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# A NOTE ON

# SOUND RADIATION INTO A UNIFORMLY FLOWING FLUID



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A NOTE ON
SOUND RADIATION INTO A UNIFORMLY FLOWING FLUID

by

Harold Levine

Introduction and Summary

There exists a considerable body of analysis pertaining to the sound radiation generated by mechanical or vibratory source arrangements and, in some instances of planar or piston types, the results encompass all magnitudes of the wavelength or frequency as well as disposition of the observation site. Fewer details are available in the circumstance of relative motion between the source and its surroundings and, particularly, with an aptness for both the compact and noncompact characterizations of a given source model. The comparative increase in average total radiated power from infinitesimally small or point sources of periodic strength, on passing from rest to steady rectilinear motion, depends only on the Mach number in the latter state; and a formally analogous rise in output is linked, by Ffowcs Williams and Lovely (1975), to the presence of a steady parallel flow past a rigid plane wall in which a compact circular piston executes normal oscillations. It is the intention here to widen the perspective of effects connected with such a background flow by regard for an elongated or strip piston, which prototype permits a straightforward and general analysis.

The time average net power output of a strip piston that vibrates normally to itself with uniform amplitude, has a finite width $\delta$ and is located in an otherwise rigid plane wall, admits different expansions according as

$$k\delta << 1, \quad M < 1, \tag{i}$$

or

$$Mk\delta << 1, \tag{ii}$$

or

$$k\delta >> 1, \quad M < 1 \tag{iii}$$

respectively, where k,M are the wave and Mach number, the last being a measure of the ratio between flow and sound speeds. In the absence of flow, M vanishes and classical (or linear acoustical) theory predicts that the total rate of energy radiation (which is distributed nonuniformly over a semicircular range) amounts to

$$P = \frac{1}{2}\rho_o kc^3 (k\delta)\left\{\int_o^{k\delta} J_o(\nu)d\nu - J_1(k\delta)\right\} \tag{1}$$

$$= \frac{1}{2}\rho_o kc^3 (k\delta)\left\{1 - \int_{k\delta}^{\infty} \frac{J_1(\nu)}{\nu} d\nu\right\} \tag{2}$$

per unit amplitude and length of the piston; here $\rho_o$ specifies the equilibrium density of the medium, c is the sound speed and $J_o$, $J_1$ are Bessel functions with the designated orders. The development which suits a compact piston,

$$P = \frac{1}{4}\rho_o kc^3 (k\delta)^2\left\{1 - \frac{(k\delta)^2}{24} + 0(k\delta)^4\right\}, \quad k\delta \to 0 \tag{3}$$

is a ready consequence of (1), while that obtained from (2),

$$P = \frac{1}{2}\rho_o kc^3 (k\delta)\left\{1 - \sqrt{\frac{2}{\pi}}(k\delta)^{-3/2}\cos\left(k\delta - \frac{\pi}{4}\right) + 0(k\delta)^{-5/2}\right\}, \quad k\delta \to \infty \tag{4}$$

represents the noncompact alternative.

It appears, in what follows, that the development containing powers of the Mach number and conforming with (ii),

$$P = \frac{1}{2}\rho_o kc^3(k\delta)\left\{\int_o^{k\delta} J_o(\nu)d\nu - J_1(k\delta)\right. \qquad Mk\delta << 1$$

$$\left. + M^2\left[\frac{1}{2}k\delta\, J_o(k\delta) + \frac{3}{2}J_1(k\delta) + \frac{1 - J_o(k\delta)}{k\delta}\right] + 0\left(M^4\right)\right\} \quad , \quad (5)$$

extends the prior result (1) and implies that

$$P = \frac{1}{4}\rho_o kc^3(k\delta)^2\left\{1 + 3M^2 + 0\left(M^4,(k\delta)^2\right)\right\} \qquad (6)$$

in the limit $k\delta \to 0$; if $k\delta >> 1$, on the other hand, the expression

