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# THE SOLAR GRAVITATIONAL FIGURE -- $J_2$ AND $J_4$

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ABSTRACT

The theory of the solar gravitational figure is derived including the effects of differential rotation. It is shown that $J_4$ is smaller than $J_2$ by a factor of about 10 rather than being of order $(J_2)^2$ as would be expected for rigid rotation. The dependence of both $J_2$ and $J_4$ on envelope mass is given. High order p-mode oscillation frequencies provide a constraint on solar structure which limits the range in envelope mass to the range $0.01 < M_e/M_\odot < 0.04$. For an assumed rotation law in which the surface pattern of differential rotation extends uniformly throughout the convective envelope, this structural constraint limits the ranges of $J_2$ and $J_4$ in units of $10^{-8}$ to $10 < J_2 < 15$ and $0.6 < -J_4 < 1.5$. Deviations from these ranges would imply that the rotation law is not constant with depth and would provide a measure of this rotation law.

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## I. INTRODUCTION

The gravitational figure of the sun can be measured directly by sending a probe carrying a drag-free guidance system to within a few solar radii of the solar center. The interpretation of such a measurement will require knowledge of how the solar structure and rotational law influence the multi-pole expansion of the external gravitational field. The study of solar oscillations limits the variations in $J_2$ due to structural uncertainties to less than a factor of 2. Refinements in the observed frequencies of the non-radial modes of high order can further reduce this uncertainty. In addition, we show how the existence of differential rotation throughout the convective zone will produce $J_4 >> J_2^2$. An accurate measurement of $J_2$ and detection of $J_4$ can substantially increase our knowledge of the solar rotation law.

The papers dealing with the theory of the gravitational figure for rotating stars divide into two types. The theory of rapidly rotation configurations is generally limited to the study of polytropic models. The classical Clairant-Legendre expansion resulting from this approach is well-describe by Tassoul (1978). Applications based on this method are discussed by James (1964), Ostriker and Mark (1968) and Hubbard, Slattery and DeVito (1975). While this method is very powerful in its ability to treat models of objects in which centripetal acceleration and gravitational acceleration approach equality, it is not easily applied to non-polytropic models because of the fundamental way in which the polytropic assumption is used.

The second type of approach is that described by Goldreich and Schubert (1972). In this approach, the stellar structure equations are analyzed in a perturbation expansion. Consequently, this method fails for objects rotating fast enough that the centripetal and gravitational accelerations are comparable. It has the advantage for application to the solar case that it is not restricted to polytropic models. Since the sun is rotating slowly, we use the Goldreich and Schubert (1972) approach. We present in the following section the theoretical analysis of a differentially rotating model and derive the equation which describes the octopole moment of the sun, $J_4$. In § III we describe the non-rotating zero order models and present the results for $J_2$ and $J_4$ in § IV.

## II. THEORY OF $J_4$

We adopt as a simple rotation law:

$$\Omega = \Omega_0 + \Omega_2 \mu^2 \qquad (1)$$

where $\Omega_0$ is the equatorial angular rotation rate and $\mu$ is the cosine of the colatitude. The term $\Omega_2 \mu^2$ represents the differential rotation. Fitting the rotation law of Howard, Boyden and Laboute (1980) to Equation (1), we find that $\Omega_2/\Omega_0 \approx -0.7$. We assume that $\Omega_2$ and $\Omega_0$ are constant in the convective envelope and that $\Omega_2 = 0$ below the convective envelope. Although this latter assumption may not be valid, $\Omega_2$ must go to zero at $r = 0$ because otherwise there would be a divergence in the shear velocity field. The centripetal acceleration $r\,\Omega^2$ appears in the equations of momentum balance. We expand $\Omega^2$ as $\Omega_0^2\,(1 + a\,\mu^2)$ where:

$$a = \frac{2\ \Omega_2}{r_0} \approx -1.4 \tag{2}$$

Following Goldreich and Schubert (1972), we write the momentum balance equation in spherical coordinates $(r, \mu)$ in terms of the gravitational potential $\phi$, the pressure $p$ and the density $\rho$. The $r$ and $\mu$ components of this equation are:

$$\rho \frac{\partial \phi}{\partial r} = \frac{\partial p}{\partial r} - \rho\ \Omega_0^2\ (1 + a\mu^2)\ r\ (1 - \mu^2) \tag{3}$$

$$\rho \frac{\partial \phi}{\partial r} = \frac{\partial p}{\partial \mu} + \rho\ \Omega_0^2\ r^2\ (1 + a\mu^2)\ \mu . \tag{4}$$

