NASA CR-166,175

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JIAA TR-28

ON CERTAIN INTEGRALS WHICH PERTAIN TO THE FORCED VIBRATION OF PLATES

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October 1979

JIAA TR - 28

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OCTOBER 1979

The work here presented has been supported by the National Aeronautics and Space Administration under NASA Grant 2215 to the Joint Institute for Aeronautics and Acoustics

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By

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Introduction

The normal velocity anywhere on an infinite plane elastic surface immersed in a uniform compressible fluid and driven by a time-periodic concentrated force or moment admits a single complex Fourier type integral representation; and its estimate at points remote from the site of excitation can be achieved with the standard procedures of asymptotic analysis. To investigate the surface response near the driving point, Crighton (1972) employs a noteworthy albeit indirect procedure which (in the case of a line force or moment) rests upon a correlation between asymptotic behaviors of the respective functions

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\zeta) e^{-i\zeta x} \, d\zeta \quad , \quad |x| \to 0 \tag{1}$$

and

$$F_+(\zeta) = \int_0^{\infty} f(x) e^{i\zeta x} \, dx \quad , \quad |\zeta| \to \infty \, , \, \mathrm{Im}\, \zeta > 0 \tag{2}$$

in the designated limits. A particular consequence of the latter relation, namely

$$F_+(i\eta) = \int_0^{\infty} f(x) e^{-\eta x} \, dx \quad , \quad \eta > 0 \tag{3}$$

associates (through Watson's lemma) terms in the asymptotic expansion of $F_+(i\eta)$ as $\eta \to +\infty$ with the Laplace transform of terms in the expansion of $f(x)$ as $x \to +0$. The determination of $F_+(\zeta)$, or half-range Fourier transform of $f(x)$, is accomplished by a split of the full range transform

$$F(\zeta) = F_+(\zeta) + F_-(\zeta) \tag{4}$$

into component parts which are analytic functions of the complex variable ζ in the overlapping domains $\text{Im}\,\zeta > -\varepsilon$, $< \varepsilon$, respectively, $F(\zeta)$ itself being analytic within the strip $|\text{Im}\,\zeta| < \varepsilon$ of the ζ-plane. As Crighton observes, however, to effect the above split (which is a characteristic of the Wiener-Hopf technique for solving integral equations) generally poses an integration problem with comparable difficulty to that presented at the outset by (1); and there are additional technicalities requiring consideration if (1) is replaced by the Fourier-Bessel integral (or Hankel transform)

$$f(x) = \int_0^\infty F(\zeta) J_0(\zeta x)\zeta \, d\zeta \tag{5}$$

that occurs in the case of point excitation. Prior to Crighton's suggested function theory approach and with a wholly different motivation, Grosjean (1965) gave precise rules and conditions for the explicit construction of convergent infinite series, around $x = 0$, which describe functions defined by the Fourier cosine/sine integrals,

$$\int_0^\infty F(\zeta) \cos \zeta x \, d\zeta \quad \text{and} \quad \int_0^\infty F(\zeta) \sin \zeta x \, d\zeta \; ;$$

since these are predicated on the hypothesis that $F(\zeta)$ is a real function, with a convergent or asymptotic expansion of the form

$$\frac{a_1}{\zeta^\alpha} + \frac{a_2}{\zeta^{\alpha+1}} + \ldots \; , \quad \begin{matrix} 0 < \alpha \leq 1 \\ a_n \text{ real} \end{matrix}$$

the results do not serve the purposes of a near field plate excitation analysis.

It is the objective of this note to directly recast the various plate integrals considered by Crighton in a manner that permits their systematic development near the source point; and thus to dispense with manipulation of the velocity transform. The advantage of such a direct procedure is manifest by the ease with which distributed forces over the plate are incorporated into the analysis, whereas Crighton's approach is rendered more difficult at the outset by any change in the velocity transform or the function $F(\zeta)$. After the procedure is introduced (§ 1) in connection with a line source problem and followed (§2) by consideration of a point source, the analogous problems relative to uniformly distributed forces over strips or circular areas of the plate receive attention (§3). There are good reasons, since the exact plate admittances possess a complicated analytical nature, to seek out useful (simple) approximate representations along the lines taken by Crighton, in fact, for the heavy fluid loading (low frequency) limit with a point force or moment.

1. The Line Source Problem

The surface velocity $v(x)e^{-i\omega t}$ of a thin elastic plate lying in the (x,z)-plane, with a uniform compressible fluid adjacent thereto (in $y > 0$, say), which results from the application of a normally directed line force $F_0\delta(x)e^{-i\omega t}$ along the z-axis, is represented by the Fourier integral

$$v(x) = -\frac{i\omega F_0}{2\pi B}\int_{-\infty}^{\infty}\frac{\sqrt{\zeta^2-k^2}\,\exp(-i\zeta x)\,d\zeta}{(\zeta^4-k_p^4)\sqrt{\zeta^2-k^2}-\mu k_p^4} \qquad (6)$$

wherein m and B denote the specific mass and bending stiffness of the plate, $\rho = \mu m$ is the fluid density, $k = \omega/c$ specifies the acoustic wave number in the fluid, and $k_p = (m\omega^2/B)^{1/4}$ designates the bending wave number of the plate in vacuum. Following a customary prescription, the path or contour of the integral (6) approaches the real axis as $|\zeta| \to \infty$ and passes above/below all singularities on the negative/positive real axis; and the function $\sqrt{\zeta^2-k^2}$, rendered single-valued with cuts from k to ∞ and $-\infty$ to $-k$, respectively, assumes the asymptotic form $|\zeta|$ at the limits of integration.

If the bending stiffness has a negligible magnitude the version of (6) for a mass-loaded plate,

$$v(x) = \frac{iF_0}{2\pi m\omega} \int_{-\infty}^{\infty} \frac{\sqrt{\zeta^2-k^2}\,\exp(-i\zeta x)}{\sqrt{\zeta^2-k^2}+\mu}\, d\zeta$$

$$= \frac{iF_0}{m\omega}\,\delta(x) - \frac{i\mu F_0}{2\pi m\omega} \int_{-\infty}^{\infty} \frac{\exp(-i\zeta x)}{\sqrt{\zeta^2-k^2}+\mu}\, d\zeta \,, \tag{7}$$

presents the specific complex-valued function

$$I(x) = \int_{-\infty}^{\infty} \frac{\exp(-i\zeta x)}{\sqrt{\zeta^2-k^2}+\mu}\, d\zeta \tag{8}$$

dealt with by Crighton in the first application of his method for estimating Fourier integrals at small values of the parametric variable; and the available estimate, which serves for later comparison, is

$$I(x) \doteq -2\log x - 2\left(\log\frac{k}{2} - \frac{i\pi}{2}\right) - 2\,\frac{\mu}{\gamma}\,\sinh^{-1}\left(\frac{\mu}{k}\right) - \frac{\pi i\mu}{\gamma} - 2C + \pi\mu x \,, \quad x \to 0 \tag{9}$$

where $\gamma = \sqrt{k^2+\mu^2}$ and $C = 0.5772\ldots$ denotes the Euler constant.

