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Slosh Dynamics of a Spin-Stabilized Spacecraft Comprising Off-Axis Tanks Filled Partially With Liquid Propellant

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ABSTRACT

The primary object of this report is to develop a linear model that describes the perturbation motion of a spinning spacecraft made of a family of tanks that are partially filled with fluid.

First the fundamental nonlinear equations of motion are derived and then specialized to a steady-state rotation of the vehicle about a given axis of rotation. Then, a thrust about the spin axis is introduced. Finally, a perturbation solution is derived which linearizes the problem. The effect of the centrifugal and coriolis accelerations together with vorticity are implicitly taken into consideration in the formulation. A variational formulation of the associated boundary conditions is presented. For most practical cases it is shown that the simple classical pendulum representation for slosh is not very appealing for a spinning spacecraft unless severe restrictions are allowed.

ACKNOWLEDGEMENT

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Appreciation is also extended to Dr. Michael El-Raheb of JPL for his preliminary work, and illuminating technical discussions on the subject.

INTRODUCTION

Problems concerning the behavior of liquid propellant motions in partially filled tanks of spin-stabilized spacecraft have gained topical importance recently, especially in those instances where the propellants constitute a large percentage of the total vehicle mass. As is known, the interaction between contained propellant and rigid body motions plays a significant role in establishing the design of the attitude control system and assessing the performance of a spacecraft. Indeed, the action of liquid propellant motions on the tank walls ("sloshing") of a spin-stabilized spacecraft may couple with its natural nutational mode of motion so as to saturate the control system and possibly cause flight instabilities, or severely impair its performance such as maintaining pointing accuracy. To predict and control the motions of a spin-stabilized spacecraft with partially filled liquid cavities, it is therefore necessary to single out the participation of sloshing propellants in the overall vehicle motions. Methods for characterizing the behavior of propellant motions in partially filled spherical tanks, as encountered with the design of the Galileo Dual-Spin Jupiter Orbiter spacecraft, were originally developed under somewhat restrictive assumptions.

In 1978 Ref. 1 devised a perturbational construct describing the small liquid motions in a partly filled spherical tank relative to a known reference state of motion. The reference motion was defined as one of uniform rotation of the liquid cavity about a fixed point on a stationary axis under the action of a constant acceleration directed negatively along that axis. In arriving at this construct, it was assumed that: 1) the propellant was an inviscid fluid; 2) the vorticity was independent of the spatial coordinates, i.e., a function of time only; 3) the steady state spin and all rotational perturbations were instantaneously communicated to the liquid; 4) coriolis accelerations were significant throughout the liquid volume but negligible at the free surface; 5) translational motions of the liquid tank systems were nonexistent.

In 1979 Ref. 2 showed that assumptions 2) and 4) gave rise to spurious instabilities for certain liquid fill ratios, and that assumption 3) was inconsistent with the mechanism of vortex transfer from the tank wall to the liquid. The inconsistency of the construct was further substantiated by Ref. 3.

The primary aim of this investigation was to develop a linear construct that describes the perturbational motions of a spin-stabilized spacecraft, comprising N tanks partially filled with liquid propellants, relative to a given state of motion. A further objective was to establish conditions under which sloshing motions in spinning tanks could be represented by simple mechanical pendulums.

The basic subject matter is kept as concise as possible. In Section 1, certain fundamental results and equations from classical mechanics and hydrodynamics are used to derive the equations of motion of a spin-stabilized spacecraft comprising N tanks partly filled with liquid propellants. The participation of the motions of the liquids in the overall vehicle motion is singled out. Diverse forms of the equations are presented. The fundamental equations are specialized to the steady-state rotation of the vehicle, as a single rigid body, about its designed spin axis under the action of a constant thrust directed along that axis. In Section 2, the nonlinear equations are specialized to a reference state of motion in which the spacecraft (solid + liquids) undergoes a uniform steady state rotation about its designed spin axis under the influence of a constant thrust directed along that axis. Then, the vehicle is slightly disturbed from the known reference state of motion and the resulting perturbational equations of motion obtained. The perturbational equations are modified to reflect the correct mechanisms of vorticity transfer from the boundaries of the tanks to the contained liquids in accordance with Ref. 2. In Section 3, the perturbational liquid motions are made to depend on the solutions of N inhomogeneous boundary value problems. The effects of centrifugal and coriolis accelerations together with vorticity are implicitly taken into consideration in the formulation. Variational formulations of the associated boundary value problems are presented. For most practical applications, it is shown that the use of mechanical pendulums to represent sloshing motions in spinning tanks is suspect and should be avoided, unless severe restrictions can be tolerated. In Section 4 the resulting perturbational equations are summarized.

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SECTION 1

EQUATIONS OF MOTION OF A SPIN-STABILIZED SPACECRAFT COMPRISING N TANKS PARTLY FILLED WITH LIQUID PROPELLANTS

PHYSICAL ABSTRACTION

Imagine a spin-stabilized spacecraft carrying N tanks partly filled with liquid propellants, as shown in Figure 1. It is assumed that: (1) the liquid cavities are arranged in a symmetrical pattern about the designed spin axis of the vehicle; (2) in the absence of liquids, the mass center of the spacecraft lies on its designed spin axis; (3) the propellants are ideal liquids; (4) capillary and mass forces acting on the liquid are negligible; and (5) mass forces acting on the solid are insignificant.

COORDINATE SYSTEMS

To describe the motion of the vehicle, take a right-handed cartesian frame of reference $\mathscr{F}$ with origin $\mathscr{O}$ and base vectors $(\bar{i}, \bar{j}, \bar{k})$ fixed relatively to the solid. To complete the specification of $\mathscr{F}$, assume that $\mathscr{O}$ is located at the mass center of the spacecraft sans liquids, and that $\bar{k}$ is coincident with its designed spin axis.

To describe the liquid motions in the nth tank, take a right-handed cartesian frame of reference $\mathscr{F}_n$ with origin $\mathscr{O}_n$ and base vectors $(\bar{i}_n, \bar{j}_n, \bar{k}_n)$ fixed relatively to the cavity, such that $\mathscr{F}_n \parallel \mathscr{F}$. The position of $\mathscr{O}_n$ relative to $\mathscr{O}$ is designated by the vector $\bar{L}_n$.

Finally, the motion of $\mathscr{F}$ with respect to inertial space is specified by the translational velocity vector $\bar{u}$ and angular velocity vector $\bar{\omega}$.

KINEMATICAL MOTIONS

Let $\bar{r}$ denote the position vector of a mass point in the solid relative to $\mathscr{F}$. Then, the velocity and acceleration vectors of the point are simply.

$$\bar{V} = \bar{u} + \bar{\omega} \times \bar{r} \ ,$$

$$\bar{u} = \dot{\bar{V}} + \bar{\omega} \times \bar{V} = \bar{\alpha} + \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r}) \ , \tag{1.1}$$

where

$$\bar{\alpha} = \dot{\bar{u}} + \bar{\omega} \times \bar{u} \tag{1.2}$$

stands for the acceleration of the spacecraft. It should be appreciated that differentiation with respect to $\mathscr{F}$ (and $\mathscr{F}_n$) is indicated by $(\dot{\cdot})$

Specify by $\bar{r}_n$ the position vector of a material point Q_n in the nth cavity relative to $\mathscr{F}_n$, i.e., relative to the cavity. Then, the velocity $V_n = \bar{V}_n(Q_n,t)$ and acceleration $\bar{a}_n = \bar{a}_n\ (Q_n,t)$ of the liquid are given by

$$\bar{V}_n = \bar{u} + \bar{\omega} \times \bar{R}_n + \bar{u}_n \ ,$$
$$\bar{a}_n = \dot{\bar{V}}_n + \bar{\omega} \times \bar{V}_n = \bar{a} + \dot{\bar{\omega}} \times \bar{R}_n + \bar{\omega} \times (\bar{\omega} \times \bar{R}_n) + 2\bar{\omega} \times \bar{v}_n + \dot{\bar{v}}_n \ , \tag{1.3}$$

where $\bar{v}_n = \bar{v}_n\ (Q_n,t)$ denotes the velocity of the particle relative to $\mathscr{F}_n$, i.e.,

$$\bar{v}_n = \dot{\bar{r}}_n \ , \tag{1.4}$$

and

$$\bar{R}_n = \bar{L}_n + \bar{r}_n \ . \tag{1.5}$$

Note that Q_n may be a point in either the liquid volume τ_n, free liquid surface σ_{F_n} or wetted boundary σ_{W_n} of the tank. (See Figure 2 for notations.)

The material derivative of a vector or scalar point function associated with the motion of liquid in the nth container is defined as

$$(\dot{\cdots}) = \frac{\partial(\cdots)}{\partial t} + (\dot{\bar{r}}_n \cdot \nabla_n)\ (\cdots) = (\cdots)_t + (\bar{v}_n \cdot \nabla_n)\ (\cdots) \tag{1.6}$$

where, of course

$$\nabla_n = \bar{i}_n \frac{\partial}{\partial x_n} + \bar{j}_n \frac{\partial}{\partial y_n} + \bar{k}_n \frac{\partial}{\partial z_n} \ , \quad \frac{\partial(\cdots)}{\partial t} = (\cdots)_t \ . \tag{1.7}$$

In particular,

$$\dot{\bar{v}}_n = \bar{v}_{n_t} + (\bar{v}_n \cdot \nabla_n)\ \bar{v}_n = \bar{v}_{n_t} + \nabla_n\ \frac{1}{2} v_n^2 + \bar{\xi}_n \times \bar{v}_n \ , \tag{1.8}$$

where $\overline{\xi}_n = \overline{\xi}_n\ (Q_n,t)$ denotes the vorticity vector of the liquid relative to $\mathcal{F}_n$, i.e.,

$$\overline{\xi}_n = \nabla_n \times \overline{v}_n \ , \tag{1.9}$$

The vorticity vector of the liquid in the nth tank relative to inertial space $\overline{\zeta}_n = \overline{\zeta}_n\ (Q_n,t)$ is defined by

$$\overline{\zeta}_n = \nabla_n \times \overline{V}_n = 2\overline{\omega} + \overline{\xi}_n \ , \tag{1.10}$$

which follows by taking the curl of (1.3_1). For reference, note that if $\nabla_n \cdot \overline{V}_n = \nabla_n \cdot \overline{v}_n = 0$ everywhere, then it can be shown that

$$\begin{aligned}
\nabla_n \times \overline{a}_n &= \dot{\overline{\zeta}}_n + \overline{\omega} \times \overline{\zeta}_n - (\overline{\zeta}_n \cdot \nabla_n)\ \overline{V}_n \\
&= \dot{\overline{\xi}}_n - (\overline{\xi}_n \cdot \nabla_n)\ \overline{v}_n - 2\ (\overline{\omega} \cdot \nabla_n)\ \overline{v}_n + 2\dot{\overline{\omega}} \\
&= \overline{\zeta}_{n_t} + \nabla_n \times (\overline{\zeta}_n \times \overline{v}_n) \\
&= \overline{\xi}_{n_t} + \nabla_n \times (\overline{\xi}_n \times \overline{v}_n) + 2\overline{\omega} \times \overline{\xi}_n - 2\nabla_n\ (\overline{\omega} \cdot \overline{v}_n) + 2\dot{\overline{\omega}}
\end{aligned} \tag{1.11}$$

EQUATIONS OF MOTION

Considering body and liquid as a single mechanical system subject to given forces, the equations of motion can, in view of the stated assumptions, be written

$$\left\{ \begin{aligned}
&M\overline{\alpha} = \overline{F} + \sum_{n=1}^{N} \overline{F}_n \ , \\
&I \cdot \dot{\overline{\omega}} + \overline{\omega} \times (I \cdot \overline{\omega}) = \overline{T} + \sum_{n=1}^{N} \overline{T}_n \ ,
\end{aligned} \right. \tag{1.12}$$

$$\left.\begin{array}{l} \bar{a}_n = -\frac{1}{\rho_n}\nabla_n p_n , \\ \\ \nabla_n \cdot \bar{V}_n = \nabla_n \cdot \bar{v}_n = 0 , \\ \\ \nabla_n \times \bar{a}_n = 0 , \end{array}\right\} \quad Q_n \in \tau_n , \qquad (n=1,2,\cdots,N) , \tag{1.13}$$

where $(\bar{F}, \bar{T})$ are the resultant vectors of external forces and moments applied to the system, $(\bar{F}_n, \bar{T}_n)$ are the resultant vectors of forces and moments arising from the action of the liquid on the nth cavity wall, $p_n(Q_n,t)$ and ρ_n denote the pressure and density of the liquid in the nth tank, M stands for the mass of the solid, and I is the momental dyadic of the solid, i.e.,

$$I = \int_{\text{solid}} [|\bar{r}|^2 E - \bar{r}\bar{r}] \, dm , \qquad (E = \bar{i}\bar{i} + \bar{j}\bar{j} + \bar{k}\bar{k}) . \tag{1.14}$$

Clearly, equation (1.12) govern the translational and rotational motions of the vehicle. Equations (1.13), together with certain boundary conditions to be established, govern the motions of the liquids in the N cavities. For a particular tank, (1.13_1) is the classical Eulerian equation of motion of a liquid in the absence of mass forces, and (1.13_2) is the mass conservation principle for an incompressible liquid, i.e., continuity equation. Another form of the mass conservation principle used implicitly throughout this investigation is given by

$$(\rho_n \, d\tau_n)^{\cdot} = 0 , \qquad (n=1,2,\cdots,N) . \tag{1.15}$$

Equation (1.13_3) is the result of taking the curl of (1.13_1), i.e., the compatibility (vorticity) equation.

