EXAMINATION OF TIME SERIES THROUGH RANDOMLY BROKEN WINDOWS

by

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ABSTRACT

In order to determine the Fourier transform of a quasi-periodic time series (linear problem), or the power spectrum of a stationary random time series (quadratic problem), it is desirable that data be recorded without interruption over a long time interval. In practice, this may not be possible. The effect of regular interruption such as the day/night cycle is well known. We here investigate the effect of irregular interruption of data collection (the "breaking" of the window function) with the simplifying assumption that there is a uniform probability p that each interval of length τ , of the total interval of length $T = N\tau$, yields no data.

For the linear case we find that the noise-to-signal ratio will have a (one-sigma) value less than ε if N exceeds $p^{-1}(1-p)\varepsilon^{-2}$. For the quadratic case, the same requirement is met by the less restrictive requirement that N exceed $p^{-1}(1-p)\varepsilon^{-1}$.

It appears that, if four observatories spaced around the earth were to operate for 25 days, each for six hours a day ($N = 100$), and if the probability of cloud cover at any site on any day is 20% ($p = 0.8$), the r.m.s. noise-to-signal ratio is 0.25% for frequencies displaced from a sharp strong signal by 15 μHz. The noise-to-signal ratio drops off rapidly if the frequency offset exceeds 15 μHz.

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I. INTRODUCTION

In many astrophysical problems one is concerned with the study of time series. It often happens that the property of particular interest is the spectrum of the time series. In principle, one may determine a time series to a prescribed accuracy by making measurements, without interruption, over a sufficiently long time interval. In practice, the length of time over which the variables may be measured will be limited. Moreover, measurements may necessarily be interrupted (or otherwise impaired) for one reason or another. The relationship of the spectrum determined by limited, interrupted measurements to the intrinsic spectrum has been the subject of many investigations, as recently reviewed by Deeming (1975).

If the original time series is denoted by x(t), one may regard the measurements y(t) as being determined by

$$y(t) = f(t)\,x(t), \tag{1.1}$$

where f(t) is the "window function." We regard x, y and f as being simple scalar functions but the procedure may be generalized to replace x, y by vectors and f by a tensor.

We use the Fourier transform notation

$$x(t) = \int d\omega\, e^{-i\omega t}\, \tilde{x}(\omega) \tag{1.2}$$

$$\tilde{x}(\omega) = \frac{1}{2\pi}\int dt e^{i\omega t}\, x(t) \tag{1.3}$$

where the limits of integration are to be taken to be $-\infty$ to $+\infty$ if other limits are not explicitly specified.

If we are interested in determining $\tilde{x}(\omega)$, the Fourier transform of the time series x(t), then we may use the relation

$$\tilde{y}(\omega) = \int d\omega' \, \tilde{f}(\omega') \, \tilde{x}(\omega - \omega') \tag{1.4}$$

to relate the Fourier transform of the measured time series y(t) to that of the original time series x(t).

We are interested in the possibility that f(t) may be regarded as a random variable, expressible as

$$f(t) = F(t;\alpha_1,\alpha_2,\ldots,\alpha_N) \ , \tag{1.5}$$

where $\alpha_1,\alpha_2,\ldots\alpha_N$ are independent random variables with specified expectation distributions. By the central limit theorem (Papoulis, 1965), we expect that the random variable f (or its Fourier transform $\tilde{f}$) will have a distribution approximately Gaussian in form if N is not a small number, so that an adequate representation of $\tilde{f}$ would be given by its mean $\langle \tilde{f} \rangle$ and its standard deviation $\sigma(\tilde{f})$.

