STUDY OF CERTAIN LAUNCHING TECHNIQUES

USING LONG ORBITING TETHERS

Grant NAG-8008

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Final Technical Report

For the period 1 July 1980 through 28 Feb. 1981

Principal Investigator

Dr. Giuseppe Colombo

March 1981

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Astrophysical Observatory
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The Smithsonian Astrophysical Observatory
and the Harvard College Observatory
are members of the
Center for Astrophysics

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Acknowledgement

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1. Introduction

This report presents a study of the basic equations governing orbital transfers using long orbiting tethers. A very simple approximation to the general transfer equation is derived for the case of short tethers and low eccentricity orbits. Numerical examples are calculated for the case of injection into a circular orbit from a platform in eccentric orbit, and injection into eccentric orbit from a platform in circular orbit. For the case of long tethers, a method is derived for reducing tether mass and increasing payload mass by tapering the tether to maintain constant stress per unit area of tether cross section. Formulas are presented for calculating the equilibrium orbital parameters taking into account the mass of the platform, tether, and payload.

2. Orbital Center of a Tethered System

In order for a tethered system to be in equilibrium in a circular orbit, the sum of the centripital and gravitational forces on the system must be zero. If the system is gravity gradient stabilized in a vertical configuration so that all parts of the system move with the same angular velocity, then the condition of equilibrium is that

$$\int_{r_1}^{r_2} \frac{GMdm}{r^2} = \int_{r_1}^{r_2} dm\ r\ \omega^2 \tag{1}$$

Equation (1) can be solved for the angular velocity which is

$$\omega = \sqrt{\int_{r_1}^{r_2} \frac{GMdm}{r^2} \int_{r_1}^{r_2} dm\ r} \tag{2}$$

A particle at a position r_c defined by

$$\frac{GM}{r_c^{\,2}} = r_c\omega^2$$

experiences no net acceleration. Solving for r_c we have

$$r_c = \left(\frac{GM}{\omega^2}\right)^{1/3} \qquad (3)$$

This value of r_c will be considered to be the orbital center of the system. The motion of this part of the system is the same as that of a free particle at that radius. This definition gives a slightly different answer than computing the center of mass of the system. Equation (1) is valid in general. However it is more convenient to evaluate the contributions to the integrals of discrete masses using the expressions

$$\sum_i \frac{GMm_i}{r_i^{\,2}}$$

and

$$\sum_i m_i r_i \quad .$$

In an eccentric orbit, the position of the orbital center will vary somewhat, but this effect can probably be neglected.* It may also be possible to neglect the mass of the tether or payload in certain siutations. The usual methods of calculating the center of mass may also be adequate for short tethers.

The transfer equations given in the next section use the orbit of the center of the system as a reference in computing the post release orbit of the payload or base launching vehicle. The release of a payload can be considered as a double launch since both the payload and the launching vehicle go into different orbits after the release. If the tether has significant mass and remains attached to the launch vehicle, the transfer equations apply to the orbital center of the launch vehicle plus tether.

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*This approximation is valid only for small eccentricity.

3. General Transfer Equations

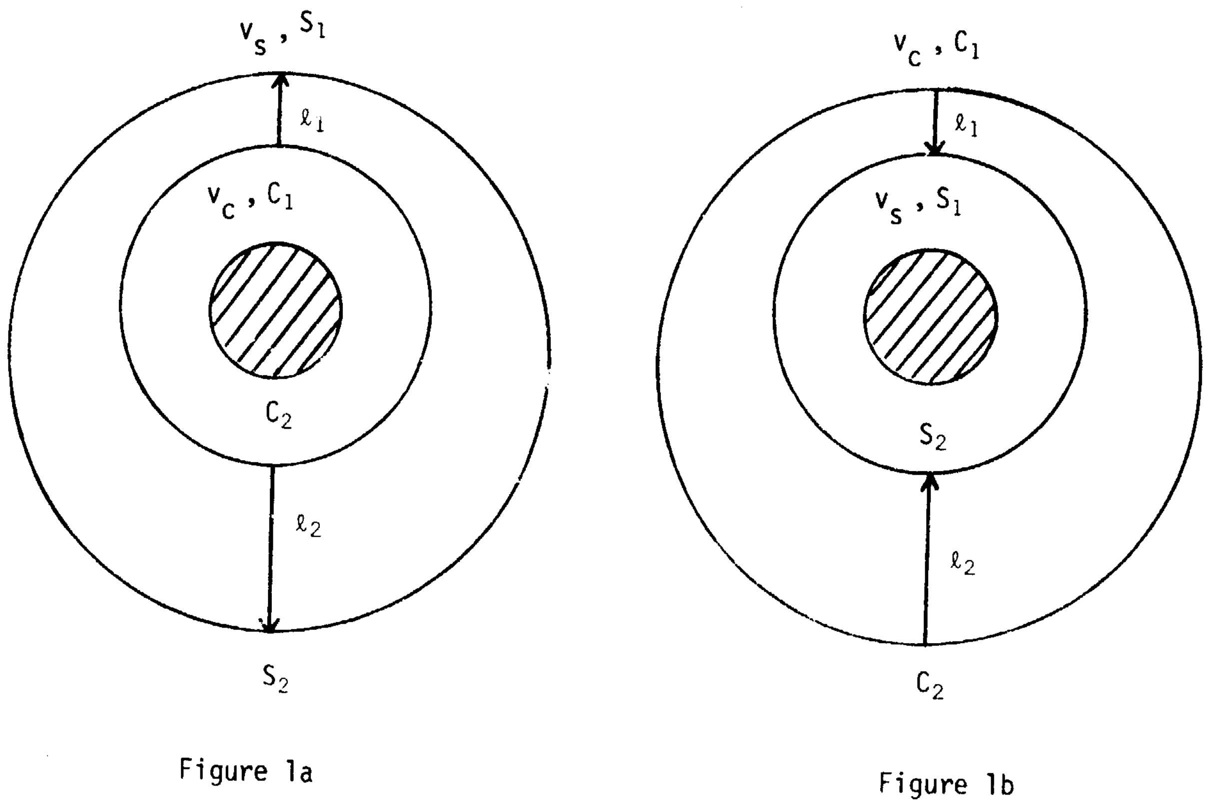

Figure 1. Launch of an end mass from a tether system. In part (a) the launch is to a higher orbit. In part (b) the launch is to a lower orbit.

In Figure 1, C_1 and C_2 are the extrema of the orbit of the orbital center of a tethered system. The derivations do not depend on whether C_1 is greater than, less than, or equal to C_2. Similarly S_1 and S_2 are the extrema of the orbit of the satellite at the end of the tether. The separation of the satellite from the center of mass at release is ℓ_1 which may be either positive or negative depending on whether the satellite being considered is at the upper or lower end of the tether. That is, S_1 may be greater than C_1 as shown in Figure 1a, or S_1 may be less than C_1 (Fig. 1b). The separation of the satellite from the orbit of the center of mass is ℓ_2 on the other side of the orbit. The sign of ℓ_2 must be the same as the sign of ℓ_1. We wish to compute S_2 as a function of C_1, C_2, and S_1.

The semi-major axis of the orbital center is

$$a_c = (C_1 + C_2) / 2$$

The semi-major axis of the orbit of the satellite after release is

$$a_s = (S_1 + S_2) / 2$$

The velocity of the center of mass at release is

$$v_c = \sqrt{GM}\sqrt{\frac{2}{C_1} - \frac{1}{a_c}}$$

The velocity of the satellite at release is

$$v_s = v_c \frac{S_1}{C_1} = \sqrt{GM}\ \frac{S_1}{C_1}\sqrt{\frac{2}{C_1} - \frac{1}{a_c}} \qquad (4)$$

Since v_s can also be written as

$$v_s = \sqrt{GM}\sqrt{\frac{2}{S_1} - \frac{1}{a_s}} \qquad (5)$$

we can equate the right sides of equations (4) and (5) to obtain the equation

$$\sqrt{GM}\sqrt{\frac{2}{S_1} - \frac{1}{a_s}} = \sqrt{GM}\ \frac{S_1}{C_1}\sqrt{\frac{2}{C_1} - \frac{1}{a_c}}$$

Solving for a_s we have

$$\frac{2}{S_1} - \frac{1}{a_s} = \frac{S_1{}^2}{C_1{}^2}\left(\frac{2}{C_1} - \frac{1}{a_c}\right)$$

or

$$a_s = \left(\frac{2}{S_1} - \frac{2S_1{}^2}{C_1{}^3} + \frac{S_1{}^2}{a_c C_1{}^3}\right)^{-1}$$

Substituting $a_s = (S_1 + S_2) / 2$ and solving for S_2, we have

$$S_2 = -S_1 + 2\left(\frac{2}{S_1} - \frac{2S_1^2}{C_1^3} + \frac{S_1^2}{a_c C_1^2}\right)^{-1} \tag{6}$$