BOUNDARY CONDITIONS

The liquid velocity relative to each cavity must satisfy certain kinematical conditions, namely, the component normal to the wetted surface σ_{W_n} of the tank must vanish (condition of non-adherence), and the component normal to the free boundary σ_{F_n} must equal the velocity of the surface normal to itself. Thus, if the equation of the free boundary is specified in the general form $F_n(Q_n,t) = 0$, $Q_n \in \sigma_{F_n}$, one can write

$$
\bar{\nu}_n \cdot \bar{v} = \begin{Bmatrix} 0 \,, & Q_n \,\varepsilon\, \sigma_{W_n} \,, \\ \\ -F_{n_t}/|\nabla_n F_n| \,, & Q_n \,\varepsilon\, \sigma_{F_n} \,, \end{Bmatrix} (n=1,2,\cdots,N) \tag{1.16}
$$

where $\bar{\nu}_n$ is the unit vector along the outward drawn normal to the surface under consideration.

The free boundary condition can also be written in the form

$$
F_{n_t} + \bar{v}_n \cdot \nabla_n F_n = 0 \,, \qquad Q_n \,\varepsilon\, \sigma_{F_n} \,, \qquad (n=1,2,\cdots,N) \,, \tag{1.17}
$$

which follows by setting the material derivative of F_n to zero. Clearly,

$$
\bar{\nu}_n = \nabla_n F_n/|\nabla_n F_n| \,, \qquad Q_n \,\varepsilon\, \sigma_{F_n} \,, \qquad (n=1,2,\cdots,N) \,. \tag{1.18}
$$

In particular, if

$$
F_n = z_n - f_n \,(x_n, y_n, t) = 0 \,, \qquad Q_n \,\varepsilon\, \sigma_{F_n} \,, \qquad (n=1,2,\cdots,N) \,. \tag{1.19}
$$

then the kinematic conditions at σ_{F_n} become

$$
\bar{\nu}_n \cdot \bar{v}_n = f_{n_t} \cos\,(\bar{\nu}_n, \bar{k}_n) \,, \qquad Q_n \,\varepsilon\, \sigma_{F_n} \,, \qquad (n=1,2,\cdots, N) \,. \tag{1.20}
$$

Forms (1.19) and conditions (1.20) are used extensively later.

Kinematical conditions (1.16), in conjunction with (1.13_2) and the divergence theorem yield the injunctions

$$
\int_{\sigma_{F_n}} (F_{n_t}/|\nabla_n F_n|) \, d\sigma_{F_n} = 0 \,, \qquad (n=1,2,\cdots,N) \,, \tag{1.21}
$$

i.e., global continuity.

The pressure inside each cavity must satisfy a condition of thermodynamic equilibrium at the free boundary, which, in the absence of capillary forces, can be taken

$$p_n = 0 \ , \quad Q_n \in \sigma_{F_N} \ , \qquad (n=1,2,\cdots,N) \ . \tag{1.22}$$

FORCES AND MOMENTS RESULTING FROM ACTION OF LIQUIDS

The forces and moments $(\bar{F}_n, \bar{T}_n)$ resulting from the action of the liquids on the walls of the cavities are given by

$$\left\{ \begin{aligned} \bar{F}_n &= \int_{\sigma_{w_n}} \bar{\nu}_n \, p_n \, d\sigma_{w_n} = \int_{\sigma_{w_n}} \bar{\nu}_n \, p_n \, d\sigma_{w_n} + \int_{\sigma_{F_n}} \bar{\nu}_n \, p_n \, d\sigma_{F_n} \\ &= \int_{\sigma_n} \bar{\nu}_n \, p_n \, d\sigma_n = \int_{\tau_n} \nabla_n p_n \, d\tau_n \ , \qquad (n=1,2,\cdots,N) \ , \\ \bar{T}_n &= \int_{\sigma_{w_n}} (\bar{R}_n \times \bar{\nu}_n) \, p_n \, d\sigma_{w_n} = \int_{\sigma_{w_n}} (\bar{R}_n \times \bar{\nu}) \, p_n d \,_{w_n} \\ &\quad + \int_{\sigma_{F_n}} (\bar{R}_n \times \bar{\nu}_n) \, p_n \, d\sigma_{F_n} \\ &= \int_{\sigma_n} (\bar{R}_n \times \bar{\nu}_n) \, p_n \, d\sigma_n = \int_{\tau_n} (\bar{R}_n \times \nabla_n p_n) \, d\tau_n \ , \qquad (n=1,2,\cdots,N) \ , \end{aligned} \right. \tag{1.23}$$

in view of (1.22) and the divergence theorem, where

$$\sigma_n = \sigma_{W_n} + \sigma_{F_n} , \qquad (n=1,2,\cdots,N) . \tag{1.24}$$

INSTANTANEOUS MASS CENTER OF SPACECRAFT

Let the instantaneous position of the spacecraft mass center (solid + liquids) relative to $\mathcal{F}$ be specified by the vector $\bar{r}_c$; then

$$\bar{r}_c = \frac{1}{M+M_l} \sum_{n=1}^{N} \int_{\tau_n} \bar{R}_n \, \rho_n \, d\tau_n , \tag{1.25}$$

in light of

$$\int_{\text{solid}} \bar{r} \, dm = 0 , \tag{1.26}$$

where the total mass of liquids

$$M_l = \sum_{n=1}^{N} \int_{\tau_n} \rho_n \, d\tau_n . \tag{1.27}$$

OTHER USEFUL FORMS OF THE EQUATIONS

Equations (1.3_2), (1.13_1), (1.23) and (1.25) make it possible to write (1.12) as

$$\left\{ \begin{aligned} &(M+M_l)[\bar{\alpha} + \dot{\bar{\omega}} \times \bar{r}_c + \bar{\omega} \times (\bar{\omega} \times \bar{r}_c)] + \sum_{n=1}^{N} \int_{\tau_n} (\dot{\bar{v}}_n + 2\omega \times \bar{v}_n)\rho_n d\tau_n = \bar{F} , \\ &K \cdot \dot{\bar{\omega}} + \bar{\omega} \times (K \cdot \bar{\omega}) + (M+M_l)(\bar{r}_c \times \bar{\alpha}) + \sum_{n=1}^{N} \int_{\tau_n} [\bar{R}_n \times (\dot{\bar{v}}_n + 2\bar{\omega} \times \bar{v}_n)]\rho_n d\tau_n = \bar{T}, \end{aligned} \right. \tag{1.28}$$

where the dyadic

$$K = I + \sum_{n=1}^{N} \int_{\tau_n} [|\bar{R}_n|^2 E - \bar{R}_n\bar{R}_n]\, \rho_n\, d\tau_n \, . \tag{1.29}$$

Solving for $\bar{\alpha}$ from (1.28_1) and substituting the resulting expression into (1.28_2),

$$\theta \cdot \dot{\bar{\omega}} + \bar{\omega} \times (\theta \cdot \bar{\omega}) + \sum_{n=1}^{N} \int_{\tau_n} [(\bar{R}_n - \bar{r}_c) \times (\dot{\bar{v}}_n + 2\bar{\omega} \times \bar{v}_n)]\rho_n d\tau_n = \bar{T} - \bar{r}_c \times \bar{F} \, , \tag{1.30}$$

where

$$\theta = K - (M + M_1)\, [|\bar{r}_c|^2 E - \bar{r}_c\bar{r}_c] \, . \tag{1.31}$$

Equation (1.30) clearly represents the rotational equation of motion of the vehicle about its instantaneous mass center. Note that I is the momental dyadic of the solid whose components have constant values with respect to $\mathscr{F}$. K − I is the momental dyadic of the liquids whose components are functions of the instantaneous forms of the liquids inside the tanks. θ −K is the inertia tensor whose components are functions of the instantaneous position of the vehicle mass center relative to $\mathscr{F}$.

Diverse forms of (1.13_1) are also possible. In particular,

$$\bar{v}_{n_t} + 2\bar{\omega} \times \bar{v}_n + \bar{\xi}_n \times \bar{v}_n + \dot{\bar{\omega}} \times \bar{r}_n + \nabla_n \phi_n = 0 \, , \qquad (n=1,2,\cdots,N) \tag{1.32}$$

where the acceleration potential

$$\phi_n = \frac{p_n}{\rho_n} + \frac{1}{2} v_n^2 + (\bar{\alpha} + \dot{\bar{\omega}} \times \bar{L}_n) \cdot \bar{R}_n - \frac{1}{2} (\bar{\omega} \times \bar{R}_n)^2 \, , \qquad (n=1,2,\cdots,N) \, , \tag{1.33}$$

can be derived from (1.3), (1.6) and (1.13_2). Taking the curl of (1.32) gives the compatibility equation

$$\bar{\xi}_{n_t} + 2\bar{\omega} \times \bar{\xi}_n + \nabla_n \times (\bar{\xi}_n \times \bar{v}_n) - 2\nabla_n (\bar{\omega} \cdot \bar{v}_n) + 2\dot{\bar{\omega}} = 0 , \quad (n=1,2,\ldots,N) . \tag{1.34}$$

STEADY STATE ROTATION OF VEHICLE UNDER ACTION OF CONSTANT THRUST

If the applied forces and constraints are such that the spacecraft (solid + liquids) performs a uniform steady state rotation ω as a single body about its designed spin axis $\bar{k}$ under the action of a constant thrust F directed along $\bar{k}$, the system is in equilibrium relative to the moving frame of reference $\mathscr{F}$ attached to the body, and (1.28), (1.30) can be written

$$\begin{cases} (M + M_1) \, [\bar{\alpha} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_c)] = \bar{F} , \\ \bar{\omega} \times (\theta \cdot \bar{\omega}) = \bar{T} - \bar{r}_c \times \bar{F} = 0 \end{cases} \tag{1.35}$$

But (1.35_2) implies

$$\begin{cases} \theta \cdot \bar{\omega} = \bar{c} \\ \bar{T} - \bar{r}_c \times \bar{F} = 0 \end{cases} \tag{1.36}$$

where $\bar{c} = c \, \bar{k}$. Equation (1.36_1) states that the angular momentum about the center of mass is conserved. If

$$\bar{r}_c = r_c \, \bar{k} , \tag{1.37}$$

then (1.36_2) is satisfied identically providing

$$\bar{T} \equiv 0 . \tag{1.38}$$

With $\bar{\omega} = \omega \bar{k}$, $\bar{r}_c = r_c \, \bar{k}$ and $\bar{F} = F \, \bar{k}$, equation (1.35_1) yields

$$\bar{\alpha} = \frac{F}{M + M_1} \bar{k} = \alpha \, \bar{k} . \tag{1.39}$$

The area parameter c in (1.36_1) is given by

$$c = I_{kk}\,\omega \tag{1.40}$$

where

$$I_{kk} = \bar{k} \cdot \theta \cdot \bar{k} \tag{1.41}$$

is the moment of inertia of the spacecraft about the axis $\bar{k}$.

