tau$ along a trajectory $\hat{\underset{\sim}{\ell}}$ perturbed by the fluctuating velocity from (2) and (3). The trajectory $\hat{\underset{\sim}{\ell}}$ obeys the dynamical equation

$$d\hat{\underset{\sim}{\ell}}/dt = \hat{\underset{\sim}{u}} \quad , \quad \text{or} \quad d\underset{\sim}{\ell}/dt = \underset{\sim}{u}\,. \tag{7-6}$$

By equating (4) and (5) and substituting into (2), we have

$$B_\chi(\tau) = \langle \chi(t, \underset{\sim}{x}_\perp)\, U(t+\tau, t)\, \chi(t) \rangle\,. \tag{7-7}$$

In Fourier form, we can write

$$\chi(t, \underset{\sim}{x}_\perp) = \iint d\omega\, d\underset{\sim}{k}\; e^{-i(\omega t - \underset{\sim}{k}\cdot \underset{\sim}{x}_\perp)}\, \chi(\omega, \underset{\sim}{k}) \tag{7-8a}$$

and

$$U(t+\tau,t)\,\chi(t) \equiv \chi\,[t, \underset{\sim}{x}_\perp - \hat{\underset{\sim}{\ell}}(\tau)]$$

$$= \iint d\omega\, d\underset{\sim}{k}\; e^{-i(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{x}_\perp + \underset{\sim}{k}\cdot\underset{\sim}{\bar{u}}\tau)}\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{\ell}(\tau)}\, \chi(\omega,\underset{\sim}{k}), \tag{7-8b}$$

and find the correlation (7) in the form:

$$B_\chi(\tau) = \iint d\omega' d\underset{\sim}{k}' \iint d\omega'' d\underset{\sim}{k}'' \exp\{-i[(\omega'+\omega'')t - (\underset{\sim}{k}'+\underset{\sim}{k}'')\cdot \underset{\sim}{x}_\perp]\} e^{-i\underset{\sim}{k}''\cdot \underset{\sim}{\bar{u}}\tau}$$
$$\times \left\langle e^{-i\underset{\sim}{k}''\cdot \underset{\sim}{\ell}(\tau)} \chi(\omega', \underset{\sim}{k}')\, \chi(\omega'', \underset{\sim}{k}'') \right\rangle$$
$$\cong \iint d\omega\, d\underset{\sim}{k}\ e^{i\underset{\sim}{k}\cdot\underset{\sim}{\bar{u}}\tau} \left\langle e^{i\underset{\sim}{k}\cdot\underset{\sim}{\ell}(\tau)} \right\rangle \Phi_\chi(\omega, \underset{\sim}{k})$$
$$= \int d\underset{\sim}{k}\ e^{i\underset{\sim}{k}\cdot\underset{\sim}{\bar{u}}\tau} \left\langle e^{i\underset{\sim}{k}\cdot\underset{\sim}{\ell}(\tau)} \right\rangle \Phi_\chi(k)\ . \tag{7-9}$$

Here the conditions of statistical stationarity and homogeneity have been used, and the hypothesis of independence

$$\left\langle e^{-i\underset{\sim}{k}''\cdot\underset{\sim}{\ell}(\tau)} \chi(\omega', \underset{\sim}{k}')\, \chi(\omega'', \underset{\sim}{k}'') \right\rangle \cong \left\langle e^{-i\underset{\sim}{k}''\cdot\underset{\sim}{\ell}(\tau)} \right\rangle \left\langle \chi(\omega', \underset{\sim}{k}')\, \chi(\omega'', \underset{\sim}{k}'') \right\rangle \tag{7-10}$$

has been applied. The spectrum $\Phi_\chi(k)$ has been defined by (3-3).