To arrive at a characterization of the (even) function $I(x)$ suitable for development when small values of x are contemplated, assume that $x > 0$ and displace the contour into the lower half of the ζ-plane where the integrand is exponentially small at infinity; then $I(x)$ is expressible in terms of a principal value integral along the branch cut between $-\infty$ and $-k$, together with a (half-) residue contribution from the pole at $\zeta = -\sqrt{k^2+\mu^2} = -\gamma$ on the lower side thereof. Consequent to a change in sign for the integration variable along the cut there obtains, accordingly,

$$I(x) = ⨍_{\infty}^{k} \frac{e^{i\tau x}(-d\tau)}{\sqrt{\tau^2-k^2}+\mu} + ⨍_{k}^{\infty} \frac{e^{i\tau x}(-d\tau)}{-\sqrt{\tau^2-k^2}+\mu} - \pi i\,\frac{\mu}{\gamma}\,e^{i\gamma x}$$

$$= 2 \fint_k^\infty e^{i\tau x} \frac{\sqrt{\tau^2-k^2}}{\tau^2-\gamma^2}\, d\tau - \pi i \frac{\mu}{\gamma} e^{i\gamma x}$$

$$= 2 \int_k^\infty \frac{e^{i\tau x}}{\sqrt{\tau^2-k^2}}\, d\tau + 2\mu^2 \fint_k^\infty \frac{e^{i\tau x}}{\sqrt{\tau^2-k^2}} \frac{d\tau}{\tau^2-\gamma^2} - \pi i \frac{\mu}{\gamma} e^{i\tau x} , \quad x > 0 \tag{10}$$

with the symbol $\fint$ used to designate a principal value. The first integral appearing in (10) can be explicitly described by means of a Hankel function, viz.

$$\int_k^\infty \frac{e^{i\tau x}}{\sqrt{\tau^2-k^2}}\, d\tau = \frac{\pi i}{2} H_0^{(1)}(kx) , \quad x > 0 \tag{11}$$

and thus the second defines a function

$$J(x) = \fint_k^\infty \frac{e^{i\tau x}}{\sqrt{\tau^2-k^2}} \frac{d\tau}{\tau^2-\gamma^2} \tag{12}$$

which satisfies the differential equation

$$\frac{d^2J}{dx^2} + \gamma^2 J = -\frac{\pi i}{2} H_0^{(1)}(kx) . \tag{13}$$

It is a simple matter to express the general solution of the latter equation, namely

$$J(x) = Ae^{i\gamma x} + Be^{-i\gamma x} - \frac{\pi i}{2\gamma} \int_0^x \sin \gamma(x-x') H_0^{(1)}(kx')\, dx' \tag{14}$$

and the consequent relations

$$A+B = J(0) = \fint_k^\infty \frac{d\tau}{\sqrt{\tau^2-k^2}\,(\tau^2-\gamma^2)} \tag{15}$$

$$\gamma(A-B) = -iJ'(0) = \fint_k^\infty \frac{\tau d\tau}{\sqrt{\tau^2-k^2}\,(\tau^2-\gamma^2)} \tag{16}$$

determine both the constants A,B . Inasmuch as the principal value integral (16) vanishes, $A = B$, and thus

$$A = B = \frac{1}{2} \int_k^\infty \frac{d\tau}{\sqrt{\tau^2-k^2}\,(\tau^2-\gamma^2)} = -\frac{1}{4\gamma\mu} \log \frac{\gamma+\mu}{\gamma-\mu} = -\frac{1}{2\gamma\mu} \sinh^{-1}(\frac{\mu}{k}) \tag{17}$$

whence

$$J(x) = -\frac{1}{2\gamma\mu} \log \frac{\gamma+\mu}{\gamma-\mu} \cos \gamma x - \frac{\pi i}{2\gamma} \int_0^x \sin \gamma(x-x')H_0^{(1)}(kx')\, dx' \ . \tag{18}$$

Combining (10), (11), and (18) supplies an alternative representation of the function defined in (8),

$$I(x) = \pi i\, H_0^{(1)}(kx) - \frac{\mu}{\gamma} \log \frac{\gamma+\mu}{\gamma-\mu} \cos \gamma x - \pi i \frac{\mu}{\gamma} e^{i\gamma x} - \pi i \frac{\mu^2}{\gamma} \int_0^x \sin \gamma(x-x')H_0^{(1)}(kx')\, dx' \ , \quad x > 0 \qquad \gamma = \sqrt{k^2+\mu^2} \tag{19}$$

which lends itself readily to series expansion for small x. The specific estimate (9) furnished by Crighton emerges from (19) once the approximations

$$H_0^{(1)}(kx) \doteq 1 + \frac{2i}{\pi} \log \frac{kx}{2} + \frac{2i}{\pi} C$$

$$\cos \gamma x \doteq 1 \quad , \quad e^{i\gamma x} \doteq 1 + i\gamma x$$

are employed and the integral (of order $x^2 \log x$) is neglected.

A slightly different procedure for passing between the representations (8) and (19) commences with the resolution

$$I(x) = \int_{-\infty}^{\infty} \frac{\sqrt{\zeta^2-k^2} - \mu}{\zeta^2 - \gamma^2} \exp(-i\zeta x)\, d\zeta = I_1(x) - \mu I_2(x) + \mu^2 I_3(x) \tag{20}$$

where

$$I_1(x) = \int_{-\infty}^{\infty} \frac{e^{-i\zeta x}}{\sqrt{\zeta^2-k^2}}\, d\zeta = 2 \int_k^\infty \frac{e^{i\tau x}}{\sqrt{\tau^2-k^2}}\, d\tau = \pi i H_0^{(1)}(kx) \tag{21}$$

$$x > 0$$

$$I_2(x) = \int_{-\infty}^{\infty} \frac{e^{-i\zeta x}}{\zeta^2 - \gamma^2}\, d\zeta = -\pi i \frac{e^{i\gamma x}}{\gamma} \tag{22}$$

and

$$I_3(x) = \int_{-\infty}^{\infty} \frac{\exp(-i\zeta x)}{\sqrt{\zeta^2-k^2}\,(\zeta^2-\gamma^2)}\, d\zeta \ . \tag{23}$$