$$P = \frac{1}{2}\rho_o kc^3(k\delta)\left\{1 + \frac{M^2}{\sqrt{1 - M^2}}\frac{1}{k\delta} + 0(k\delta)^{-3/2}\right\} , \quad k\delta \to \infty \qquad (7)$$

takes the place of (4). The principal effect of flow in (6) is merely to raise the long wave power estimate by a numerical factor $1 + 3M^2$, whereas the comparison of (4) and (7) discloses that the flow brings about a significant change in the relative order of magnitude of the second term in the short wave estimate, namely from $(k\delta)^{-3/2}$ to $(k\delta)^{-1}$. Inasmuch as the second term of (7) is, in the absolute sense, independent of the piston geometry (and, specifically, of its breadth), a corresponding feature may be presumed for other shapes. A further estimate,

$$P = \frac{1}{4}\rho_o kc^3(k\delta)^2\frac{1 + \frac{1}{2}M^2}{\left(1 - M^2\right)^{5/2}} + 0(k\delta)^4 \quad , \quad k\delta << 1 \quad , \quad 0 < M < 1 \; , \; (8)$$

which accompanies the stipulation (1), has a more extensive validity in respect to the Mach number than does (6).

## The Analysis

Let the undisplaced piston occupy a strip $|x| < a$, $-\infty < y < \infty$, in the fixed plane (or wall) $z = 0$, and assume the existence of a steady flow with uniform speed $U = Mc$ in the parallel (or x-) direction. Given the normal piston displacement (exclusive of a periodic time factor $e^{-i\omega t}$),

$$z = \eta(x) = \int_{-\infty}^{\infty} A(\zeta) e^{i\zeta x} d\zeta \tag{9}$$

where

$$A(\zeta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \eta(x) e^{-i\zeta x} dx = \frac{\sin \zeta a}{\pi \zeta} \tag{10}$$

for the 'top-hat' profile

$$\eta(x) = \begin{cases} 1 , & |x| < a \\ 0 , & |x| > a \end{cases} ; \tag{11}$$

then the corresponding acoustical excitation, derivable from a velocity potential $\phi(x,z)e^{-i\omega t}$, is implicit in the convected wave equation

$$\nabla^2 \phi = \left(-ik + M \frac{\partial}{\partial x}\right)^2 \phi \quad , \quad k = \omega/c$$

or (12)

$$\left(1 - M^2\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} + 2iMk \frac{\partial \phi}{\partial x} + k^2 \phi = 0 \ , \quad z > 0 \ ,$$

together with the boundary or wall condition

$$\frac{\partial \phi}{\partial z}\Big|_{z=0} = -i\omega\eta + U\frac{\partial \eta}{\partial x} \tag{13}$$

and an outgoing wave condition at infinity.

A solution of the linear partial differential equation (12) has the form

$$\phi(x,z) = \int_{-\infty}^{\infty} B(\zeta)\ \exp\{i\zeta x + i\kappa(\zeta)z\}d\zeta \tag{14}$$

where

$$\kappa^2(\zeta) = (k - M\zeta)^2 - \zeta^2 , \qquad \arg \kappa(\zeta) \geq 0 \tag{15}$$

and compliance with the boundary condition (13) necessitates a proportionality

$$B(\zeta) = \frac{\zeta U - \omega}{\kappa(\zeta)} A(\zeta) \tag{16}$$

between the weighting factors $A(\zeta)$, $B(\zeta)$ of the respective integrals (9), (14). The normal derivative of $\phi$, as computed from (14), (15) and (10), proves to be

$$\frac{\partial \phi}{\partial z}\Big|_{z=0} = i\int_{-\infty}^{\infty} B(\zeta)\kappa(\zeta)e^{i\zeta x}d\zeta = -i\int_{-\infty}^{\infty}(\omega - \zeta U)A(\zeta)e^{i\zeta x}d\zeta$$