If ℓ_1 is small compared to the radius of the orbit, it is useful to find the rate of change of S_2 with respect to S_1. This will give us the ratio of ℓ_2 to ℓ_1 for small values of ℓ_1. Differentiating equation (6), we have

$$\frac{dS_2}{dS_1} = -1 - 2\left(\frac{2}{S_1} - \frac{2S_1^2}{C_1^3} + \frac{S_1^2}{a_c C_1^2}\right)^{-2}\left(-\frac{2}{S_1^2} - \frac{4S_1}{C_1^3} + \frac{2S_1}{a_c C_1^2}\right)$$

Evaluating the derivative at $S_1 = C_1$, we have

$$\frac{dS_2}{dS_1} = -1 - 2\left(\frac{2}{C_1} - \frac{2}{C_1} + \frac{1}{a_c}\right)^{-2}\left(-\frac{2}{C_1^2} - \frac{4}{C_1^2} + \frac{2}{a_c C_1}\right)$$

$$\frac{dS_2}{dS_1} = -1 - 2\left(\frac{1}{a_c}\right)^{-2}\left(-\frac{6}{C_1^2} + \frac{2}{a_c C_1}\right) \tag{7}$$

If the orbital eccentricity is low so that $C_1 \approx a_c$, equation (7) becomes

$$\frac{dS_2}{dS_1} \approx -1 - 2\, a_c^2 \left(-\frac{4}{a_c^2}\right)$$

$$\frac{dS_2}{dS_1} \approx -1 + 8 = 7 \tag{8}$$

Therefore for small ℓ_1 and small eccentricity we have

$$\ell_2 \approx 7\ell_1 \tag{9}$$

If the orbit of the center of mass is circular so that $C_1 = a_c$, equation (6) for S_2 becomes

$$S = -S_1 + 2\left(\frac{2}{S_1} - \frac{2S_1^2}{a_c^3} + \frac{S_1^2}{a_c^3}\right)^{-1}$$

$$= -S_1 + 2\left(\frac{2}{S_1} - \frac{S_1^2}{a_c^3}\right)^{-1}$$

$$= -S_1 + 2\left(\frac{2a_c^3 - S_1^3}{S_1 a_c^3}\right)^{-1}$$

$$S_2 = -S_1 + \frac{2a_c^3 S_1}{2a_c^3 - S_1^3} = \frac{S_1^4}{2a_c^3 - S_1^3} \qquad (10)$$

The apogee S_2 goes to infinity if $2a_c^3 = S_1^3$ or $P_2/a_c = 2^{1/3} = 1.2599$. If a_c is that of a 200 km orbit, ℓ_1 must be 1710 km to achieve escape velocity.

If the orbit of the payload is circular we can set $S_1 = S_2 \equiv S$ in equation (6) for S_2 to obtain

$$S = -S + 2\left(\frac{2}{S} - \frac{2S^2}{C_1^3} + \frac{S^2}{a_c C_1^2}\right)^{-1}$$

$$\frac{1}{2S} = \frac{1}{2}\left(\frac{2}{S} - \frac{2S^2}{C_1^3} + \frac{S^2}{a_c C_1^2}\right)$$

$$-\frac{1}{S} = S^2\left(\frac{1}{a_c C_1^2} - \frac{2}{C_1^3}\right) = \frac{S^2}{C_1^2}\left(\frac{1}{a_c} - \frac{2}{C_1}\right)$$

$$\frac{C_1^2}{S^3} = \frac{C_1 - 2a_c}{a_c C_1}$$

Substituting $a_c = (C_1 + C_2)/2$ and solving for C_2 we have

$$\frac{C_1^3}{S^3} = \frac{C_1 - (C_1 + C_2)}{(C_1 + C_2)/2} = \frac{2C_2}{C_1 + C_2}$$

$$C_1^4 + C_2C_1^3 = 2C_2S^3$$

$$C_1^4 = C_2(2S^3 - C_1^3)$$

$$C_2 = C_1^4/(2S^3 - C_1^3) \qquad (11)$$

Using equation (11), we can calculate the required prigee height to achieve an orbit of radius S using a tether of length ℓ_1 from the payload to the orbital center. The distance from the launching vehicle to the orbital center will depend on the relative masses of the payload, launch vehicle and tether.

4. Injection Into Circular Orbit

Figure 2 shows a plot calculated using equation (11) for injecting a payload into a circular orbit from the Shuttle in an eccentric orbit. Figure 2 plots ℓ_1 against the perigee height of the orbital center of the system before release. Equation (9), which is $\ell_2 \approx 7\ell_1$ is reasonably well satisfied in the plot. For the last point at $\ell_1 = 44$ km, the value of ℓ_2 is 500 - 200 = 300 km. This is within about 2-1/2% of the value given by equation (9), namely, $\ell_2 = 7\ell_1 = 7 \times 44 = 308$ km.

5. Injection Into Eccentric Orbit

Figure 3 shows a plot calculated using equation (10) for injecting a payload into eccentric orbit from the Shuttle in a circular orbit. The distance ℓ_1 of the payload from the orbital center at 200 km is plotted against the apogee height of the payload after release. The approximate formula $\ell_2 \approx 7\ell_1$ is satisfied at the beginning of the plot. The curve becomes quite non-linear toward the end of the plot which is terminated at approximately synchronous altitude. The payload could be circularized at synchronous height using an apogee kick motor. The mass of the tether system necessary to reach synchronous altitude is substantial, and the payload small. Tapered tethers must be used for the reasons described in the next two sections. With a small payload the Shuttle orbit would not be greatly altered by the release.

6. Constant Diameter Tethers

In previous sections it has been noted that tether lengths of 1250 and 1700 km would be required to launch payloads to synchronous altitude or escape velocity from Shuttle altitude. In this section, the tether stress per unit area of cross section is computed as a function of tether length with no payload

1.0	0.4930043593E+03
2.0	0.4660174273E+03
3.0	0.4790391895E+03
4.0	0.4720696311E+03
5.0	0.4651087377E+03
6.0	0.4581564945E+03
7.0	0.4512128871E+03
8.0	0.4442779010E+03
9.0	0.4373515218E+03
10.0	0.4304337349E+03
11.0	0.4235245259E+03
12.0	0.4166238806E+03
13.0	0.4097317845E+03
14.0	0.4028482234E+03
15.0	0.3959731830E+03
16.0	0.3891066490E+03
17.0	0.3822486072E+03
18.0	0.3753990435E+03
19.0	0.3685579436E+03
20.0	0.3617252935E+03
21.0	0.3549010791E+03
22.0	0.3480852863E+03
23.0	0.3412779011E+03
24.0	0.3344789095E+03
25.0	0.3276882976E+03
26.0	0.3209060513E+03
27.0	0.3141321568E+03
28.0	0.3073666003E+03
29.0	0.3006093679E+03
30.0	0.2938604457E+03
31.0	0.2871198200E+03
32.0	0.2803874771E+03
33.0	0.2736634032E+03
34.0	0.2669475846E+03
35.0	0.2602400077E+03
36.0	0.2535406589E+03
37.0	0.2468495245E+03
38.0	0.2401665910E+03
39.0	0.2334918448E+03
40.0	0.2268252724E+03
41.0	0.2201668603E+03
42.0	0.2135165952E+03
43.0	0.2068744635E+03
44.0	0.2002404519E+03