Since

$$\left(\frac{d^2}{dx^2}+\gamma^2\right) I_3(x) = -\pi i\ H_0^{(1)}(kx)$$

and

$$I_3(x) = \alpha e^{i\gamma x} + \beta e^{-i\gamma x} - \frac{\pi i}{\gamma} \int_0^x \sin \gamma(x-x') H_0^{(1)}(kx')\ dx' \tag{24}$$

it follows that

$$\alpha+\beta = I_3(0) = \int_{-\infty}^{\infty} \frac{d\zeta}{\sqrt{\zeta^2-k^2}\,(\zeta^2-\gamma^2)} \tag{25}$$

and

$$\gamma(\alpha-\beta) = -iI_3'(0) = -\int_{-\infty}^{\infty} \frac{\zeta d\zeta}{\sqrt{\zeta^2-k^2}\,(\zeta^2-\gamma^2)} = 0 \ ,$$

the null value of the last integral being a consequence of odd symmetry for the integrand. On deforming the contour of the integral (25) around the branch cut extending from k to ∞ and observing the cancellation of half-residues from semi-circular arcs traced in the same sense above and below the point $\zeta = \gamma$ thereupon, the connection

$$\int_{-\infty}^{\infty} \frac{d\zeta}{\sqrt{\zeta^2-k^2}\,(\zeta^2-\gamma^2)} = 2 ⨍_k^{\infty} \frac{d\tau}{\sqrt{\tau^2-k^2}\,(\tau^2-\gamma^2)} \tag{26}$$

results; hence (15) and (25) jointly indicate that $\alpha = \beta = 2A = 2B$ and the identity of (24) and (19) follows.

It is appropriate to record, after noting the relations

$$\int_0^{\infty} \sin \gamma x\ H_0^{(1)}(kx)\ dx = \frac{1}{\mu} + \frac{2i}{\pi\mu} \log\left(\frac{\gamma-\mu}{k}\right)$$

and

$$\int_0^\infty \cos \gamma x \, H_0^{(1)}(kx) \, dx = -\frac{i}{\mu} \qquad \gamma > k$$

a complementary form of the representation (19), namely

$$I(x) = \pi i \, H_0^{(1)}(kx) + \pi i \frac{\mu^2}{\gamma} \int_x^\infty \sin \gamma(x-x') H_0^{(1)}(kx') \, dx' \tag{27}$$

which permits a systematic approximation for $I(x)$ at large values of x .

Upon rewriting the response integral (6), that incorporates both the effect of plate mass and stiffness, in the fashion adopted by Crighton,

$$v(x) = -\frac{i\omega F_0}{2\pi B} \int_{-\infty}^\infty \left\{ \frac{(\zeta^4 - k_p^4)(\zeta^2 - k^2)}{P(\zeta)} + \mu k_p^4 F(\zeta) \right\} e^{-i\zeta x} \, d\zeta \tag{28}$$

where

$$P(\zeta) = (\zeta^4 - k_p^4)^2 (\zeta^2 - k^2) - \mu^2 k_p^8 = \prod_{n=1}^{5} (\zeta^2 - \zeta_n^2) \tag{29}$$

and

$$F(\zeta) = \frac{\sqrt{\zeta^2 - k^2}}{P(\zeta)} \tag{30}$$

attention focusses on the second contribution thereto, since the first can be immediately evaluated in terms of the residues at the relevant poles of the rational function $1/P(\zeta)$. Further to the use of the partial fraction decomposition

$$\frac{1}{P(\zeta)} = \sum_{n=1}^{5} \frac{2\alpha_n \zeta_n}{\zeta^2 - \zeta_n^2} \,, \quad \frac{1}{\alpha_n} = \left.\frac{dP}{d\zeta}\right|_{\zeta=\zeta_n} \,, \quad \sum_{n=1}^{5} \alpha_n \zeta_n = 0 \tag{31}$$

the function

$$G(x) = \int_{-\infty}^\infty F(\zeta) e^{-i\zeta x} \, d\zeta = \int_{-\infty}^\infty \frac{\sqrt{\zeta^2 - k^2}}{P(\zeta)} \, e^{-i\zeta x} \, d\zeta \tag{32}$$

is linked with previously encountered ones, namely $I_3(x)$ and $J(x)$ [cf. (23) and (12)]; thus

$$G(x) = \sum_{n=1}^{5} 2\alpha_n\zeta_n \int_{-\infty}^{\infty} \frac{\sqrt{\zeta^2-k^2}}{\zeta^2-\zeta_n^2} \exp(-i\zeta x)\, d\zeta$$

$$= 2 \sum_{n=1}^{5} \alpha_n\zeta_n(\zeta_n^2-k^2) \int_{-\infty}^{\infty} \frac{\exp(-i\zeta x)}{\sqrt{\zeta^2-k^2}} \frac{d\zeta}{\zeta^2-\zeta_n^2} . \tag{33}$$

Suppose that the roots ζ_n , $n = 1\text{-}5$, of the polynomial equation $P(\zeta) = 0$ lie above the integration contour in the ζ-plane, while those of opposite sign are located below the contour. Then, if a particular ζ_n is complex-valued, additive and distinct contributions to the pertinent integral in (33) arise from a pole (at $\zeta = \mp\zeta_n$) and a branch line (between $\mp\infty$ and $\mp k$, according as $x \gtrless 0$), viz.