Now $\bar{v}_n = 0$ in each tank, and (1.32), (1.33), it follows

$$\frac{p_n}{\rho_n} + \bar{\alpha} \cdot \bar{R}_n - \frac{1}{2} (\bar{\omega} \times R_n)^2 = \phi_n = c_n \ , \ Q_n \in \tau_n \ , \qquad (n=1,2,\cdots,N) \ , \tag{1.42}$$

where c_n are constants. If the equations of the free boundaries are specified as $F_n = z_n - f_n = 0$, then (1.42) can be written

$$p_n = \alpha\rho_n\,(f_n - z_n) \ , \quad Q_n \in \tau_n \ , \qquad (n=1,2,\cdots,N) \ , \tag{1.43}$$

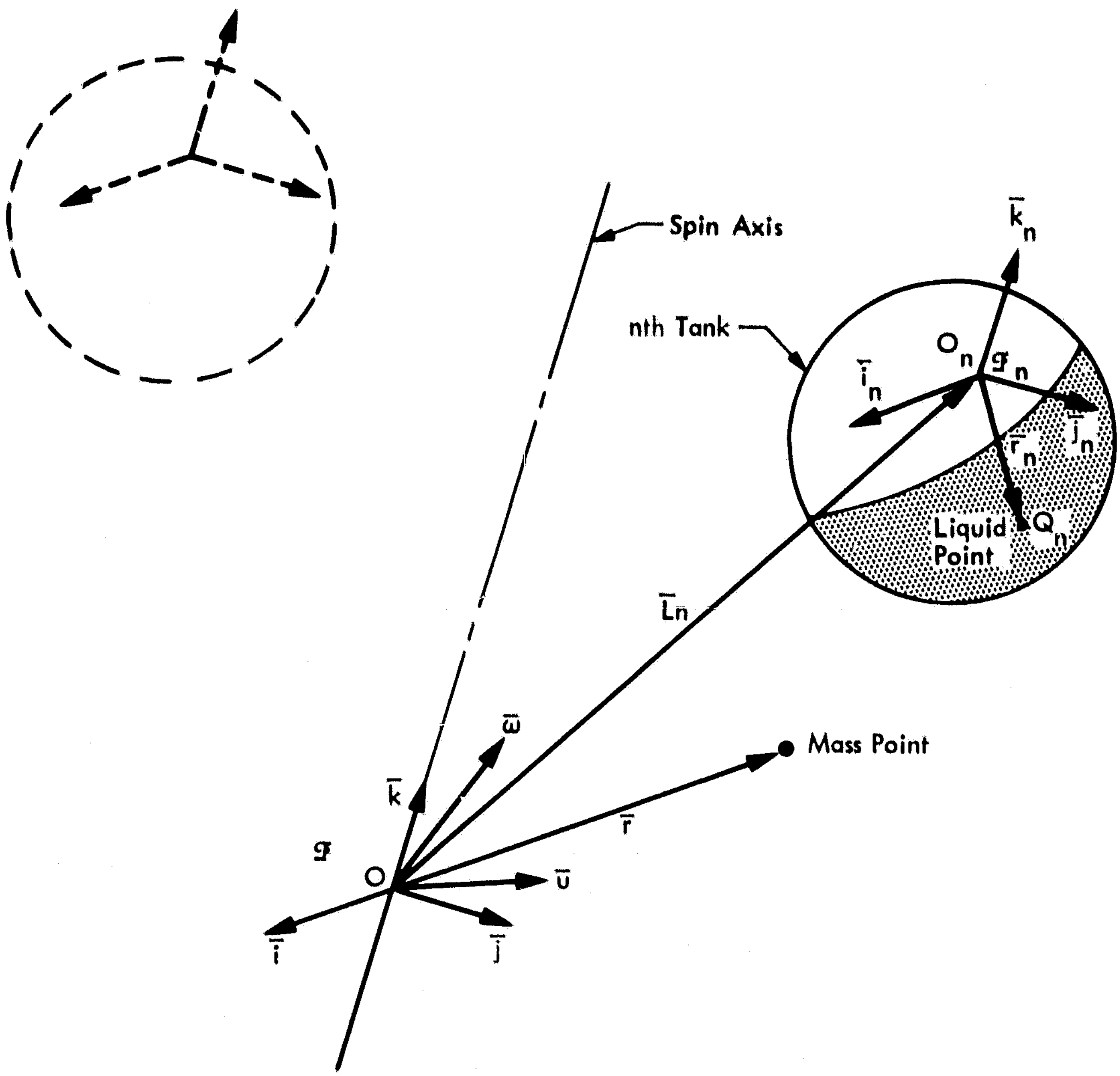
in light of conditions (1.22), where

$$f_n = \frac{1}{\alpha}\,\{c_n + \frac{1}{2}\,\omega^2\,[(L_{1n} + x_n)^2 + (L_{2n} + y_n)^2]\} + L_{3n} \ , \qquad (n=1,2,\cdots,N) \ , \tag{1.44}$$

and (L_{1n}, L_{2n}, L_{3n}) are the components of $\bar{L}_n$ with respect to $\mathscr{F}$. The equations

$$F_n = z_n - f_n = 0 \ , \quad Q_n \in \tau_{F_n} \ , \qquad (n=1,2,\cdots,N) \ , \tag{1.45}$$

where f_n are given by (1.44), are N paraboloids of revolution whose axes coincide with the vehicle spin axis. The constants c_n can be determined from the known liquid volumes.



$\mathcal{F}_n \parallel \mathcal{F}$, $(n = 1, 2, \ldots N)$

Figure 1. Spacecraft and Tank Geometry

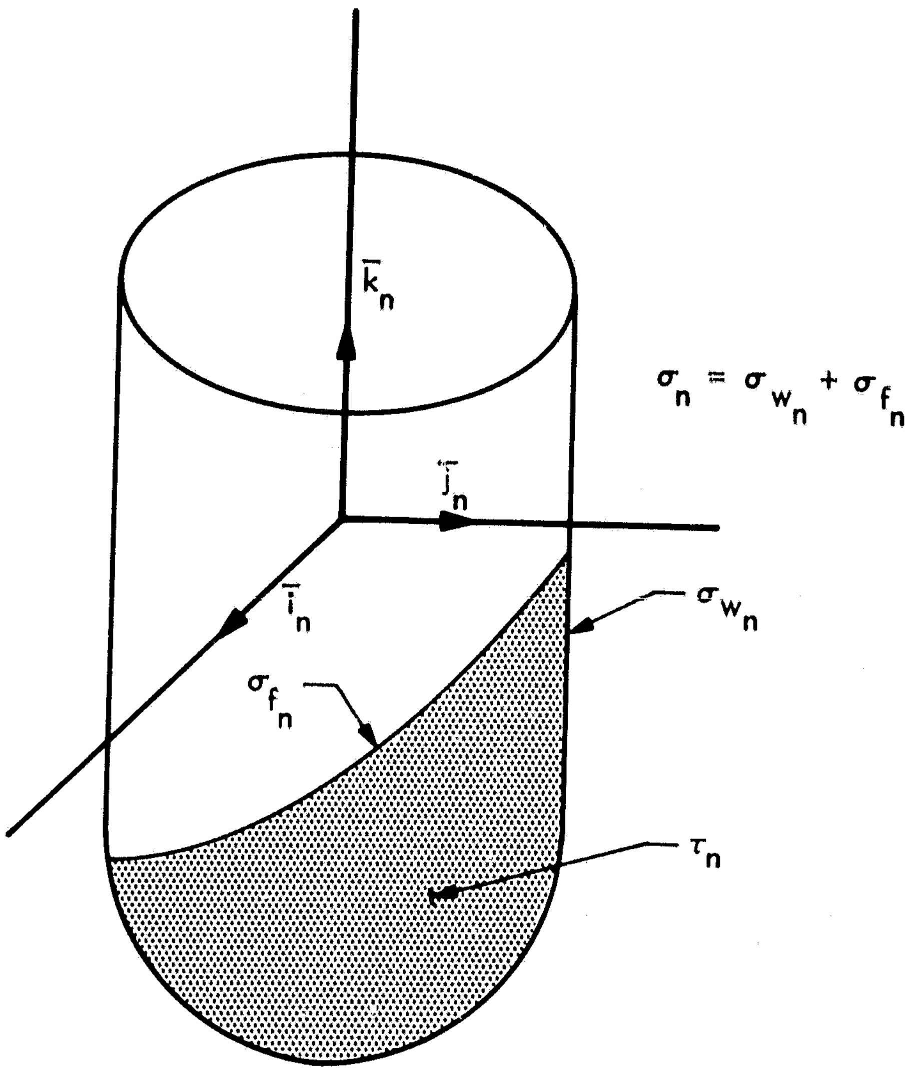


Figure 2. Cavity Geometry

SECTION 2

SMALL PERTURBATION EQUATIONS OF MOTION OF A SPIN-STABILIZED SPACECRAFT COMPRISING N TANKS PARTLY FILLED WITH LIQUID PROPELLANTS

GENERAL CONSIDERATIONS

The equations that govern the motion of a spacecraft carrying N tanks partly filled with liquid propellants are nonlinear, and do not admit general, exact solutions. Nevertheless, it is sometimes possible to obtain or approximate particular solutions, such as steady state rotation of the vehicle about its designed spin axis under the action of a constant thrust derived above. Then, if the system is supposed to be slightly disturbed from this known state of motion, the resulting small motion of deviation can usually be described by a set of linear ordinary and partial differential equations with appropriate initial and boundary conditions.

Such equations are usually constructed in the following manner. Suppose the complete nonlinear equations of motion of the vehicle and liquids to be written for two states of motion: reference and disturbed. The reference or 'undisturbed' state describes the path of the mass center, attitude of the vehicle, and the shapes of the free liquid surfaces in the cavities. The disturbed state describes the actual motions of the vehicle and liquids, and consists of the reference motions plus small deviant motions therefrom. The difference between the governing equations of the perturbed and reference states, upon neglecting all terms involving products of deviant quantities and/or their derivatives, yields the perturbation equations (Ref. 4).

REFERENCE STATE OF MOTION

The reference state of motion embodying the particular desired attitude of the spacecraft is, in general, one of nonuniformity. For many practical situations, the reference state of motion may be taken as one in which the spacecraft (solid + liquids) undergoes a uniform steady state rotation as a single rigid body about its designed spin axis under the influence of a constant thrust directed along that axis. With this approximation to the actual reference state of motion, and assuming further no velocities of liquids relative to the cavities, it follows that

$$\left\{\begin{aligned} &\bar{\omega} = \omega\bar{k}\ ,\ \bar{T} = 0\ ,\ \bar{\omega}\times(\theta\cdot\bar{\omega}) = \bar{\omega}\times(K\cdot\bar{\omega})\ , \\ &\bar{\alpha} = \alpha\bar{k}\ ,\ \bar{F} = F\bar{k}\ ,\ \bar{\omega}\times(I\cdot\bar{\omega}) = -\,\bar{\omega}\times[(K-I)\cdot\bar{\omega}]\ , \\ &\dot{\bar{\omega}} = 0\ ,\ \bar{r}_c = r_c\,\bar{k}\ ,\ \sum_{n=1}^{N}\bar{F}_n = -M_1\bar{\alpha}\ , \end{aligned}\right. \tag{2.1}$$

$\approx$

$$\left\{\begin{array}{l} \displaystyle\sum_{n=1}^{N} \bar{T}_n = -\bar{\omega} \times [(K-I)\cdot\bar{\omega}] \quad , \quad J - K = -(M+M_1)\left[|r_c|^2 E - \bar{r}_c\bar{r}_c\right] \quad , \\ \displaystyle K - I = \sum_{n=1}^{N} \int_{\tau_n} \left[|\bar{R}_n|^2 \, E - \bar{R}_n\bar{R}_n\right] \rho_n d\tau_n \quad , \quad \bar{R}_n = \bar{L}_n + \bar{r}_n \quad ; \end{array}\right.$$

$$\left\{\begin{array}{l} \bar{v}_n = 0 \quad , \quad \bar{\xi}_n = 0 \quad , \quad \bar{\zeta}_n = 2\bar{\omega} \quad , \quad p_n = \alpha\rho_n(f_n - z_n) , \\ \bar{V}_n = \bar{u} + \bar{\omega} \times \bar{R}_n \quad , \quad \bar{a}_n = \bar{\alpha} + \bar{\omega} \times (\bar{\omega} \times R_n) \quad , \end{array}\right\} \begin{array}{l} Q_n \in \tau_n , \\ (n=1,2,\ldots,N) ; \end{array} \tag{2.2}$$

$$\left\{\begin{array}{l} z_n - f_n = 0, \\ f_n = \dfrac{1}{\alpha}\left\{c_n + \dfrac{1}{2}\,\omega^2\left[\left(L_{1n}+x_n\right)^2 + \left(L_{2n}+y_n\right)^2\right]\right\} + L_{3n} \quad , \\ \bar{v}_n = \dfrac{\alpha}{g_n}\,\bar{k}_n - \dfrac{(L_{1n}+x_n)}{g_n}\,\bar{i}_n + \dfrac{(L_{2n}+y_n)}{g_n}\,\bar{j}_n \quad , \\ g_n = g_n\,(x_n , y_n) = \left\{\alpha^2 + \omega^4\left[(L_{1n}+x_n)^2 + (L_{2n}+y_n)^2\right]\right\}^{1/2} , \end{array}\right\} \begin{array}{l} Q_n \in \sigma_{F_n} , \\ (n=1,2,\ldots,N) ; \end{array} \tag{2.3}$$