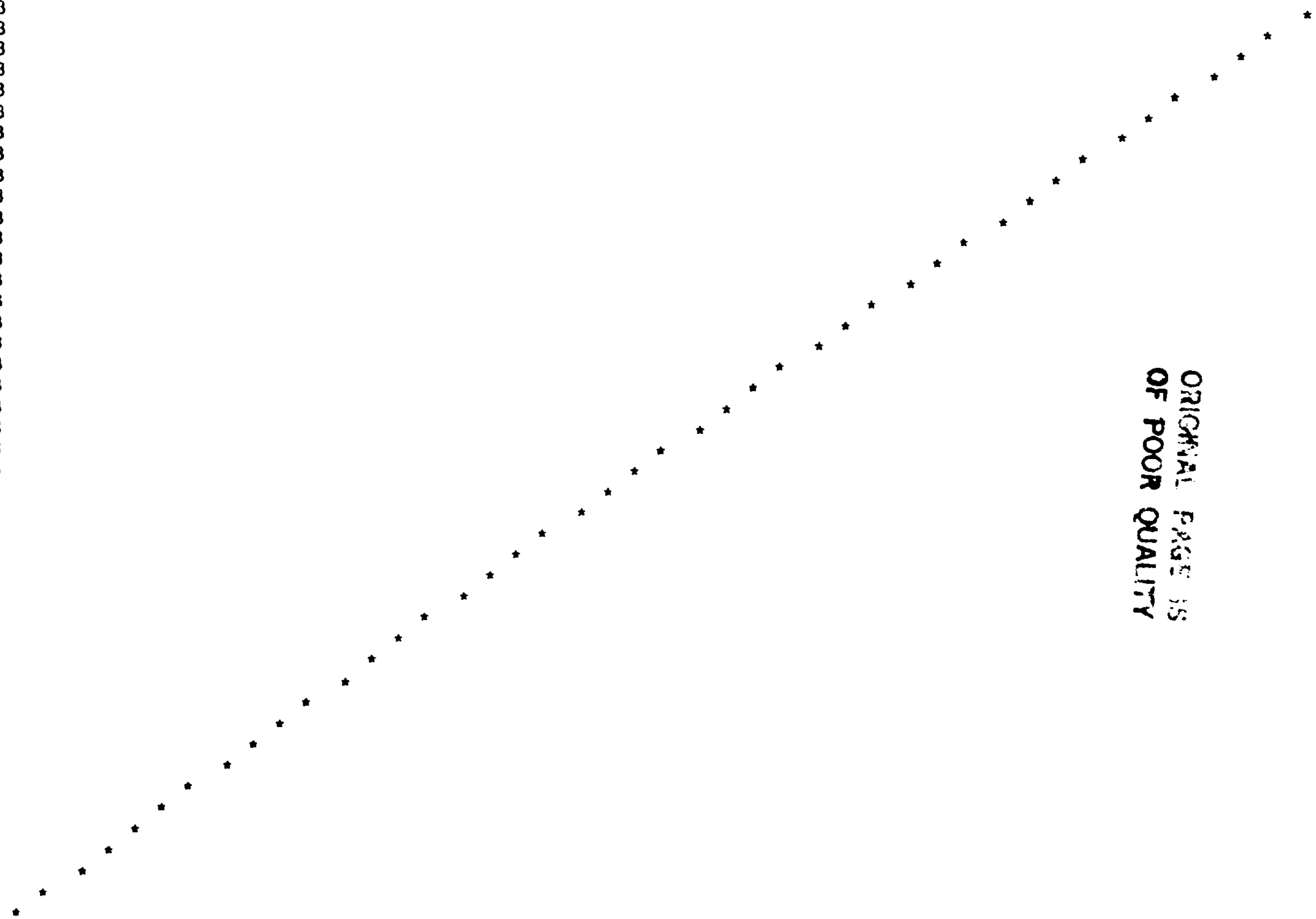

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Figure 2. Plot of perigee height of the center of mass in km vs. ℓ_1 in km for inserting a payload into a circular orbit of 500 km altitude. The first column lists ℓ_1 and the second column is the periger height of the center of mass of the system before launch.

10.0	[illegible]4587623E+03
20.0	0.[illegible]4596311E+03
30.0	[illegible]41782655E+03
40.0	0.4874727318E+03
50.0	0.5617466148E+03
60.0	0.6370184710E+03
70.0	0.7133059732E+03
80.0	0.7906281238E+03
90.0	0.8690041688E+03
100.0	0.9484538322E+03
110.0	0.1028997331E+04
120.0	0.1110655390E+04
130.0	0.1193449260E+04
140.0	0.1277400732E+04
150.0	0.1362532154E+04
160.0	0.1448866455E+04
170.0	0.1536427154E+04
180.0	0.1625238188E+04
190.0	0.1715324930E+04
200.0	0.1806712205E+04
210.0	0.1899426320E+04
220.0	0.1993494080E+04
230.0	0.2088943015E+04
240.0	0.2185801405E+04
250.0	0.2284098304E+04
260.0	0.2383863567E+04
270.0	0.2485127879E+04
280.0	0.2587922781E+04
290.0	0.2692280705E+04
300.0	0.2798235002E+04
310.0	0.2905819975E+04
320.0	0.3015070915E+04
330.0	0.3126024137E+04
340.0	0.3238717012E+04
350.0	0.3353188015E+04
360.0	0.3469476757E+04
370.0	0.3587624034E+04
380.0	0.37076[illegible]868E+04
390.0	0.3829663555E+04
400.0	0.3953643710E+04
410.0	0.4079658325E+04
420.0	0.4207754814E+04
430.0	0.4337982077E+04
440.0	0.4470390551E+04
450.0	0.4605032275E+04
460.0	0.4741960953E+04
470.0	0.4881232021E+04
480.0	0.5022902721E+04
490.0	0.5167032167E+04
500.0	0.5313681431E+04
510.0	0.5462913619E+04
520.0	0.5614791959E+04
530.0	0.5769389890E+04
540.0	0.5926771157E+04
550.0	0.6087009911E+04
560.0	0.6250180810E+04
570.0	0.6416361135E+04
580.0	0.6585630902E+04
590.0	0.6758072983E+04
600.0	0.6933773239E+04
610.0	0.7112820653E+04
620.0	0.7295307473E+04
630.0	0.7481329366E+04
640.0	0.7670985573E+04
650.0	0.7864379083E+04
660.0	0.8061616808E+04
670.0	0.8262809771E+04
680.0	0.8468073310E+04
690.0	0.8677527286E+04
700.0	0.8891296306E+04
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720.0	0.9332303089E+04
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740.0	0.9792194873E+04
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870.0	0.1333150[illegible]191E+05
880.0	0.13655[illegible]E+05
890.0	0.1398296881E+05
900.0	0.1431923422E+05
910.0	[illegible]14664[illegible]E+05
920.0	[illegible]959[illegible]E+05
930.0	[illegible]
940.0	[illegible]
950.0	[illegible]
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970.0	0.1693410066E+05
980.0	0.173495402[illegible]E+05
990.0	[illegible]
1000.0	0.1821626231E+05
1010.0	0.1866856960E+05
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1030.0	0.1961386440E+05
1040.0	0.2010806738E+05
1050.0	0.2061751178E+05
1060.0	0.2114290149E+05
1070.0	0.2168490494E+05
1080.0	0.2224455700E+05
1090.0	0.2282246451E+05
1100.0	0.2341960975E+05
1110.0	0.2403695510E+05
1120.0	0.2467552845E+05
1130.0	0.2533642823E+05
1140.0	0.2602083025E+05
1150.0	0.2672999425E+05
1160.0	0.2746527167E+05
1170.0	[illegible]2822811407E+05
1180.0	[illegible]2902[illegible]8261E+05
1190.0	0.2988285[illegible]E+05
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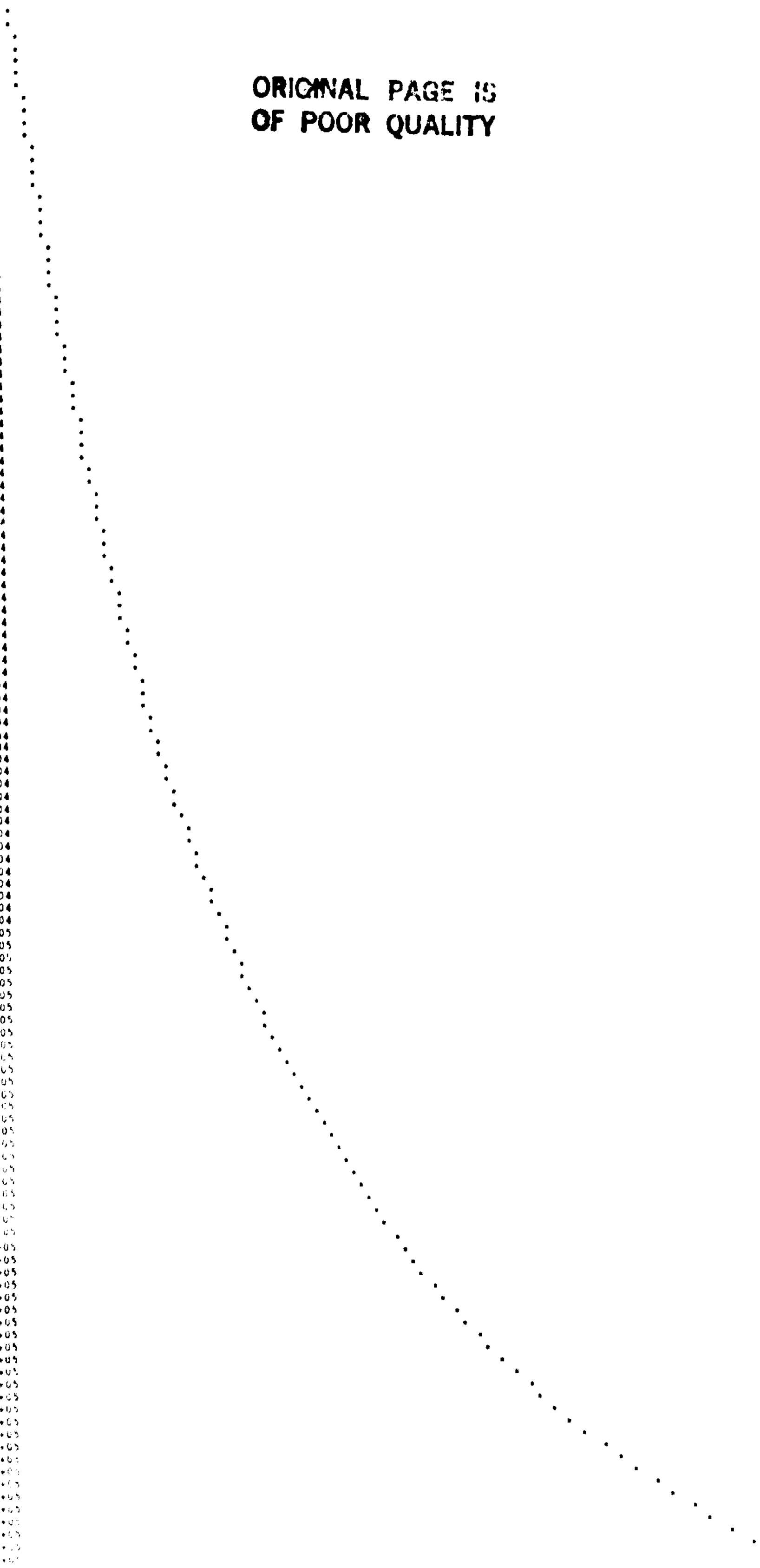