$$= -\frac{i\omega}{\pi}\int_{-\infty}^{\infty}\frac{\sin \zeta a}{\zeta}e^{i\zeta x}d\zeta + \frac{iU}{\pi}\int_{-\infty}^{\infty}\sin \zeta a\ e^{i\zeta x}d\zeta$$

$$= -\frac{i\omega}{2}\{\mathrm{sgn}(a+x) + \mathrm{sgn}(a-x)\} + U\{\delta(x+a) - \delta(x-a)\} \tag{17}$$

where $\text{sgn}\,\nu = \pm 1$, $\nu \gtrless 0$ and $\delta$ signifies a Dirac delta function; it is thus evident that $\partial\phi/\partial z$ vanishes on the rigid section of wall ($|x| > a$), has a uniform magnitude over the piston ($|x| < a$) and manifests the joint presence of a line sink/source at the leading/trailing edges of the piston ($|x| = \mp a$).

The time average power output of the piston is conveniently found through its rate of local working on the adjacent medium (with an equilibrium density $\rho_o$), viz.

$$P = \frac{1}{2} \rho_o \omega \, \text{Im} \int_{-\infty}^{\infty} \phi^*(x,o) \, \frac{\partial\phi(x,o)}{\partial z} \, dx \tag{18}$$

and, consequent to the use of the representation (14) for $\phi$ and the connection (16), this becomes

$$P = \pi\rho_o\omega \, \text{Re} \int_{-\infty}^{\infty} |B(\zeta)|^2 \kappa(\zeta) d\zeta$$

$$= \pi\rho_o\omega(kc)^2 \, \text{Re} \int_{-\infty}^{\infty} \frac{\left(1 - \frac{M\zeta}{k}\right)^2}{\left\{(k - M\zeta)^2 - \zeta^2\right\}^{1/2}} |A(\zeta)|^2 \, d\tau \quad ; \tag{19}$$

the same result obtains if the transformation of (18) commences with a version

$$P = \frac{1}{2} \rho_o \omega \, \text{Im} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} B^*(\zeta) e^{-i\zeta x} \left\{- \frac{i\omega}{2} [\text{sgn}(a + x) + \text{sgn}(a - x)] \right.$$

$$\left. + U[\delta(x + a) - \delta(x - a)]\right\} d\zeta$$

that incorporates the explicit determination of $\partial\phi/\partial z$ given in (17). On substituting the expression (10) for $A(\zeta)$ and simplifying (19), it appears that

$$P = \frac{\rho_o kc^3}{\pi} \int_{-\frac{1}{1-M}}^{\frac{1}{1-M}} \frac{(1 - M\tau)^2}{\{(1 - M\tau)^2 - \tau^2\}^{1/2}} \frac{\sin^2(ka\tau)}{\tau^2} d\tau$$

$$= \frac{\rho_o kc^3}{\pi\sqrt{1 - M^2}} \int_{-1}^{1} \frac{(1 - M\xi)^2}{\sqrt{1 - \xi^2}} \frac{\sin^2\left(ka \frac{\xi - M}{1 - M^2}\right)}{(\xi - M)^2} d\xi \quad , \quad M < 1$$

in which the change of variable $\tau = (\xi - M)/(1 - M^2)$ figures.

To reduce the latter integral put $\xi = \cos\Theta$ and introduce the abbreviations

$$\alpha = ka \ , \quad \beta = \frac{1}{1 - M^2} \tag{20}$$

whence

$$P = \frac{\rho_o}{\pi} kc^3\beta^{1/2} \left\{F_o(\alpha,M) - 2MF_1(\alpha,M) + M^2F_2(\alpha,M)\right\} \tag{21}$$

with

$$F_n(\alpha,M) = \int_0^{\pi} \cos^n\Theta \frac{\sin^2(\alpha\beta(\cos\Theta - M))}{(\cos\Theta - M)^2} d\Theta, \quad n = 0,1,2 \ . \tag{22}$$

An alternative single integral representation for each of the functions $F_n(\alpha,M)$, which is the more suitable in seeking estimates appropriate to different hypotheses regarding the magnitude of the argument variables, rests on the fact that the second order $\alpha$ - derivatives of the $F_n$ are explicitly known in terms of trigonometric and Bessel functions, viz.