$$\int_{-\infty}^{\infty} \frac{\exp(-i\zeta x)}{\sqrt{\zeta^2-k^2}} \frac{d\zeta}{\zeta^2-\zeta_n^2} = \frac{\pi i}{\zeta_n} \frac{e^{i\zeta_n x}}{\sqrt{\zeta_n^2-k^2}} + 2 \int_{k}^{\infty} \frac{\exp(i\tau x)}{\sqrt{\tau^2-k^2}} \frac{d\tau}{\tau^2-\zeta_n^2} , \quad x > 0 ; \tag{34}$$

and estimates of the latter function for small values of x can be found in a manner described above. The pole contribution is absent if ζ_n is real and larger than k in magnitude, whence

$$\int_{-\infty}^{\infty} \frac{\exp(-i\zeta x)}{\sqrt{\zeta^2-k^2}} \frac{d\zeta}{\zeta^2-\zeta_n^2} = 2 ⨍_{k}^{\infty} \frac{\exp(i\tau x)}{\sqrt{\tau^2-k^2}} \frac{d\tau}{\tau^2-\zeta_n^2}$$

$$= 4 \cos \zeta_n x ⨍_{k}^{\infty} \frac{d\tau}{\sqrt{\tau^2-k^2}} \frac{1}{\tau^2-\zeta_n^2} - \frac{\pi i}{2\zeta_n} \int_0^x \sin \zeta_n(x-x')H_0^{(1)}(kx')\, dx' \qquad , \; x > 0$$

$$= - \frac{\cos \zeta_n x}{\zeta_n\sqrt{\zeta_n^2-k^2}} \log\left(\frac{\zeta_n}{k} + \sqrt{\frac{\zeta_n^2}{k^2} - 1}\right) - \frac{\pi i}{2\zeta_n} \int_0^x \sin \zeta_n(x-x')H_0^{(1)}(kx')\, dx' . \tag{35}$$

2. The Point Source Problem

A corresponding version of the analysis hitherto given is available for dealing with fluid-loaded plate excitation by point forces; in this circumstance the plate velocity resulting from a normal force $F_0\delta(r)e^{-i\omega t}$ is characterized by the Hankel transform integral

$$v(r) = -\frac{i\omega F_0}{2\pi B}\int_0^\infty \frac{\sqrt{\zeta^2-k^2}\; J_0(\zeta r)\zeta d\zeta}{(\zeta^4-k_p^4)\sqrt{\zeta^2-k^2} - \mu k_p^4} \qquad (36)$$

where r denotes the radial distance from the point of application of the force, J_0 identifies the zero-order Bessel function and the other symbols retain their earlier significance. Utilizing the same rearrangement of integrand factors which brings the response integral (6) to the form (28), it now follows that

$$v(r) = \frac{\omega F_0}{2B}\sum_{n=1}^{5} \alpha_n\zeta_n(\zeta_n^4-k_p^4)(\zeta_n^2-k^2)H_0^{(1)}(\zeta_n r) - \frac{i\omega F_0}{2\pi B}\int_0^\infty F(\zeta)J_0(\zeta r)\zeta d\zeta \qquad (37)$$

where the sum contribution, involving the Hankel function $H_0^{(1)}$, arises from the pole singularities of the function $1/P(\zeta)$ defined in (31) (namely, at the points ζ_n, $n = 1$-5, in the upper half of the ζ-plane) and the integral term involves the function $F(\zeta)$ defined by (30), (31). The Hankel transform of $F(\zeta)$, namely

$$G(r) = \int_0^\infty F(\zeta)J_0(\zeta r)\zeta d\zeta \qquad (38)$$

is next rewritten [compare (32)-(33)] as

$$G(r) = \sum_{n=1}^{5} 2\alpha_n\zeta_n \int_0^\infty \frac{\sqrt{\zeta^2-k^2}}{\zeta^2 - \zeta_n^2} J_0(\zeta r)\zeta d\zeta$$

$$= 2 \sum_{n=1}^{5} \alpha_n\zeta_n(\zeta_n^2-k^2) \int_0^\infty \frac{J_0(\zeta r)}{\sqrt{\zeta^2-k^2}} \frac{\zeta d\zeta}{\zeta^2 - \zeta_n^2} \tag{39}$$

whence the point force admittance, or ratio $v(0)/F_0$, acquires the representation

$$v(0)/F_0 = \frac{i\omega}{\pi B} \left\{ \sum_{n=1}^{5} \alpha_n\zeta_n \log \zeta_n \; (\zeta_n^4-k_p^4)(\zeta_n^2-k^2) - \sum_{n=1}^{5} \alpha_n\zeta_n(\zeta_n^2-k^2) \int_0^\infty \frac{\zeta d\zeta}{\sqrt{\zeta^2-k^2}} \frac{1}{\zeta^2 - \zeta_n^2} \right\} . \tag{40}$$

After subdividing the range of integration and employing suitable changes of variable it turns out that

$$\int_0^\infty \frac{\zeta d\zeta}{\sqrt{\zeta^2-k^2}} \frac{1}{\zeta^2 - \zeta_n^2} = \int_k^\infty \frac{\zeta d\zeta}{\sqrt{\zeta^2-k^2}} \frac{1}{\zeta^2 - \zeta_n^2} + i \int_0^k \frac{\zeta d\zeta}{\sqrt{k^2-\zeta^2}} \frac{1}{\zeta^2 - \zeta_n^2}$$

$$= \int_0^\infty \frac{d\tau}{\tau^2 - \zeta_n^2 + k^2} - i \int_0^k \frac{d\tau}{\tau^2 + \zeta_n^2 - k^2} \tag{41}$$

and thus the individual integrals of (40) are readily evaluated in accordance with the specifications (real, complex) of the ζ_n ; if ζ_n is real and greater than k , the first (a principal value integral) vanishes and the second equals

$$-\frac{1}{2} (\zeta_n^2-k^2)^{-1/2} \log\left(\frac{\sqrt{\zeta_n^2-k^2} + ik}{\sqrt{\zeta_n^2-k^2} - ik}\right) .$$

A knowledge of the function

$$H(r) = \int_0^\infty \frac{J_0(\zeta r)}{\sqrt{\zeta^2-k^2}} \frac{\zeta d\zeta}{\zeta^2 - \zeta_n^2} \tag{42}$$

(which has the particular value (41) at $r = 0$) is a prerequisite to the eventual determination of $v(r)$ from (37) - (39); and, inasmuch as

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \zeta_n^2\right)H(r) = -\int_0^\infty \frac{J_0(\zeta r)\zeta d\zeta}{\sqrt{\zeta^2-k^2}} = -\frac{e^{ikr}}{r} \tag{43}$$

there follows a representation,

$$H(r) = AJ_0(\zeta_n r) + BN_0(\zeta_n r) + \frac{\pi}{2} J_0(\zeta_n r) \int_0^r N_0(\zeta_n r')e^{ikr'}\, dr' - \frac{\pi}{2} N_0(\zeta_n r) \int_0^r J_0(\zeta_n r')e^{ikr'}\, dr' , \tag{44}$$

which involves both the zero order Bessel and Neumann functions and is rendered precise by the assignments

$$A = H(0) \quad , \quad B = 0 \, . \tag{45}$$

Since finite range integrals occur in (44) the latter permits ready estimates for $H(r)$ at small magnitudes of r .

