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TOWARDS SUB-OPTIMAL STOCHASTIC CONTROL OF PARTIALLY OBSERVABLE STOCHASTIC SYSTEMS

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ABSTRACT

The paper deals with a class of multi-dimensional stochastic control problems with noisy data and bounded controls encountered in aerospace design. The emphasis is on sub-optimal design, the optimality being taken in quadratic mean sense. To that effect the problem is viewed as a stochastic version of the Lurie problem known from nonlinear control theory. The main result is a separation theorem (involving a nonlinear Kalman-like filter) suitable for Lurie-type approximations (Theorem 30). The theorem allows for discontinuous characteristics. As a byproduct we prove the existence of strong solutions to a class of non-Lipschitzian stochastic differential equations in n dimensions.

I PROBLEM FORMULATION

In order to motivate as well as justify the problem formulation (1)-(4) to be dealt with in this paper, let us start by noting the following particular case encountered in flight control design. The longitudinal motion of an aircraft in turbulence can be described (at some level of approximation) by the equations

(1') $$\dot{\bar{X}}_t = \bar{A}\bar{X}_t + \bar{b}\zeta_t + \bar{f}\eta_t, \quad \cdot = \frac{d}{dt},$$
$$\dot{\zeta}_t = \kappa \,\mathrm{Sat}(V_t - \zeta_t), \quad 0 < \kappa < \infty,$$

where $\bar{X}_t$ is a state vector (made of the angle of attack, pitch attitude and pitch rate), ζ_t is the elevator deflection (scalar), V_t is the control variable (scalar), the state noise η_t representing turbulence is assumed Gaussian with the spectral density $k_1/(\omega^2 + k_2)$; $\bar{A}$ is a matrix, $\bar{b}, \bar{f}$ are vectors, and Sat(.) denotes the standard saturation function.
The admissible control policies are to be based on the observation data only:

(3') $$V_t \in \mathcal{V} \equiv \{\text{"feedback functionals" of } [Y_s, s \le t]\},$$

where the data $\{Y_t\}$ are generated by the equations

(2') $$Y_t = C\begin{bmatrix}\bar{X}_t \\ \zeta_t \\ \eta_t\end{bmatrix} + \bar{G}N_t$$

with suitable matrices C and $\bar{G}$, and a vector valued Gaussian white noise $\{N_t\}$ independent of $\{\eta_t\}$.
The objective is to suppress the effects of turbulence by choosing an admissible control $\{V_t\}$ such that

(4') $$E\int_0^T (\text{normal acceleration})^2 dt = \min_{\{V_t\}\in\mathcal{V}}$$

In order to become amenable to modern techniques, the above problem can be easily recast within the framework of the following canonical problem formulation:

(1) $$dX_t = (AX_t + bU_t)dt + F\,dW_t,$$

(2) $$dY_t = CX_t\,dt + G\,dW_t, \quad 0 < t < T < \infty,$$

(3) $$\{U_t\} \in \mathcal{U} \equiv \{\{U_t\} : U_t \text{ adapted } \mathcal{B}_t^Y, \ |U_t| \le \kappa\},$$

(4) $$J = \int_0^T E(X_s^* Q X_s)\,ds \longrightarrow \min_{U \in \mathcal{U}}$$

where X_t denotes a new state vector, U_t a new control variable, $\{W_t\}$ is a vector valued Wiener process, A, F, C, G, Q are suitable matrices, b a vector, κ a positive bound.
It is implicit that we work on an underlying probability space $\{\Omega, \mathcal{B}, P\}$ with elements denoted by ω ; the symbol $\mathcal{B}_t^Y$ in (30) denotes the σ-algebra of events from $\mathcal{B}$ generated by the data process $\{Y_s, s \le t\}$.
Also the following technical conditions are assumed throughout:

(5) X_0 is Gaussian, with Cov $X_0 > 0$, and is independent of $W_t, t > 0$;
$GG^* = I$, $Y_0 \equiv 0$;
U_t is independent of $\{W_s - W_\sigma, s, \sigma \ge t\}$.