If, on the other hand, x(t) is a random time series, we will be concerned with the autocorrelation function $R_x(t)$, defined by

$$R_x(t) = \langle x(t') \, x(t'+t) \rangle \ , \tag{1.6}$$

and the power spectrum of the time series, defined as the Fourier transform of $R_x(t)$:

$$R_x(t) = \int d\omega \, e^{-i\omega t} S_x(\omega) , \tag{1.7}$$

$$S_x(\omega) = \frac{1}{2\pi} \int dt e^{i\omega t} \, R_x(t) . \tag{1.8}$$

On noting that

$$\langle \tilde{x}(\tilde{\omega})\, \tilde{x}(\omega') \rangle = S_x(\omega)\ \delta(\omega+\omega') \tag{1.9}$$

and evaluating $\langle \tilde{y}(\omega)\, \tilde{y}(\omega') \rangle$, we may verify that

$$S_y(\omega) = \int d\omega'\ w(\omega')\ S_x(\omega-\omega') \tag{1.10}$$

where

$$w(\omega) = \tilde{f}(\omega)\, \tilde{f}(-\omega). \tag{1.11}$$

Clearly the function $w(\omega)$ represents the capability of the measurement process, described by the "window function" $f(t)$, to determine the power spectrum $S_x(\omega)$. The function $w(\omega)$ may be expressed in terms of the independent random variables

$$w(\omega) = W(\omega;\alpha_1,\alpha_2,\ldots,\alpha_N). \tag{1.12}$$

Once again, unless N is a small number, we expect that the distribution of w will be approximately Gaussian so that it may be characterized by its mean value $\langle w \rangle$ and standard deviation $\sigma(w)$.

This article was prompted by a problem related to the determination of normal modes of oscillation of the sun, as determined by measurement of the photospheric velocity field. Measurements have been presented by Deubner (1975) and by Rhodes et al. (1977), and their theoretical interpretation discussed by Ulrich and Rhodes (1977) and by Ulrich et al. (1978). For optimum determination of the power spectrum of the velocity field (expressed as a function of wave number), it is clearly desirable to make observations without interpretation over as long an interval as possible. Away from polar regions, observations from a single station are interrupted by the day-night cycle which leads to unacceptable aliasing of the data. Observations made from a spacecraft in polar orbit would obviously yield un-aliased data of higher quality and higher frequency resolution. Observations made from the south pole during austral midsummer can lead to several days of uninterrupted observation and to still longer intervals with occasional, irregular interruption. It is also possible to select three or four stations around the earth which, in the absence of any cloud cover, could give continual coverage of the sun for many weeks. However, one must anticipate that some of the data would be lost by cloud cover.

It is clearly desirable that one should be able to make some estimate of the accuracy with which oscillation modes may be determined when it appears possible to observe the sun over a long interval of time losing some blocks of time because of cloud cover. The purpose of this article is to develop a model which enables us to address problems of this type. After presenting a few general formulas, we shall simplify the problem considerably by supposing that observations are made over a large number N of equal time intervals, each of length τ, so that the total time interval T is given by

$$T = N\tau . \tag{1.13}$$

With certain additional simplifying assumptions, we shall consider the statistical properties of the functions $\tilde{f}(\omega)$ and $w(\omega)$ which are representative of "randomly broken" window functions.

II. MATHEMATICAL MODEL

In the case that the window function $f(t)$ is expressible in the form (1.5), in terms of a number of random variables, we wish to study the distribution of the functions $\tilde{f}(\omega)$, $w(\omega)$, entering equations (1.4) and (1.10). We suppose that the distribution of the variables α_1 to α_N is given by the probability function $P(\alpha_1, \ldots, \alpha_N)$ such that $P(\alpha_1, \ldots, \alpha_N)d\alpha_1 \ldots, d\alpha_N$ is the probability of finding α_1 in the range α_1 to $\alpha_1 + d\alpha_1$, etc. Then the expectation value of the quantity $\tilde{f}(\omega)$ is given by

$$\langle \tilde{f}(\omega) \rangle = \int d^N\alpha \; P_N(\alpha) \; F(\omega;\alpha_1,\ldots,\alpha_N), \tag{2.1}$$