Figure 3. Plot of the apogee height vs. ℓ_1 in km for launching a payload into an eccentric orbit with the center of mass of the system at 200 km before launch.

at the end.

Figure 4. A tether of constant cross section attached to a launching platform. The orbital center of the system is at a distance r_1 from the center of the earth.

In Figure 4, a tether of constant cross section and density ρ is connected to a platform at distance r_0 from the center of the earth. The orbital center of the system is at r_1. We wish to calculate the stress in the tether necessary to support the weight of the tether itself. The force on each element of the tether is the difference between the centripital force and the gravitation force. The total force F at the point r_1 is given by

$$F = \int_{r_1}^{r_2} (dm\ r\ \omega^2 - GM\ dm/r^2) \tag{12}$$

Substituting $dm = \rho\ A\ dr$ where A is the cross sectional area, and setting $F = C\ A$ where C is the stress per unit area in the tether we have

$$C\ A = \rho\ A \int_{r_1}^{r_2} (r\omega^2 - GM/r^2)dr$$

or

$$C = \rho \int_{r_1}^{r_2} (r\omega^2 - GM/r^2)dr$$

Performing the integration gives

$$C = \rho \left[\frac{1}{2}\ \omega^2\ (r_2{}^2 - r_1{}^2) + GM\ (\frac{1}{r_2} - \frac{1}{r_1}) \right] \tag{13}$$

0.0	0.0000000000E+00
20.0	0.1215899947E+08
40.0	0.4853810978E+08
60.0	0.1089918260E+09
80.0	0.1933763915E+09
100.0	0.3015497718E+09
120.0	0.4333716288E+09
140.0	0.5887032959E+09
160.0	0.7674077524E+09
180.0	0.9693496001E+09
200.0	0.1194395038E+10
220.0	0.1442411842E+10
240.0	0.1713269336E+10
260.0	0.2006836375E+10
280.0	0.2322991321E+10
300.0	0.2661602019E+10
320.0	0.3022545777E+10
340.0	0.3405699347E+10
360.0	0.3810940901E+10
380.0	0.4238150010E+10
400.0	0.4687207627E+10
420.0	0.5157996069E+10
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600.0	0.1035421017E+11
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1080.0	0.3218717113E+11
1100.0	0.3333535100E+11
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1360.0	0.4990232425E+11
1380.0	0.5130029531E+11
1400.0	0.5271556339E+11
1420.0	0.5414805968E+11
1440.0	0.5559771602E+11
1460.0	0.5706446496E+11
1480.0	0.5854823970E+11
1500.0	0.6004897411E+11
1520.0	0.6156660272E+11
1540.0	0.6310106069E+11
1560.0	0.6465228383E+11
1580.0	0.6622020857E+11
1600.0	0.6780477197E+11
1620.0	0.6940591170E+11
1640.0	0.7102356604E+11
1660.0	0.7265767387E+11
1680.0	0.7430817465E+11
1700.0	0.7597500844E+11

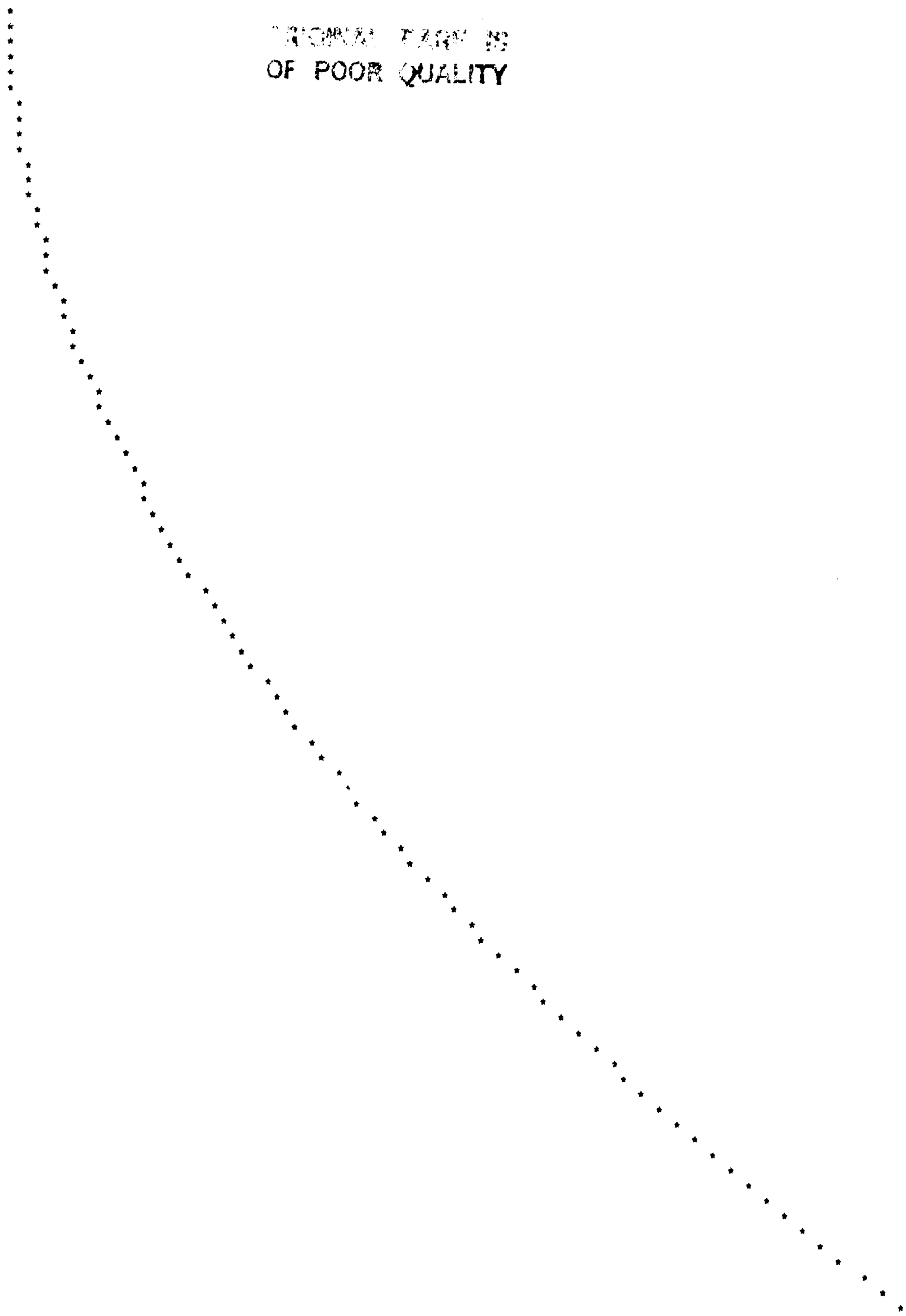

Figure 5. Stress per unit area vs. $(r_2 - r_1)$ for a constant diameter tether. The length in km is listed in the first column, and the stress per unit area in dynes/cm^2 at the point r_1 is listed in the second column.

Figure 5 shows a plot of C in dynes/cm^2 vs. (r_2-r_1) with $r_1 = 200$ km + R_e where R_e is the radius of the earth. The angular velocity ω is set equal to $\sqrt{GM/r_1^3}$ since r_1 is the orbital center of the system. To be in equilibrium the tether has to extend below r_1 and be connected to a heavy mass to counterbalance the force on the part of the tether above r_1. A value of 1.45 g/cc was assumed for ρ. The break strength of kevlar is about 2.7×10^{10} dynes/cm^2. We see from the plot that tether lengths of 1250 and 1700 km are impossible with this material even without a payload. Depending on the safety factor used, tether length appear to be limited to several hundred kilometers. The next section discusses a way around this problem.