Finally we underline that the solutions to stochastic differential equations arising in the above formulation and below are required to be taken in the strong sense, i.e. causal functionals of the given noise sample. The absence of strong solutions rules out the feedback solutions to the control problem.

Notational remark. The generic variables are suppressed from the notation; the symbol I means the identity matrix; * means the transpose; E(.), E(.|.) denote the expectation, resp. the conditional expectation; the notation for random processes $\{W_t\}, \{U_t\}$, etc., is often shortened to just W, U, etc.

II SEPARATION

The separation principle has been so far a key tool for solving stochastic control problems with noisy observation data. It enables us to transform the latter problem into the one with complete observations of the state - the separated problem (see for instance [Won 1], [Fle]). In this section we improve on the separation result of [Ru2] by relaxing the assumption Dim X = Dim Y, Det C $\neq$ 0 in (1), (2). This assumption has been a subject of concern in the literature on separation (see [Won 1], [Da 1], where an approximate method of dealing with this problem has been suggested). By doing this we at the same time establish a basis for the development in Section IV.

In what follows we use the notation consistent with [Ru 2]. Working with (1) - (5), let $\{\tilde X_t\}, \{\tilde Y_t\}$, $t \geq 0$ be defined by

(6) $$d\tilde X_t = A\tilde X_t\,dt + F\,dW_t, \quad \tilde X_0 = X_0,$$
$$d\tilde Y_t = C\tilde X_t\,dt + G\,dW_t, \quad \tilde Y_0 = 0,$$

and let $\tilde{\mathcal{U}}$ be the set of controls

(7) $$\tilde{\mathcal{U}} \equiv \{\{U_t\} : U_t \text{ adapted } \mathcal{B}_t^{\tilde Y},\ |U_t| \leq K\}.$$

The definition of the separated problem is as follows:

(8) Definition (Separated problem corresponding to (1) - (4)).

(9) $$d\hat X_t = (A\hat X_t + bU_t)dt + P_tC^*dZ_t^0, \quad \hat X_0 = E\{X_0\},$$

$\{Z_t^0, \mathcal{B}_t^{\tilde Y}\}$ ... the innovation (Wiener) process corresponding to $\tilde X, \tilde Y$;

(10) $$\mathcal{J} = \int_0^T E(\hat X_s^* Q \hat X_s)\,ds \longrightarrow \underset{U\in\tilde{\mathcal{U}}}{\mathrm{Min}};$$

the matrix function P_t is given by
$$\dot P_t = AP_t + P_tA^* + FF^* - P_tC^*CP_t, \quad P_0 = \mathrm{Cov}\, X_0.$$

We have the following separation theorem

(11) Theorem. Consider (1) - (10).
(i) Assume the pair (C,A) is observable
Then

(12) $$\underset{U\in\mathcal{U}}{\mathrm{Min}}\, J \geq \underset{U\in\tilde{\mathcal{U}}}{\mathrm{Min}}\, \mathcal{J}$$

(ii) If moreover $\{U_{0t}\}$ is the optimal control for the separated problem, and if it is adapted to the σ-algebra
$$\mathcal{B}_t^{Y^u} = \sigma\{Y_s^u,\ s \leq t\}$$
where $\{Y_t^u\}$ denotes the data corresponding to $\{U_{0t}\}$, then $\{U_{0t}\}$ is also optimal for the original problem, i.e. $\{U_{0t}\} \in \mathcal{U}$,
$$\underset{U\in\mathcal{U}}{\mathrm{Min}}\, J = J(U_0).$$

In this case

(13) $$\hat X_t^u \equiv \hat X_t^{U_0} \equiv E\{X_t^u \mid \mathcal{B}_t^{Y^u}\},$$
$$Z_t^0 = Z_t^{U_0} = Y_t^u - \int_0^t C\hat X_s^u\,ds.$$