where $d^N\alpha$ denotes $d\alpha_1 \ldots, d\alpha_N$, and $P_N(\alpha)$ denotes $P(\alpha_1, \ldots, \alpha_N)$. If we use the following notation for the variance of a complex variable of a complex variable z,

$$\sigma^2(z) = \sigma^2(z_r) + \sigma^2(z_i), \tag{2.2}$$

where z_r and z_i are the real and imaginary parts of z, then noting that $\tilde{f}(-\omega)$ is the complex conjugate of $\tilde{f}(\omega)$, we see that

$$\sigma^2\left(\tilde{f}(\omega)\right) = \langle \tilde{f}(\omega)\, \tilde{f}(-\omega) \rangle - \langle \tilde{f}(\omega) \rangle \langle \tilde{f}(-\omega) \rangle \; . \tag{2.3}$$

The first term on the right-hand side may be evaluated from

$$\langle w(\omega) \rangle \equiv \langle \tilde{f}(\omega)\, \tilde{f}(-\omega) \rangle = \int d^N\alpha \; P_N(\alpha) \tilde{F}(\omega;\alpha_1,\ldots,\alpha_N)\, \tilde{F}(-\omega;\alpha_1,\ldots,\alpha_N) \tag{2.4}$$

We see that equation (2.4) also gives the expectation value of the "window spectrum" $w(\omega)$ which appears in equation (1.10) and is appropriate for the discussion of stationary random time series. The variance of this function is given by

$$\sigma^2(w) = \langle w(\omega)\, w(-\omega) \rangle - \langle w(\omega) \rangle \langle w(-\omega) \rangle \tag{2.5}$$

where

$$\langle w(\omega)\, w(-\omega)\rangle = \int d^N\alpha P_N(\alpha) \left\{\tilde{F}(\omega;\alpha_1,\ldots,\alpha_N\right\}^2 \left\{\tilde{F}(-\omega,\alpha_1,\ldots,\alpha_N)\right\}^2 . \quad (2.6)$$

As indicated in the introduction, we intend to consider the case that the observing time t_0 to t_N, of length T, is divided into N equal intervals bounded by times t_1, t_2, ... where

$$t_n = t_0 + n\tau \quad (2.7)$$

so that we may adopt the form

$$F(t;\alpha_1,\ldots,\alpha_N) = \sum_{n=1}^{N} \alpha_n \left\{h(t-t_{n-1}) - h(t-t_n)\right\} \quad (2.8)$$

where h(t) is the Heavyside function:

$$\left.\begin{aligned} h(t) &= 0 \quad \text{if } t<0, \\ &= 1 \quad \text{if } t>0. \end{aligned}\right\} \quad (2.9)$$

We also assume that the intervals are statistically independent, so that we may write

$$P_N(\alpha)d^N\alpha \equiv P(\alpha_1,\ldots,\alpha_N)d\alpha_1,\ldots,d\alpha_N = \left\{P_1(\alpha_1)\; d\alpha_1\right\}\cdots\left\{P_N(\alpha_N)d\alpha_N\right\} . \quad (2.10)$$

If we assume that, for each interval, there is a (uniform) probability p that the window is open and probability $1-p$ that it is closed, then

$$P(\alpha_n) = p\;\delta(\alpha_n - 1) + (1-p)\delta(\alpha_n) . \quad (2.11)$$

In evaluating $\langle\tilde{f}(\omega)\rangle$, given by (2.1), we will use

$$\langle\alpha_n\rangle = p . \quad (2.12)$$

In evaluating the quantity given by equation (2.4), we will need to evaluate $\langle\alpha_m\, \alpha_n\rangle$ which is clearly given by p^2 if $m \neq n$ but by p if $m = n$.