$$\frac{\partial^2 F_o}{\partial \alpha^2} = 2\pi\beta^2 \cos(2\alpha\beta M) J_o(2\alpha\beta) \tag{23}$$

$$\frac{\partial^2 F_1}{\partial \alpha^2} = 2\pi\beta^2 \sin(2\alpha\beta M) J_1(2\alpha\beta) \tag{24}$$

and

$$\frac{\partial^2 F_2}{\partial \alpha^2} = 2\pi\beta^2 \cos(2\alpha\beta M) \left\{J_o(2\alpha\beta) - \frac{1}{2\alpha\beta} J_1(2\alpha\beta)\right\} \quad . \tag{25}$$

The two-stage integration of (23), which serves as a model for that of (23), (24) and is rendered definite by the universal conditions $F_n(0,M) = \frac{\partial F_n(0,M)}{\partial \alpha} = 0$, begins with the expression

$$\frac{\partial F_o}{\partial \alpha} = 2\pi\beta^2 \int_0^\alpha \cos(2\beta M\tau) J_o(2\beta\tau)\, d\tau$$

and carries on via the sequential formulas

$$F_o(\alpha,M) = 2\pi\beta^2 \int_0^\alpha d\nu \int_0^\nu \cos(2\beta M\tau) J_o(2\beta\tau)\, d\tau$$

$$= 2\pi\beta^2 \left\{\alpha \int_0^\alpha \cos(2\beta M\tau) J_o(2\beta\tau) d\tau - \int_0^\alpha \tau \cos(2\beta M\tau) J_o(2\beta\tau) d\tau \right\}$$

$$= \pi\alpha\beta \int_0^{2\alpha\beta} J_o(\nu) \cos M\nu\, d\nu - \frac{\pi}{2} \int_0^{2\alpha\beta} \nu J_o(\nu) \cos M\nu\, d\nu$$

$$= \pi\alpha\beta \int_0^{2\alpha\beta} J_o(\nu) \cos M\nu\, d\nu - \frac{\pi}{2} \left\{\frac{d}{dM} \int_0^{2\alpha\beta} J_o(\nu) \sin M\nu\, d\nu - 4\alpha\beta^2 M \sin(2\alpha\beta M) J_o(2\alpha\beta)\right\}$$

$$= \pi\alpha\beta\, P(\alpha,M) - \frac{\pi}{2} \frac{d}{dM} Q(\alpha,M) + 2\pi\alpha\beta^2 M \sin(2\alpha\beta M) J_o(2\alpha\beta) \tag{26}$$

where

$$P(\alpha,M) = \int_0^{2\alpha\beta} J_o(\nu)\cos M\nu \, d\nu = \beta^{\frac{1}{2}} - \int_{2\alpha\beta}^{\infty} J_o(\nu)\cos M\nu \, d\nu \qquad (27)$$

and $M < 1$

$$Q(\alpha,M) = \int_0^{2\alpha\beta} J_o(\nu)\sin M\nu \, d\nu = -\int_{2\alpha\beta}^{\infty} J_o(\nu)\sin M\nu \, d\nu \quad . \qquad (28)$$

The analogues of (26) for $F_1$ and $F_2$ are

$$F_1(\alpha,M) = \pi\alpha\beta M P(\alpha,M) - \frac{\pi}{2}Q(\alpha,M) - \frac{\pi}{2}M\frac{d}{dM}Q(\alpha,M) + 2\pi\alpha\beta^2M^2 \sin(2\alpha\beta M)J_o(2\alpha\beta) \qquad (29)$$

and

$$F_2(\alpha,M) = \pi\alpha\beta M^2P(\alpha,M) - \frac{\pi}{2}MQ(\alpha,M) - \frac{\pi}{2}\frac{d}{dM}Q(\alpha,M) + \pi\alpha\beta^2M\left(1 + M^2\right)\sin(2\alpha\beta M)J_o(2\alpha\beta)$$
$$+ \pi\alpha\beta \cos(2\alpha\beta M)J_o(2\alpha\beta) + \frac{\pi}{2}\{1 - \cos(2\alpha\beta M)J_o(2\alpha\beta)\} \qquad (30)$$

respectively.