$$
\left\{
\begin{aligned}
& M\bar{a} = \bar{F} + \sum_{n=1}^{N} \bar{F}_n \\
& I \cdot \dot{\bar{\omega}} + \bar{\omega} \times (I \cdot \bar{\omega}) = \bar{T} + \sum_{n=1}^{N} \bar{T}_n \\
& \left.
\begin{aligned}
& \bar{a}_n = -\frac{1}{\rho_n} \nabla_n p_n \ , \\
& \nabla_n \cdot \bar{V}_n = \nabla_n \cdot \bar{v}_n = 0, \\
& \nabla_n \times \bar{a}_n = 0 \ ,
\end{aligned}
\right\} \quad Q_n \in \tau_n, \\
& \bar{\nu}_n \cdot \bar{v}_n = \begin{cases} 0 \ , & Q_n \in \sigma_{W_n} \\ f_{n_t} \cos(\bar{\nu}_n, \bar{k}_n), \ (f_{n_t} = 0), & Q_n \in \sigma_{F_n} \ , \ (z_n = f_n) \end{cases} \\
& p_n = 0 \ , \quad Q_n \in \sigma_{F_n} \ , \quad (z_n = f_n) \ , \qquad\qquad (n=1,2,\ldots,N) \\
& \bar{F}_n = \int_{\sigma_{W_n}} \nu_n p_n \, d\sigma_{W_n} = \int_{\sigma_{W_n}} \bar{\nu}_n p_n \, d\sigma_{W_n} + \int_{\sigma_{F_n}} \bar{\nu}_n p_n \, d\sigma_{F_n} \\
& \qquad = \int_{\sigma_n} \bar{\nu}_n p_n \, d\sigma_n = \int_{\tau_n} \nabla_n p_n \, d\tau_n \ , \\
& \bar{T}_n = \int_{\sigma_{W_n}} (\bar{R}_n \times \bar{\nu}_n) \, p_n \, d\sigma_{W_n} = \int_{\sigma_{W_n}} (\bar{R}_n \times \bar{\nu}_n) \, p_n \, d\sigma_{W_n} \\
& \qquad + \int_{\sigma_{F_n}} (\bar{R}_n \times \bar{\nu}_n) \, p_n \, d\sigma_{F_n} = \int_{\sigma_n} (\bar{R}_n \times \bar{\nu}_n) \, p_n \, d\sigma_n \\
& \qquad = \int_{\tau_n} (\bar{R}_n \times \nabla_n p_n) \, d\tau_n \ , \quad \int_{\sigma_{F_n}} f_{n_t} \cos(\bar{\nu}_n, \bar{k}_n) \, d\sigma_{F_n} = 0, \\
& \sigma_n = \sigma_{W_n} + \sigma_{F_n} \, .
\end{aligned}
\right. \tag{2.4}
$$

The constants c_n appearing in (2.3$_2$) can be determined from the known liquid volume in each cavity.

The foregoing completely define the reference state of motion of spacecraft and liquids.

PERTURBATION EQUATIONS

In the reference state of motion defined above, the spacecraft with partly filled liquid cavities can be associated with a certain solid vehicle, called transformed consisting of the given vehicle and frozen liquids with free boundaries (2.3). Let us find the small deviant motions of the transformed spacecraft in passing from a position corresponding to steady state rotation under constant thrust to a neighboring perturbed position. One can conceive of this transition as proceeding in two stages: 1) displacement of the whole spacecraft, as a single rigid body, to a perturbed position; 2) deformation of the forms $z_n = f_n$ $(n=1,2,\ldots,N)$ of the liquids (by covering the free boundaries with layers τ_n' of zero volumes) into forms $z_n = f_n + f_n'$ $(n=1,2,\ldots,N)$, where

$$f_n' = df_n \quad , \quad (n=1,2,\ldots,N) \tag{2.5}$$

are small deviations relative to the f_n, and τ_n' denote differences in the forms of the instantaneous and reference volumes.

Regarding the nth cavity, let $\mathscr{R}_n$ stand for the region of the $x_n y_n$-plane bounded by the projection of a closed curve representing the locus of the intersection points of the reference free surface with cavity wall(s) σ_{wn}. Then, in the first approximation Ref. 4,

$$\int_{\tau_n+\tau_n'} (\cdots)\, d\tau_n \approx \int_{\tau_n} (\cdots)\, d\tau_n + \iint_{\mathscr{R}_n} dx_n dy_n \int_{f_n}^{f_n+f_n'} (\cdots)\, dz_n , \tag{2.6}$$

where $(\cdots)$ can be a scalar, vector or dyadic point function. In particular, we observe

$$\iint_{\mathscr{R}_n} \rho_n\, f_n'\, dx_n\, dy_n = 0 \quad , \tag{2.7}$$

owing to the incompressibility of the liquid. Consider the disturbed form of $\overline{F}_n$, namely,

$$\overline{F}_n + \overline{F}'_n = \int_{\tau_n+\tau'_n} \left(V_n p_n + V_n p'_n\right) d\tau_n \tag{2.8}$$

$$= \int_{\tau_n} V_n p_n \, d\tau_n + \int_{\tau_n} V_n p'_n \, d\tau_n + \iint_{\mathscr{R}_n} dx_n dy_n \int_{f_n}^{f_n + f'_n} V_n p_n \, dz_n$$

$$+ \iint_{\mathscr{R}_n} dx_n dy_n \int_{f_n}^{f_n+f'_n} V_n p'_n \, dz_n \quad ,$$

in view of (2.6), where $\overline{F}_n$, p_n, τ_n are reference motion values, and

$$\begin{cases} \overline{F}'_n = d\overline{F}_n \\ p'_n = dp_n \end{cases} \tag{2.9}$$

are small deviations.

Subtracting out the reference value of $\overline{F}_n$, i.e.,

$$\overline{F}_n = \int_{\tau_n} V_n p_n \, dp_n \quad ,$$

from (2.8), it follows

$$\overline{F}'_n = \int_{\tau_n} \nabla_n p'_n \, d\tau_n + \iint_{\mathscr{R}_n} dx_n dy_n \int_{f_n}^{f_n+f'_n} \nabla_n p_n \, dz_n$$
$$+ \iint_{\mathscr{R}_n} dx_n dy_n \int_{f_n}^{f_n+f'_n} \nabla_n p'_n \, dz_n \quad , \tag{2.10}$$

which, to the first order of small quantities, becomes

$$\overline{F}'_n = \int_{\tau_n} \nabla_n p'_n \, d\tau_n + \iint_{\mathscr{R}_n} dx_n dy_n \int_{f_n}^{f_n+f'_n} \nabla_n p_n \, dz_n \quad . \tag{2.11}$$

But, from (2.2_2) and (2.4_3),

$$\nabla_n p_n = -\rho_n \, [\overline{\alpha} + \overline{\omega} \times (\overline{\omega} \times \overline{R}_n)] \quad . \tag{2.12}$$

Substituting (2.12) into (2.11) and performing the indicated integration, one obtains, to the same degree of approximation,

$$\overline{F}'_n = \int_{\tau_n} \nabla_n p'_n \, d\tau_n - \overline{\omega} \times [\overline{\omega} \times \iint_{\mathscr{R}_n} \overline{R}_n f'_n \, \rho_n \, dx_n dy_n] \tag{2.13}$$
$$= \int_{\tau_n} \nabla_n p'_n \, d\tau_n - \overline{\omega} \times [\overline{\omega} \times \iint_{\mathscr{R}_n} \overline{r}_n f'_n \, \rho_n \, dx_n \, dy_n] \quad ,$$

in view of (2.7).

Using similar arguments, it can easily be shown that

$$T_n' = \int_{\tau_n} (\overline{R}_n \times V_n p_n')d\tau_n - \iint_{\mathscr{R}_n} (\overline{R}_n \times \overline{\alpha})\ \rho_n\ f_n'\ dx_n dy_n \tag{2.14}$$

$$- \iint_{\mathscr{R}_n} \left\{ \overline{R}_n \times [\overline{\omega} \times (\overline{\omega} \times \overline{R}_n)] \right\} f_n'\ \rho_n\ dx_n\ dy_n \quad ,$$

$$\overline{r}_c' = \frac{1}{M+M_1} \sum_{n=1}^{N} \iint_{\mathscr{R}_n} \overline{R}_n f_n'\ \rho_n\ dx_n dy_n \tag{2.15}$$

$$= \frac{1}{M+M_1} \sum_{n=1}^{N} \iint_{\mathscr{R}_n} \overline{r}_n f_n'\ \rho_n\ dx_n dy_n \quad ,$$

$$K' = \sum_{n=1}^{N} \iint_{\mathscr{R}_n} [\,|\overline{R}_n|^2 E - \overline{R}_n \overline{R}_n]\ f_n'\ \rho_n\ dx_n dy_n \quad . \tag{2.16}$$

In arriving at these results terms involving products of deviant quantities and/or their derivatives as well as local positional changes of liquid particles were neglected.

Working to the same degree of approximation,

$$\left\{ \begin{aligned} \overline{V}_n' &= \overline{u}' + \overline{\omega}' \times \overline{R}_n + \overline{v}_n' \quad , \\ \dot{\overline{V}}_n' &= \dot{\overline{u}}' + \dot{\overline{\omega}}' \times \overline{R}_n + \overline{\omega} \times \overline{v}_n' + \overline{v}_{n_t}' \quad , \\ \overline{a}_n' &= \dot{\overline{V}}_n' + \overline{\omega}' \times \overline{V}_n + \overline{\omega} \times \overline{V}_n' \quad , \\ &= \overline{\alpha}' + \dot{\overline{\omega}}' \times \overline{R}_n + \overline{\omega}' \times (\overline{\omega} \times \overline{R}_n) + \overline{\omega} \times (\overline{\omega}' \times \overline{R}_n) \\ &\quad + 2\overline{\omega} \times \overline{v}_n' + \overline{v}_{n_t}' \quad , \end{aligned} \right. \tag{2.17}$$

where

$$\bar{\alpha}' = \bar{u}' + \bar{\omega}' \times \bar{u} + \bar{\omega} \times \bar{u}' \quad . \tag{2.18}$$

The perturbational equations that govern the small liquid motions in the N cavities are obtained in a similar fashion. They are

$$\left\{\begin{array}{l} \left.\begin{array}{l} \bar{a}_n' = -\dfrac{1}{\rho_n} \nabla_n p_n' \ , \\ \nabla_n \cdot \bar{V}_n' = \nabla_n \cdot \bar{v}_n' = 0 \ , \\ \nabla_n \times \bar{a}_n' = 0 \ , \end{array}\right\} Q_n \in \tau_n \ , \\ \bar{\nu}_n \cdot \bar{v}_n' = \left\{\begin{array}{lll} 0 & , \ Q_n \in \sigma_{W_n} & , \\ f_{n_t}' \, (\bar{\nu}_n \cdot \bar{k}_n) & , \ Q_n \in \sigma_{F_n} & , \ (z_n = f_n) \ , \end{array}\right. \\ p_n' - \rho_n \, \alpha \, f_n' = 0 \ , \quad Q_n \in \sigma_{F_n} \ , \quad (z_n = f_n) \ , \\ \displaystyle\int_{\sigma_{F_n}} f_{n_t}' \cos(\bar{\nu}_n \ , \ \bar{k}_n) \, d\sigma_{F_n} = \iint_{\mathscr{R}_n} f_{n_t}' \, dx_n dy_n = 0 \ , \end{array}\right\} (n=1,2,\cdots,N). \tag{2.19}$$

To the same degree of approximation, it can be shown that

$$\bar{\nu}_n \cdot \bar{v}_{n_t}' = \left\{\begin{array}{lll} 0 & , \ Q_n \in \sigma_{W_n} & , \\ f_{n_{tt}}' \, (\bar{\nu}_n \cdot \bar{k}_n) & , \ Q_n \in \sigma_{F_n} & , \ (z_n = f_n) \ , \end{array}\right\} (n=1,2,\cdots,N), \tag{2.20}$$

and

$$\bar{\nu}_n \cdot \bar{v}'_{n_{ttt}} = \left\{ \begin{array}{lll} 0 & , \; Q_n \in \sigma_{W_n} & , \\ & & \\ f'_{n_{tttt}} (\bar{\nu}_n \cdot \bar{k}_n) & , \; Q_n \in \sigma_{F_n} & , \; (z_n = f_n) \; , \end{array} \right\} (n=1,2,\cdots,N). \tag{2.21}$$