By following a procedure analogous to that which has led to (3-11a), we transform (9) into

$$B_\chi(\tau) = \int_0^L dx' \int_0^\infty dk\, k\, J_0(k\bar{u}\tau)\, \Phi_n(k)\, m_\chi^2(x',k) \langle e^{i\underset{\sim}{k}\cdot\underset{\sim}{\ell}(\tau)} \rangle .$$

(7-11)

By identifying with (5-5) and noting that

$$\int_0^L dx'\, m_\chi^2(x',k) = 2\pi^2 k_0^2 L\, f_\chi(k), \tag{7-12}$$

we determine

$$\zeta(k,\tau) \equiv \langle e^{i\underset{\sim}{k}\cdot\underset{\sim}{\ell}(\tau)} \rangle . \tag{7-13}$$

It is thus easy to verify (5-6) in a frozen-turbulence. Thus the distinction between the perturbed trajectory in non-frozen turbulence and the free trajectory in frozen turbulence lies in

$\zeta \neq 1$ and $\zeta = 1$, respectively.

We conclude that the Eulerian temporal correlation is thus obtained from the Eulerian spatial correlation that is represented by its spectral function $\Phi_\chi(k)$.

7.2. Calculation of the Perturbation Kernel

The perturbation kernel

$$\zeta(k,\tau) \equiv \langle e^{i\underset{\sim}{k}\cdot\underset{\sim}{\ell}(\tau)} \rangle = \int_{-\infty}^{\infty} d\underset{\sim}{\ell}\; p(\tau,\underset{\sim}{\ell})\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{\ell}} \tag{7-14}$$

can be calculated by means of a probability of transition $p(\tau, \underset{\sim}{l})$ that governs the random path $\underset{\sim}{l}$ in a time τ, and is determined by the Fokker-Planck equation

$$\frac{\partial p}{\partial \tau} = K \frac{\partial^2 p}{\partial \underset{\sim}{l}^2} \tag{7-15}$$

in isotropic turbulence. The eddy diffusivity K, which is characteristic of $\underset{\sim}{\tilde{u}}$ -turbulence, in accordance with the trajectory perturbation (40), takes the Kolmogoroff-Richardson form

$$K = a_K \varepsilon^{1/3} k^{-4/3}, \tag{7-16}$$

when the spectrum of velocity fluctuations obeys the Kolmogoroff law, with a rate of energy dissipation ε.

In Fourier space, (15) is

$$\frac{\partial p(\tau, \underset{\sim}{k})}{\partial \tau} = - K k^2 p(\tau, \underset{\sim}{k}), \tag{7-17}$$

where

$$p(\tau, \underset{\sim}{k}) = \frac{1}{(2\pi)^2} \int d\underset{\sim}{l}\, e^{-i \underset{\sim}{k} \cdot \underset{\sim}{l}} p(\tau, \underset{\sim}{l}) \tag{7-18}$$

is the Fourier component. Noting that

$$p(\tau, k=0) = \frac{1}{(2\pi)^2} \tag{7-19}$$

by the condition of normalization

$$\int d\underset{\sim}{l}\, p(\tau, \underset{\sim}{l}) = 1 , \tag{7-20}$$

we find the following solution

$$p(\tau, \underset{\sim}{k}) = \frac{1}{(2\pi)} e^{-Kk^2\tau} , \tag{7-21}$$

and the perturbation kernel

$$\zeta(k, \tau) = \langle e^{ik\cdot l(\tau)} \rangle = e^{-k^2 K\tau} , \tag{7-22}$$

from (18), (19) and (21).

7.3. Frequency Spectrum

Now we substitute (22) into (11), to obtain

$$B_\chi(\tau) = \int_0^L dx' \int_0^\infty dk\; k\, J_0(k\bar{u}\tau)\, \Phi_n(k)\, e^{-Kk^2\tau} m_\chi^2(x', k) , \tag{7-23}$$

or, approximately,

$$B_\chi(\tau) \cong 2\pi^2 k_0^2 L \int_0^\infty dk\; k\, J_0(k\bar{u}\tau)\, \Phi_n(k)\, f_\chi(k)\, e^{-Kk^2\tau} , \tag{7-24}$$

from (11) and (12). The approximation assumes uniform K and $\bar{u}$ in the space.