The proof follows from Lemma 2.2 and Theorem 3.1 of [Ru 2] in the case Dim X = Dim Y, Det C $\neq$ 0. The latter condition was needed to justify the following implication:

For a $\mathcal{B}_t^{Y^U}$-adapted control $\{U_t\}$ let

(14) $$CE(\tilde X_t \mid \mathcal{B}_t^{\tilde Y}) \overset{\text{def}}{=} C\,E(\tilde X_t \mid \mathcal{B}_t^{Y^U}) \quad \text{w.p.1}$$

then

(15) $$E(\tilde X_t \mid \mathcal{B}_t^{\tilde Y}) = E(\tilde X_t \mid \mathcal{B}_t^{Y^U}) \quad \text{w.p.1}.$$

This implication, however, can be seen to hold under the much weaker assumption of observability, as shown in the APPENDIX I.

III SEPARATED PROBLEM

Theorem (11) calls for an optimal solution to the separated problem (8). This relatively simple Markovian problem has been approached by many techniques; none of them, however, produced a solution, except for the case when Dim $\hat X$ = 1 (or equivalent). Our approach has been via the sample path optimality conditions, [Ru 2]:

(16) Theorem.
(i) The optimal control to the separated problem exists.
(ii) Assume $U \in \tilde{\mathcal{U}}$ is such that w.p.1

(17) $$\varrho_t^U \overset{\text{def}}{=} E\Big(b^*\int_t^T e^{A^*(s-t)}Q\hat X_s^U\,ds \,\Big|\, \mathcal{B}_t^{Z^0}\Big) \neq 0 \quad \text{a.e. } t$$

Then U is optimal iff

(18) $$U_t = -K\,\varrho_t^U / \|\varrho_t^U\|$$

(19) Special case: Dim X = 1. Using the theorem it has been shown in [Ru 1] that the optimal U is given by

(20) $$U_t = -K \ \mathrm{signum}\ \hat X_t$$

and that the resulting stochastic DE has a strong solution. This control moreover satisfies the separation theorem of the previous section ([Ru 2]), and is therefore optimal for the corresponding one-dimensional version of (1) - (4).

(21) Multidimensional case: bang-bang character of optimal control. This lesser result follows directly from the above Theorem (16) by imposing essentially the controllability of the pair (A,b). We need to show that with probability 1, $\varrho_t^U \neq 0$ on any interval of positive length. To this end let us expand the expression for $\hat X_t$ as follows:
$$\hat X_t = e^{At}E(X_0) + \int_0^t e^{A(t-s)}bU_s\,ds + \int_0^t e^{A(t-s)}P_sC^*dZ_s^0.$$

Upon a substitution into (17), and using the properties of stochastic integrals, we obtain

$$\varrho_t^U = b^* K(t)\, e^{At} \int_0^t e^{-As} P_s C^* dz_s^\circ +$$

+ absolutely continuous path,

where K(t) is given by

$$K(t) = \int_t^T e^{A^*(s-t)} Q\, e^{A(s-t)} ds$$

If C is square and nonsingular, then the stochastic integral above behaves as a scaled down Brownian motion. If in addition $b^*K(t)\exp At$ is not identically zero on any interval of positive length, then the whole first term on the right is a scaled down Brownian path and consequently ϱ^U has the desired property. Now

$$b^* K(t)\, e^{At} \equiv 0 \;\Rightarrow\; b^* K(t) \equiv 0 \;\Rightarrow$$

$$\Rightarrow\; b^* K(t)\, b \equiv 0 \;\Rightarrow\; \int_t^T \| \sqrt{Q}\, e^{A(s-t)} b \|^2 ds \equiv 0$$

$$\Rightarrow\; \sqrt{Q}\, e^{At} b \equiv 0 \;\Rightarrow$$

$$\Rightarrow\; b, Ab, \ldots, A^{n-1} b \in \mathcal{N}ullspace(Q).$$

The latter can not happen if $Q \neq 0$ and if the pair (A,b) is controllable. It is likely that the condition on C could be relaxed; but we dont pursue this any further.