Hence

$$\langle \alpha_m \alpha_n \rangle = p^2 + (p - p^2)\,\delta_{mn}, \tag{2.13}$$

where δ_{mn} is the Kronecker function. In evaluating the quantity given by (2.6), we need to evaluate the expectation value of $\alpha_m \alpha_n \alpha_p \alpha_q$. By considering the various possibilities (m,n,p,q all different: two of them the same, etc.) we find that

$$\begin{aligned}\langle \alpha_m \alpha_n \alpha_p \alpha_q \rangle = p^4 &+ (p^3 - p^4)(\delta_{mn} + \delta_{mp} + \delta_{mq} + \delta_{np} + \delta_{nq} + \delta_{pq}) \\ &+ (p^2 - 3p^3 + 2p^4)(\delta_{npq} + \delta_{mpq} + \delta_{mnq} + \delta_{mnp}) \\ &+ (p - 4p^2 + 6p^3 - 3p^4)\,\delta_{mnpq},\end{aligned} \tag{2.14}$$

where $\delta_{mnp} = 1$ if $m = n = p$ otherwise 0, and δ_{mnpq} is defined similarly.

III. EVALUATION OF MODEL

For simple (non-random) time series, equation (1.4) gives the relationship between the Fourier transforms of the original and measured time series. In this context, the properties of the random window function f(t) may be characterized by $\langle \tilde{f}(\omega) \rangle$ and $\sigma^2(\tilde{f})$.

On substituting the form (2.8) into (1.5), we find that

$$F(\omega;\alpha_1, \ldots, \alpha_N) = \frac{1}{\pi\omega} \sum_{n=1}^{N} \alpha_n \sin\left(\frac{1}{2}\omega\tau\right) e^{i\omega\left|t_o + (n - \frac{1}{2})\tau\right|} . \quad (3.1)$$

On using (2.1) and (2.12), we obtain

$$< \tilde{f}(\omega) > = \frac{1}{2\pi} \; T \; p \; \text{sinc}(\frac{1}{2} \; \omega \; T) \; e^{\frac{1}{2} i\omega (t_o + t_N)} , \quad (3.2)$$

where $\text{sinc}\,\theta = \theta^{-1} \sin \theta$.

On using (2.4), we find that

$$<w(\omega)> = (\frac{1}{2\pi})^2 \; T^2 \left[p^2 \text{sinc}^2 (\frac{1}{2}\omega T) + N^{-1} p(1-p) \text{sinc}^2 (\frac{1}{2}\omega\tau)\right] . \quad (3.3)$$

Hence, using (2.3), we obtain

$$\sigma^2(\tilde{f}) = \left(\frac{1}{2\pi}\right)^2 T^2 N^{-1} \; r \quad - p) \; \text{sinc}^2\left(\frac{1}{2} \; \omega\tau\right). \quad (3.4)$$

For evaluating the effects of "breaking" of the window function, it is convenient to normalize the standard deviation with respect to the maximum value of $\tilde{f}(\omega)$, which is the value at $\omega = 0$. Accordingly, we introduce the definition

$$\Sigma_1(\omega) = \frac{\sigma(\tilde{f}(\omega))}{<\tilde{f}(0)>} . \quad (3.5)$$

For the case under consideration, this has the form

$$\Sigma_1(\omega) = N^{-1/2} \, p^{-1/2} \, (1-p)^{1/2} \; \left|\text{sinc} \; \frac{1}{2} \; \omega\tau\right| . \quad (3.6)$$

For discussion of the properties of randomly broken windows in the study of stationary random time series, it is necessary to evaluate the mean value and standard deviation of $w(\omega)$. The former is given by equation (3.3). The first term inside the brackets has the same form as arises in the non-random case ($p = 1$). The second term represents a change in the mean spectrum, so it is convenient to introduce the symbol Δ_2 for the ratio of the additional term to the maximum value of the principal term:

$$\Delta_2 = N^{-1} p^{-1} (1-p) \operatorname{sinc}^2 (\tfrac{1}{2}\omega T) \quad . \tag{3.7}$$