Assuming that the acceleration of the rotation is small compared to the acceleration of gravity, we may expand all quantities into unperturbed quantities $\phi_0$, $p_0$ and $\rho_0$ and their perturbations $\phi_1$, $p_1$ and $\rho_1$. The unperturbed equations are standard and will not be repeated here. The first-order perturbation equations are:

$$\rho_0 \frac{\partial \phi_1}{\partial r} + \rho_1 \frac{\partial \phi_0}{\partial r} = \frac{\partial p_1}{\partial r} - \rho_0\ \Omega_0^2\ (1 + a\mu^2)\ r\ (1 - \mu^2) \tag{5}$$

$$\rho_0 \frac{\partial \phi_1}{\partial \mu} = \frac{\partial p_1}{\partial \mu} + \rho_0\ \Omega_0\ r^2\ (1 + a\mu^2)\ \mu . \tag{6}$$

We now expand $\phi_1$, $\rho_1$ and $p_1$ in spherical harmonics, e.g.:

$$\phi_1 = \sum_{\ell=0,2,4} \phi_{1\ell}\ P_\ell(\mu) \tag{7}$$

where $P_\ell(\mu)$ is the Legendre polynomial of order $\ell$. Using standard recursion relations for the $P_\ell$, the momentum equations have $\ell = 2$ and $\ell = 4$ components which are:

$$\ell = 2: \quad \rho_0 \frac{\partial \phi_{12}}{\partial r} + \rho_{12} \frac{\partial \phi_0}{\partial r} = \frac{\partial p_{12}}{\partial r} + \frac{2}{3} \rho_0 \Omega_0^2 r(1 - \frac{a}{7}). \tag{8}$$

$$\rho_0 \phi_{12} = p_{12} + \frac{r^2 \rho_0 \Omega_0^2}{3} \cdot (1 + \frac{3a}{7}) \tag{9}$$

and

$$\ell = 4: \quad \rho_0 \frac{\partial \phi_{14}}{\partial r} + \rho_{14} \frac{\partial \phi_0}{\partial r} = \frac{\partial p_{14}}{\partial r} + \frac{8a}{35} \rho_0 \Omega_0^2 r \tag{10}$$

$$\rho_0 \phi_{14} = p_{14} + \frac{2a}{35} r^2 \rho_0 \Omega_0^2 \tag{11}$$

The above equations must be supplemented by the Poisson equation:

$$\frac{\partial^2 \phi_{1\ell}}{\partial r^2} + \frac{2}{r} \frac{\partial \phi_{1\ell}}{\partial r} - \frac{\ell(\ell + 1)}{r^2} \phi_{1\ell} = - 4\pi G\rho_{1\ell} \tag{12}$$

Equations (8) and (9) and the $\ell = 2$ component of Equation (12) combine to give the analogue of the standard Goldreich and Schubert (1972) equation for $\phi_{12}$. This equation is:

$$\frac{\partial^2 \phi_{12}}{\partial r^2} + \frac{2}{r} \frac{\partial \phi_{12}}{\partial r} - \frac{6\phi_{12}}{r^2} = \frac{4\pi r^2}{M_r} \left[\phi_{12} \frac{d\rho_0}{dr} - \frac{4}{21} a r\rho_0 \Omega_0^2 - \frac{r^2}{3} \frac{d}{dr} \left(\rho_0 \Omega_0^2 + \frac{3}{7} \rho_0 a\Omega_0^2\right)\right]. \tag{13}$$

This differs from the standard equation in having the extra term involving 4/21 and in replacing $\Omega_0^2$ with $\Omega_0^2 \cdot (1 + 3a/7)$. The equation for the $\ell = 4$ component of Equation (12) is similarly derived and is:

$$\frac{\partial^2 \phi_{14}}{\partial r^2} + \frac{\partial \phi_{14}}{\partial r} - \frac{20}{r^2} \phi_{14} = \frac{4\pi r^2}{M_r} \left[ \phi_{14} \frac{d\rho_0}{dr} + \frac{4}{35} ar \rho_0 \Omega_0^2 - \frac{2}{35} r \frac{d}{dr} \left( a\rho_0 \Omega_0^2 \right) \right] \tag{14}$$

Equations (13) and (14) must be solved subject to the boundary conditions that $\phi_{1\ell} \propto r^{\ell}$ at $r = 0$ and $\phi_{1\ell} \propto r^{-\ell-1}$ as $r \to \infty$ Numerically, these conditions were satisfied by expanding $\phi_{1\ell}$ as $C_\ell r^\ell$ at the innermost mass shell. Equations (13) and (14) were then integrated from the center to the outermost mass point. The outer boundary condition was imposed as an equation of condition and $C_\ell$ was adjusted through a Newton-Raphson iteration procedure. The distribution of $a(r)$ was chosen so that $a(r) = 0$ for $r < r_e$ and $a(r) = -1.4$ for $r > r_e$ where $r_e$ is the radius at the inner edge of the convection zone. In crossing this radius, the following jump condition is applied to the $\phi_{14}$ solution:

$$\frac{\partial \phi_{14}}{\partial r} \Big|_{r_e+\varepsilon} = \frac{\partial \phi_{14}}{\partial r} \Big|_{r_e-\varepsilon} - \frac{8\pi r^4 \rho_0 \Omega_0^2 a}{35 M_r} \tag{15}$$

where $\varepsilon$ is a small distance encompassing the gradient in a. The exterior gravitational field is described by the normalized values of $\phi_{1\ell}$:

$$J_\ell = \frac{r^{\ell+1}}{GM} \phi_{1\ell}. \tag{16}$$

## III. THE SOLAR MODELS

The static solar model has been calculated using the input physics described in the paper by Bahcall et al. (1980). The method differs from previous solar model calculations that have been done using the UCLA code in two respects:

1) The atmosphere code discussed by Ulrich and Rhodes (1977), Ulrich, Rhodes and Deubner (1979) and Lubow, Rhodes and Ulrich (1980), has been extended to include all of the interior physics. By adjusting the mixing length and hydrogen abundance, the atmosphere integration has been extended inward to a radius of $1.5 \times 10^9$ cm. The final interior zone is treated analytically with a theory accurate to better than 1 percent. The interior distribution of hydrogen abundance has been taken from the current standard solar model. This method yields neutrino fluxes which exceed those in the standard model by 10 percent.

2) Partial ionization of all heavy elements is now included in an approximate way following Bodenheimer, Forbes, Gould and Henyey (1965). Further details of the calculational methods will be given by Ulrich and Rhodes (1981).

Two fully converged models were computed: The standard model and a model with the interior heavy element abundance reduced from $Z = 0.018$ to $Z = 0.005$. The convective envelope abundances were normal. The mix of both cases was that given by Ross and Aller (1976). The second model is referred to as the low Z model. In addition to these models, a series of incomplete envelopes was calculated with a specified mixing length parameter. These envelopes were stopped when either 99 percent of the mass or radius was used in the inward integration leaving a large residual of the other variable. An approximate analytic continuation to the center was added by assuming that $M_r \propto r^3$, $(P_0 - P) \propto r^2$, $(\rho_0 - \rho)$

$\propto r^2$ and $(T_0 - T) \propto r^2$. Almost none of the perturbation in is produced in the poorly treated portion of the model.

IV. RESULTS AND DISCUSSION

There are two factors which can influence the values of $J_2$ and $J_4$: 1) the distribution of mass through the sun and 2) the distribution of $\Omega$ (r, $\theta$) through the solar interior. This latter function is observed at $r = R_o$, but there is as yet only a preliminary indication of its variation with r. A range in structure variation is possible in principle so that both factors must be considered in interpreting any potential measurement of $J_2$ and $J_4$. Fortunately, the structural variations are not constrained by the measurements of the eigenfrequencies of high order non-radial oscillations. The recent observations by Rhodes, Harvey and Duvall (1981) limit the variations in the envelope mass to $0.01 < M_e/M_o < 0.04$. As a measure of the importance of structure in determining $J_2$ and $J_4$, we have assumed that $\Omega (r, \theta) = \Omega (R, \theta)$ for $r > r_e$ and that $\Omega (r, \theta) = \Omega (R)$ for $r < r_e$. We have used the rotation law given by Howard, Boyden and La Bonte (1980) in calculating $J_2$ and $J_4$. Our results are given in Figure 1. The above limits on $M_e/M_o$ translate into permitted ranges in $J_2$ and $J_4$:

$$\begin{aligned} 1.0 \times 10^{-7} &< J_2 < 1.5 \times 10^{-7} \\ 6 \quad \times 10^{-9} &< - J_4 < 1.5 \times 10^{-8} \end{aligned} \tag{17}$$

Measured values of $J_2$ or $J_4$ outside of these ranges would indicate a deviation from our simple assumptions for $\Omega$ (r, $\theta$).

The subsurface rotation rate can be measured with the method described by Rhodes, Deubner and Ulrich (1979) and by Deubner, Rhodes and Ulrich (1979).

This second paper gives some first results which show $\Omega$ (r, $\theta$) increasing as r decreases. If this result is borne out by subsequent observations, we would expect larger values for $J_2$ and $|J_4|$. The theory of the solar dynamo depends on such a variation of $\Omega$ with r (Stix 1977). Ultimately, it may be possible to probe the rotation law using the oscillations; however, there are two critical uncertainties which must be resolved before we will know the effectivness of the oscillation method: 1) the oscillations must be adequately long lived and 2) it must be possible to identify correctly individual modes of oscillation for all spatial scales. Also, the oscillations cannot provide a good measure of the rotation rate off the solar equator because foreshortening limits the viewing region. Thus, the measurement of both $J_2$ and $J_4$ would provide very important constraints on $\Omega$ (r, $\theta$), which are different from the constraints imposed by the oscillations.