Of course the lack of more specific characterization of the structure of optimal control law inhibits the usefullness of the separation theorem (11). As we have noted above, other approaches have not fared better. We mention at this point

(22) <u>Martingale based optimality criterion</u> of [E]1], [Da 3];

(23) <u>Comparison theorem</u> method of [Wa];

(24) <u>Bayes formula</u> approach of [Ba];

(25) <u>Dynamic programming</u>, [Won 3].

The above considerations as well as the experience with the corresponding deterministic control problem point to a rather complex nonlinear structure of optimal control in more than one dimension, thereby suggesting to turn to approximations which could be handled mathematically and which also would be compatible with the nature of our control problem:

(26) <u>Sub-optimal control policies.</u>
We will consider Markovian control policies of the form

(27) $$U_t = \kappa\, \varphi(\gamma^* \hat{X}_t), \quad \gamma \in R^n,$$

where $\varphi(\xi)$ are functions like Sat ξ, Sign ξ, and similar. They are simple enough to instrument and to deal with mathematically.
A sub-optimal design of these "Lurie-type" controls is described in [Won 3], the optimality being taken in quadratic mean sense. The approach combines the dynamic programming and statistical linearization techniques to obtain a quadratic-form-type approximation to the solution of the Bellman equation. Although the statistical linearization is difficult to justify and the approximation used for solving Bellman equation is rather crude, the computational experience and experiments with simple examples seem favorable.
A direct utilization of the statistical linearization for Lurie-type optimal stochastic control has been reported in [Lim].

The Lurie-type policies are also strongly suggested by taking a second look at the equation (1') of the aircraft problem statement: With $V(t) = k^*\bar{X}_t$ (k being a vector) and $\bar{f} = 0$, (1') become the well known Lurie system

(28) $$\dot{\bar{X}}_t = \bar{A}\bar{X}_t + \bar{b}\zeta_t, \quad \dot{\zeta}_t = \kappa\, \mathrm{Sat}(k^*\bar{X}_t - \zeta_t),$$

whose stability has been extensively studied (see [Lef]). Some simple examples as well as some rather deep investigations (see [Won 2]) indicate that a good stable system design of the unperturbed mode of a stochastic system with complete observations will produce a good stochastic control (quadratic mean sense).

The aim of this paper is now to answer the following :
Consider the separated problem (9), (10) corresponding to the original problem (1) - (4). Let $\{U_t\}$ denote a sub-optimal Lurie-type control process for the separated problem:

$$U_t = \varphi(\gamma^* \hat{X}_t) \in \tilde{\mathcal{U}}$$

such that

$$|J(U) - \min_{\tilde{\mathcal{U}}} J| < \varepsilon.$$

(i) Does this control law realize a feedback , i.e. does the corresponding stochastic DE have a strong solution?

(ii) If so, is this control admissible for the original problem (1) - (4), i.e. is $U \in \mathcal{U}$?

(iii) If so, can we estimate the approximation error

$$| J(U) - \min_{\mathcal{U}} J | \qquad ?$$

These questions are answered in the following section.

IV SEPARATION THEOREM FOR LURIE-TYPE APPROXIMATIONS

Let us denote by Φ the class of functions

(29) $$\Phi \equiv \{ \varphi(\xi) : R^1 \to R^1 : \varphi(-\xi) = -\varphi(\xi),\ \varphi \text{ nondecreasing},\ |\varphi(\xi)| \le \kappa, \text{ all } \xi \}$$

The separation theorem below is the main result of this section

(30) <u>Theorem.</u> Consider the stochastic control problem (1)-(4) and the related separated problem (9), (10). Let

(31) $$U_t = \varphi(\gamma^* \hat{X}_t)$$

for some $\varphi \in \Phi$ and a vector γ (* means transpose). Assume further

(32) (C,A) observable if φ is Lipschitz
or
(33) $\operatorname{Det} C \neq 0$, $(\operatorname{Dim}X = \operatorname{Dim}Y)$, if φ is discontinuous.