7. Tapered Tethers

In very long tethers considerable tether mass can be saved, and payload mass increased by tapering the tether so as to maintain constant stress per unit area along the tether.

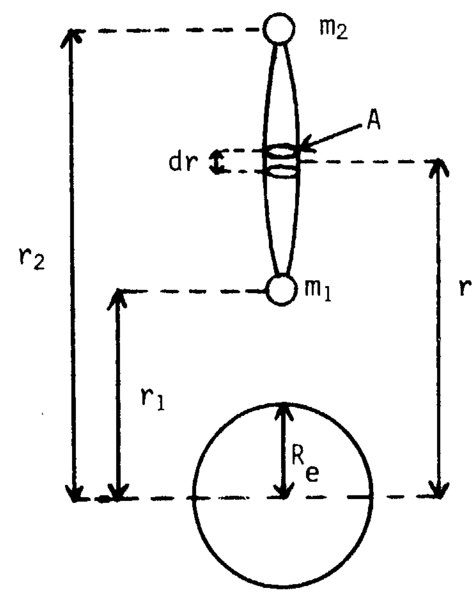

Figure 6. Tether cross section vs. position along the tether for maintaining constant stress per unit area.

In Figure 6, two masses m_1 and m_2 are connected by a tether of variable cross section A. We wish to derive an equation for the cross section A as a function of the distance r from the center of the earth such that the stress per unit area of cross section is constant. The tension is therefore

$$T = c\ A \tag{14}$$

where c is the maximum safe stress for the tether material. The mass dm of a section of tether of length dr is

$$dm = \rho\ A\ dr \tag{15}$$

where ρ is the density of the material. The gravitational force dF_g on the element is

$$dF_g = -dm\ GM/r^2 \tag{16}$$

and the centrifugal force dF_c is

$$dF_c = dm\ r\ \omega^2 \tag{17}$$

where ω is the orbital angular velocity. The net tension force dT on the wire element is

$$dT = \frac{dT}{dr}\ dr \tag{18}$$

Differentiating equation (14) and substituting into (18) gives

$$dT = c\ \frac{dA}{dr}\ dr \tag{19}$$

For the system to be in equilibrium, the sum of the gravitational, centrifugal and tension forces on each element of wire must be zero. Adding equations (16), (17), and (18) gives

$$dF_g + dF_c + dT = -dmGM/r^2 + dmr\omega^2 + C\ \frac{dA}{dr}\ dr = 0 \tag{20}$$

Substituting equation (15) into (20), gives

$$\rho\ A\ dr\ (-GM/r^2 + r\omega^2) + c\ \frac{dA}{dr}\ dr = 0 \quad ,$$

$$\frac{dA}{A} = -\ \frac{\rho}{c}\ (-GM/r^2 + r\ \omega^2)\ dr \tag{21}$$

The differential equation (21) can be integrated to give A as a function of r. The result is

$$\ell n\ A = -\ \frac{\rho}{c}\ (GM/r + \frac{1}{2}\ r^2\ \omega^2) + k \tag{22}$$

or

$$A = \ell\ -\ \frac{\rho}{c}\ (GM/r + \frac{1}{2}\ r^2\ \omega^2) + k \tag{23}$$

The integration constant k can be computed by specifying the cross section at some convenient point. For example if $A = A_1$ at $r = r_1$, then from equation (22),

$$k = \ell n\ A_1 + \frac{\rho}{c}\ (GM/r_1 + \frac{1}{2}\ r_1{}^2\ \omega^2) \tag{24}$$

Specifying the cross section and therefore the tension [through equation (14)] at either end of the wire fixes the equilibrium angular velocity ω. For example, the sum of the forces on mass 1 in equilibrium is

$$-m_1\ GM/r_1{}^2 + m_1 r_1 \omega^2 + C\ A_1 = 0 \tag{25}$$

Solving equation (25) for ω gives

$$\omega = \sqrt{GM/r_1{}^3 - \frac{c\ A_1}{m_1 r_1}} \tag{26}$$

Note that $c\ A_1$ must be less than the gravitational force for ω to have a real solution. If the cross section A_2 is given at the upper end, the sum of forces on mass 2 in equilibrium is

$$-m_2\ GM/r_2{}^2 + m_2 r_2 \omega^2 - c\ A_2 = 0 \tag{27}$$

Solving for ω gives

$$\omega = \sqrt{GM/r_2{}^3 + \frac{c\ A_2}{m_2 r_2}} \tag{28}$$

In this case, ω is always real, however it may be larger than the orbital angular velocity of an orbit above the earth's surface.

The cross sectional area given by equation (23) has its maximum value at the "orbital center" of the system. That is, the value of r where the gravitational and centrifugal accelerations are equal. This can be shown analytically using equation (23). The maximum value of A occurs when the exponent in equation (23) has its maximum. The maximum is obtained by differentiating equation (22) with respect to r, which gives equation (21), and setting the result to zero. We have therefore

$$-GM/r_m{}^2 + r_m\ \omega^2 = 0$$

$$GM/r_m{}^2 = r_m\ \omega^2 \tag{29}$$

where r_m is the thickest part of the tether. The left side of equation (29) is the gravitational acceleration, and the right is the centrifugal acceleration. Equation (29) is the equation for a free particle in a circular orbit of radius r_m. The net acceleration is away from the orbital center in both directions so that the system is supported from the orbital center, and the tether is thickest at this point.

The total mass M of the tether obtained by integration of equation (2) is

$$M = \int_{r_1}^{r_2} \rho\ A\ dr \tag{30}$$

where A is given by equation (23).

The net acceleration a_2 of the upper end of the tether is

$$a_2 = - GM/r_2{}^2 + r_2 \omega^2 \tag{31}$$

The tension at the end is $T_2 = C A_2$. This payload mass m_2 is therefore $T_2/\bar{a}_2$. Note that r_2 must be above the orbital center or a_2 and m_2 will be negative.

A computer program has been written to evaluate the integral in equation (30) and plot the cross section as a function of r. Figure 7 shows a plot of tether cross section (cm^2) vs. r (km) for the following parameters: r_1 = 6580 km (202 km altitude), tether length = 1250 km, m_1 = 200 kg, ρ = 1.45 g/cc, $C = 1.5 \times 10^{10}$ dynes/cm^2, and tether diameter = .06 cm at r_1. The tension at the lower end is $.424 \times 10^8$ dynes. The orbital center is 600 km from the lower end. The tether mass is 776 kg. The tension at the upper end is $.413 \times 10^8$ dynes and m_2 = 214 kg. The orbital angular velocity is 1.0377204×10^{-3} radians/sec. The lower and upper end velocities are 6.8282 and 8.1253 km/sec.

Figure 8 shows the tether cross section vs. r for a 1250 km tether attached to an 86 ton Shuttle at 202 km. The tether diameter at the Shuttle end is 3mm, and total tether mass is 6.9 tons. The payload mass is 160 kg. The tether is thickest at the orbital center which is 29.6 km from the Shuttle. The orbital angular velocity is 1.174905×10^{-3} radians/sec. The lower and upper end velocities are 7.7309 and 9.1995 km/sec respectively.

Figure 9 shows tether cross section for a 1700 km tether attached to the Shuttle at 202 km. The diameter at the Shuttle end is 4mm. The total tether mass is 13 tons and the payload is 32.6 kg. The orbital center is 53 km from the Shuttle. The orbital angular velocity is 1.168687×10^{-3} radians/sec. The lower and upper end velocities are 7.6900 and 9.6767 km/sec.