On writing equation (2.6) in the simpler form

$$\langle w(\omega) w(-\omega) \rangle = \langle \tilde{F}(\omega;\alpha) \tilde{F}(\omega;\alpha) \tilde{F}(-\omega;\alpha) \tilde{F}(-\omega;\alpha) \rangle \tag{3.8}$$

and using equations (1.3) and (2.8), we see that

$$\langle w(\omega) w(-\omega) \rangle = \frac{1}{(2\pi\omega)^4} \sum_{mnpq} \langle \alpha_m \alpha_n \alpha_p \alpha_q \rangle \left(e^{i\omega t_m} - e^{i\omega t_{m-1}} \right) \left(e^{i\omega t_n} - e^{i\omega t_{n-1}} \right)$$
$$\times \left(e^{-i\omega t_p} - e^{-i\omega t_{p-1}} \right) \left(e^{-i\omega t_q} - e^{-i\omega t_{q-1}} \right) \quad . \tag{3.9}$$

On using equation (2.14), we see that this may be expressed in the form

$$\langle w(\omega) w(-\omega) \rangle = \frac{1}{(2\pi\omega)^4} \Big\{ p^4 E_1 + (p^3 - p^4)(E_2 + E_2^* + 4E_3)$$
$$+ 2(p^2 - 3p^3 + 2p^4)(E_4 + E_4^*) \tag{3.10}$$
$$+ (p - 4p^2 + 6p^3 - 3p^4) \, E_5 \Big\}$$

where

$$
\left.
\begin{aligned}
E_1 &= \left|\sum_m \left(e^{i\omega t_m} - e^{i\omega t_{m-1}}\right)\right|^4 , \\
E_2 &= \left\{\sum_m \left(e^{i\omega t_m} - e^{i\omega t_{m-1}}\right)^2\right\}\left\{\sum_n \left(e^{-i\omega t_n} - e^{-i\omega t_{n-1}}\right)\right\}^2 , \\
E_3 &= \left\{\sum_m \left(e^{i\omega t_m} - e^{i\omega t_{m-1}}\right)\left(e^{-i\omega t_m} - e^{-i\omega t_{m-1}}\right)\right\}\left|\sum_n \left(e^{i\omega t_n} - e^{i\omega t_{n-1}}\right)\right|^2 , \\
E_4 &= \left\{\sum_m \left(e^{i\omega t_m} - e^{i\omega t_{m-1}}\right)\right\}\left\{\sum_n \left(e^{i\omega t_n} - e^{i\omega t_{n-1}}\right)\left(e^{-i\omega t_n} - e^{-i\omega t_{n-1}}\right)^2\right\} , \\
E_5 &= \sum_m \left(e^{i\omega t_m} - e^{i\omega t_{m-1}}\right)^2 \left(e^{-i\omega t_m} - e^{-i\omega t_{m-1}}\right)^2 .
\end{aligned}
\right\} \quad (3.11)
$$

On evaluating these sums, we find that

$$
\left.
\begin{aligned}
E_1 &= 16 \sin^4\left(\frac{1}{2}\omega T\right), \\
E_2 &= 16 \sin^2\left(\frac{1}{2}\omega T\right)\sin^2\left(\frac{1}{2}\omega\tau\right)\frac{\sin\omega T}{\sin\omega\tau} , \\
E_3 &= 16\, N \sin^2\left(\frac{1}{2}\omega T\right)\sin^2\left(\frac{1}{2}\omega\tau\right) , \\
E_4 &= 16 \sin^2\left(\frac{1}{2}\omega T\right)\sin^2\left(\frac{1}{2}\omega\tau\right) \\
E_5 &= 16\, N \sin^4\left(\frac{1}{2}\omega\tau\right) .
\end{aligned}
\right\} \quad (3.12)
$$