The point admittance and velocity profile for a fluid loaded plate have, as a matter of fact, been determined by Filippi and Saadat (1972), independently of Crighton, using a straightforward though lengthier transform/inversion procedure; none of these authors, however, consider distributed force plate excitations, whose analysis is taken up next.

3. Distributed Source Problems

If the thin plate adjacent to an infinite expanse of fluid, is subject to a uniform normal pressure, with the magnitude $P_0 e^{-i\omega t}$, over a strip domain $|x| < \ell$, $-\infty < z < \infty$ its velocity factor possesses the one-dimensional Fourier integral representation

$$v(x) = -\frac{i\omega P_0}{\pi B} \int_{-\infty}^{\infty} \frac{\sqrt{\zeta^2-k^2}\,\exp(-i\zeta x)}{(\zeta^4-k_p^4)\sqrt{\zeta^2-k^2}-\mu k_p^4} \frac{\sin \zeta \ell}{\zeta} d\zeta , \tag{45}$$

which reverts to (6) in the limits $\ell \to 0$, $2P_0\ell \to F_0$. On neglecting the bending stiffness at first, a simplified version of this integral,

$$\begin{aligned} v(x) &= \frac{iP_0}{\pi m\omega} \int_{-\infty}^{\infty} \frac{\sqrt{\zeta^2-k^2}\,\exp(-i\zeta x)}{\sqrt{\zeta^2-k^2}+\mu} \frac{\sin \zeta\ell}{\zeta} d\zeta \\ &= \frac{iP_0}{\pi m\omega} \int_{-\infty}^{\infty} \cos \zeta x \frac{\sin \zeta\ell}{\zeta} d\zeta - \frac{i\mu P_0}{\pi m\omega} \int_{-\infty}^{\infty} \frac{\exp(-i\zeta x)}{\sqrt{\zeta^2-k^2}+\mu} \frac{\sin \zeta\ell}{\zeta} d\zeta \\ &= \frac{iP_0}{2m\omega} [\mathrm{sgn}(\ell+x)+\mathrm{sgn}(\ell-x)] - \frac{i\mu P_0}{\pi m\omega} L(x) \end{aligned} \tag{46}$$

places in evidence the (even) function of x ,

$$L(x) = \int_{-\infty}^{\infty} \frac{\exp(-i\zeta x)}{\sqrt{\zeta^2-k^2}+\mu} \frac{\sin \zeta\ell}{\zeta} d\zeta \tag{47}$$

whose counterpart, $I(x)$, is defined in (8). Pursuant to a resolution of the sine function into exponentials, there obtains

$$L(x) = \frac{1}{2i} \int_{-\infty}^{\infty} \frac{1-e^{-i\zeta\ell} - (1-e^{i\zeta\ell})}{\zeta} \frac{\exp(-i\zeta x)\, d\zeta}{\sqrt{\zeta^2-k^2}+\mu} = L_1(x) + L_2(x) + L_3(x) \tag{48}$$

with the component parts

$$\begin{aligned} L_1(x) &= \frac{1}{2i} \int_{-\infty}^{\infty} \frac{e^{-i\zeta x} - e^{-i\zeta(x+\ell)}}{\zeta} \frac{d\zeta}{\sqrt{\zeta^2-k^2}+\mu} \\ L_2(x) &= \frac{1}{2i} \int_{-\infty}^{\infty} \frac{1 - e^{-i\zeta x}}{\zeta} \frac{d\zeta}{\sqrt{\zeta^2-k^2}+\mu} \end{aligned} \tag{49}$$

and

$$L_3(x) = -\frac{1}{2i} \int_{-\infty}^{\infty} \frac{1-e^{-i\zeta(\ell-x)}}{\zeta} \frac{d\zeta}{\sqrt{\zeta^2-k^2}+\mu} = \frac{1}{2i} \int_{-\infty}^{\infty} \frac{1-e^{-i\zeta(\ell-x)}}{\zeta} \frac{d\zeta}{\sqrt{\zeta^2-k^2}+\mu} .$$

When $0 < x < \ell$ all the latter can be inferred from the single function

$$F(\alpha,\beta) = \int_{-\infty}^{\infty} \frac{e^{-i\alpha\zeta} - e^{-i\beta\zeta}}{\zeta} \frac{d\zeta}{\sqrt{\zeta^2-k^2} + \mu} \quad , \quad \alpha,\beta > 0 \tag{50}$$

which, on adopting the transformation scheme of §2, is expressible in the form

$$F(\alpha,\beta) = 2 \int_k^{\infty} \frac{d\tau}{\sqrt{\tau^2-k^2}} \left(\frac{e^{i\beta\tau}-e^{i\alpha\tau}}{\tau}\right) + 2\mu^2 \;⨍_k^{\infty} \frac{d\tau}{\sqrt{\tau^2-k^2}} \frac{1}{\tau^2-\gamma^2} \left(\frac{e^{i\beta\tau}-e^{i\alpha\tau}}{\tau}\right)$$

$$+ \pi i \frac{\mu}{\gamma^2} (e^{i\alpha\gamma}-e^{i\beta\gamma}) \; . \tag{51}$$

Let

$$P(\alpha) = \int_k^{\infty} \frac{d\tau}{\sqrt{\tau^2-k^2}} \frac{e^{i\alpha\tau}}{\tau} \tag{52}$$

and

$$\frac{dP}{d\alpha} = - \frac{\pi}{2} H_0^{(1)}(k\alpha) \quad ,$$

whence

$$P(\alpha) = P(0) - \frac{\pi}{2} \int_0^{\alpha} H_0^{(1)}(k\alpha')d\alpha' = \frac{\pi}{2k} [1 - \int_0^{k\alpha} H_0^{(1)}(\xi) \; d\xi] \; . \tag{53}$$

Write

$$Q(\alpha) = ⨍_k^{\infty} \frac{d\tau}{\sqrt{\tau^2-k^2}} \frac{1}{\tau^2-\gamma^2} \frac{e^{i\alpha\tau}}{\tau} \tag{54}$$

and (cf. (12), (18))

$$\frac{dQ}{d\alpha} = - \frac{i}{2\gamma\mu} \log \frac{\gamma+\mu}{\gamma-\mu} \cos \gamma\alpha + \frac{\mu}{2\gamma} \left\{ \sin \gamma\alpha \int_0^{\alpha} \cos \gamma x' H_0^{(1)}(kx') \; dx' \right.$$

$$\left. - \cos \gamma\alpha \int_0^{\alpha} \sin \gamma x' H_0^{(1)}(kx') \; dx' \right\}$$

so that

$$Q(\alpha) = Q(0) - \frac{i}{2\gamma^2\mu} \log \frac{\gamma+\mu}{\gamma-\mu} \sin \gamma\alpha + \frac{\pi}{2k\gamma^2} \int_0^{k\alpha} H_0^{(1)}(\xi) \; d\xi$$

$$- \frac{\pi}{2k\gamma^2} \int_0^{k\alpha} \cos \gamma(\alpha - \frac{\xi}{k}) H_0^{(1)}(\xi) \; d\xi \; . \tag{55}$$