Figure 10 has an 86 ton platform at the top of a 1700 km tether. The tether diameter is 4mm at the top. The total tether mass is 17 tons and the payload at the lower end is 187.5 kg. The ratio of payload to tether mass is about 4.4 times as great as in Figure 9. The orbital angular velocity is

0.0	0.2827433388E-02
12.5	0.2899970986E-02
25.0	0.2972636245E-02
37.5	0.3045353868E-02
50.0	0.3118047068E-02
62.5	0.3190637709E-02
75.0	0.3263046455E-02
87.5	0.3335192913E-02
100.0	0.3406995795E-02
112.5	0.3478373068E-02
125.0	0.3549242126E-02
137.5	0.3619519948E-02
150.0	0.3689123271E-02
162.5	0.3757968760E-02
175.0	0.3825973181E-02
187.5	0.3893053576E-02
200.0	0.3959127436E-02
212.5	0.4024112879E-02
225.0	0.4087928823E-02
237.5	0.4150495162E-02
250.0	0.4211732936E-02
262.5	0.4271564504E-02
275.0	0.4329913713E-02
287.5	0.4386706063E-02
300.0	0.4441868866E-02
312.5	0.4495331408E-02
325.0	0.4547025103E-02
337.5	0.4596883637E-02
350.0	0.4644843115E-02
362.5	0.4690842199E-02
375.0	0.4734822235E-02
387.5	0.4776727382E-02
400.0	0.4816504729E-02
412.5	0.4854104402E-02
425.0	0.4889479670E-02
437.5	0.4922587038E-02
450.0	0.4953386335E-02
462.5	0.4981840789E-02
475.0	0.5007917102E-02
487.5	0.5031585505E-02
500.0	0.5052819813E-02
512.5	0.5071597466E-02
525.0	0.5087899566E-02
537.5	0.5101710898E-02
550.0	0.5113019947E-02
562.5	0.5121818904E-02
575.0	0.5128103666E-02
587.5	0.5131873818E-02
600.0	0.5133132620E-02
612.5	0.5131886972E-02
625.0	0.5128147381E-02
637.5	0.5121927911E-02
650.0	0.5113246129E-02
662.5	0.5102123045E-02
675.0	0.5088583043E-02
687.5	0.5072653800E-02
700.0	0.5054366204E-02
712.5	0.5033754266E-02
725.0	0.5010855017E-02
737.5	0.4985708411E-02
750.0	0.4958357211E-02
762.5	0.4928846878E-02
775.0	0.4897225454E-02
787.5	0.4863543435E-02
800.0	0.4827853646E-02
812.5	0.4790211110E-02
825.0	0.4750672913E-02
837.5	0.4709298071E-02
850.0	0.4666147388E-02
862.5	0.4621283316E-02
875.0	0.4574769813E-02
887.5	0.4526672201E-02
900.0	0.4477057021E-02
912.5	0.4425991891E-02
925.0	0.4373545358E-02
937.5	0.4319786761E-02
950.0	0.4264786085E-02
962.5	0.4208613821E-02
975.0	0.4151340828E-02
987.5	0.4093038197E-02
1000.0	0.4033777111E-02
1012.5	0.3973628722E-02
1025.0	0.3912664013E-02
1037.5	0.3850953682E-02
1050.0	0.3788568009E-02
1062.5	0.3725576749E-02
1075.0	0.3662049010E-02
1087.5	0.3598053143E-02
1100.0	0.3533656644E-02
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1162.5	0.3207954359E-02
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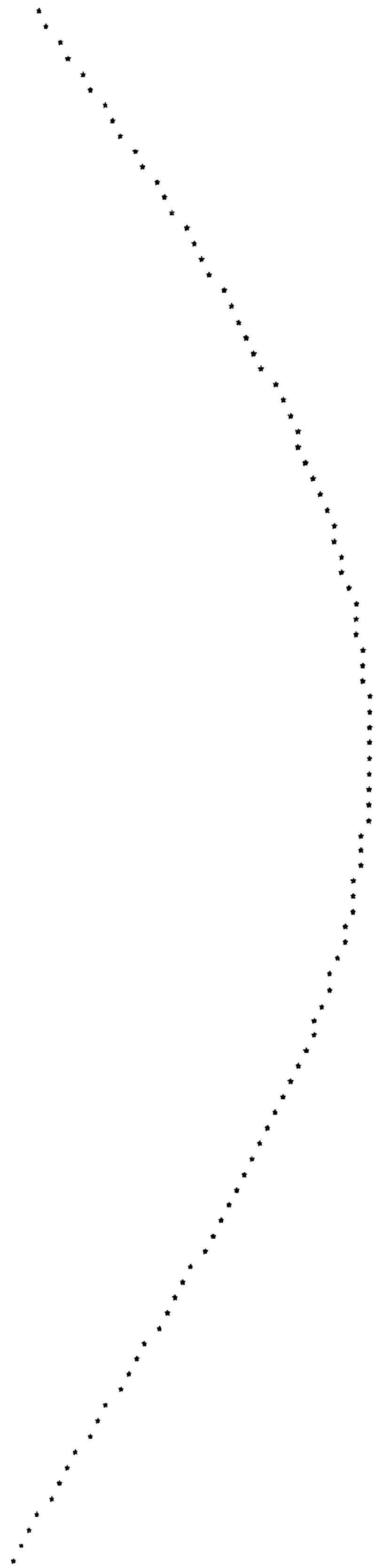

Figure 7. Tether cross section vs. position along the tether for a 1250 km tether with 200 kg at the lower end and 214 kg at the upper end. The position in km starting at the lower end is listed in the first column, and the cross section in cm^2 in the second column.

0.0	0.7066583471E-01
12.5	0.7076890882E-01
25.0	0.7080754584E-01
37.5	0.7080184152E-01
50.0	0.7075197382E-01
62.5	0.7065820178E-01
75.0	0.7052086424E-01
87.5	0.7034037833E-01
100.0	0.7011723764E-01
112.5	0.6985201033E-01
125.0	0.6954533687E-01
137.5	0.6919792774E-01
150.0	0.6881056081E-01
162.5	0.6838407868E-01
175.0	0.6791938575E-01
187.5	0.6741744523E-01
200.0	0.6687927597E-01
212.5	0.6630594918E-01
225.0	0.6569858509E-01
237.5	0.6505834945E-01
250.0	0.6438645003E-01
262.5	0.6368413297E-01
275.0	0.6295267921E-01
287.5	0.6219340072E-01
300.0	0.6140763691E-01
312.5	0.6059675082E-01
325.0	0.5976212554E-01
337.5	0.5890516043E-01
350.0	0.5802726755E-01
362.5	0.5712986805E-01
375.0	0.5621438863E-01
387.5	0.5528225896E-01
400.0	0.5433490382E-01
412.5	0.5337374876E-01
425.0	0.5240020791E-01
437.5	0.5141568539E-01
450.0	0.5042157140E-01
462.5	0.4941923940E-01
475.0	0.4841004329E-01
487.5	0.4739531487E-01
500.0	0.4637636138E-01
512.5	0.4535446308E-01
525.0	0.4433087120E-01
537.5	0.4330680578E-01
550.0	0.4228345388E-01
562.5	0.4126196781E-01
575.0	0.4024346355E-01
587.5	0.3922901934E-01
600.0	0.3821967440E-01
612.5	0.3721642782E-01
625.0	0.3622023758E-01
637.5	0.3523201974E-01
650.0	0.3425264777E-01
662.5	0.3328295201E-01
675.0	0.3232371927E-01
687.5	0.3137569261E-01
700.0	0.3043957117E-01
712.5	0.2951601019E-01
725.0	0.2860562117E-01
737.5	0.2770897205E-01
750.0	0.2682658761E-01
762.5	0.2595894988E-01
775.0	0.2510649875E-01
787.5	0.2426963260E-01
800.0	0.2344870902E-01
812.5	0.2264404566E-01
825.0	0.2185592110E-01
837.5	0.2108457584E-01
850.0	0.2033021331E-01
862.5	0.1959300097E-01
875.0	0.1887307143E-01
887.5	0.1817052366E-01
900.0	0.1748542418E-01
912.5	0.1681780833E-01
925.0	0.1616768156E-01
937.5	0.1553502072E-01
950.0	0.1491977539E-01
962.5	0.1432186922E-01
975.0	0.1374120124E-01
987.5	0.1317764726E-01
1000.0	0.1263106117E-01
1012.5	0.1210127629E-01
1025.0	0.1158810668E-01
1037.5	0.1109134849E-01
1050.0	0.1061078121E-01
1062.5	0.1014616899E-01
1075.0	0.9697261849E-02
1087.5	0.9263796939E-02
1100.0	0.8845499714E-02
1112.5	0.8442085110E-02
1125.0	0.8053258672E-02
1137.5	0.7678717658E-02
1150.0	0.7318152097E-02
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1212.5	0.5713727006E-02
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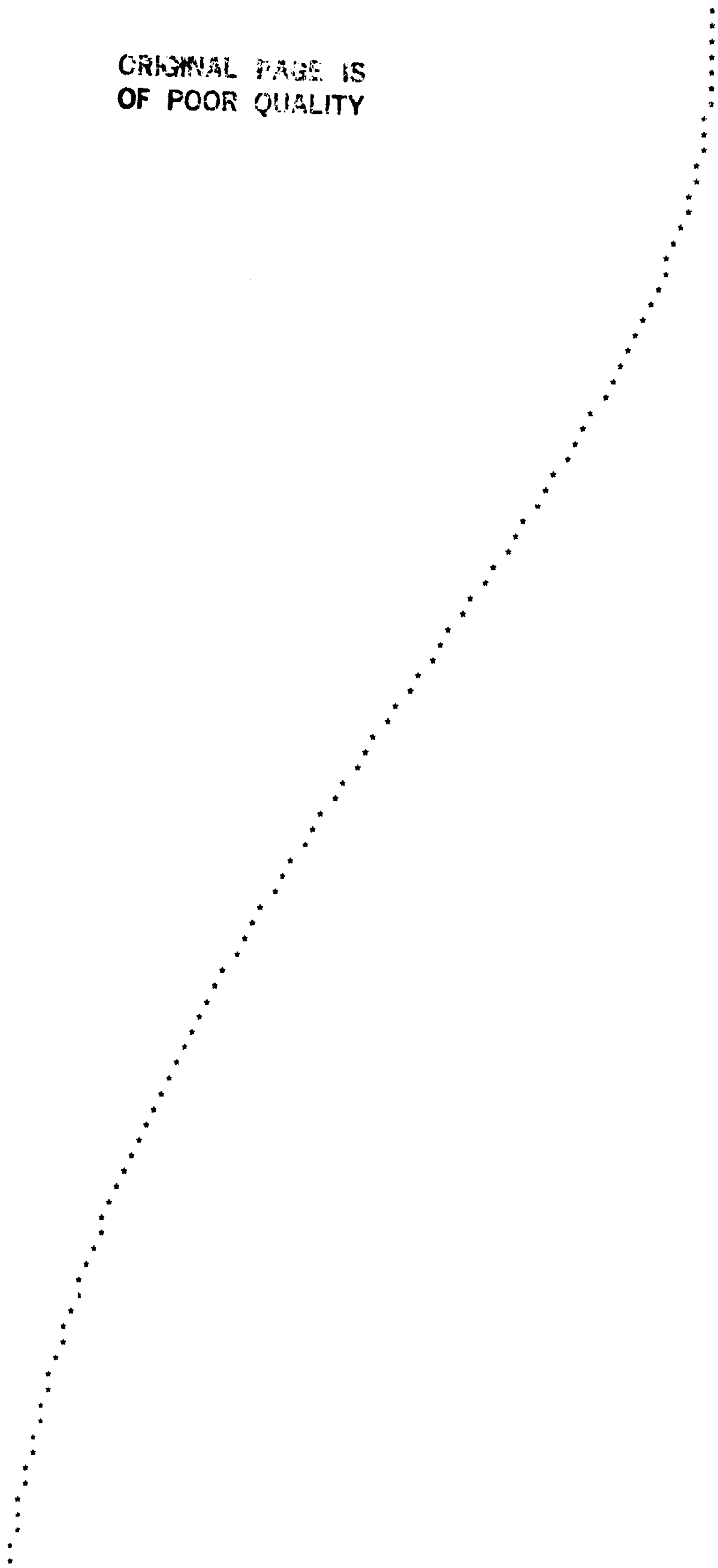