Then

(34) $$U_t \in \tilde{\mathcal{U}} \cap \mathcal{U}$$

and hence

$$\hat{\hat{X}}_t \equiv \hat{X}_t \equiv E(X_t \mid \mathcal{B}^Y_t).$$

(35) If U is ε-optimal in $\tilde{\mathcal{U}}$ (i.e. for the separated problem), it is also ε-optimal in $\mathcal{U}$ (i.e. for the original control problem).

(i) Proof of $U \in \tilde{\mathcal{U}}$. The inclusion follows readily by showing that the Ito equation (9) admits a strong solution. If φ is Lipschitz then there is no problem and the standard Ito theory applies. Otherwise we have the following lemma, which is of interest on its own:

(36) Lemma. Consider the stochastic (Ito) differential equation

(37) $$dX_t = \left(f(X_t) + g(X_t)\right)dt + F_t\, dW_t \qquad t \in [0,T],$$

where $\operatorname{Dim} X_t = n$, $\{W_t\}$ is an n-dimensional Wiener process,

(38) $g: R^n \to R^n$ is a measurable, bounded and monotone mapping, i.e.
$$(\bar{X} - \bar{\bar{X}})^*(g(\bar{X}) - g(\bar{\bar{X}})) \leq 0 ,$$

(39) $f: R^n \to R^n$ is Lipschitz and satisfies the growth condition
$$\|f(X)\| \leq \text{const} \cdot (1 + \|X\|) ,$$

(40) F_t is an $(n \times n)$-matrix function, nonsingular and continuous on $[0,T]$.

Then (37) has a strong solution.
For proof see APPENDIX 2.

In order to apply this lemma, we may assume $b = -\gamma$ in (9), (otherwise we take a nonsingular transformation $\bar{X} = KX$ such that $K^*Kb = -\gamma$). It is easy to verify that

$$g(X) = -b\varphi(b^*X)$$

is monotone:

$$-(\bar{X} - \bar{\bar{X}})^* b\,[\varphi(b^*\bar{X}) - \varphi(b^*\bar{\bar{X}})] = -(\bar{X}^*b - \bar{\bar{X}}^*b)[\varphi(\bar{X}^*b) - \varphi(\bar{\bar{X}}^*b)] \leq 0$$

Clearly $f(X) = AX$ and $F_t = P_t C^*$ also satisfy the conditions of the lemma (see the hypothesis of the theorem).

(ii) Proof of $U \in \mathcal{U}$. Consider again (9), (31):

(41) $$d\hat{\hat{X}}_t = (A\hat{\hat{X}}_t + b\varphi(\gamma^*\hat{\hat{X}}_t))\,dt + P_t C^* dZ^o_t .$$

By (i) above there is a unique solution $\hat{\hat{X}}_t$ adapted the σ-algebra $\mathcal{B}^{Z^o}_t$ and so $U_t = \varphi(\gamma^*\hat{\hat{X}}_t) \in \tilde{\mathcal{U}}$ is well defined. Next for $U \in \tilde{\mathcal{U}}$ we have that (see (2.23) of [Ru 2])

(42) $$dZ^o_t = d\tilde{Y}_t - C\hat{\hat{X}}_t\, dt = dY_t - C\hat{\hat{X}}_t\,dt ,$$

and hence $\hat{\hat{X}}_t$ is also a solution to

(43) $$d\hat{\hat{X}}_t = (A - P_t C^* C)\hat{\hat{X}}_t\, dt + b\varphi(\gamma^*\hat{\hat{X}}_t) + P_t C^* dY_t .$$

We will now use this equation to verify that $\hat{\hat{X}}_t$ is in fact adapted $\mathcal{B}^Y_t$. This then will imply that $U_t = \varphi(\gamma^*\hat{\hat{X}}_t)$ belongs to $\mathcal{U}$.