On the basis of (51) - (55) a finite integral characterization of $F(\alpha,\beta)$ is reached, viz.

$$F(\alpha,\beta) = -\frac{\pi k}{\gamma^2}\int_{k\alpha}^{k\beta} H_0^{(1)}(\xi)d\xi + \pi i\,\frac{\mu}{\gamma^2}(e^{i\alpha\gamma}-e^{i\beta\gamma}) - i\,\frac{\mu}{\gamma^2}\log\frac{\gamma+\mu}{\gamma-\mu}(\sin\gamma\beta-\sin\gamma\alpha)$$
$$-\pi\,\frac{\mu^2}{k\gamma^2}\left[\int_0^{k\beta}\cos\gamma(\beta-\frac{\xi}{k})H_0^{(1)}(\xi)\,d\xi - \int_0^{k\alpha}\cos\gamma(\alpha-\frac{\xi}{k})H_0^{(1)}(\xi)\,d\xi\right] \tag{56}$$

and thence follows for the bounded velocity $v(0)$ along the mid-line of the strip

$$v(0) = \frac{iP_0}{m\omega} - \frac{i\mu P_0 L(0)}{\pi m\omega} = \frac{iP_0}{m\omega} - \frac{\mu P_0}{\pi m\omega}F(0,\ell)$$
$$= \frac{iP_0}{m\omega} + \frac{\mu P_0 k}{m\omega\gamma^2}\int_0^{k\ell} H_0^{(1)}(\xi)\,d\xi - \frac{i\mu^2 P_0}{m\omega\gamma^2}(1-e^{i\gamma\ell}) + \frac{i\mu^2 P_0}{\pi m\omega\gamma^2}\log\frac{\gamma+\mu}{\gamma-\mu}\sin\gamma\ell$$
$$+\frac{\mu^3 P_0}{mk\gamma^2\omega}\int_0^{k\ell}\cos\gamma(\ell-\frac{\xi}{k})H_0^{(1)}(\xi)\,d\xi\;. \tag{57}$$

It is also possible to express, in analogous fashion, the mean velocity over the width of the strip,

$$\frac{1}{2\ell}\int_{-\ell}^{\ell} v(x)\,dx = \frac{iP_0}{m\omega} - \frac{i\mu P_0}{2\pi\ell m\omega}\int_{-\ell}^{\ell} L(x)\,dx$$
$$= \frac{iP_0}{m\omega} - \frac{i\mu P_0}{\pi\ell m\omega}\int_{-\ell}^{\ell}\left(\frac{\sin\zeta\ell}{\zeta}\right)^2\frac{d\zeta}{\sqrt{\zeta^2-k^2}+\mu}\,, \tag{58}$$

having regard for the result

$$\int_{-\infty}^{\infty}\left(\frac{\sin\zeta\ell}{\zeta}\right)^2\frac{d\zeta}{\sqrt{\zeta^2-k^2}+\mu} = i\pi\,\frac{k}{\gamma^2}\left\{\ell\int_0^{2k\ell} H_0^{(1)}(\xi)\,d\xi - 2k\ell H^{(1)}(2k\ell)\right\} + \pi\frac{\mu\ell}{\gamma^2}$$
$$-i\,\frac{\pi\mu}{2\gamma^3}(1-e^{2i\gamma\ell}) - \frac{\mu}{2\gamma^3}\log\frac{\gamma+\mu}{\gamma-\mu}(1-\cos 2\gamma\ell) \tag{59}$$
$$+\,i\pi\,\frac{\mu^2}{2k\gamma^3}\int_0^{2k\ell}\sin\gamma(2\ell-\frac{\xi}{k})H_0^{(1)}(\xi)\,d\xi$$

and this enables a characterization of the distributed force admittance/impedance of the plate.

For points on the plate external to the strip, say $x > \ell$, the relation

$$L(x) = \frac{1}{2i} F(x-\ell, x+\ell) \tag{60}$$

obtains and, applying (56), this yields

$$\begin{aligned} L(x) = \frac{i\pi k}{2\gamma^2} \int_{k(x-\ell)}^{k(x+\ell)} H_0^{(1)}(\xi)\, d\xi + \frac{\pi\mu}{2\gamma^2} (e^{i\gamma(x-\ell)} - e^{i\gamma(x+\ell)}) \\ - \frac{\mu}{2\gamma^2} \log \frac{\gamma+\mu}{\gamma-\mu} (\sin \gamma(x+\ell) - \sin \gamma(x-\ell)) \\ + i\pi \frac{\mu^2}{2k\gamma^2} \{ \int_0^{k(x+\ell)} \cos \gamma(x+\ell - \frac{\xi}{k}) H_0^{(1)}(\xi) d\xi - \int_0^{k(x-\ell)} \cos \gamma(x-\ell - \frac{\xi}{k}) H_0^{(1)}(\xi) d\xi \} . \end{aligned} \tag{61}$$

In the limit $\ell \to 0$,

$$\begin{aligned} L(x) &\doteq \ell (\frac{\partial}{\partial \ell} L)_{\ell=0} + O(\ell^2) \\ &= \ell \{ i \frac{\pi k^2}{\gamma} H_0^{(1)}(kx) - i \frac{\pi\mu}{\gamma} e^{i\gamma x} - \frac{\mu}{\gamma} \log \frac{\gamma+\mu}{\gamma-\mu} \cos \gamma x + i \frac{\pi\mu^2}{\gamma^2} H_0^{(1)}(kx) \\ &\qquad - i \frac{\pi\mu^2}{k\gamma} \int_0^{kx} \sin \gamma(x - \frac{\xi}{k}) H_0^{(1)}(\xi)\, d\xi \} \\ &= \ell I(x) + O(\ell^2) \end{aligned}$$

and the recovery of the line force expressions (7), (8) from (46) is immediate.