The disturbed form of the translational equation of motion of the spacecraft is given by

$$M\,(\bar{\alpha} + \bar{\alpha}') = \bar{F} + \bar{F}' + \sum_{n=1}^{N} (\bar{F}_n + \bar{F}'_n) \tag{2.22}$$

where, as before, $\bar{\alpha}$, $\bar{F}$, $\bar{F}_n$ are reference motion values given by (2.1) and (2.4_8), and

$$\left\{ \begin{array}{l} \bar{\alpha}' = d\bar{\alpha} \\ \bar{F}' = d\bar{F} \\ \bar{F}'_n = d\bar{F}_n \end{array} \right. \tag{2.23}$$

are small deviations therefrom. In the reference state of motion

$$M\bar{\alpha} = \bar{F} + \sum_{n=1}^{N} F_n \quad , \tag{2.24}$$

in view of (2.4_1). Subtracting (2.2_4) from (2.22), one obtains

$$M\bar{\alpha}' = \bar{F}' + \sum_{n=1}^{N} \bar{F}'_n \quad , \tag{2.25}$$

for the perturbational equation of translational motion of the vehicle. Now, from (2.13),

$$\sum_{n=1}^{N} \bar{F}_n' = \sum_{n=1}^{N} \int_{\tau_n} V_n p_n' \, d\tau_n - \bar{\omega} \times [\bar{\omega} \times \sum_{n=1} \iint_{\mathscr{R}_n} \bar{R}_n f_n' \rho_n \, dx_n dy_n] \tag{2.26}$$

$$= \sum_{n=1}^{N} \int_{\tau_n} V_n p_n' \, d\tau_n - (M+M_1) \, \bar{\omega} \times (\bar{\omega} \times \bar{r}_c') \quad , \qquad ,$$

in view of (2.15). In addition, from (2.19_1) and (2.17_3),

$$V_n p_n' = - \rho_n \left\{ \bar{\alpha}' + \dot{\bar{\omega}}' \times \bar{R}_n + \bar{\omega}' \times (\bar{\omega} \times \bar{R}_n) \right.$$
$$\left. + \bar{\omega} \times (\bar{\omega}' \times \bar{R}_n) + 2\bar{\omega} \times \bar{v}_n' + \bar{v}_{n_t}' \right\} \quad ,$$

so that

$$\sum_{n=1}^{N} \int_{\tau_n} V_n p_n' \, d\tau_n = - M_1 \, \bar{\alpha}' - (M+M_1) \, \dot{\bar{\omega}}' \times \bar{r}_c$$
$$- (M+M_1) \, \bar{\omega}' \times (\bar{\omega} \times \bar{r}_c) - (M_1+M) \, \bar{\omega} \times (\bar{\omega}' \times \bar{r}_c)$$
$$- \sum_{n=1}^{N} \int_{\tau_n} (\bar{v}_{n_t}' + 2\bar{\omega} \times \bar{v}_n') \, \rho_n \, d\tau_n \tag{2.27}$$
$$= - M_1 \, \bar{\alpha}' - (M_1+M) \, \dot{\bar{\omega}}' \times \bar{r}_c - (M_1+M) \, \bar{\omega} \times (\bar{\omega}' \times \bar{r}_c)$$
$$- \sum_{n=1}^{N} \int_{\tau_n} (\bar{v}_{n_t}' + 2\bar{\omega} \times \bar{v}_n') \, \rho_n \, d\tau_n \quad ,$$

where

$$\bar{r}_c = \frac{1}{M+M_1} \sum_{n=1}^{N} \int_{\tau_n} \bar{R}_n \rho_n \, d\tau_n = r_c \bar{k} \quad , \quad \bar{\omega} = \omega \bar{k} \tag{2.28}$$

Owing to (2.26) and (2.27), equation (2.25) becomes

$$(M+M_1) \, [\bar{\alpha}' + \dot{\bar{\omega}}' \times \bar{r}_c + \bar{\omega} \times (\bar{\omega}' \times \bar{r}_c) + \bar{\omega} \times (\bar{\omega} \times \bar{r}_c')] + \sum_{n=1}^{N} \int_{\tau_n} (\bar{v}'_{n_t} + 2\bar{\omega} \times \bar{v}_n') \rho_n \, d\tau_n = \bar{F}' \quad . \tag{2.29}$$

Equation (2.29) can also be obtained by perturbing (1.28_1) directly.

The disturbed form of the rotational equation of motion of the spacecraft can be written

$$I \cdot (\dot{\bar{\omega}} + \dot{\bar{\omega}}') + (\bar{\omega} + \bar{\omega}') \times [I \cdot (\bar{\omega} + \bar{\omega}')] = \bar{T} + \bar{T}' + \sum_{n=1}^{N} (\bar{T}_n + \bar{T}_n') \quad , \tag{2.30}$$

which, upon discarding nonlinearities involving $\bar{\omega}'$ and subtracting out reference equation (2.4_2), yields

$$I \cdot \dot{\bar{\omega}}' + \bar{\omega}' \times (I \cdot \bar{\omega}) + \bar{\omega} \times (I \cdot \bar{\omega}') = \bar{T}' + \sum_{n=1}^{N} \bar{T}_n' \quad , \tag{2.31}$$

for the perturbational motion. But, from (2.14)

$$\sum_{n=1}^{N} \bar{T}'_n = \sum_{n=1}^{N} \int_{\tau_n} (\bar{R}_n \times V_n p'_n)\, d\tau_n$$

$$- \sum_{n=1}^{N} \iint_{\mathscr{R}_n} (\bar{R}_n \times \bar{\alpha})\, \rho_n\, f'_n\, dx_n dy_n$$

$$- \sum_{n=1}^{N} \iint_{\mathscr{R}_n} \left\{ \bar{R}_n \times [\bar{\omega} \times (\bar{\omega} \times \bar{R}_n)] \right\} f'_n\, \rho_n\, dx_n dy_n \tag{2.32}$$

$$= \sum_{n=1}^{N} \int_{\tau_n} (\bar{R}_n \times V_n p'_n)\, d\tau_n - (M+M_1)\, (\bar{r}'_c \times \bar{\alpha})$$

$$- \bar{\omega} \times (K' \cdot \bar{\omega}) \quad ,$$

in view of (2.15), (2.16) and the identity

$$\sum_{n=1}^{N} \iint_{\mathscr{R}_n} \left\{ \bar{R}_n \times [\bar{\omega} \times (\bar{\omega} \times \bar{R}_n)] \right\} f'_n\, \rho_n\, dx_n\, dy_n = K' \quad . \tag{2.33}$$

Moreover,

$$\sum_{n=1}^{N} \int_{\tau_n} (\bar{R}_n \times V_n p'_n)\, d\tau_n = - (M+M_1)\, (\bar{r}_c \times \bar{\alpha}') - (K-I) \cdot \bar{\omega}' \tag{2.34}$$

$$- \bar{\omega}' \times [(K-I) \cdot \bar{\omega}] - \bar{\omega} \times [(K-I) \cdot \bar{\omega}']$$

$$- \sum_{n=1}^{N} \int_{\tau_n} \left\{ \bar{R}_n \times (\bar{v}'_{n_t} + 2\bar{\omega} \times \bar{v}'_n) \right\} \rho_n\, d\tau_n,$$

in light of the identity

$$\sum_{n=1}^{N} \int_{\tau_n} \left\{ \bar{R}_n \times [\bar{\omega}' \times (\bar{\omega} \times \bar{R}_n)] \right\} \rho_n \, d\tau_n + \sum_{n=1}^{N} \int_{\tau_n} \left\{ \bar{R}_n \times [\bar{\omega} \times (\bar{\omega}' \times \bar{R}_n)] \right\} \rho_n \, d\tau_n = \bar{\omega}' \times [(K-I) \cdot \bar{\omega}] + \bar{\omega} \times [(K-I) \cdot \bar{\omega}'] \quad . \tag{2.35}$$

Because of (2.32) and (2.34), equation (2.31) assumes the form

$$K \cdot \dot{\bar{\omega}}' + \bar{\omega}' \times (K \cdot \bar{\omega}) + \bar{\omega} \times (K \cdot \bar{\omega}') + \bar{\omega} \times (K' \cdot \bar{\omega}) + (M+M_1)\,(\bar{r}_c' \times \bar{\alpha}) + (M+M_1)\,(\bar{r}_c \times \bar{\alpha}') + \sum_{n=1}^{N} \int_{\tau_n} [\bar{R}_n \times (\bar{v}'_{n_t} + 2\bar{\omega} \times \bar{v}_n')] \rho_n \, d\tau_n = \bar{T}' \quad . \tag{2.36}$$

Equation (2.36) can also be deduced by perturbing (1.28_2) directly.

The foregoing results are predicated on the assumption that the steady state spin and all rotational and translational perturbations are instantaneously communicated to the liquids inside the cavities. As noted in Ref. 2, this assumption is at variance with the mechanisms of vorticity transfer from the boundaries of the tanks to the contained liquids. This can best be understood by the following.

Suppose that the cavities are motionless for $t < t_0$ and that the rotation of the vehicle about its spin axis attains its steady state value by accelerating smoothly so as not to cause any ripples on the free boundaries. Then, under these circumstances, vorticity is, for the most part, transmitted by coriolis and centrifugal effects in thin Ekman layers (Ref. 5). On the other hand, these layers do not develop for periodic motions at spacecraft control frequencies. Consequently, even though steady state spin equilibria may be attained throughout the viscous liquids after some transient time, small periodic rotations about local axes of the tanks can never be transmitted to the cores of the liquids and should, therefore, be neglected.

The above arguments require that the terms $\dot{\bar{\omega}}' \times \bar{r}_n$, $\bar{\omega}' \times \bar{r}_n$ in (2.17) be discarded. With this injunction, the perturbational equations that govern the motions of spacecraft and liquids become

$$
\begin{cases}
M\bar{\alpha}' = \bar{F}' + \displaystyle\sum_{n=1}^{N} \bar{F}_n' \, , \\[2ex]
I \cdot \dot{\bar{\omega}}' + \bar{\omega}' \times (I \cdot \bar{\omega}) + \bar{\omega} \times (I \cdot \bar{\omega}') = \bar{T}' + \displaystyle\sum_{n=1}^{N} \bar{T}_n' \, , \\[2ex]
\left.\begin{array}{l}
\bar{a}_n' = -\dfrac{1}{\rho_n} \nabla_n p_n' \, , \\[1ex]
\nabla_n \cdot \bar{V}_n' = \nabla_n \cdot \bar{v}_n' = 0 \, , \qquad Q_n \in \tau_n \, , \\[1ex]
\nabla_n \quad \bar{a}_n' = 0 \, , \\[1ex]
\bar{\nu}_n \cdot \bar{v}_n' = \begin{cases} 0 \, , & Q_n \in \sigma_{W_n} \, , \\[2ex] f_{n_t}' \, (\bar{\nu}_n \cdot \bar{k}_n) \, , & Q_n \in \sigma_{F_n} \, , \quad (z_n = f_n) \, , \end{cases} \\[4ex]
p_n' - \alpha\rho_n \, f_n' = 0 \, , \quad Q_n \in \sigma_{F_n} \, , \quad (z_n = f_n) \, , \\[2ex]
\displaystyle\int_{\sigma_{F_n}} f_{n_t}' \cos(\bar{\nu}_n , \bar{k}_n) \, d\sigma_{F_n} = \iint_{\mathscr{R}_n} f_{n_t}' \, dx_n dy_n = 0 \, , \\[3ex]
\bar{F}_n' = \displaystyle\int_{\tau_n} \nabla_n p_n' \, d\tau_n - \bar{\omega} \times \Big[\bar{\omega} \times \iint_{\mathscr{R}_n} \bar{R}_n \, f_n' \, \rho_n \, dx_n dy_n\Big] \\[3ex]
\qquad = \displaystyle\int_{\sigma_{W_n}} \bar{\nu}_n \, p_n' \, d\sigma_{W_n} - \bar{\omega} \times \Big[\bar{\omega} \times \iint_{\mathscr{R}_n} \bar{R}_n \, f_n' \, \rho_n \, dx_n dy_n\Big] \, , \\
\approx
\end{array}\right\} (7=1,2,\cdots,N), \\
\approx
\end{cases}
\qquad (2.37)
$$

$$
\left.\begin{aligned}
\bar{T}'_n = &\int_{\iota_n} (\bar{R}_n \times \nabla_n p'_n)\, d\iota_n \\
&- \iint_{\mathscr{R}_n} \left\{ \bar{R}_n \times [\bar{\alpha} + \bar{\dot{\omega}} \times (\bar{\omega} \times \bar{R}_n)] \right\} f'_n\, \rho_n\, dx_n dy_n \\
= &\int_{\sigma_{w_n}} (\bar{R}_n \times \bar{\nu}_n)\, p'_n\, d\sigma_{w_n} \\
&- \iint_{\mathscr{R}_n} \left\{ \bar{R}_n \times [\bar{\alpha} + \bar{\dot{\omega}} \times (\bar{\omega} \times R_n)] \right\} f'_n\, \rho_n\, dx_n dy_n ,
\end{aligned}\right)
$$

where

$$
\left\{\begin{aligned}
\bar{V}'_n &= \bar{u}' + \bar{\omega}' \times \bar{L}_n + \bar{v}'_n , \\
\dot{\bar{V}}'_n &= \dot{\bar{u}}' + \dot{\bar{\omega}}' \times \bar{L}_n + \bar{\omega} \times \bar{v}'_n + \bar{v}'_{n_t} , \\
\bar{a}'_n &= \dot{\bar{V}}'_n + \bar{\omega}' \times \bar{V}_n + \bar{\omega} \times \bar{V}'_n \\
&= \bar{\alpha}' + \dot{\bar{\omega}}' \times \bar{L}_n + \bar{\omega}' \times (\bar{\omega} \times \bar{L}_n) \\
&\qquad + \bar{\omega} \times (\bar{\omega}' \times \bar{L}_n) + 2\bar{\omega} \times \bar{v}'_n + \dot{\bar{v}}'_{n_t} , \\
\bar{\alpha}' &= \dot{\bar{u}}' + \bar{\omega}' \times \bar{u} + \bar{\omega} \times \bar{u}' ,
\end{aligned}\right\} \quad (n=1,2,\cdots,N) \qquad (2.38)
$$

