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AN ALGORITHM FOR THE RAPID LOCATION OF AN EXTREMUM OF A FUNCTION SUBJECT ONLY TO GEOMETRIC RESTRICTIONS

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ABSTRACT

If a function of a single variable is convex and symmetric in a neighborhood of an extremum, the extremum may be approximated to a precision that increases by at least a power of two per functional evaluation. This procedure may be used to drive a complex optimization procedure (such as the Davidon-Fletcher-Powell) in the kind of multivariate area estimation problem encountered in remote sensing.

Key words: Optimization, convex functions
TECHNICAL MEMORANDUM

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OF AN EXTREMUM OF A FUNCTION SUBJECT
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by

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1. INTRODUCTION

Let \( f(X) \) have a minimum on an interval \([X_0, X_2]\) and assume further that \( f \) is convex upward there and symmetric around its minimum. Then we know the following fact about the minimum: (Let \( X_1 = \frac{X_0 + X_2}{2} \)).

Theorem: Assume without loss of generality that \( f(X_0) \leq f(X_2) \). Then \( f \) assumes its minimum at a point between

\[
\frac{1}{2}(X_0 + X_1) + \frac{1}{2} (X_2 - X_1) \frac{f(X_0) - f(X_1)}{f(X_2) - f(X_1)}
\]

and whichever of \( X_0 \) and \( X_1 \) that has smaller functional value \( f(X) \).
2. PROOF OF THEOREM

Case I: \( f(X_1) \leq f(X_0) \leq f(X_2) \).

Let \( X^* \) be such that \( f(X^*) = f(X_0) \) and \( X_1 \leq X^* \leq X_2 \). It exists by symmetry about the minimum and convexity upward. Similarly, by upward convexity \( \{X^*, f(X^*)\} \) is below a segment joining the points \( \{X_1, f(X_1)\} \) and \( \{X_2, f(X_2)\} \) in the graph of \( f \), so

\[
f(X_0) = f(X^*) \leq f(X_1) + \frac{(X^* - X_1)}{(X_2 - X_1)}[f(X_2) - f(X_1)]
\]

So \( X^* \geq X_1 + \frac{f(X_0) - f(X_1)}{f(X_2) - f(X_1)} (X_2 - X_1) \)

By symmetry of \( f \) around its minimum

\[
X_{\text{min}} = \frac{X_0 + X^*}{2}
\]

So

\[
X_{\text{min}} \geq \frac{1}{2} (X_0 + X_1) + \frac{1}{2} (X_2 - X_1) \frac{f(X_0) - f(X_1)}{f(X_2) - f(X_1)}
\]

Now \( X^* \leq X_2 \) implies \( X_{\text{min}} = \frac{X_0 + X^*}{2} \leq \frac{X_0 + X_2}{2} = X_1 \)

So \( X_1 \geq X_{\text{min}} \) and we have case I.

Case II: \( f(X_0) \leq f(X_1) \leq f(X_2) \)

Again, let \( X^* \) be such that \( f(X^*) = f(X_0) \) but \( X_0 \neq X^* \). \( X^* \leq X_1 \) by convexity. Also by upward convexity, \( \{X_1, f(X_1)\} \) is below the segment connecting \( \{X^*, f(X^*)\} \) to \( \{X_2, f(X_2)\} \), so

\[
f(X_1) \leq f(X^*) + \frac{(X_1 - X^*)}{(X_2 - X^*)} (f(X_2) - f(X^*))
\]
and this may be manipulated to

\[ x^* \leq x_1 + (x_2 - x_1) \frac{f(x_0) - f(x_1)}{f(x_2) - f(x_1)} \]

Again by symmetry

\[ x_{\min} = \frac{x_0 + x^*}{2}, \text{ so} \]

\[ x_{\min} \leq \frac{1}{2} [x_0 + x_1] + \frac{1}{2}(x_2 - x_1) \frac{f(x_0) - f(x_1)}{f(x_2) - f(x_1)} \]

But \( x_0 \leq x_{\min} \) by assumption, so we have case II.

The case \( f(x_0) \leq f(x_2) < f(x_1) \) violates convexity upward, so

Q.E.D.

**Corollary:** The new sub-interval containing the minimum of \( f \) is at most one fourth the length of \([x_0, x_2]\).

**Proof:** The computed boundary in the formula is clearly from its formula nearer the other boundary than is

\[ \frac{x_0 + x_1}{2} = 3/4x_0 + 1/4x_2. \]

Q.E.D.
3. APPLICATION

These results become an algorithm for the minimization or maximization of a function meeting or nearly meeting the requirements of symmetry and convexity. This method involves replacing \([X_0, X_2]\) by the new interval at successive iterations. Convergence is at least by powers of one fourth at a cost of two functional evaluations per iteration.