Hence equation (3.10) is found to be expressible as

$$\langle w(\omega)\, w(-\omega)\rangle = \left(\frac{T}{2\pi}\right)^4 \Big\{ p^4 \operatorname{sinc}^4\left(\frac{1}{2}\omega T\right)$$

$$+ 2\left(p^3 - p^4\right) N^{-2} \operatorname{sinc}^2\left(\frac{1}{2}\omega T\right) \operatorname{sinc}^2\left(\frac{1}{2}\omega \tau\right) \frac{\operatorname{sinc}\omega T}{\operatorname{sinc}\omega\tau}$$

$$+ 4\left(p^3 - p^4\right) N^{-1} \operatorname{sinc}^2\left(\frac{1}{2}\omega T\right) \operatorname{sinc}^2\left(\frac{1}{2}\omega \tau\right)$$

$$+ 4\left(p^2 - 3p^3 + 2p^4\right) N^{-2} \operatorname{sinc}^2\left(\frac{1}{2}\omega T\right) \operatorname{sinc}^2\left(\frac{1}{2}\omega \tau\right)$$

$$+ \left(p - 4p^2 + 6p^3 - 3p^4\right) N^{-3} \operatorname{sinc}^4\left(\frac{1}{2}\omega \tau\right) \Big\} . \tag{3.13}$$

On using equations (2.3) and the definition

$$\Sigma_2(\omega) = \frac{\sigma\left(w(\omega)\right)}{\left(\frac{1}{2\pi}\, Tp\right)^2} , \tag{3.14}$$

we find that

$$\Sigma_2(\omega) = \Big\{ 2\, N^{-1} p^{-1} (1-p) \operatorname{sinc}^2\left(\frac{1}{2}\omega T\right) \operatorname{sinc}^2\left(\frac{1}{2}\omega \tau\right)$$

$$+ 2N^{-2} p^{-1} (1-p) \operatorname{sinc}^2\left(\frac{1}{2}\omega T\right) \operatorname{sinc}^2\left(\frac{1}{2}\omega \tau\right) \frac{\sin\omega T}{\sin\omega\tau}$$

$$+ N^{-2} p^{-2} (3 - 10p + 7p^2) \operatorname{sinc}^2\left(\frac{1}{2}\omega T\right) \operatorname{sinc}^2\left(\frac{1}{2}\omega \tau\right)$$

$$+ N^{-3} p^{-3} (1 - 4p + 6p^2 - 3p^3) \operatorname{sinc}^4\left(\frac{1}{2}\omega \tau\right) \Big\}^{1/2} . \tag{3.15}$$

IV. DISCUSSION

We see from the preceding sectior that the effect of a random "breaking" of the window function is to produce an aliasing of any signal. This effect is described by the function Σ_1 or by Δ_2 and Σ_2 as given by equations (3.6), (3.7) and (3.15).

For the "linear" problem, as described by equation (1.4) the mean Fourier transform of the window function, as given by equation (3.2), has the same form as it does in the non-random case, although it is reduced by a factor p. The standard deviation is characterized by Σ_1, defined by equation (3.5) and given by equation (3.6).

It is convenient to introduce the notation

$$\omega_T = 2T^{-1}, \quad \omega_\tau = 2\tau^{-1}. \tag{4.1}$$

On noting that sinc $\theta \lesssim 1$ for $\theta \lesssim 1$, and that $|\mathrm{sinc}\ \theta| \lesssim \theta^{-1}$ for $\theta \gtrsim 1$, we see that $\Sigma_1 \lesssim S_1$ where

$$\left.\begin{aligned} S_1(\omega) &= N^{-1/2}p^{-1/2}(1-p)^{1/2}, \quad \omega \lesssim \omega_\tau \\ S_1(\omega) &= N^{-1/2}p^{-1/2}(1-p)^{1/2}(\omega_\tau/\omega), \quad \omega \gtrsim \omega_\tau. \end{aligned}\right\} \tag{4.2}$$

Hence the aliasing is most severe within the range of a few times ω_τ of the strongest signal.