By the Fourier transform in the definition (5-8), we can transform the time correlation function (24) into the following frequency spectrum

$$W_\chi(\omega) = 8\pi^2 k_0^2 L \int_0^\infty dk\, k\, \Phi_n(k)\, f_\chi(k)\, M(\omega, k; K, \bar{u}) . \tag{7-25}$$

In an analogous manner, we derive the frequency spectrum of phase fluctuations

$$W_S(\omega) = 8\pi^2 k_0^2 L \int_0^\infty dk\, k\, \Phi_n(k)\, f_S(k) M(\omega, k; K, \bar{u}) . \tag{7-26}$$

Here

$$M(\omega, k; K, \bar{u}) = \int_0^\infty d\tau\, e^{-Kk^2\tau} J_0(k\bar{u}\tau) \cos\omega\tau \tag{7-27}$$

is a weight function. It takes the following two asymptotic values:

(i) frozen turbulence with $\bar{u} \gg Kk$, or $k \gg k_u$

$$M(\omega, k; K=0, \bar{u}) = \begin{cases} \left[(k\bar{u})^2 - \omega^2\right]^{-\frac{1}{2}}, & k\bar{u} > \omega \quad (7\text{-}28a) \\ 0, & k\bar{u} < \omega , \quad (7\text{-}28b) \end{cases}$$

(ii) non-frozen turbulence, with $\bar{u} \ll Kk$, or $k \ll k_u$

$$M(\omega, k; K, \bar{u}=0) = \frac{Kk^2}{(Kk^2)^2 + \omega^2} . \tag{7-29}$$

The critical wave number k_u which separates the two cases

(i) and (ii) is

$$k_u = a_K^3 \varepsilon \bar{u}^{-3} \tag{7-30}$$

The numerical coefficient

$$a_K = 0.96 \tag{7-31}$$

has been determined by Tchen, using a kinetic method.[8-10]

Note that the integrands

$$k \, \Phi_n(k) \, f_{\chi}(k, k_F) \, M(\omega, k, k_u)$$

and

$$k \, \Phi_n(k) \, f_S(k, k_F) \, M(\omega, k, k_u)$$

in (25) and (26) are functions of the following two dimensionless variables

$$k/k_F \qquad \text{and} \qquad k_u/k_F \, , \tag{7-32}$$

which characterize the three regions (3-24a)-(3-24c) of diffraction and distinguish between the frozen turbulence (i) and the non-frozen turbulence (ii).

For a spectrum Φ_n lying in the diffraction region (3-24b), we have

$$k/k_F \sim 1 \, . \tag{7-33}$$

The remaining parameter k_u/k_F from (32) characterizes the following three cases relative to the degree of frozen turbulence:

$$k_u/k_F \simeq 1, \quad k_u/k_F \ll 1, \text{ and } k_u/k_F \gg 1.$$

In the asymptotic limit of high frequency ($\omega \to \infty$), we have:

(A) $k_u \simeq k_F$

$W_S \sim \omega^{-2}$, corresponding to (ii) (7-34a)

$W_\chi \sim \omega^{-8/3}$, corresponding to (i), (7-34b)

(B) $k_u \ll k_F$

$W_S, W_\chi \sim \omega^{-8/3}$, corresponding to (i), (7-35)

(C) $k_u \gg k_F$

$W_S, W_\chi \sim \omega^{-2}$, corresponding to (ii). (7-36)

In the asymptotic limit of low frequency ($\omega \to 0$), we have

$$W_\chi(\omega \to 0) = 8\pi^2 k_0^2 L \int_0^\infty dk\, k\, \Phi_n(k)\, f_\chi(k)\, M(\omega = 0, k; K, \bar{u}),$$
(7-37)
$$W_S(\omega \to 0) = 8\pi^2 k_0^2 L \int_0^\infty dk\, k\, \Phi_n(k)\, f_S(k)\, M(\omega = 0, k; K, \bar{u}),$$

with

$$M(\omega=\upsilon,k;K,\bar{u})=\int_0^\infty d\tau\, e^{-Kk^2\tau} J_0(k\bar{u}\tau), \qquad (7\text{-}38)$$

from (25), (26) and (27).