In addition, observe

$$
\left.\begin{aligned}
&\bar{\nu}_n \cdot \bar{v}'_{n_t} = \begin{cases} 0 , & Q_n \in \sigma_{w_n} , \\[2ex] f'_{n_{tt}} \, (\bar{\nu}_n \cdot \bar{k}_n) , & Q_n \in \sigma_{F_n} , \quad (z_n = f_n) , \end{cases} \\[2ex]
&\bar{\nu}_n \cdot \bar{v}'_{n_{ttt}} = \begin{cases} 0 , & Q_n \in \sigma_{w_n} , \\[2ex] f'_{n_{tttt}} \, (\bar{\nu}_n \cdot \bar{k}_n) , & Q_n \in \sigma_{F_n} , \quad (z_n = f_n) , \end{cases} \\[2ex]
&\nu_n \cdot \bar{v}'_{n_t} = \nu \cdot \bar{v}'_{n_{ttt}} = 0 , \quad Q_n \in \tau_n , \\[2ex]
&\iint_{\mathscr{R}_n} \bar{R}_n \, f'_n \, \rho_n \, dx_n dy_n \\
&\qquad = \iint_{\mathscr{R}_n} \bar{r}_n \, f'_n \, \rho_n \, dx_n dy_n \\
&\qquad = \int_{\sigma_{F_n}} \bar{R}_n \, f'_n \, \rho_n \, (\bar{\nu}_n \cdot \bar{k}_n) \, d\sigma_{F_n} \\
&\qquad = \int_{\sigma_{F_n}} \bar{r}_n \, \rho_n \, f'_n \, (\bar{\nu}_n \cdot \bar{k}_n) \, d\sigma_{F_n} ,
\end{aligned}\right\} (n=1,2,\cdots,N) \qquad (2.39)
$$

and

$$
\begin{aligned}
\bar{r}_c' &= \frac{1}{M_1+M} \sum_{n=1}^{N} \iint_{\mathscr{R}_n} \bar{R}_n \, f_n' \, \rho_n \, dx_n dy_n \\
&= \frac{1}{M_1+M} \sum_{n=1}^{N} \iint_{\mathscr{R}_n} \bar{r}_n \, f_n' \, \rho_n \, dx_n \, dy_n \qquad (2.40) \\
&= \frac{1}{M_1+M} \sum_{n=1}^{N} \int_{\sigma_{F_n}} \rho_n \bar{R}_n \, (\bar{\nu}_n \cdot \bar{k}_n) \, d\sigma_{F_n} \quad .
\end{aligned}
$$

SECTION 3

PERTURBATIONAL LIQUID MOTIONS REDUCED TO SOLUTIONS OF N INHOMOGENEOUS BOUNDARY VALUE PROBLEMS

PERTURBATIONAL EQUATIONS DESCRIBING LIQUID MOTIONS

In light of (2.37) and (2.38), the perturbational equations that govern the motions of liquids inside the N cavities can be written

$$
\left.\begin{array}{l}
\left.\begin{array}{l}
\bar{v}'_{n_t} + 2\bar{\omega} \times \bar{v}'_n + \nabla_n \phi'_n = 0 \;, \\[1ex]
\nabla_n \cdot \bar{v}'_n = 0, \\[1ex]
\bar{\xi}'_{n_t} + 2\bar{\omega} \times \bar{\xi}'_n - 2\nabla_n(\bar{\omega}\cdot\bar{v}'_n) = 0 \;, \\[1ex]
\phi'_n = p'_n/\rho_n + \bar{\beta}_n \cdot \bar{R}_n \;, \\[1ex]
\bar{\beta}_n = \bar{\alpha}' + \dot{\bar{\omega}}' \times \bar{L}_n + \bar{\omega}' \times (\bar{\omega}\times\bar{L}_n) + \bar{\omega} \times (\bar{\omega}'\times\bar{L}_n) \;,
\end{array}\right\} Q_n \varepsilon \tau_n \\[1ex]
\bar{\omega} = \omega\bar{k} = \omega\bar{k}_n \;, \; (\omega \sim \text{ constant}) \\[1ex]
\bar{\nu}_n \cdot v'_n = \begin{cases} 0 \;, & Q_n \,\varepsilon\, \sigma_{W_n} \;, \\ f'_{n_t}\,(\bar{\nu}_n\cdot\bar{k}_n) \;, & Q_n \,\varepsilon\, \sigma_{F_n} \;, \; (z_n = f_n) \;, \end{cases} \\[1ex]
p'_n - \alpha\rho_n f'_n = 0 \;, \; Q_n \,\varepsilon\, \sigma_{F_n} \;, \; (z_n = f_n) \;, \\[1ex]
\displaystyle\int_{\sigma_{F_n}} f'_{n_t}\,(\bar{\nu}_n\cdot\bar{k}_n)\, d\sigma_{F_n} = \iint_{\mathscr{R}_n} f'_{n_t}\, dx_n dy_n = 0 \;,
\end{array}\right\} (n=1,2,\ldots,N) \tag{3.1}
$$

In addition,

$$
\left.\begin{aligned}
&\bar{\nu}_n \cdot \bar{v}'_{n_t} = \begin{cases} 0 \;, & Q_n \in \sigma_{W_n}, \\ f'_{n_{tt}} \, (\bar{\nu}_n \cdot \bar{k}_n) \;, & Q_n \in \sigma_{F_n}, \; (z_n = f_n), \end{cases} \\
&\nu_n \cdot v'_{n_{ttt}} = \begin{cases} 0 \;, & Q_n \in \sigma_{W_n}, \\ f'_{n_{tttt}} \, (\bar{\nu}_n \cdot \bar{k}_n) \;, & Q_n \in \sigma_{F_n}, \; (z_n = f_n), \end{cases} \\
&Vn \cdot \bar{v}'_{n_t} = Vn \cdot v'_{n_{ttt}} = 0 \;, \quad Q_n \in \tau_n \;, \\
&\int_{\sigma_{F_n}} f'_n \, (\bar{\nu}_n \cdot \bar{k}_n) \, d\sigma_{F_n} = \iint_{Q_n} f'_n \, dx_n \, dy_n = 0, \\
&\bar{F}'_n = \int_{\tau_n} V_n p'_n \, d\tau_n - \bar{\omega} \times \Big[\bar{\omega} \times \iint_{\mathscr{R}_n} \bar{R}_n \, f'_n \, \rho_n \, dx_n \, dy_n\Big] \\
&\quad = \int_{\sigma_{W_n}} \bar{\nu}_n \, p'_n \, d\sigma_{W_n} - \bar{\omega} \times \Big[\bar{\omega} \times \int_{\sigma_{F_n}} \bar{R}_n \, f'_n \, \rho_n \, (\bar{\nu}_n \cdot \bar{k}_n) \, d\sigma_{F_n}\Big], \\
&T'_n = \int_{\tau_n} (\bar{R}_n \times V_n p'_n) \, d\tau_n - \iint_{R_n} \{\bar{R}_n \times [\bar{\alpha} + \bar{\omega} \times (\bar{\omega} \times \bar{R}_n)]\} \\
&\qquad f'_n \, \rho_n \, dx_n \, dy_n = \int_{\sigma_{W_n}} (\bar{R}_n \times \bar{\nu}_n) \, p'_n \, d\sigma_{W_n} - \int_{\sigma_{F_n}} \{\bar{R}_n \times [\bar{\alpha} \\
&\qquad + \bar{\omega} \times (\bar{\omega} \times \bar{R}_n)]\} \, f'_n \, \rho_n \, (\bar{\nu}_n \cdot \bar{k}_n) \, d\sigma_{F_n}, \\
&\int_{\sigma_{F_n}} p'_n \, (\bar{\nu}_n \cdot \bar{k}_n) \, d\sigma_{F_n} = \int_{\mathscr{R}_n} p'_n \, dx_n \, dy_n = 0,
\end{aligned}\right\} (n=1,2,\ldots,N) \qquad (3.2)
$$

DERIVATION OF BOUNDARY VALUE PROBLEMS

The determination of perturbational liquid motions inside the N cavities can be made to depend on the solutions of N boundary value problems with acceleration potentials ϕ_n' as dependent variables. This can be shown in the following manner.

Consider the nth cavity. Take the divergence of (3.1_1),

$$\nabla_n \cdot \left(\bar{v}'_{n_t} + 2\bar{\omega} \times \bar{v}'_n + \nabla_n \phi'_n\right) = 0 \quad ,$$

obtaining

$$\nabla^2 \phi'_n = 2\bar{\omega} \cdot \bar{\xi}'_n \quad , \tag{3.3}$$

in view of

$$\nabla_n \cdot \bar{v}'_{n_t} = 0 \quad , \quad \nabla_n \cdot \nabla_n \phi'_n = \nabla_n^{2} \phi'_n \quad , \nabla_n \cdot (2\bar{\omega}\ \bar{v}'_n) = -\ 2\bar{\omega} \cdot \bar{\xi}'_n \ .$$

Differentiate (3.3) partially with respect to time,

$$\nabla^2 \phi'_{n_t} = 2\bar{\omega} \cdot \bar{\xi}'_{n_t} \quad . \tag{3.4}$$

Take the scalar product of $\bar{\omega}$ and equation (3.1_1)

$$2\bar{\omega} \cdot \left[\xi'_{n_t} + 2\bar{\omega} \times \bar{\xi}'_{n_t} - 2\nabla_n(\bar{\omega}\cdot\bar{v}'_n)\right] = 0 \quad ,$$

obtaining

$$2\bar{\omega} \cdot \bar{\xi}'_{n_t} = 4\bar{\omega} \cdot \nabla_n \ (\bar{\omega}\cdot\bar{v}'_n) \quad , \tag{3.5}$$

in light of

$$2\bar{\omega} \cdot (2\bar{\omega} \times \bar{\xi}'_n) = 0 \ .$$

Substitute (3.5) into (3.4) and differentiate the resulting expression partially with respect to time,

$$\nabla_n^2 \phi'_{n_{tt}} = 4\bar{\omega} \cdot \nabla_n \left(\bar{\omega} \cdot \bar{v}'_{n_t} \right) . \tag{3.6}$$

Take the scalar product of $\bar{\omega}$ and equation (3.1_1)

$$\bar{\omega} \cdot \left[\bar{v}'_{n_t} + 2\bar{\omega} \times \bar{v}'_n + \nabla_n \phi'_n \right] = 0 \quad ,$$

obtaining

$$\bar{\omega} \cdot \bar{v}'_{n_t} = - \bar{\omega} \cdot \nabla_n \phi'_n \quad , \tag{3.7}$$

owing to

$$\bar{\omega} \cdot (2\bar{\omega} \times \bar{v}'_n) = 0 \ .$$

Substitute (3.7) into (3.6),

$$\nabla_n^2 \phi'_{n_{tt}} + 4\omega^2 \phi'_{n_{z_n z_n}} = 0 \quad , \quad Q_n \in T_n \quad . \tag{3.8}$$

In view of

$$\bar{\omega} \cdot \nabla_n \ (\bar{\omega} \cdot \nabla_n \phi'_n) = \omega^2 \phi'_{n_{z_n z_n}} \quad .$$

Expression (3.8) is the desired partial differential equation for ϕ'_n.