Hence, the three integrals $F_o$, $F_1$, $F_2$ are characterized, apart from explicit terms, through the simpler pair, namely $P(\alpha,M)$ and $Q(\alpha,M)$; and the representation for the power which involves the latter turns out to be

$$P = \rho_o kc^3\Bigg\{\alpha\beta^{-\frac{1}{2}} P(\alpha,M) + \beta^{\frac{1}{2}} M\left(1 - \frac{M^2}{2}\right)Q(\alpha,M) - \frac{1}{2}\beta^{-\frac{1}{2}}\frac{d}{dM}Q(\alpha,M)$$
$$+ \alpha\beta^{\frac{3}{2}} M\left(2 - M^2\right)\sin(2\alpha\beta M)J_o(2\alpha\beta) + \alpha\beta^{\frac{3}{2}} M^2 \cos(2\alpha\beta M)J_1(2\alpha\beta)$$
$$+ \frac{1}{2}\beta^{\frac{1}{2}} M^2\left(1 - \cos(2\alpha\beta M)J_o(2\alpha\beta)\right)\Bigg\} \quad . \qquad (31)$$

Multiple differentiation of $P(\alpha,M)$ and $Q(\alpha,M)$ with respect to M, at the specific value $M = 0$, permits the formation of developments containing powers of the Mach number, which commence in the fashion

$$P(\alpha,M) = \int_0^{2\alpha} J_o(\nu)d\nu + \frac{1}{2} M^2 \left\{-4\alpha^2 J_1(2\alpha) + 2\alpha J_o(2\alpha) + \int_0^{2\alpha} J_o(\nu)d\nu\right\} + O\left(M^4\right)$$

and

$$Q(\alpha,M) = 2\alpha M J_1(2\alpha) + \frac{1}{3} M^3 \left\{-4\alpha^3 J_1(2\alpha) + 8\alpha^2 J_o(2\alpha) + 4\alpha J_1(2\alpha)\right\} + O\left(M^5\right) \quad .$$

Employing these estimates and approximating the other terms of (31) in accordance with the hypothesis that $\alpha\beta M << 1$, or simply $Mka << 1$, the development (5) cited in the introduction emerges after the dimensionless magnitude

$$2\alpha = 2ka = k\delta$$

is redefined in terms of the strip breadth $\delta$.

The same development for the average rate of energy radiation can be secured, as a matter of technical interest, by the judicious extraction of finite parts for divergent integrals; thus, recalling the Legendre polynomial generating function

$$\left(1 - 2zt + t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(z)t^n \quad , \qquad |t| < 1 \tag{32}$$

and writing

$$\left\{(k - M\zeta)^2 - \zeta^2\right\}^{-\frac{1}{2}} = \left(k^2 - \zeta^2\right)^{-\frac{1}{2}} \left(1 - 2zt + t^2\right)^{-\frac{1}{2}}$$

with the identifications

$$t = \frac{M\zeta}{\sqrt{k^2 - \zeta^2}} , \quad z = \frac{k}{\sqrt{k^2 - \zeta^2}} \tag{33}$$

it is formally inferred from (19) that

$$P = \frac{\rho_o \omega (kc)^2}{\pi} \operatorname{Re} \int_{-\infty}^{\infty} \frac{\left(1 - \frac{M\zeta}{k}\right)^2}{\sqrt{k^2 - \zeta^2}} \frac{\sin^2 \zeta a}{\zeta^2} \sum_{n=0}^{\infty} \left(\frac{M\zeta}{\sqrt{k^2 - \zeta^2}}\right)^n P_n \left(\frac{k}{\sqrt{k^2 - \zeta^2}}\right) d\zeta \tag{34}$$