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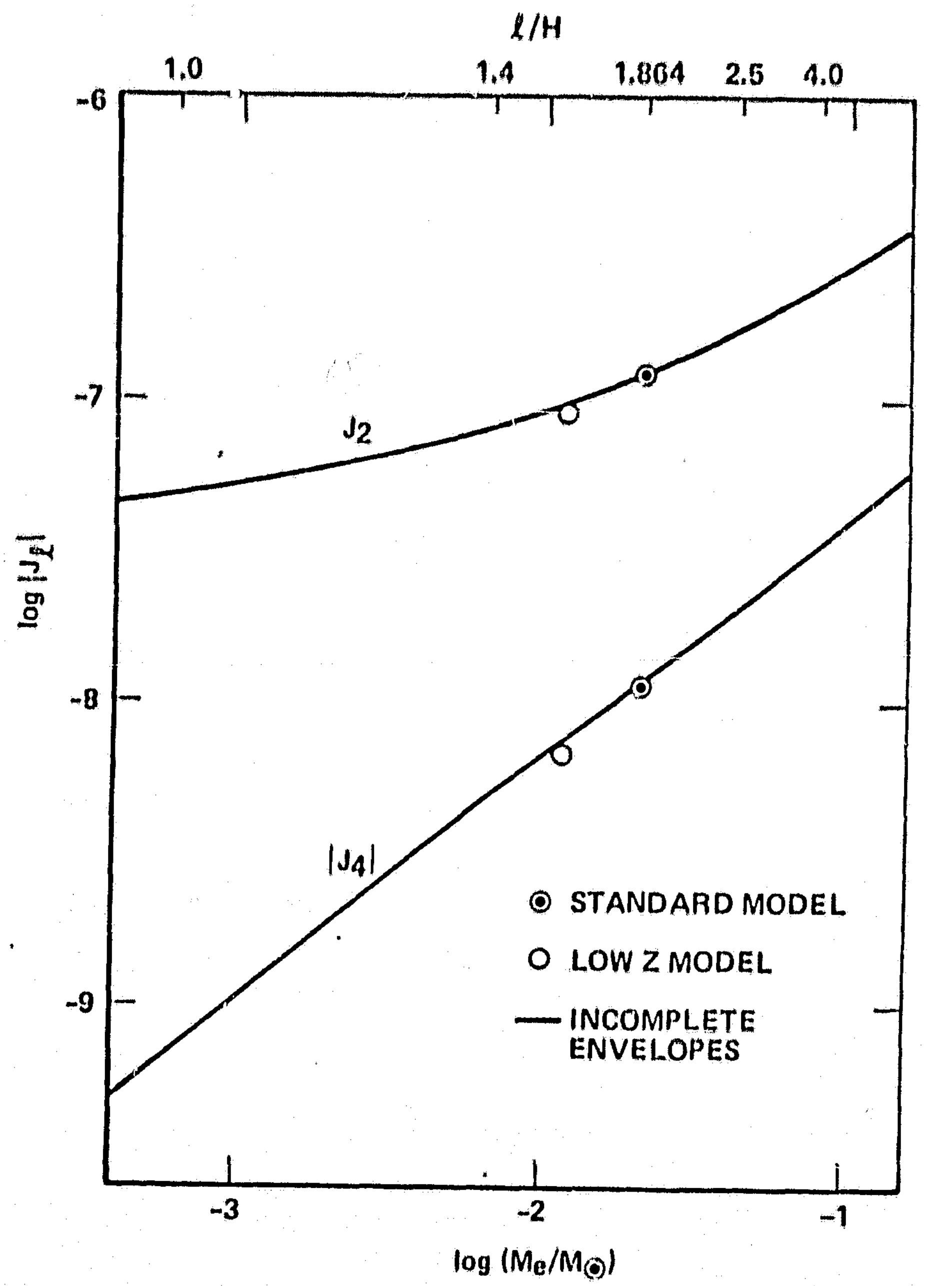
## FIGURE CAPTION

Figure 1. The dependence of $J_2$ and $J_4$ on fractional envelope mass $M_e/M_\oplus$.

Corresponding values of $\ell/H$ are indicated on the top of the figure. These values are dependent on a number of computational details and are less significant than $M_e/M_\oplus$. The characteristics of the standard model and the low Z model are discussed in the text.



ℓ/H
1.0
1.4
1.864
2.5
4.0
-6
-7
-8
-9
log |$J_\ell$|
$J_2$
|$J_4$|
STANDARD MODEL
LOW Z MODEL
INCOMPLETE ENVELOPES
-3
-2
-1
log ($M_e/M_\odot$)


Figure 1