If φ is Lipschitz, then $\mathcal{B}^Y_t$-adaptedness is immediate. Otherwise $\mathcal{B}^Y_t$-adaptedness readily follows from pathwise uniqueness of solutions to (43), and that is shown the same way as in (i) above (we note that the lemma above remains in force for $f = f(t,X)$ such that the properties (39) hold uniformly). The rest of the argument is a technicality which we delegate to APPENDIX 3.

(iii) Proof of (35) is immediate from the inequality (12) of the theorem (11):
Let U^ε_t be such that

$$J(U^\varepsilon) - \min_{\tilde{\mathcal{U}}} J \leq \varepsilon .$$

By (12)

$$\min_{\tilde{\mathcal{U}}} J \leq \min_{\mathcal{U}} J$$

and so

$$J(U^\varepsilon) - \min_{\mathcal{U}} J \leq J(U^\varepsilon) - \min_{\tilde{\mathcal{U}}} J \leq \varepsilon .$$

V SUMMARY

Assume the hypothesis of Theorem (30). The sub-optimal stochastic control of Lurie type for the control system (1) - (4) is generated by the following feedback structure:

$$dX_t = (AX_t + bU_t)dt + F\, dW_t$$
$$dY_t = CX_t\, dt + G\, dW_t$$
$$d\hat{X}_t = [(A - P_t C^*C)\hat{X}_t + b\,\varphi(\gamma^*\hat{X}_t)]\,dt + P_t C^* dY_t$$
$$\dot{P}_t = AP_t + P_t A^* + FF^* - P_t C^* C P_t$$
$$U_t = \varphi(\gamma^*\hat{X}_t)$$

The choice of φ, γ is based on the performance of the auxiliary hypothetic system (separated system in which $\{Z^o_t\}$ is a Wiener process)

$$d\hat{\hat{X}}_t = [A\hat{\hat{X}}_t + b\,\varphi(\gamma^*\hat{\hat{X}}_t)]\,dt + P_t C^* C dZ^o_t .$$

$\hat{X}_t$ has the interpretation of the best mean square estimate of the state X_t given the past data $Y_s, s \leq t$. The performance of the system is at least as good as that of the separated system.

APPENDIX 1

Fix a $\tau > 0$ and let us take the conditional expectation given the σ-algebra $\mathcal{B}_\tau^{Y^u}$ in (14); noting the relations (see [Ru 2] (2.30),(3.7),(3.8))

$$\{U_t\} \in \mathcal{U} \Rightarrow \mathcal{B}_t^{\tilde{Y}} \equiv \mathcal{B}_t^{Z^\circ} \subseteq \mathcal{B}_t^{Y^u} \perp\!\!\!\perp \mathcal{B}_{t+}^{Z^\circ}$$

where $\perp\!\!\!\perp$ means independence and

$$\mathcal{B}_{t+}^{Z^\circ} = \sigma\{Z_s^\circ - Z_u^\circ,\ s,u \geq t\},$$

let us evaluate first the left hand side in (14) :

$$C\,E(E(\tilde{X}_t|\mathcal{B}_t^{\tilde{Y}})|\mathcal{B}_\tau^{Y^u}) =$$
$$= C\,E\Big(e^{At}E(X_0) + \int_0^t e^{A(t-s)}P_s C^* dZ_s^\circ \Big| \mathcal{B}_\tau^{Y^u}\Big)$$
$$= \ldots$$
$$= C e^{A(t-\tau)} E(\tilde{X}_\tau|\mathcal{B}_\tau^{\tilde{Y}}),\quad \tau \leq t.$$

The right-hand side of (14) becomes next ($\tau \leq t$)

$$C E(E(\tilde{X}_t|\mathcal{B}_t^{Y^u})|\mathcal{B}_\tau^{Y^u}) = C\,E(\tilde{X}_t|\mathcal{B}_\tau^{Y^u}) =$$
$$= C\,E\Big(e^{At}X_0 + \int_0^t e^{A(t-s)} F\,dW_s \Big| \mathcal{B}_\tau^{Y^u}\Big) =$$
$$= \ldots$$
$$= C e^{A(t-\tau)} E(\tilde{X}_\tau|\mathcal{B}_\tau^{Y^u}).$$