From the properties (3-12) and (3-21), it is seen that $f_{1\chi}$ weighs toward small scales and f_{1S} weighs toward large scales, so that the frequency spectrum W_χ is dominated by the frozen turbulence, while the frequency spectrum is dominated by the non-frozen turbulence. These physical features and the spectra (34a,b) have been observed in atmospheric environments. See Fig. 5.[11]

The time-space transformation in non-frozen turbulence is developed above by using the approximation of independence (10). A correction has been estimated,[10] and is found small in isotropic turbulence.

8. METHODS OF REDUCTION OF TURBULENCE EFFECTS IN REMOTE SENSING

8.1. Principle of the Methods of Reduction of Turbulence Effects

In remote sensing, an unperturbed field $u_0(\rho')$ from a source or object at the transmitter plane ρ' is perturbed by turbulence, so that the field $u(\rho)$ at the receiver takes the form

$$u(\rho) = \int_{-\infty}^{\infty} d\rho' \; u_0(\rho') \, h(\rho, \rho') , \tag{8-1}$$

where $h(\rho, \rho')$ is a kernel function representative of the perturbation. For interpretation of measured data, the principle is:

A. to solve for u_0, or some statistical characteristics of $u_0(\rho')$, if the kernel h can be found,

B. to minimize the perturbation by means of measurements at two wavelengths.

We shall describe below four methods under principle A, and one method under principle B.

8.2. Speckle Interferometry

This method was developed by Gezari, Labeyrie and Stachnik.[12]

The perturbed field (1) is transformed into irradiance, yielding a relation between the perturbed and unperturbed irradiances I, I_o, in the form of convolution integral

$$I(\rho) = I_o(\rho) * H(\rho) \, , \tag{8-2}$$

which is solved by Fourier transformation, yielding

$$\langle |I(K)|^2 \rangle = |I_o(K)|^2 \langle |H(K)|^2 \rangle \tag{8-3a}$$

where $H(\rho)$ and H(K) are called "spread function" and "transfer function," respectively, and K is wave number.

Since a star can be considered as a point source, i.e. a δ-function, one can determine H(K) by pointing the receiver to a star, and calculate $|I_o(K)|^2$ by

$$|I_o(K)|^2 = \langle |I(K)|^2 \rangle \Big/ \langle |H(K)|^2 \rangle \, . \tag{8-3b}$$

from (3a).

8.3. Intensity Correlation

This method was developed by Korff.[13]

Instead of developing an equation for irradiances at one point, as with (2), it was suggested to consider the correlation function of irradiances at two points:

$$C(r) = \langle I(\rho) I(\rho - r) \rangle \, . \tag{8-4}$$

A convolution is developed for correlations $C_o(r)$ and $C(r)$ at the source and receiver, in the form:

$$C(K) = \int_{-\infty}^{\infty} dK' \, C_o(K') \, H(K-K') . \qquad (8\text{-}5)$$

If $C_o(K)$ is a slowly varying function, and H(K) is a rapidly varying function, (4) can be approximated by the relation

$$C(K) \cong C_o(K) \int_{-\infty}^{\infty} dK' \, H(K') , \qquad (8\text{-}6)$$

which determines the spectral structure of $C_o(K)$. Then a subsequent inversion will yield the correlation $C_o(r)$ at the source.