When the applied pressure acts on a circular area (with radius = a) the velocity distribution corresponding to (46) takes the form

$$v(r) = \frac{iaP_0}{m\omega} \int_0^\infty \frac{\sqrt{\zeta^2-k^2}}{\sqrt{\zeta^2-k^2}+\mu} \frac{J_1(\zeta a)}{\zeta} J_0(\zeta r)\zeta d\zeta$$

$$= \frac{iaP_0}{m\omega} \int_0^\infty J_0(\zeta r)J_1(\zeta a)\, d\zeta - \frac{ia\mu P_0}{m\omega} \int_0^\infty \frac{J_0(\zeta r)J_1(\zeta a)}{\sqrt{\zeta^2-k^2}+\mu} d\zeta \qquad (62)$$

as a function of the radial distance from the center of this domain. The simplification of (62) commences by noting that

$$\int_0^\infty J_0(\zeta r)J_1(\zeta a)\, d\zeta = \begin{cases} 0 , & 0 < a < r \\ 1/2a , & 0 < a = r \\ 1/a , & a > r \end{cases} \qquad (63)$$

and adopting the rearrangement

$$\int_0^\infty \frac{J_0(\zeta r)J_1(\zeta a)}{\sqrt{\zeta^2-k^2}+\mu} d\zeta = \int_0^\infty \frac{J_0(\zeta r)J_1(\zeta a)}{\sqrt{\zeta^2-k^2}} d\zeta + \mu^2 \int_0^\infty \frac{J_0(\zeta r)J_1(\zeta a)}{\sqrt{\zeta^2-k^2}(\zeta^2-\gamma^2)} d\zeta - \mu \int_0^\infty \frac{J_0(\zeta r)J_1(\zeta a)}{\zeta^2-\gamma^2} d\zeta$$

(64)

for the second term. Let

$$M(r) = \int_0^\infty \frac{J_0(\zeta r)J_1(\zeta a)}{\sqrt{\zeta^2-k^2}} d\zeta = \frac{1}{2\pi a} \int_0^a r'dr' \int_0^{2\pi} d\psi \frac{\exp(ik\sqrt{r^2+r'^2-2rr'\cos\psi})}{\sqrt{r^2+r'^2-2rr'\cos\psi}}$$

(65)

and define

$$N(r) = \int_0^\infty \frac{J_0(\zeta r)J_1(\zeta a)}{\sqrt{\zeta^2-k^2}\,(\zeta^2-\gamma^2)} d\zeta \qquad (66)$$

whence the relation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \gamma^2\right)N(r) = -M(r)$$

follows, with a consequent representation

$$N(r) = AJ_0(\gamma r) + \frac{\pi}{2} J_0(\gamma r) \int_0^r r'N_0(\gamma r')M(r')\, dr' - \frac{\pi}{2} N_0(\gamma r) \int_0^r r'J_0(\gamma r')M(r')dr' \tag{67}$$

wherein

$$A = N(0) = \int_0^\infty \frac{J_1(\zeta a)}{\sqrt{\zeta^2-k^2}} \frac{d\zeta}{\zeta^2-\gamma^2} . \tag{68}$$

Since

$$\left(\frac{d^2}{da^2} + \frac{1}{a}\frac{d}{da} + \gamma^2 - \frac{1}{a^2}\right)A = - \int_0^\infty \frac{J_1(\zeta a)}{\sqrt{\zeta^2-k^2}} d\zeta = \frac{i}{ka} (e^{ika}-1)$$

it turns out that

$$A = \alpha J_1(\gamma a) - \frac{i\pi}{2k} J_1(\gamma a) \int_0^a N_1(\gamma \tau)(e^{ik\tau}-1)\, d\tau + \frac{i\pi}{2k} N_1(\gamma a) \int_0^a J_1(\gamma\tau)(e^{ik\tau}-1)\, d\tau$$

$$= \alpha J_1(\gamma a) - \frac{i}{ka\gamma^2} (1-e^{ika}) + \frac{\pi}{2\gamma} J_1(\gamma a) \int_0^a N_0(\gamma\tau)e^{ik\tau}\, d\tau$$

$$- \frac{\pi}{2\gamma} N_1(\gamma a) \int_0^a J_0(\gamma\tau)e^{ik\tau}\, d\tau \tag{69}$$

after invoking some Bessel function properties. A vanishes, independently of α, when $a = 0$ and

$$\alpha \frac{\gamma}{2} = \left(\frac{dA}{da}\right)_{a=0} = \frac{1}{2} \int_0^\infty \frac{\zeta d\zeta}{\sqrt{\zeta^2-k^2}} \frac{1}{\zeta^2-\gamma^2}$$

so that

$$\alpha = \frac{1}{\gamma} \left\{ ⨍_0^\infty \frac{d\tau}{\tau^2-\mu^2} - i \int_0^k \frac{d\tau}{\tau^2+\mu^2} \right\} = - \frac{i}{\gamma\mu} \tan^{-1} \frac{k}{\mu} \tag{70}$$

and the determination of A is complete.

The third and last integral in (64), say

$$P(r) = \int_0^\infty \frac{J_0(\zeta r)J_1(\zeta a)}{\zeta^2-\gamma^2} d\zeta , \tag{71}$$

can also be recast by means of its differential equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \gamma^2\right)P(r) = - \int_0^\infty J_0(\zeta r)J_1(\zeta a)\, d\zeta = - \frac{1}{a} , \quad a > r \tag{72}$$

and this yields

$$P(r) = \beta J_0(\gamma r) + \frac{\pi}{2a} J_0(\gamma r) \int_0^r r' N_0(\gamma r')\, dr' - \frac{\pi}{2a} N_0(\gamma r) \int_0^r r' J_0\, (\gamma r')\, dr'$$

$$= \beta J_0(\gamma r) - \frac{1}{a\gamma^2}$$

or

$$P(r) = P(0) J_0(\gamma r) - \frac{1}{a\gamma^2} (1 - J_0(\gamma r)) \tag{73}$$

where

$$P(0) = \int_0^\infty \frac{J_1(\zeta a)\, d\zeta}{\zeta^2 - \gamma^2} = \frac{1}{2\gamma} \int_0^\infty J_1(\gamma a)\, [\frac{1}{\zeta - \gamma} - \frac{1}{\zeta + \gamma}]\, d\zeta \tag{74}$$

and a principal value is understood. Differentiation with respect to a in the formulas (cf. Luke (1962))

$$\int_0^\infty \frac{J_0(\zeta a)\, d\zeta}{\zeta + \gamma} = \frac{\pi}{2} [H_0(\gamma a) - N_0(\gamma a)]$$

and (75)

$$⨍_0^\infty \frac{J_0(\zeta a)\, d\zeta}{\gamma - \zeta} = \frac{\pi}{2} [H_0(\gamma a) + N_0(\gamma a)] ,$$

where H_0 denotes the Struve function, yields

$$\int_0^\infty \frac{J_1(\zeta a)}{\zeta + \gamma}\, d\zeta = \frac{1}{\gamma a} + \frac{\pi}{2} [H_0'(\gamma a) - N_0'(\gamma a)]$$

and (76)

$$⨍_0^\infty \frac{J_1(\zeta a)}{\zeta - \gamma}\, d\zeta = \frac{-1}{\gamma a} + \frac{\pi}{2} [H_0'(\gamma a) + N_0'(\gamma a)]$$