$$= \frac{2}{\pi} \rho_o \omega (kc)^2 \int_o^k \frac{\sin^2 \zeta a}{\zeta^2} \frac{1}{\sqrt{k^2 - \zeta^2}} \left\{ 1 + M^2 \frac{\zeta^2}{k^2} - 2M^2 \frac{\zeta^2}{k^2 - \zeta^2} + \frac{1}{2} \frac{M^2 \zeta^2}{k^2 - \zeta^2} \left(3 \frac{k^2}{k^2 - \zeta^2} - 1\right) + o\left(M^4\right) \right\} d\zeta \quad . \tag{35}$$

Since the restriction $|t| < 1$ which assures convergence of the expansion (32) does not hold uniformly in the context of the pertinent formulas (33), (34) all save a pair of terms in (35) possess nonintegrable singularities (at $\zeta = k$). The proper integrals are

$$F(ka) = k^2 \int_o^k \frac{\sin^2 \zeta a}{\zeta^2} \frac{d\zeta}{\sqrt{k^2 - \zeta^2}} = \frac{\pi}{2} ka \left\{ \int_o^{2ka} J_o(\nu) d\nu - J_1(2ka) \right\}$$

and

$$G(ka) = \int_o^k \frac{\sin^2 \zeta a}{\sqrt{k^2 - \zeta^2}} d\zeta = \frac{\pi}{4} \left(1 - J_o(2ka)\right) \quad ;$$

after noting an alternative expression

$$\int_o^k \frac{\sin^2 \zeta a}{\zeta^2} \frac{1}{\sqrt{k^2 - \zeta^2}} \frac{\zeta^2}{k^2 - \zeta^2} d\zeta = - \int_o^k \frac{\sin^2 \zeta a}{\zeta^2} \frac{d}{dk} \left(\frac{k}{\sqrt{k^2 - \zeta^2}}\right) d\zeta$$

for the first improper integral, its finite part may be identified as

$$- \frac{d}{dk}\left\{k \int_0^k \frac{\sin^2\zeta a}{\zeta^2} \frac{d\zeta}{\sqrt{k^2 - \zeta^2}}\right\} = - \frac{d}{dk}\left\{\frac{F(ka)}{k}\right\} = - \frac{\pi a}{2k} J_1(2ka) \quad .$$

Likewise, the prescription

$$\int_0^k \frac{\sin^2\zeta a}{\zeta^2} \frac{1}{\sqrt{k^2 - \zeta^2}} \left(3 \frac{k^2}{k^2 - \zeta^2} - 1\right) d\zeta$$

$$= \int_0^k \frac{\sin^2\zeta a}{\zeta^2} \frac{1}{\sqrt{k^2 - \zeta^2}} \left(1 - 4 \frac{k^2}{k^2 - \zeta^2} + 3 \frac{k^4}{\left(k^2 - \zeta^2\right)^2}\right) d\zeta$$

$$= \frac{d^2}{dk^2}\left\{k^2 \int_0^k \frac{\sin^2\zeta a}{\zeta^2\sqrt{k^2 - \zeta^2}} d\zeta\right\} - \frac{d}{dk}\left\{k \int_0^k \frac{\sin^2\zeta a}{\zeta^2\sqrt{k^2 - \zeta^2}} d\zeta\right\}$$

$$= \frac{d^2}{dk^2}\left\{F(ka)\right\} - \frac{d}{dk}\left\{\frac{1}{k} F(ka)\right\}$$

$$= \pi a^2 J_o(2ka) - \frac{\pi a}{2k} J_1(2ka)$$

supplies a value for the other improper integral which appears in (35). When the various integral determinations are brought together with (35), the resultant power estimate,

$$P = \rho_o a k^2 c^3 \left\{\int_0^{2ka} J_o(\nu) d\nu - J_1(2ka) + M^2\left[kaJ_o(2ka) + \frac{3}{2} J_1(2ka) + \frac{1 - J_o(2ka)}{2ka}\right] + O\, M^4\right\}$$

$$Mka \ll 1,$$

agrees with (5).