Thus (14) becomes

$$C e^{A(t-\tau)}\big(E(\tilde{X}_\tau|\mathcal{B}_\tau^{\tilde{Y}}) - E(\tilde{X}_\tau|\mathcal{B}_\tau^{Y^u})\big) = 0,\ \text{all } t \geq \tau.$$

Next, taking derivatives with respect to t, and putting $t = \tau$ afterwards, we obtain

$$C\ (\ E(\tilde{X}_\tau|\mathcal{B}_\tau^{\tilde{Y}}) - E(\tilde{X}_\tau|\mathcal{B}_\tau^{Y^u})\ ) = 0,$$
$$CA\ (\qquad \text{ditto} \qquad) = 0,$$
$$\ldots\ldots$$
$$CA^{n-1}(\qquad \text{ditto} \qquad) = 0.$$

Therefore (15) follows if

$$\text{Rank}\,[C, CA, \ldots, CA^{n-1}] = \text{Dim}\,A,$$

that is,observability.

APPENDIX 2 (Proof of (36))

The existence of strong solutions will be verified via a device from [Ya], namely by showing that

(a) a weak solution exists

and

(b) the path-wise uniqueness holds.

(For simplicity we take $X_0 = 0$.)

(a) Recall that by a weak solution of (37) we mean a probability space

$$\{\bar{\Omega}, \bar{\mathcal{B}}, \bar{P}, \bar{\mathcal{B}}_t\}$$

and a pair of processes $\{\bar{X}_t\},\{\bar{W}_t\}$ such that

$\{\bar{X}_t\}$ is $\bar{\mathcal{B}}_t$-adapted, continuous path,

$(\{\bar{W}_t\}, \bar{\mathcal{B}}_t)$ is a Wiener process,

(44) $d\bar{X}_t = f(\bar{X}_t)dt + g(\bar{X}_t)dt + F_t\,d\bar{W}_t$,w.p.1, all t.

(A strong solution of (37) is a sample-path continuous process $\{X_t\}$ adapted $\mathcal{B}_t^W$ and such that (37) holds w.p.1.)

The existence of weak solution is now shown as follows. Define

$$\bar{X}_t = \int_0^t F_s\,dW_s,$$
$$\psi_t = F_t^{-1}(f(\bar{X}_t) + g(\bar{X}_t)).$$

ψ_t is clearly $\mathcal{B}_t^W$-adapted and moreover

$$E\Big(\exp\Big[\int_0^T \psi_t\,dW_t - (1/2)\int_0^T \|\psi_t\|^2 dt\Big]\Big) = 1$$

as a consequence of φ being bounded and f satisfying the growth condition (39). Therefore we can use the Girsanov theorem and conclude that there is a new Wiener process $\{\bar{W}_t\}$ on the probability space

$$(\Omega, \mathcal{B}, \bar{P}, \mathcal{B}_t)$$
$$d\bar{P} = dP\,\exp\Big[\int_0^T \psi\,dW - \tfrac{1}{2}\int_0^T \|\psi\|^2 dt\Big]$$

such that

$$W_t - \int_0^t \psi_s\,ds = \bar{W}_t.$$

The last equation can be rewritten as

$$dW_t = F_t^{-1}\big(f(\bar{X}_t) + g(\bar{X}_t)\big)dt + d\bar{W}_t$$

or

$$(F_t\,dW_t =)\ \ d\bar{X}_t = [f(\bar{X}_t) + g(\bar{X}_t)]dt + F_t\,d\bar{W}_t$$

which is (44).

(b) While weak existence follows essentially from the growth of f and boundedness of g, the path-wise uniqueness is a consequence of f being Lipschitz and g being monotone.