8.4. Phase Retrieval

The methods described in 8.2 and 8.3 do not provide the phase information. Therefore Knox[14] suggested the retrieval of the phase information by the following formula:

$$\langle I(K) I^*(K+\Delta K) \rangle = I_o(K) I_o^*(K+\Delta K) \langle H(K) H^*(K+\Delta K) \rangle . \qquad (8\text{-}7)$$

8.5. Image Compensation

This method was developed by Deitz.[15] Instead of one single processing of the field u_o at the source through turbulence with kernel h to measure u_I at the receiver, following the formula

$$u_I(\rho'') = \int_{-\infty}^{\infty} d\rho' \; u_o(\rho')\, h(\rho''-\rho'), \tag{8-8a}$$

as in (1), this image is subjected to a second processing through a lens, or "imagery," with a known kernel h_I , to yield a second image u_F at a focal distance, according to a similar formula:

$$u_F(\rho) = \int_{-\infty}^{\infty} d\rho'' \; u_I(\rho'')\, h_I(\rho-\rho'') . \tag{8-8b}$$

The Fourier transform of both formulas (8a) and (8b), i.e.

$$u_I(K) = u_o(K)\, h(K) , \tag{8-9a}$$

$$u_F(K) = u_I(K)\, h_I(K) , \tag{8-9b}$$

reduces to

$$u_F(K) = u_o(K)\, h_I(K)\, h(K) . \tag{8-10}$$

An optimum reception is then obtained by a possible adjustment of h_I such that

$$h_I(K) \simeq \left[h(K) \right]^{-1} . \tag{8-11}$$

Similar processings can be made for irradiances in the place of fields, giving the relation

$$\langle |I_F(K)|^2 \rangle = |I_0(K)|^2 * \langle |\tau(K)|^2 \rangle . \tag{8-12}$$

This relation can be approximated by a simpler one

$$\langle |I_F(K)|^2 \rangle \cong |I_0(K)|^2 \langle |\tau(K)|^2 \rangle , \tag{8-13}$$

if $|I_0(K)|^2$ varies slowly.

8.6. Multi-Wavelength Coherence for Reduction of Turbulence Effects

The method was developed by Kjelaas, Nordal and Bjerkestrand,[7] and has been discussed in Sec. 6. We give its principle in the following lines.

In laser absorption at two wavelengths, the method by Kjelaas can show that performance degradation by turbulent scintillation is reduced by exploiting the coherence between the fluctuations at two wavelengths.

The experiment finds fluctuations of concentration $c'(t)$ by measuring the log amplitude fluctuations χ_1, χ_2 at two wavelengths λ_1, λ_2, separated by a time interval Δ, i.e.

$$c'(t) = \beta \left[\chi_2(t+\Delta) - \chi_1(t) \right] , \tag{8-14a}$$

where β is a constant, dependent on scintillation path and absorption coefficient. Similarly,

$$c(t+\tau) = \beta\left[\chi_2(t+\tau+\Delta) - \chi_1(t+\tau)\right]. \qquad (8\text{-}14b)$$

From (14a) and (14b), the correlation function $C(\tau)$ can be obtained, and is transformed into a spectral function for c', which takes the form

$$W_c(\omega) = W_{\chi_1}(\omega) + W_{\chi_2}(\omega) - W_*(\omega) \qquad (8\text{-}15)$$

where $W_{\chi_1}(\omega)$ and $W_{\chi_2}(\omega)$ are spectral functions of χ_1, χ_2 -fluctuations, and

$$W_*(\omega) \cong C_{012}(\omega) \cos \omega\Delta \qquad (8\text{-}16)$$

is a co-spectrum dependent on Δ and λ_2/λ_1 . This method shows a reduction of performance degradation by minimizing $W_c(\omega)$, that is by maximizing $W_*(\omega)$. This is achieved by letting $\Delta \to 0$ and $\lambda_2/\lambda_1 \to 1$. The method has been applied to atmospheric experiments.

9. EFFECTS OF ABSORPTION BY WATER VAPOR ON LIDAR PERFORMANCE AT 10.6 μm

Recently experiments by Schwiesow and Calfee[16] have shown that the fluctuations of water vapor concentration are equally important, if not more important than the fluctuations of temperature in contributing to the refractive index fluctuations at 10 μm. Experiments have shown that simple arguments of these new effects by water vapor are not sufficient, and would require a careful re-examination of the problem by an analysis of the propagation of light through the atmospheric turbulence.