Turning attention next to a noncompact piston, typified by the inequality $\alpha = ka \gg 1$, and having regard for the ultimate stage of the successive equalities

$$\int_{\gamma}^{\infty} J_o(\nu)\cos M\nu \, d\nu = \int_{\gamma}^{\infty} d\left(\nu J_1(\nu)\right) \frac{\cos M\nu}{\nu}$$

$$= - J_1(\gamma)\cos M\gamma + \int_{\gamma}^{\infty} \nu J_1(\nu)\left\{\frac{\cos M\nu}{\nu^2} + M \frac{\sin M\nu}{\nu}\right\} d\nu$$

$$= - J_1(\gamma)\cos M\gamma + \int_{\gamma}^{\infty} \frac{J_1(\nu)}{\nu} \cos M\nu \, d\nu + M \int_{\gamma}^{\infty} d\left(-J_o(\nu)\right) \sin M\nu$$

$$= - J_1(\gamma)\cos M\gamma + \int_{\gamma}^{\infty} \frac{J_1(\nu)}{\nu} \cos M\nu \, d\nu + MJ_o(\gamma)\sin M\gamma + M^2 \int_{\gamma}^{\infty} J_o(\nu)\cos M\nu \, d\nu$$

it follows that

$$\int_{\gamma}^{\infty} J_o(\nu)\cos M\nu \, d\nu = \frac{1}{1 - M^2} \left\{-J_1(\gamma)\cos M\gamma + MJ_o(\gamma)\sin M\gamma + \int_{\gamma}^{\infty} \frac{J_1(\nu)}{\nu} \cos M\nu \, d\nu\right\} \tag{35}$$

whence the function $P(\alpha,M)$ defined in (30) is expressible as

$$P(\alpha,M) = \frac{1}{\beta^2} + \beta\left\{J_1(2\alpha\beta)\cos(2\beta M) - MJ_o(2\alpha\beta)\sin(2\alpha\beta M) - \int_{2\alpha\beta}^{\infty} \frac{J_1(\nu)}{\nu} \cos M\nu \, d\nu\right\} . \tag{36}$$

Correspondingly,

$$\int_{\gamma}^{\infty} J_o(\nu)\sin M\nu \, d\nu = \frac{1}{1 - M^2} \left\{-J_1(\gamma)\sin M\gamma - MJ_o(\gamma)\cos M\gamma + \int_{\gamma}^{\infty} \frac{J_1(\nu)}{\nu} \sin M\nu \, d\nu\right\} \tag{37}$$

and

$$Q(\alpha,M) = \beta\left\{J_1(2\alpha\beta)\sin(2\alpha\beta M) + MJ_o(2\alpha\beta)\cos(2\alpha\beta M) - \int_{2\alpha\beta}^{\infty} \frac{J_1(\nu)}{\nu} \sin M\nu \, d\nu\right\} \tag{38}$$

from which it is deduced (on employing the result $d\beta/dM = 2\beta^2 M$) that

$$\frac{dQ}{dM} = \beta^2 M^2 J_o(2\alpha\beta)\cos(2\alpha\beta M) + \beta^2 M J_1(2\alpha\beta)\sin(2\alpha\beta M)$$

$$+ 2\alpha\beta^2 M J_o(2\alpha\beta)\sin(2\alpha\beta M) + 2\alpha\beta^2 J_1(2\alpha\beta)\cos(2\alpha\beta M) - M\beta^2 \int_{2\alpha\beta}^{\infty} \frac{J_1(\nu)}{\nu} \sin M\nu \, d\nu \tag{39}$$