(with primes indicating an argument derivative), whence

$$P(r) = -\frac{1}{a\gamma^2} - \frac{\pi}{2\gamma} N_1(\gamma a) J_0(\gamma r) , \quad r < a . \tag{77}$$

Collectively, (65), (67), (69), (70), (71), and (77) provide an efficient basis for estimating the last integral in (64)--and thus the normal velocity $v(r)$ --within the area directly subjected to external surface pressure; and they furnish, in particular, an expression

$$v(0) = \frac{iP_0}{m\omega} - \frac{ia\mu P_0}{m\omega}\{\frac{1}{ika}(e^{ika}-1) + \mu^2 A + \frac{\mu}{a\gamma^2} + \frac{\mu\pi}{2\gamma}N_1(\gamma a)\}$$

$$= \frac{iP_0}{m\omega}\{1 + i\frac{k\mu}{\gamma^2}(e^{ika}-1) + i\frac{a\mu^2}{\gamma}\tan^{-1}(\frac{k}{\mu})J_1(\gamma a) - \frac{\mu^2}{\gamma^2} - \frac{\pi a\mu^2}{2\gamma^2}N_1(\gamma a) \quad (78)$$

$$- \frac{\pi a\mu^3}{2\gamma k}J_1(\gamma a)\int_0^{ka} N_0(\frac{\gamma}{k}\xi)e^{i\xi}d\xi + \frac{\pi a\mu^3}{2\gamma k}N_1(\gamma a)\int_0^{ka} J_0(\frac{\gamma}{k}\xi)e^{i\xi}d\xi$$

for the velocity at the center thereof. The mean value of $v(r)$ on this area

$$\bar{v} = \frac{2}{a^2}\int_0^a rv(r)\,dr$$

$$= \frac{iP_0}{m\omega} - \frac{2i\mu P_0}{m\omega}\int_0^\infty \frac{J_1^2(\zeta a)}{\zeta}\frac{d\zeta}{\sqrt{\zeta^2-k^2}+\mu} \quad (79)$$

can be related to previous integrals whose display in convenient forms is available; and the link appears after a sequence of manipulations

$$\int_0^\infty \frac{J_1^2(\zeta a)}{\zeta}\frac{d\zeta}{\sqrt{\zeta^2-k^2}+\mu} = \frac{2}{\pi}\int_0^{\pi/2} d\theta \int_0^\infty \frac{J_2(2\zeta a\sin\theta)}{\zeta}\frac{d\zeta}{\sqrt{\zeta^2-k^2}+\mu}$$

$$= \frac{2}{\pi}\int_0^{\pi/2} d\theta \int_0^\infty \left\{-\frac{1}{2a\sin\theta}\frac{d}{d\zeta}\left(\frac{J_1(2\zeta a\sin\theta)}{\zeta}\right)\frac{1}{\sqrt{\zeta^2-k^2}+\mu}\right\}d\zeta$$

$$= \frac{2}{\pi}\int_0^{\pi/2} d\theta\left\{\frac{1}{2(\mu-ik)} - \frac{1}{2a\sin\theta}\int_0^\infty \frac{J_1(2\zeta a\sin\theta)}{\sqrt{\zeta^2-k^2}}\frac{d\zeta}{(\sqrt{\zeta^2-k^2}+\mu)^2}\right\}$$

$$= \frac{1}{2}\frac{1}{\mu-ik} - \frac{1}{\pi a}\int_0^{\pi/2}\frac{d\theta}{\sin\theta}\int_0^\infty \frac{J_1(2\zeta a\sin\theta)}{\sqrt{\zeta^2-k^2}}\frac{\zeta^2-\gamma^2+2\mu^2-2\mu\sqrt{\zeta^2-k^2}}{(\zeta^2-\gamma^2)^2}d\zeta$$

$$= \frac{1}{2}\frac{1}{\mu-ik} - \frac{1}{\pi a}\int_0^{\pi/2}\frac{d\theta}{\sin\theta}\left\{\int_0^\infty \frac{J_1(2\zeta a\sin\theta)}{\sqrt{\zeta^2-k^2}}\frac{d\zeta}{\zeta^2-\gamma^2}\right.$$

$$\left. + \frac{\mu^2}{\gamma}\frac{d}{d\gamma}\int_0^\infty \frac{J_1(2\zeta a\sin\theta)}{\sqrt{\zeta^2-k^2}}\frac{d\zeta}{\zeta^2-\gamma^2} - \frac{\mu}{\gamma}\frac{d}{d\gamma}\int_0^\infty \frac{J_1(2\zeta a\sin\theta)}{\zeta^2-\gamma^2}d\zeta\right\} \quad (80)$$

which turns up representatives of the integrals defined in (68) and (71).

There is, likewise, sufficient information at hand to obtain forthwith the corresponding measures of the plate velocity when bending stiffness plays a role. The integrals

$$\int_{-\infty}^{\infty} F(\zeta) \frac{\sin \zeta\ell}{\zeta} e^{-i\zeta x} d\zeta = 2 \sum_{n=1}^{5} \alpha_n \zeta_n (\zeta_n^2 - k^2) \int_{-\infty}^{\infty} \frac{\sin \zeta\ell}{\zeta} \frac{e^{-i\zeta x} d\zeta}{\sqrt{\zeta^2 - k^2}\,(\zeta^2 - \zeta_n^2)}$$

and

$$\int_{0}^{\infty} F(\zeta) \frac{J_1(\zeta a)}{\zeta} J_0(\zeta r)\zeta d\zeta = 2 \sum_{n=1}^{5} \alpha_n \zeta_n (\zeta_n^2 - k^2) \int_{0}^{\infty} \frac{J_0(\zeta r) J_1(\zeta a)\, d\zeta}{\sqrt{\zeta^2 - k^2}\,(\zeta^2 - \zeta_n^2)}$$

in particular, make an appearance and the above presentations reveal that functions of the type (54) and (66) are involved.

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