To establish boundary conditions for ϕ'_n, first differentiate (3.1_1) partially with respect to time,

$$\bar{v}'_{n_{tt}} + 2\bar{\omega} \times \bar{v}_{n_t} + \nabla_n \phi'_{n_t} = 0 \quad . \tag{3.9}$$

Now, from (3.1_1)

$$\bar{v}'_{n_t} = - \nabla_n \phi'_n - 2\bar{\omega} \times \bar{v}'_n \quad . \tag{3.10}$$

Substitute (3.10) into 3.9),

$$\bar{v}'_{n_{tt}} + 4\omega^2 \bar{v}'_n - 4\omega^2 \bar{k}_n \, (\bar{k}_n \cdot \bar{v}'_n) + \nabla_n \phi'_{n_t} + \nabla_n \times (2\bar{\omega}\phi'_n) = 0 \quad , \tag{3.11}$$

in view of

$$\nabla_n \times (2\bar{\omega}\phi'_n) = - 2\bar{\omega} \times \nabla_n \phi'_n \quad .$$

Differentiate (3.11) partially with respect to time,

$$\bar{v}'_{n_{ttt}} + 4\omega^2 \bar{v}'_{n_t} - 4\omega^2 \bar{k}_n \left(\bar{k}_n \cdot \bar{v}_{n_t}\right) + \nabla_n \phi'_{n_{tt}} + \nabla_n \times \left(2\bar{\omega}\phi'_{n_t}\right) = 0 \quad . \tag{3.12}$$

Take the scalar product of $\bar{k}_n$ and equation (3.1_1),

$$\bar{v}_{n_t} \cdot \bar{k}_n = - \phi'_{n_{z_n}} \tag{3.13}$$

because

$$\bar{k}_n \cdot (\bar{\omega} \times \bar{v}'_n) \quad 0 \quad ,$$

$$\bar{k}_n \cdot \nabla_n \phi'_n = \phi'_{n_{z_n}} \quad .$$

Substitute (3.13) into (3.12) and rearrange.

$$\nabla_n \phi'_{n_{tt}} + \nabla_n \times \left(2\bar{\omega}\phi'_{n_t}\right) + 4\omega^2 \phi'_{n_{z_n}} \; \bar{k}_n = - \left(\bar{v}'_{n_{ttt}} + 4\omega^2 \bar{v}'_{n_t}\right) \tag{3.14}$$

Project (3.14) along the outward directed normals to σ_{w_n} and σ_{F_n},

$$\bar{\nu}_n \cdot \left[V_n \phi'_{n_{tt}} + V_n \times \left(2\bar{\omega}\, \phi'_{n_t}\right) + 4\omega^2 \phi_{n_{z_n}} \bar{k}_n \right] = 0 \;,\; Q_n \varepsilon\, \sigma_{w_n} \tag{3.15}$$

$$= -(\bar{\nu}_n \cdot \bar{k}_n) \left(f'_{n_{tttt}} + 4\omega^2 f'_{n_{tt}}\right) \;,\; Q_n \varepsilon\, \sigma_{F_n} \;,\; (z_n = f_n) \;,$$

in view of (3.2_1) and (3.2_2). To complete the specification of the boundary conditions, solve (3.14) for p'_n

$$p'_n = \rho_n \left(\phi'_n - \bar{\beta}_n \cdot \bar{R}_n\right)$$

and substitute into condition (3.1_8), obtaining

$$\phi'_n - \alpha f'_n = \bar{\beta}_n \cdot \bar{R}_n \;,\; Q_n \varepsilon\, \sigma_{F_n} \;. \tag{3.16}$$

The free surface displacement can be eliminated from the problem as follows. Solve (3.16) for f'_n and substitute the resulting expression into (3.15),

$$\bar{\nu}_n \cdot \left[V_n \phi'_{n_{tt}} + V_n \times \left(2\bar{\omega}\phi_{n_t}\right) + 4\omega^2 \phi'_{n_{z_n}} \bar{k}_n \right] = 0 \;,\; Q_n \varepsilon\, \sigma_{w_n} \;, \tag{3.17}$$

$$= -\frac{1}{g_n}\left(\phi'_{n_{tttt}} + 4\omega^2 \phi_{n_{tt}}\right) + \frac{1}{g_n} \bar{R}_n \cdot \left(\bar{\beta}_{n_{tttt}} + 4\omega^2 \bar{\beta}_{n_{tt}}\right),\; Q_n \varepsilon \sigma_{F_n} \;,\; (z_n = f_n) \;,$$

where

$$f'_n = \frac{1}{\alpha}\phi'_n - \frac{1}{\alpha}\bar{\beta}_n \cdot \bar{R}_n,\; Q_n \varepsilon\, \sigma_{F_n} \;, \tag{3.18}$$

$$g_n = \left(\frac{\bar{\nu}_n \cdot \bar{k}_n}{\alpha}\right) \quad \text{(Equation } (2.3_3)) \;,\; Q_n \varepsilon\, \sigma_{F_n} \;. \tag{3.19}$$

Take the divergence of (3.14) and make use of (3.2_3),

$$\nabla_n \cdot \left[\nabla_n \phi'_{n_{tt}} + \nabla_n \times \left(2\bar{\omega}\, \phi'_{n_t} \right) + 4\omega^2 \phi'_{n_{z_n}} \bar{k}_n \right] =$$

$$\nabla_n^2 \phi'_{n_{tt}} + 4\omega^2 \phi'_{n_{z_n z_n}} = 0 \quad , \quad Q_n \varepsilon \tau_n \quad , \tag{3.20}$$

in agreement with (3.8).

The inhomogeneous boundary value problem defined by (3.8) and (3.17) governs the perturbational liquid motions inside the nth tank. The effects of centrifugal and coriolis accelerations together with vorticity are implicitly taken into consideration in the formulation.

Observe that the excitation vector $\bar{\beta}_n$ is a function of the perturbational translational and rotational accelerations and rotational velocities of the spacecraft. They, in turn, are related to the motions of the liquids in the remaining N-1 tanks via the perturbational equations of translational and rotational motion of the spacecraft.

The force and moment resulting from the action of the perturbational liquid motions on the tank wall can be written

$$\bar{F}'_n = \int_{\sigma_n} \bar{\nu}_n \phi'_n \rho_n \, d\sigma_n - \int_{\sigma_{F_n}} \frac{\bar{\omega} \times (\bar{\omega} \times \bar{R}_n)}{g_n} \phi'_n \rho_n \, d\sigma_{F_n} - \bar{\beta}_n \int_{\tau_n} \rho_n d\tau_n$$

$$+ \int_{\sigma_{F_n}} \frac{\bar{\omega} \times (\bar{\omega} \times \bar{R}_n)}{g_n} (\bar{\beta}_n \cdot \bar{R}_n) \rho_n \, d\sigma_{F_n} \quad , \tag{3.21}$$

$$\bar{T}'_n = \int_{\sigma_n} (\bar{R}_n \times \bar{\nu}_n) \phi'_n \rho_n \, d\sigma_n - \int_{\sigma_{F_n}} \frac{\left\{\bar{R}_n \times [\bar{\alpha} + \bar{\omega} \times (\bar{\omega} \times \bar{R}_n)]\right\}}{g_n} \phi'_n \rho_n \, d\sigma_{F_n}$$

$$- \int_{\tau_n} (\bar{R}_n \times \bar{\beta}_n) \rho_n \, d\tau_n + \int_{\sigma_{F_n}} \frac{\left\{\bar{R}_n \times [\bar{\alpha} + \bar{\omega} \times (\bar{\omega} \times \bar{R}_n)]\right\}}{g_n} (\bar{\beta}_n \cdot \bar{R}_n) \rho_n \, d\sigma_{F_n} \quad , \tag{3.22}$$

in light of (3.1_4) (3.2_5) (3.2_6), (3.2_7), (3.18), (3.19) and the divergence theorem.

VARIATIONAL FORMULATION OF ASSOCIATED HOMOGENEOUS PROBLEM

Consider the associated homogeneous boundary value problem pertinent to the nth tank,

$$\nabla_n \cdot \left[\nabla_n \phi'_{n_{tt}} + \nabla_n \times \left(2\bar{\omega}\phi'_{n_t}\right) + 4\omega^2 \phi'_{n_{z_n}} \bar{k}_n\right] = 0 \quad , Q_n \varepsilon \tau_n \quad ,$$

$$\bar{\nu}_n \cdot \left[\nabla_n \phi'_{n_{tt}} + \nabla_n \times \left(2\omega\phi'_{n_t}\right) + 4\omega^2 \phi'_{n_{z_n}} \bar{k}_n\right] = 0 \quad , Q_n \varepsilon \sigma_{w_n} \quad , \tag{3.23}$$

$$= - \frac{(\bar{\nu}_n \cdot \bar{k}_n)}{\alpha} \left(\phi'_{n_{tttt}} + 4\omega^2 \phi'_{n_{tt}}\right), \quad Q_n \varepsilon \sigma_{F_n} \quad , \quad (z_n = f_n) \quad .$$

It is fairly straightforward to show that the first variation of the functional ($\delta I_n = 0$)

$$I_n = \frac{1}{2} \int_{t_0}^{t} \left\{ \int_{\tau_n} \left[\left(\nabla_n \phi'_{n_t}\right)^2 - 4\omega^2 \phi'^{\,2}_{n_{z_n}} - 2\omega\left(\phi'_{n_{ty_n}} \phi'_{n_{x_n}} - \phi'_{n_{tx_n}} \phi'_{n_{y_n}}\right)\right] d\tau_n \right.$$

$$\left. - \frac{1}{\alpha} \int_{\sigma_{F_n}} \left[\left(\phi'_{n_{tt}}\right)^2 - 4\omega^2 \left(\phi'_{n_t}\right)^2\right] (\bar{\nu}_n \cdot \bar{k}_n) \, d\sigma_{F_n} \right\} dt \tag{3.24}$$

subject to

$$\delta\phi'_n \Big|_{t_0}^{t} = \delta\phi'_t \Big|_{t_0}^{t} = 0 \quad ,$$

gives the boundary value problem (3.23).

The functional (3.24) together with the methods of Rayleigh and Ritz make it possible to obtain approximate solutions of (3.23) in terms of eigenfunctions and eigenvalues of simpler, but related, boundary value problems.

For instance, the eigenfunctions ψ_{n_i} and eigenvalues K_{n_i} satisfying the boundary value problem

$$\begin{cases} \nabla^2 \psi_{n_i} = 0 \quad , \quad Q_n \in \tau_n \quad , \\ \bar{\nu}_n \cdot \nabla_n \psi_{n_i} = \begin{cases} 0 \quad , \quad Q_n \in \sigma_{W_n} \quad , \\ \dfrac{K_{n_i}}{g_n} \psi_{n_i} \quad , \quad Q_n \in \sigma_{F_n} \quad , \quad (z_n = f_n) \end{cases} \end{cases} \tag{3.25}$$

and orthogonality conditions

$$\begin{cases} \displaystyle\int_{\tau_n} \nabla_n \psi_{n_i} \cdot \nabla_n \psi_{n_j} \, d\tau_n = \begin{cases} K_{n_i} \displaystyle\int_{\sigma_{F_n}} \frac{1}{g_n} \psi_{n_i}{}^2 \, d\sigma_{F_n} = \|\psi_{n_i}\|^2 K_{n_i} , & j=i, \\ 0 & , j \neq i, \end{cases} \\ \displaystyle\int_{\sigma_{F_n}} \frac{1}{g_n} \psi_{n_i} \psi_{n_j} \, d\sigma_{F_n} = \begin{cases} \displaystyle\int_{\sigma_{F_n}} \frac{1}{g_n} \psi_{n_i}{}^2 \, d\sigma_{F_n} = \|\psi_{n_i}\|^2 \quad , & j=i \\ 0 \, , & j \neq i \end{cases} \end{cases} \tag{3.26}$$

where

$$g_n = (\bar{\nu}_n \cdot \bar{k}_n)/\alpha \quad , \tag{3.27}$$

are available for a spherical container, Ref. 1. Assume that ϕ_n' can be expanded in a series

$$\phi_n' = \sum_i \psi_{n_i} \; q_{n_i}(t) \quad , \tag{3.28}$$

where $q_{n_i}(t)$ are to be determined. Substituting (3.28) into (3.24) and performing the first variation of the resulting expression ($\delta I_n = 0$) subject to

$$\delta \, \dot{q}_{n_i}\Big|_{t_0}^{t} = \delta \, q_{n_i}\Big|_{t_0}^{t} = 0 \quad ,$$

one obtains the system of ordinary differential equations

$$\begin{aligned} &\dddot{q}_{n_i} + \left(K_{n_i} + 4\omega^2\right) \ddot{q}_{n_i} + \frac{2\omega}{\|\psi_{n_i}\|^2} \sum_j \left[\int_{\tau_n} \frac{\partial\psi_{n_j}}{\partial y_n} \frac{\partial\psi_{n_i}}{\partial x_n} - \right. \\ &\left. \frac{\partial\psi_{n_i}}{\partial y_n} \frac{\partial\psi_{n_j}}{\partial x_n} \, d\tau_n \right] \dot{q}_{n_j} + \\ &\frac{4\omega^2}{\|\psi_{n_i}\|^2} \sum_j \left[\int_{\tau_n} \frac{\partial\psi_{n_i}}{\partial z_n} \frac{\partial\psi_{n_j}}{\partial z_n} \, d\tau_n \right] q_{n_j} = 0 \quad , \quad i=1,2,\cdots\cdots , \end{aligned} \tag{3.29}$$

for the determination of the q_{n_i}. Other examples can be adduced.