Most theories of propagation deal with a real refractive index fluctuation. But because the absorption by water vapor on lidar performance at 10.6 μm is important, we include a complex refractive index by writing

$$n + i m \tag{9-1}$$

While a detailed analysis for a weak scintillation theory can be made in a separate task, we can predict the following log-amplitude fluctuations:

$$\langle \chi^2 \rangle = 2\pi^2 k_0^2 L \int_0^\infty dk\, k \sum_i \Phi_i(k)\, f_{\chi i}(k)\,, \tag{9-2}$$

where χ is log-amplitude fluctuation, $k_0 = 2\pi/\lambda$ is optical wavenumber, λ is optical wavelength, L is path length, $\Phi(k)$ is the spectrum of atmospheric inhomogeneities.

The index

$$i = 1, 2, 3 \tag{9-3}$$

denotes the contributions from nn , mm and nm couplings. The filter functions $f_{\chi i}$ are functions of k/k_F , where

$$k_F = (k_0/L)^{\frac{1}{2}} \tag{9-4}$$

is the Fresnel wavenumber, and determine at what scales the spectrum $\Phi(k)$ should be effective.

By writing

$$\langle \chi^2 \rangle \equiv \sum_i \langle \chi_i^2 \rangle = 2\pi^2 k_0^2 L \sum_i \Psi_i \ , \tag{9-5}$$

with

$$\Psi_i \equiv \int_0^\infty dk \ k \ \Phi_i(k) \ f_{\chi_i}(k) \ , \tag{9-6}$$

we expect to find the following:

(a) For i=1 in nn coupling, the filter function $f_{\chi_1}(k)$ determines that those scales in the Kolmogoroff inertia subrange are most effective, so that Ψ_1 depends on

$$\left(\frac{\delta n}{\delta T}\right)^2 C_T^{\ 2} + \left(\frac{\delta n}{\delta q}\right)^2 C_q^{\ 2} \tag{9-7a}$$

and

$$\langle \chi_1^2 \rangle \sim k_0^{7/6} L^{11/6} , \qquad (9\text{-}7b)$$

in agreement with the weak scintillation theory without absorption. Here T is temperature and q is density of water vapor.

(b) For i=2 in mm coupling, the filter function $f_{\chi_2}(k)$ determines that scales larger than the Kolmogoroff scales may not be negligible as in (a). The results (7a) and (7b) are not valid, and will show additional contributions from

$$(\delta m/\delta T)^2 , \quad (\delta m/\delta q)^2 .$$

(c) For i=3 in mn coupling, we expect a law of $\langle \chi_3^2 \rangle$ which will consist of a mixture of (a) and (b), by contributions from

$$\frac{\delta n}{\delta T} \frac{\delta m}{\delta T} , \quad \frac{\delta n}{\delta q} \frac{\delta m}{\delta q} .$$

The large scale fluctuations of the refractive index, temperature and water vapor are not well studied in the literature.

The complex nature (1) due to the presence of water vapor will find a pronounced effect in the reduction of turbulence effect when two optical wave lengths are used, as suggested by Kjelaas.[7]

10. DISCUSSIONS

By means of a nonlinear analysis of the perturbed trajectory along which an optical variable is transported by turbulence, we have derived a space-time relation which greatly improves the statistics of wave propagation. As an illustration, we have analyzed the frequency spectra for χ and S-fluctuations, and generalized the formulas of remote sensings by extending to non-frozen turbulence (cross-wind determination, multiwavelength sensing).