Assume that on some probability space $(\Omega, \mathcal{B}, P, \mathcal{B}_t)$ we have two pairs of processes (X_t, W_t) and $(\bar{X}_t, W_t)$ such that $X_0 = \bar{X}_0 = 0$ and $(W_t, \mathcal{B}_t)$ is a Wiener process. Define

$$Z_t = X_t - \bar{X}_t.$$

Then Z_t is differentiable and we can write

$$\frac{d}{dt}\|Z_t\|^2 = 2Z_t^* \dot{Z}_t =$$
$$= (X_t - \bar{X}_t)^*([f(X_t) - f(\bar{X}_t)] + [g(X_t) - g(\bar{X}_t)])$$

Now by the Lipschitz property of f

$$(X_t - \bar{X}_t)^*(f(X_t) - f(\bar{X}_t)) \leq \text{const}\,\|X_t - \bar{X}_t\|^2,\quad \text{const} > 0,$$

and by monotonicity of g

$$(X_t - \bar{X}_t)^*(g(X_t) - g(\bar{X}_t)) \leq 0\ .$$

Hence

$$\frac{d}{dt}\|Z_t\|^2 \leq \text{const}\,\|Z_t\|^2,\ \|Z_0\|^2 = 0$$

Thus $Z_t \equiv 0$ for all t w.p.1 and path uniqueness follows.

APPENDIX 3

(Proof that $\{X_t\}$ of (43) is adapted $\mathcal{B}_t^Y$ in the case of non-Lipschitz φ .)

From (ii) of the proof of Theorem (30), the path-wise uniqueness for (43) holds. Let $\mathcal{D}_t$ denote the set of all possible paths of $\{X_t\}$ restricted to the interval (0,t) and denote $\mathcal{D} = \mathcal{D}_T$. Then

$$\mathcal{D} \in \mathcal{B}_{C^n(0,T)},\ \mathcal{D}_t \in \mathcal{B}_{C^n(0,t)},$$

where we have denoted by $\mathcal{B}_{C^n(0,T)}$ the σ-algebra of Borel sets of the space of n-dimensional continuous functions on (0,T) resp. (0,t) - $C^n(0,T)$ resp. $C^n(0,t)$.
Define the mappings

$$f : \mathcal{D} \to C^n(0,T)$$

$$f_t : \mathcal{D}_t \to C^n(0,t)$$

by

$$f(\hat{X})(s) \equiv \hat{X}_s - \hat{X}_0 - \int_0^s \{(A - \underline{P}_\sigma C^* C)\hat{X}_\sigma - \varphi(\gamma^* \hat{X}_\sigma)\} d\sigma$$

Clearly

$$f_t = \pi_{(0,t)} f$$

where $\pi_{(0,t)}$ denotes the restriction to $C^n(0,t) \to C^n(0,t)$. By uniqueness, f and f_t are one-to-one mappings w.p.1, and hence the inverses f^{-1}, f_t^{-1} exist and

$$f_t^{-1} = \pi_{(0,t)} f^{-1}.$$

Next f and f_t are measurable $\mathcal{B}_{C^n(0,T)} / \mathcal{B}_{C^n(0,T)}$ resp. $\mathcal{B}_{C^n(0,t)} / \mathcal{B}_{C^n(0,t)}$ for all t, hence by the Kuratowski theorem of measure theory the inverse mappings f^{-1}, f_t^{-1} are measurable.
Let π_t denote the coordinate mapping; then

$$\pi_t f^{-1}(\xi) : R^1 \times C^n(0,T) \to R^1$$

is measurable $\mathcal{F} \times \mathcal{B}_{C^n(0,T)}$ ($\mathcal{F}$ is the σ-algebra of Borel sets on (0,T)), and moreover

$$\pi_t f^{-1}(\xi) = \pi_t \pi_{(0,t)} f^{-1}(\xi)$$

so that

$$\pi_t f^{-1}(\xi) \text{ is measurable } \mathcal{B}_{C^n(0,t)} .$$

The proof is finished by taking

$$\xi = \{\xi_t\} \; , \; \xi_t = \int_0^t P_s C^* dY_s .$$

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