Substituting the expressions (36), (38) and (39) for $P(\alpha,M)$, $Q(\alpha,M)$ and $dQ/dM$ into the formula (31) constitutes the last step of our power analysis, and yields the generally valid representations

$$P/\rho_o kc^3 = \alpha + \frac{1}{2}\beta^{\frac{1}{2}} M^2 - \alpha\beta^{\frac{1}{2}} \int_{2\alpha\beta}^{\infty} \frac{J_1(\nu)}{\nu} \cos M\nu \, d\nu + \frac{1}{2} M\beta^{\frac{1}{2}} J_1(2\alpha\beta)\sin(2\alpha\beta M) - \frac{1}{2}\beta^{\frac{1}{2}} M \int_{2\alpha\beta}^{\infty} \frac{J_1(\nu)}{\nu} \sin M\nu \, d\nu \tag{40}$$

and

$$\alpha = \frac{k\delta}{2} = ka, \quad \beta = \frac{1}{1 - M^2}$$

$$P/\rho_o kc^3 = \alpha\beta^{\frac{1}{2}} \int_{0}^{2\alpha\beta} \frac{J_1(\nu)}{\nu} \cos M\nu \, d\nu + \frac{1}{2} M\beta^{\frac{1}{2}} J_1(2\alpha\beta)\sin(2\alpha\beta M) + \frac{1}{2}\beta^{\frac{1}{2}} M \int_{0}^{2\alpha\beta} \frac{J_1(\nu)}{\nu} \sin M\nu \, d\nu \tag{41}$$

where the former lends itself to estimation if $\alpha\beta >> 1$ and the latter if $\alpha\beta << 1$. In particular, when $\alpha \to 0$ and M does not assume values close to unity (or $\beta$ close to infinity) the representation (41) implies

$$P/\rho_o kc^3 = \alpha^2\beta^{\frac{5}{2}} \left(1 + \frac{1}{2} M^2\right) + O\left(\alpha^4\right) = \frac{1}{4} (k\delta)^2 \frac{1 + \frac{1}{2} M^2}{\left(1 - M^2\right)^{\frac{5}{2}}} + O(k\delta)^4$$

$$= \frac{1}{4}\,(k\delta)^2\left\{1 + 3M^2 + O\left(M^4\right)\right\} + O(k\delta)^4\ , \qquad M \to 0$$

as stated previously. For small values of M, furthermore, the developments

$$\int_0^{2\alpha\beta} \frac{J_1(\nu)}{\nu}\cos M\nu\, d\nu = \int_0^{2\alpha} \frac{J_1(\nu)}{\nu}\, d\nu + \frac{1}{2}\, M^2\left\{- \int_0^{2\alpha} \nu J_1(\nu)\, d\nu + 2J_1(2\alpha)\right\} + O\ M^4$$

$$= \int_0^{2\alpha} J_o(\nu)\, d\nu - J_1(2\alpha) + \frac{1}{2}\, M^2\left\{2\alpha J_o(2\alpha) + 2J_1(2\alpha) - \int_0^{2\alpha} J_o(\nu)\, d\nu\right\} + O\left(M^4\right)$$

and

$$\int_0^{2\alpha\beta} \frac{J_1(\nu)}{\nu}\sin M\nu\, d\nu = M \int_0^{2\alpha} J_1(\nu)\, d\nu + O\left(M^3\right)$$

$$= M\left\{1 - J_o(2\alpha)\right\} + O\left(M^3\right)$$

support, in conjunction with (41), the propriety of the result (5).

The terms of (40) are arranged in a sequence that befits their relative importance in the limit $\alpha \to \infty$, $M \neq 0$; namely, the first and second are $O(\alpha)$ and $O(1)$, respectively, while the third and fourth are $O(\alpha^{-\frac{1}{2}})$ and the last is $O(\alpha^{-\frac{3}{4}})$. Of particular note, as previously stated, is that the flow manifests it presence, to the leading order in the average power output at short wavelengths, by a contribution which does not involve the piston scale. Estimates for the terms of (40) which contain Bessel functions are readily gained through the utilization of large argument asymptotic forms pertinent to these functions.

## Reference

Ffowcs Williams, J. E. and Lovely, D. J. 1975 J. Fluid Mech. 71, 689.

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