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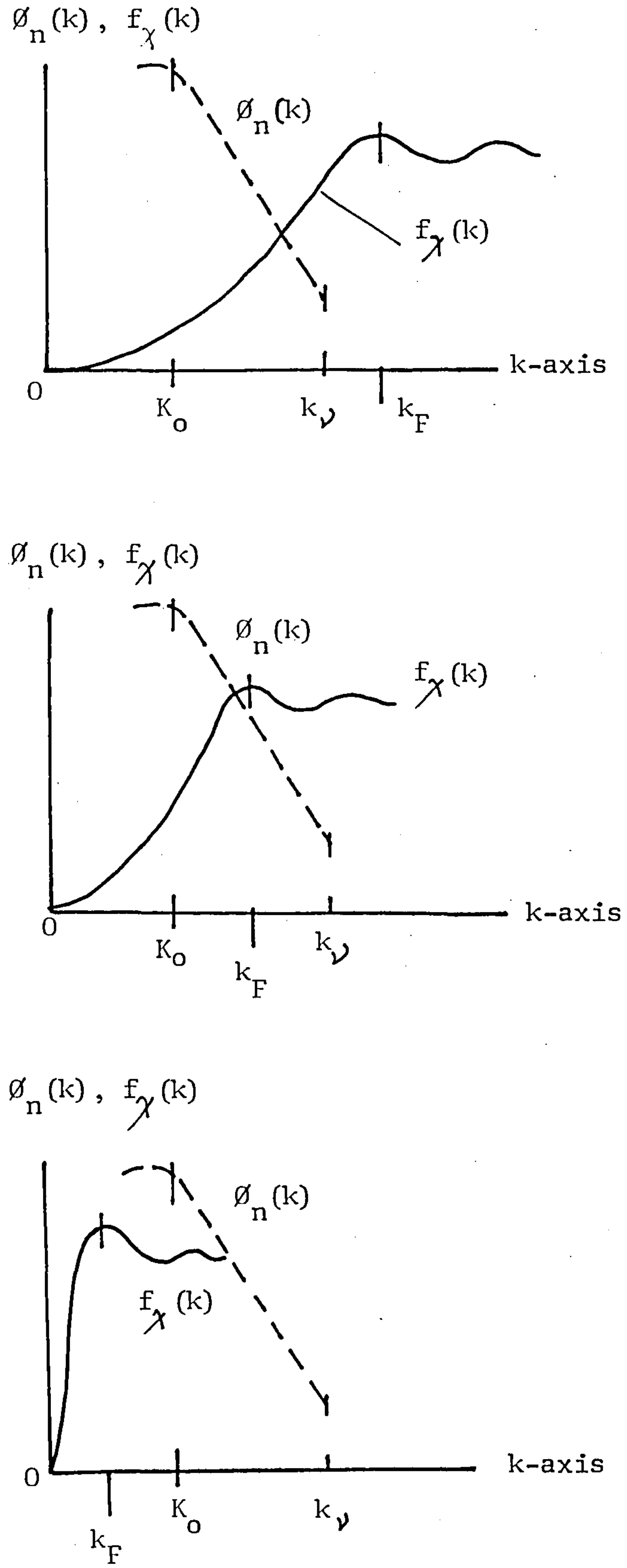

(a) $k_\nu < k_F$

geometric optics region

(b) $K_o < k_F < k_\nu$

diffraction region

(c) $k_F < K_o$

large eddy region

Fig.1 Three regions of propagation

(filter-functions f_χ, $f_S = 2 - f_S$)

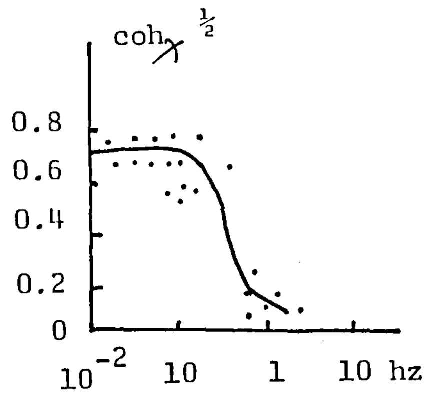

Fig. 3 Experiment in Hawaii.