If the excitation vectors $\overline{\beta}_n$ are regarded as preassigned functions of time, the inhomogeneous boundary value problem defined by (3.8) and (3.17) can be deduced from the first variation of the functional

$$I_n = \frac{1}{2} \int_{t_0}^{t} \left\{ \iiint_{\tau_n} \left[\left(\nabla_n \phi'_{n_t}\right)^2 - 4\omega^2 \phi'^{\,2}_{n_{z_n}} - 2\omega \, \phi'_{n_{y_n}} \phi'_{n_{x_n}} - \phi'_{n_{tx_n}} \phi'_{n_{y_n}} \right] d\tau_n \right.$$

$$- \frac{1}{\alpha} \iint_{\sigma_{F_n}} \left[\left(\phi'_{n_{tt}}\right)^2 - 4\omega^2 \left(\phi'_{n_t}\right)^2 \right] (\bar{\nu}_n \cdot \bar{k}_n) \, d\sigma_{F_n} \tag{3.30}$$

$$\left. + \frac{2}{\alpha} \iint_{\sigma_{F_n}} \left(\bar{\beta}_{n_{tttt}} + 4\omega^2 \beta_{n_{tt}} \right) \cdot \bar{R}_n \, (\bar{\nu}_n \cdot \bar{k}_n) \phi'_n \, d\sigma_{F_n} \right\} dt .$$

CHARACTERIZATION OF THE BOUNDARY VALUE PROBLEMS

For a single frequency excitation

$$\left\{ \begin{aligned} \bar{\beta}_n &= \bar{c}_n \, e^{i\Omega t} \, , \\ \phi'_n &= \psi_n \, e^{i\Omega t} \, , \end{aligned} \right. \tag{3.31}$$

the inhomogeneous boundary value problem spelled out by (3.8) and (3.17) becomes

$$\frac{\partial^2 \psi_n}{\partial x_n^2} + \frac{\partial^2 \psi_n}{\partial y_n^2} + \left(1 - \frac{4\omega^2}{\Omega^2}\right) \frac{\partial^2 \psi_n}{\partial z_n^2} = 0 \quad , \quad Q_n \in \tau_n \tag{3.32}$$

$$\bar{\nu}_n \cdot \left[\frac{\partial \psi_n}{\partial x_n} \bar{i}_n + \frac{\partial \psi_n}{\partial y_n} \bar{j}_n + \left(1 - \frac{4\omega^2}{\Omega^2}\right) \frac{\partial \psi_n}{\partial z_n} \bar{k}_n - i \frac{2\omega}{\Omega} \nabla_n \times (\bar{k}_n \psi_n) \right] = 0 \ ,$$

$$Q_n \in \sigma_{w_n} \ , \tag{3.33}$$

$$= \frac{\Omega^2}{g_n} \left(1 - \frac{4\omega^2}{\Omega^2}\right) \psi_n - \frac{\Omega^2}{g_n} \left(1 - \frac{4\omega^2}{\Omega^2}\right) (\bar{c}_n \cdot \bar{R}_n) \ , \quad Q_n \ \sigma_{F_n} \ .$$

If $2\omega/\Omega < 1$, the partial differential equation (3.32) can be brought to an elliptic form by the transformation

$$x_n = x_n,\; y_n = y_n,\; z_n = \sqrt{1-4\omega^2/\Omega^2}\,\bar{z}_n \quad ,$$

namely,

$$\frac{\partial^2\psi_n}{\partial x_n^2} + \frac{\partial^2\psi_n}{\partial y_n^2} + \frac{\partial^2\psi_n}{\partial \bar{z}^2} = 0 \quad , \quad Q_n \in \tau_n^* \tag{3.34}$$

with boundary conditions

$$\bar{\nu}_n \cdot \left[\frac{\partial\psi_n}{\partial x_n}\bar{i}_n + \frac{\partial\psi_n}{\partial y_n}\bar{j}_n + \sqrt{1 - \frac{4\omega^2}{\Omega^2}}\;\frac{\partial\psi_n}{\partial \bar{z}_n}\bar{k}_n - i\,\frac{2\omega}{\Omega}\,\nabla_n \times (\bar{k}_n\,\psi_n)\right]$$

$$= 0 \; , \quad Q_n \in \sigma_{W_n}^* \quad , \tag{3.35}$$

$$= \frac{\Omega^2}{g_n}\left(1 - \frac{4\omega^2}{\Omega^2}\right)\psi_n - \frac{\Omega^2}{g_n}\left(1 - \frac{4\omega^2}{\Omega^2}\right)\left(\bar{c}_n \cdot \bar{R}_n\right) , \quad Q_n \in \sigma_{F_n}^* \quad ,$$

where τ_n^*, $\sigma_{W_n}^*$, $\sigma_{F_n}^*$ are the stretched values of τ_n, σ_{W_n}, σ_{F_n}. On the other hand, if $2\omega/\Omega > 1$, the partial differential equation (3.32) can be brought to an hyperbolic form by the transformation

$$x_n = x_n,\; y_n = y_n,\; z_n = \sqrt{\frac{4\omega^2}{\Omega^2} - 1}\;\bar{z}_n \quad ,$$

namely,

$$\frac{\partial^2\psi_n}{\partial x_n^2} + \frac{\partial^2\psi_n}{\partial y_n^2} - \frac{\partial^2\psi_n}{\partial z_n^2} = 0 \quad , \quad Q_n \in \tau_n^* \quad , \tag{3.36}$$

with boundary conditions

$$\bar{\nu}_n \cdot \left[\frac{\partial\psi_n}{\partial x_n}\ \bar{i}_n + \frac{\partial\psi_n}{\partial y_n}\ \bar{j}_n - \sqrt{\frac{4\omega^2}{\Omega^2} - 1}\ \frac{\partial\psi_n}{\partial z_n}\ \bar{k}_n - i\ \frac{2\omega}{\Omega}\ \nu_n \times (\bar{k}_n\psi_n)\right] = 0\ ,$$

$$Q_n \in \sigma^*_{w_n}\ , \tag{3.37}$$

$$= -\frac{\Omega^2}{g_n}\left(\frac{4\omega^2}{\Omega^2} - 1\right)\psi_n + \frac{\omega^2}{g_n}\left(\frac{4\omega^2}{\Omega^2} - 1\right)\left(\bar{c}_n \cdot \bar{R}_n\right)\ ,\quad Q_n \in \sigma^*_{F_n}$$

For low spin rates the liquid motions are governed by an elliptic partial differential equation with mixed boundary conditions. As the spin rate is decreased indefinitely the governing equations degenerate to the classical lateral slosh equations. This strongly suggests that the custom of representing sloshing motions in spinning tanks by simple mechanical pendulums is valid only for $0 \leq 2\omega/\Omega < 1$.

For ratios of $2\omega/\Omega$ in excess of unity, the sloshing motions are governed by an hyperbolic partial differential equation with mixed boundary conditions. The nature of the hyperbolic differential system together with the form displayed in (3.19) point out that the representation of sloshing motions in spinning tanks by mechanical pendulums is, in general incorrect.

Yet, for most practical applications $2\omega/\Omega > 1$. Consequently, the use of mechanical pendulums to reproduce sloshing motions in spinning tanks is suspect and should be avoided, unless severe restrictions can be tolerated.

The nature of the solutions of the above boundary value problems in the case of a completely filled spherical tank is discussed in Ref. 5.

SECTION 4

SUMMARY OF PERTURBATIONAL EQUATIONS

The perturbational equations that describe the small motions of a spacecraft with N partly filled liquid propellant tanks relative to a reference state of motion wherein the vehicle undergoes steady state rotations, as a single rigid body, about its designed spin axis in a constant acceleration field are collected below.

$$
\left\{
\begin{aligned}
& M\ddot{\overline{x}}' = \overline{F} + \sum_{n=1}^{N} \overline{F}_n' \\
& I \cdot \dot{\overline{\omega}}' + \overline{\omega}' \times (I \cdot \overline{\omega}) + \overline{\omega} \times (I \cdot \overline{\omega}') = \overline{T} + \sum_{n=1}^{N} \overline{T}_n' \\
& \left.\begin{aligned}
& \nabla_n \cdot \left[\nabla_n \phi'_{n_{tt}} + \nabla_n \times \left(2\overline{\omega}\phi'_{n_t} \right) + 4\omega^2 \phi'_{n_{z_n}} \overline{k}_n \right] = 0, \quad Q_n \epsilon \tau_n \, , \\
& \overline{\nu}_n \cdot \left[\nabla_n \phi'_{n_{tt}} + \nabla_n \times (2\overline{\omega}\phi'_{n_t}) + 4\omega^2 \phi'_{n_{z_n}} \overline{k}_n \right] = 0, \quad Q_n \epsilon \sigma_{W_n} \, , \\
& = -\frac{1}{g_n} \left(\phi'_{n_{tttt}} + 4\omega^2 \phi'_{n_{tt}} \right) + \frac{1}{g_n} \Big(\overline{\beta}_{n_{tttt}} \\
& \quad + 4\omega^2 \overline{\beta}_{n_{tt}} \Big) \cdot R_n, \quad Q_n \epsilon \sigma_{F_n}, \quad (z_n = f_n) \\
& \overline{F}_n' = \int_{\sigma_n} \overline{\nu}_n \phi_n \rho_n d\sigma_n - \int_{\sigma_{F_n}} \frac{\overline{\omega} \times \overline{\omega} \times \overline{R}_n}{g_n} \phi_n' \rho_n d\sigma_{F_n} - \overline{\beta}_n \int_{\tau_r} \rho_n d\tau_n \\
& \quad + \int_{\sigma_{F_n}} \frac{\overline{\omega} \times (\overline{\omega} \times \overline{R}_n)}{g_n} (\overline{\beta}_n \cdot \overline{R}_n) \rho_n d\sigma_{F_n} \, ,
\end{aligned}\right\} (n=1,2,\ldots,N), \\
& \approx \qquad\qquad\qquad\qquad \approx
\end{aligned}
\right.
\tag{4.1}
$$

$$\left[\begin{aligned} \overline{T}'_n = & \int_{\sigma_n} \left(\overline{R}_n \times \nu_n\right) \phi'_n \rho_n \, d\sigma_n - \int_{\sigma_{F_n}} \left\{ \frac{\overline{R}_n \times [\overline{\alpha} + \overline{\omega} \times (\overline{\omega} \times \overline{R}_n)]}{g_n} \right\} \phi'_n \rho_n \, d\sigma_{F_n} \\ & - \int_{\tau_n} (\overline{R}_n \times \overline{\beta}_n) \, \rho_n \, d\tau_n + \int_{\sigma_{F_n}} \left\{ \frac{\overline{R}_n \times [\overline{\alpha} + \overline{\omega} \times (\overline{\omega} \times \overline{R}_n)]}{g_n} \right\} (\overline{\beta}_n \cdot \overline{R}_n) \rho_n \, d\sigma_{F_n} , \end{aligned}\right]$$

where

$$\left\{\begin{aligned} f'_n &= \frac{1}{\alpha} (\phi'_n - \overline{\beta}_n \cdot \overline{R}_n), \quad Q_n \in \sigma_{F_n} , \\ p'_n &= \rho_n (\phi'_n - \overline{\beta}_n \cdot \overline{z}_n), \quad Q_n \in \tau_n , \\ \overline{\beta}_n &= \overline{\alpha}' + \dot{\overline{\omega}}' \times \overline{L}_n + \overline{\omega}' \times (\overline{\omega} \times L_n) + \overline{\omega} \times (\overline{\omega}' \times \overline{L}_n), \quad Q_n \in \tau_n , \\ g_n &= \frac{(\overline{\nu}_n \cdot \overline{k}_n)}{\alpha}, \quad Q_n \in \sigma_{F_n} , \\ \sigma_n &= \sigma_{W_n} + \sigma_{F_n} , \end{aligned}\right\} (n=1,2,\ldots,N), \qquad (4.2)$$

and the reference state parameters are spelled out in (2.1, 2, 3, 4).

CONCLUDING REMARKS

The perturbational liquid motions in the nth cavity are governed by the boundary value problem displayed in (4.13) and (4.14). Further meaningful progress is not possible until a method is devised for solving this differential system.

For a single frequency excitation and low spin rates, the problem reduces to an elliptic partial differential equation with mixed boundary conditions. This differential system can be solved numerically in a fairly straightforward manner. However, this case does not seem too important in most practical applications.

For a single frequency excitation and high spin rates, the problem reduces to an hyperbolic partial differential equation with mixed boundary conditions. Numerically, the solution process of this differential system is formidable. However, a numerical approach using Green's function appears promising. This would, of course, require extensive analyses to develop the computational algorithms.

Another approach to the general problem is to employ a variational formulation in conjunction with the methods of Rayleigh and Ritz. Basically, the technique depends on being able to represent ϕ_n' in the form

$$\phi_n' = \sum_i \psi_{n_i} \, q_{n_i}(t),$$

where the ψ_{n_i} are eigenfunctions of a related, but simpler, boundary value problem, and the q_{n_i} are generalized coordinates to be determined from the solution of an infinite set of fourth order ordinary differential equations. This set of equations is obtained by minimizing the variational integral with respect to the independent coordinates q_{n_i}. Such a method was used to arrive at system (3.29). In light of the discontinuous nature of the problem, this approach should be used with caution.

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