Figure 8. Tether cross section in cm^2 (second column) vs. position in km along the tether (first column) for a 1250 km tether attached to the Shuttle at the lower end.

0.0	0.1256637061E+00
17.0	0.1260439383E+00
34.0	0.1262790949E+00
51.0	0.1263691145E+00
68.0	0.1263144325E+00
85.0	0.1261159738E+00
102.0	0.1257751436E+00
119.0	0.1252938155E+00
136.0	0.1246743173E+00
153.0	0.1239194139E+00
170.0	0.1230322887E+00
187.0	0.1220165227E+00
204.0	0.1208760718E+00
221.0	0.1196152426E+00
238.0	0.1182386667E+00
255.0	0.1167512737E+00
272.0	0.1151582636E+00
289.0	0.1134650774E+00
306.0	0.1116773685E+00
323.0	0.1098009728E+00
340.0	0.1078418785E+00
357.0	0.1058061966E+00
374.0	0.1037001310E+00
391.0	0.1015299493E+00
408.0	0.9930195437E-01
425.0	0.9702245581E-01
442.0	0.9469774333E-01
459.0	0.9233406048E-01
476.0	0.8993757976E-01
493.0	0.8751437898E-01
510.0	0.8507041895E-01
527.0	0.8261152264E-01
544.0	0.8014335590E-01
561.0	0.7767140967E-01
578.0	0.7520098392E-01
595.0	0.7273717315E-01
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629.0	0.6784867236E-01
646.0	0.6543303744E-01
663.0	0.6304211049E-01
680.0	0.6067980037E-01
697.0	0.5834975857E-01
714.0	0.5605537619E-01
731.0	0.5379978232E-01
748.0	0.5158584383E-01
765.0	0.4941616651E-C1
782.0	0.4729309749E-01
799.0	0.4521872877E-01
816.0	0.4319490190E-01
833.0	0.4122321362E-01
850.0	0.3930502242E-01
867.0	0.3744145598E-01
884.0	0.3563341921E-01
901.0	0.3388160307E-01
918.0	0.3218649382E-01
935.0	0.3054838283E-01
952.0	0.2896737671E-01
969.0	0.2744340771E-01
986.0	0.2597624442E-01
1003.0	0.2456550249E-01
1020.0	0.2321065555E-01
1037.0	0.2191104601E-01
1054.0	0.2066589587E-01
1071.0	0.1947431737E-01
1088.0	0.1833532348E-01
1105.0	0.1724783619E-01
1122.0	0.1621070648E-01
1139.0	0.1522270403E-01
1156.0	0.1428254657E-01
1173.0	0.1338889888E-01
1190.0	0.1254038335E-01
1207.0	0.1173558820E-01
1224.0	0.1097307523E-01
1241.0	0.1025138708E-01
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1275.0	0.8924601073E-02
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1309.0	0.7743438597E-02
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1343.0	0.6696194329E-02
1360.0	0.6219195885E-02
1377.0	0.5771404098E-02
1394.0	0.5351444318E-02
1411.0	0.4957972258E-02
1428.0	0.4589674560E-02
1445.0	0.4245273128E-02
1462.0	0.3923525819E-02
1479.0	0.3623228409E-02
1496.0	0.3343215758E-02
1513.0	0.3082362811E-02
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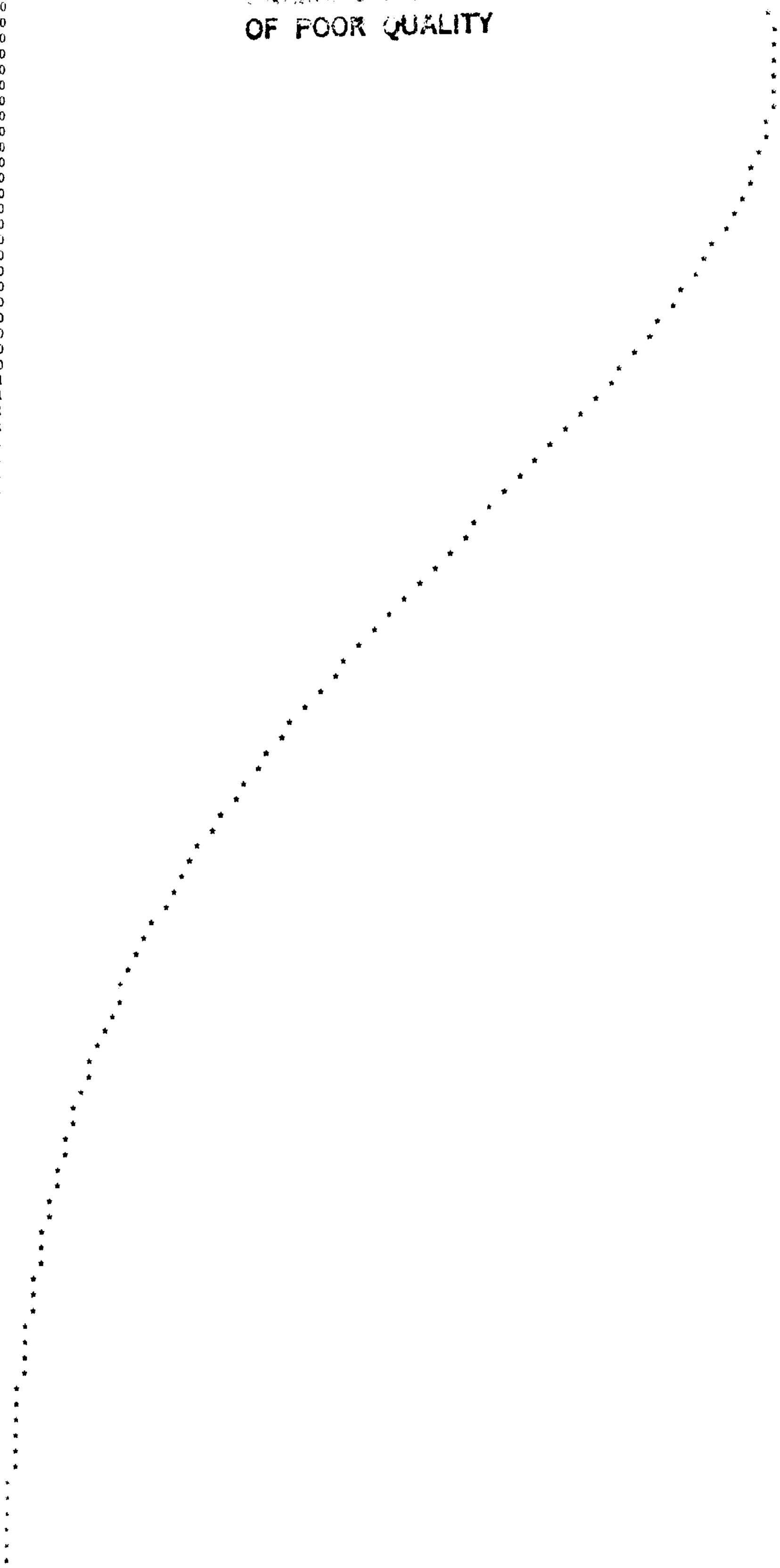

Figure 9. Tether cross section in cm^2 (second column) vs. position in km along the tether (first column) for a 1700 km tether attached to the Shuttle at the lower end.