$k_{01} = 34.5$ GHz

$k_{02} = 9.6$ GHz

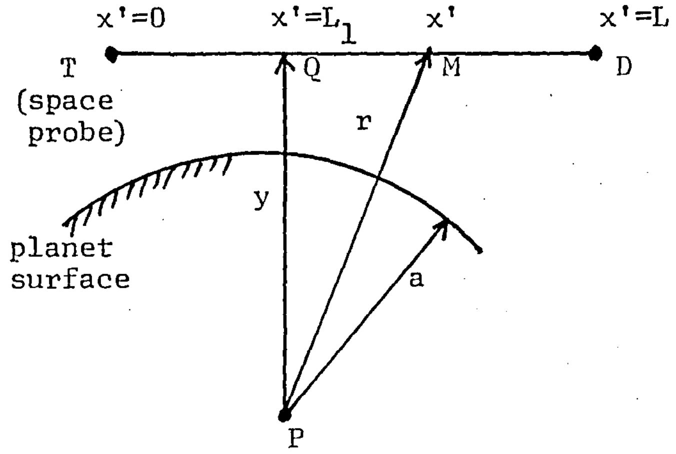

Fig. 2 Space probing of planetary atmosphere

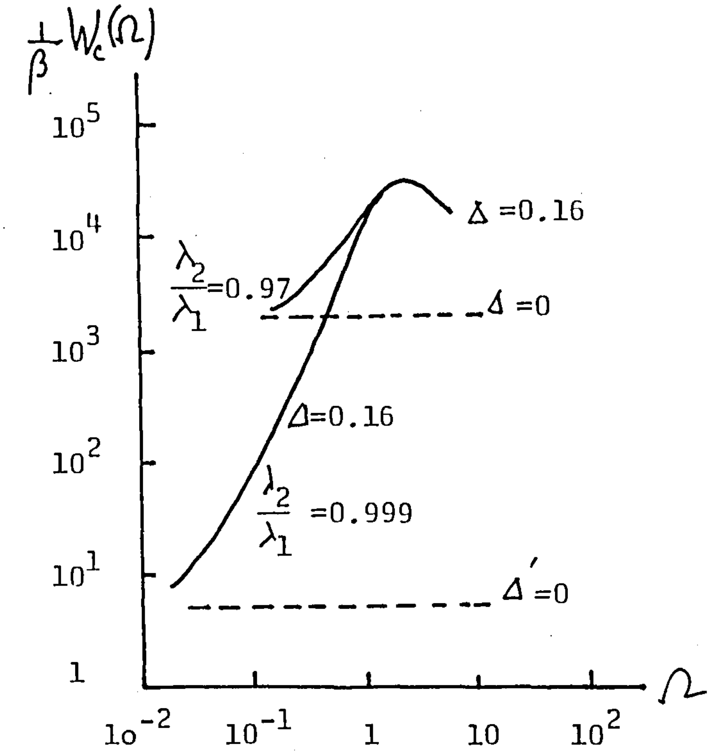

Fig. 4 Frequency spectrum of concentration fluctuations of a gas

$$\Omega = \frac{\omega}{u}\frac{\sqrt{\lambda_2 L}}{2\pi}, \quad \beta = (2/\alpha L)^2 L^{11/6}$$

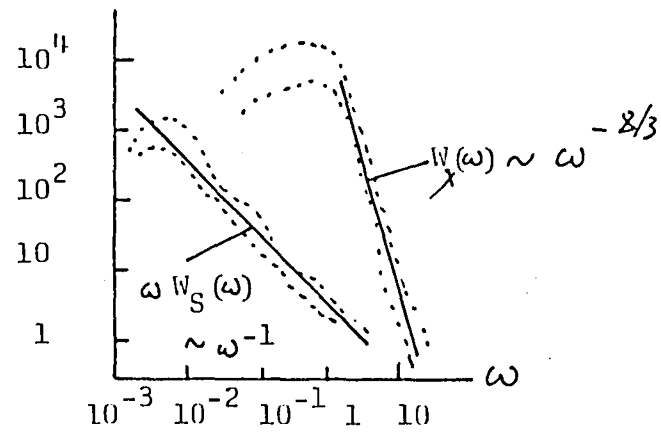

Fig. 5 Frequency spectrum

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