We may infer from the above restriction the minimum number N_1 of intervals necessary to ensure that Σ_1 is below an assigned level ε for a given value of p.

We see from (4.2) that we require $N \geqslant N_1$ where

$$N_1 = (1-p)p^{-1}\varepsilon^{-2}. \tag{4.3}$$

If, for instance, $p = 0.8$ and we require that $\Sigma_1 \leq 0.05$, N must be at least 100.

For the "quadratic" problem in which we are determining the spectrum of a stationary random time series, the aliasing is described by the functions $\Delta_2(\omega)$ and $\Sigma_2(\omega)$ given by equations (3.7) and (3.15).

We see from equation (3.3) that the second term in brackets is smaller than the envelope of the first term, and so may be neglected, for $\omega < \omega_\Delta$, where

$$\omega_\Delta = N^{1/2}(1-p)^{1/2}p^{-1/2}\omega_T \tag{4.4}$$

For $\omega > \omega_\Delta$, the second term produces a "tail" to the principal contribution to $\langle w(\omega) \rangle$. By an argument similar to that leading to equation (4.2), we find that $\Delta_2 \lesssim D_2$, where

$$\left.\begin{aligned} D_2(\omega) &= N^{-1}p^{-1}(1-p) \;,\quad \omega < \omega_\tau, \\ D_2(\omega) &= N^{-1}p^{-1}(1-p)(\omega_\tau/\omega)^2 \;,\quad \omega > \omega_\tau. \end{aligned}\right\} \tag{4.5}$$

Now consider the four terms in the second bracket in equation (3.15). It is clear that the third term may be ignored by comparison with the first since it has a similar dependence on ω but includes an extra factor N^{-1}. The second term may also be ignored by comparison with the first: the extra factor N^{-1} $(\sin \omega T)/(\sin \omega\tau)$ has a maximum value of unity, and an RMS value of order of $N^{-1/2}$.

In comparing the fourth term by comparison to the first, we see that the ratio is given by

$$R = Q(p)\,\frac{1}{N^2}\,\frac{\operatorname{sinc}^2\left(\frac{1}{2}\omega T\right)}{\operatorname{sinc}^2\left(\frac{1}{2}\omega\tau\right)} \tag{4.6}$$

where

$$Q(p) = \frac{1 - 3p + 3p^2}{2p^2} \quad . \tag{4.7}$$

It is easily verified that $Q(p) \leqslant 0.5$ in the range $0.5 \leqslant p \leqslant 1$. Hence R has a maximum value of order 0.5 at values of ω for which $\omega\tau = 2n\pi$. We find that, when R is averaged over frequency, it varies with N as N^{-1}. Hence we may, to sufficient approximation ignore the fourth term and so replace (3.15) by

$$\Sigma_2(\omega) \approx 2^{1/2} N^{-1/2} p^{-1/2} (1-p)^{1/2} \left|\operatorname{sinc}\left(\tfrac{1}{2}\omega T\right)\right| \left|\operatorname{sinc}\left(\tfrac{1}{2}\omega\tau\right)\right| . \tag{4.8}$$

We find that $\Sigma_2 \lesssim S_2$, where

$$\left.\begin{aligned} S_2(\omega) &= 2^{1/2} N^{-1/2} p^{-1/2} (1-p)^{1/2}, && \omega \leqslant \omega_T , \\ S_2(\omega) &= 2^{1/2} N^{-1/2} p^{-1/2} (1-p)^{1/2} \frac{\omega_T}{\omega} , && \omega_T \leqslant \omega \leqslant \omega_\tau , \\ S_2(\omega) &= 2^{1/2} N^{-3/2} p^{-1/2} (1-p)^{1/2} \left(\frac{\omega_\tau}{\omega}\right)^2, && \omega \geqslant \omega_\tau . \end{aligned}\right\} \tag{4.9}$$

We see that, for the same values of N and p, the maximum value of Σ_2 is $2^{1/2}$ times larger than the maximum value of Σ_1. Hence the minimum number N_2 of intervals necessary to ensure that Σ_2 is below an assigned level, for various values of p, is twice the corresponding value of N_1, given by equation (4.3).