0.0	0.5515324607E-02
17.0	0.5927121614[illegible]E-02
34.0	0.6363444455E-02
51.0	0.6825233163E-02
68.0	0.7313454859E-02
85.0	0.7829068170E-02
102.0	0.8373026647E-02
119.0	0.8946275505E-02
136.0	0.9549748221E-02
153.0	0.1018436299E-01
170.0	0.1085161906E-01
187.0	0.1155059297E-01
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221.0	0.1305186293E-01
238.0	0.1385516288E-01
255.0	0.1469458008E-01
272.0	0.1557081726E-01
289.0	0.1648453000E-01
306.0	0.1743632238E-01
323.0	0.1842674288E-01
340.0	0.1945628008E-01
357.0	0.2052535856E-01
374.0	0.2163433480E-01
391.0	0.2278349312E-01
408.0	0.2397304183E-01
425.0	0.2520310938E-01
442.0	0.2647374077E-01
459.0	0.2778489406E-01
476.0	0.2913643705E-01
493.0	0.3052814423E-01
510.0	0.3195969394E-01
527.0	0.3343066571E-01
544.0	0.3494053794E-01
561.0	0.3648868589E-01
578.0	0.3807437984E-01
595.0	0.3969678372E-01
612.0	0.4135495397E-01
629.0	0.4304783879E-01
646.0	0.4477427774E-01
663.0	0.4653300167E-01
680.0	0.4832263315E-01
697.0	0.5014168713E-01
714.0	0.5198857207E-01
731.0	0.5386159152E-01
748.0	0.5575894597E-01
765.0	0.5767873524E-01
782.0	0.5961896113E-01
799.0	0.6157753062E-01
816.0	0.6355225929E-01
833.0	0.6554087527E-01
850.0	0.6754102350E-01
867.0	0.6955027033E-01
884.0	0.7156610850E-01
901.0	0.7358596248E-01
918.0	0.7560719408E-01
935.0	0.7762710841E-01
952.0	0.7964296005E-01
969.0	0.8165195956E-01
986.0	0.8365128020E-01
1003.0	0.8563806483E-01
1020.0	0.8760943305E-01
1037.0	0.8956248847E-01
1054.0	0.9149432612E-01
1071.0	0.9340203998E-01
1088.0	0.9528273056E-01
1105.0	0.9713351261E-01
1122.0	0.9895152271E-01
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1156.0	0.1024779288E+00
1173.0	0.1041807780E+00
1190.0	0.1058397689E+00
1207.0	0.1074522672E+00
1224.0	0.1090156971E+00
1241.0	0.1105275587E+00
1258.0	0.1119854325E+00
1275.0	0.1133869860E+00
1292.0	0.1147299798E+00
1309.0	0.1160122742E+00
1326.0	0.1172318340E+00
1343.0	0.1183867347E+00
1360.0	0.1194751670E+00
1377.0	0.1204954417E+00
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1428.0	0.1231323153E+00
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1479.0	0.125[illegible]66E+00
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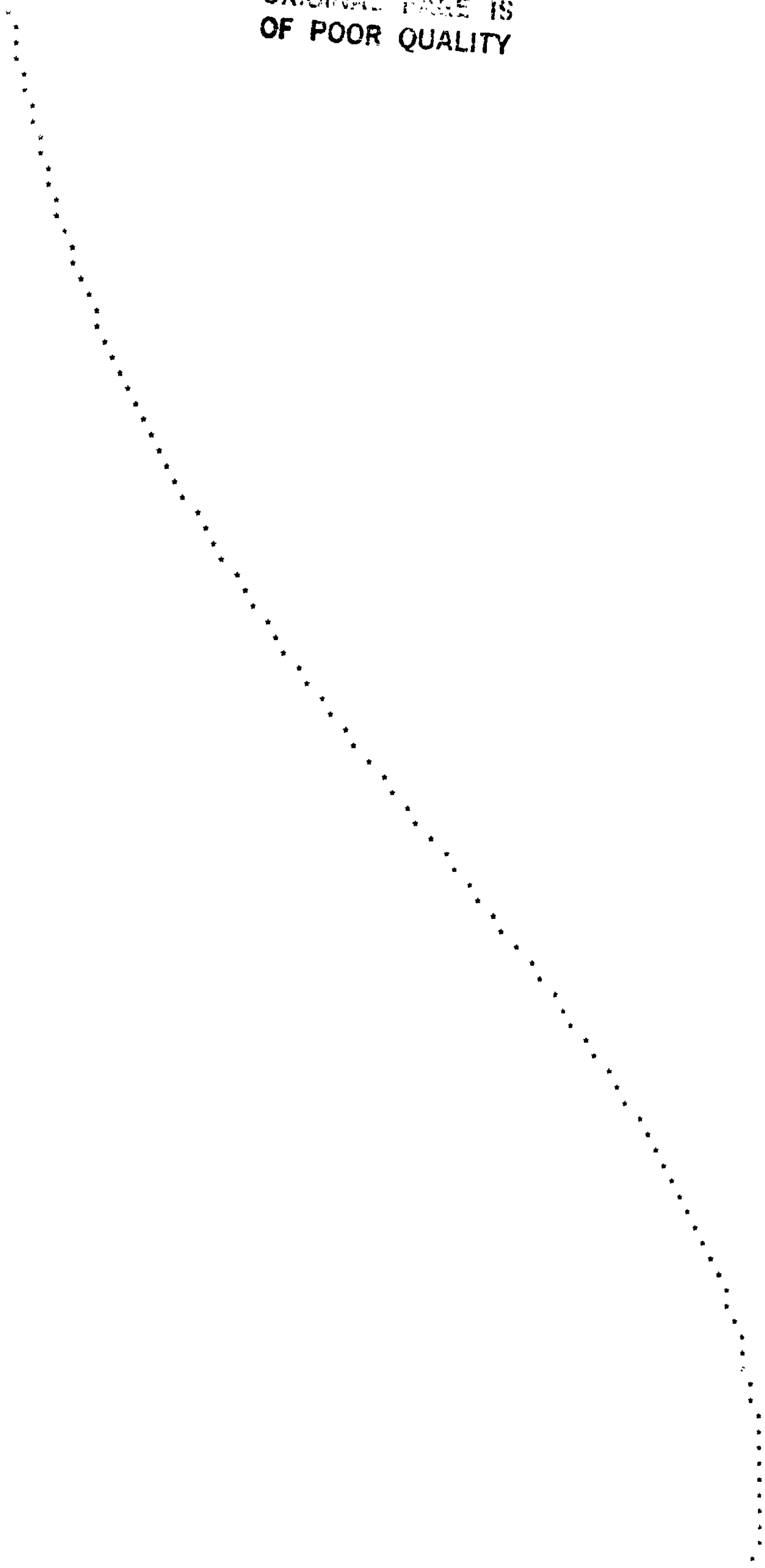

Figure 10. Tether cross section in cm^2 (second column) vs. position in km along the tether (first column) for a 1700 km tether attached to an 86 ton platform at the upper end.

8.5360992×10^{-3} radians/sec. The lower and upper end velocities are 5.6618 and 7.0679 km/sec.

For any of the cases discussed above, heavier payloads may be launched by scaling all the masses and cross sections. As an alternative to increasing the Shuttle or platform mass, the Shuttle may be moved further from the orbital center so that less mass is required.

8. Conclusions

Equations have been derived for calculating the orbital parameters required for the Shuttle or Space Platform in order to launch payloads into a particular orbit using a tether of length ℓ. The effect of the mass of the payload and tether have been considered and a method derived for tapering long tethers in order to reduce tether mass and increase payload mass. The general transfer equations developed make it possible to determine the post release orbit of both the upper and lower mass from the orbital parameter of the center of the system before release. This is important particularly in the case of launching heavy payloads from the Shuttle in low earth orbit in order to avoid re-entry, or assist in a planned re-entry.