However, the quadratic case is more complicated than the linear case in that $\Sigma_2(\omega)$ is more complicated than $\Sigma_1(\omega)$, and $\Delta_2(\omega)$ is nonzero (whereas $\Delta_1(\omega)$ is zero and has been neglected).

On noting that the dominant term of (3.3) (that which survives in the nonrandom case that $p = 1$) varies as $(\omega_T/\omega)^2$ for $\omega > \omega_T$, we see that $\Sigma_2(\omega)$ is less than the tail of the dominant term, and so may be neglected, for $\omega < \omega_\Sigma$, where

$$\omega_\Sigma = 2^{-1/2} N^{1/2} p^{1/2} (1-p)^{-1/2} \omega_T \; . \qquad (4.10)$$

On the other hand, we find that $S_2(\omega)$ is less than $D_2(\omega)$ for $\omega > \omega_c$, where

$$\omega_c = 2^{1/2} N^{1/2} p^{1/2} (1-p)^{-1/2} \omega_T \equiv 2\omega_\Sigma \; . \qquad (4.11)$$

Hence we may, to adequate approximation, ignore $\Sigma_2(\omega)$ in assessing the aliasing which occurs in the quadratic case.

We see from (4.5) that the minimum value of N necessary to ensure that $\Delta_2(\omega)$ is less than some maximum value ε is given by $N > N_2$, where

$$N_2 = p^{-1}(1-p)\varepsilon^{-1} \; . \qquad (4.12)$$

We see from (4.3) that N_2 is smaller than N_1 by the factor ε. Hence aliasing is likely to be less serious in the quadratic case than it is in the linear case.

In order to assess the implications of the present model concerning ground-based observations of solar oscillations, one will need to have detailed estimates of the expected spectrum (in particular, the spacing and relative power of nearby lines) and the expected cloud cover at three or four observatories positioned round the world. It is also desirable that the present model should be extended by considering separate values of p for each of the observatories, and possibly by taking into account the correlation between cloud cover on consecutive days.

Nevertheless, we can illustrate the results of this model by considering a hypothetical situation. Suppose that four observatories are located around the world in such a way as to give continuous coverage (in the absence of cloud cover), and that these observatories are operated for 25 days. Then $N = 100$. Suppose that, for any observatory on any day, there is 20% probability of cloud cover so that $p = 0.80$. We find from (4.1) that (using $\nu = \omega/2\pi$), $\nu_\tau = 15\mu$ Hz. For frequencies less than this value, (4.5) shows that the aliasing amounts to 0.25% or less. For frequencies above 15μ Hz, the aliasing drops off rapidly.

Although it will be necessary to make more detailed and specific calculations to draw definite conclusions, it appears from the above simple example that it may be possible to carry out high-quality studies of solar oscillations from a chain of ground-based observatories.

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REFERENCES

Bendat, J.S., and Piersol, A.G. 1966 (New York: Wiley), p. 200.

Deeming, T.J. 1975, Ap. Sp. Sci., 36, 137.

Deubner, F.L. 1975, Astr. Ap., 44, 371.

Papoulis, A. 1965, Probability, Random Variables and Stochastic Processes (New York: McGraw-Hill), p. 266.

Rhodes, E.J., Ulrich, R.K., and Simon, G.W. 1977, Ap. J., 218, 901.

Ulrich, R.K., and Rhodes, E.J. 1977, Ap. J., 218, 521.

Ulrich, R.K., Rhodes, E.J., and Deubner, F.L. 1979, Ap. J., 227, 638.