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ABSTRACT

We present approximate analytic expressions for evaluation of the frequency and angular dependence of synchrotron emissivity from mildly relativistic particles with arbitrary energy spectrum and pitch angle distribution in a given magnetic field. Eqs. (11) to (14) give our result in its most general form. We show that eq. (14) can be simplified considerably as in eq. (22) and that for most practical cases eq. (11) accepts simpler forms (eqs. 29 or 35). These results are valid for particle distribution which are not extremely anisotropic.

Our result agrees with previous expressions for a non-relativistic Maxwellian particle distribution, and when extrapolated to non-relativistic and extreme relativistic regimes, it agrees with the previous expressions obtained under those limiting conditions.

We compare the result from our analytic expression with results from detailed numerical evaluations. We find excellent agreement not only at frequencies large compared to the gyro-frequency but also at lower frequencies, in fact, all the way down to the gyro-frequency, where the analytic approximations are expected to be less accurate.

Subject headings: Synchrotron Radiation; Sun: Flares, Radio Radiation; Stars: Accretion, White Dwarfs; Plasmas

I. INTRODUCTION

Synchrotron radiation from highly relativistic particles is an old hat worn by astrophysicists successfully for many decades. At extreme non-relativistic energies the primary contribution to emissivity is the well known cyclotron radiation at the first and perhaps few low harmonics. In the intermediate range there does not exist a simple formula for evaluaiton of the emissivity (or absorption coefficient) from particles with an arbitrary energy spectrum and pitch angle distribution. The usual practice is to use lengthy numerical methods for evaluation of these quantities at mildly relativistic energies (cf.e.g., Ramaty 1969 and Takakura 1972, in connection with solar flare problems and Lamb and Master 1979, in connection with accreting white dwarfs). Earlier, in connection with fusion plasma, Trubnikov (1958) and Drummond and Rosenbluth (1963) derived equations for emissivity (or absorption coefficient) of synchrotron radiation from a thermal plasma (cf., e.g., Bekefi 1966, ch. 6, and references cited there).

In most astrophysical problems the low harmonics of cyclotron radiation will be either self-absorbed, or absorbed by surrounding plasma, or suppressed by Razin-Tsytovich effect. Consequently, only radiation at high harmonics where optical depth $\tau_\nu \leq 1$ is of interest. We find that at such high harmonics there exists a simple approximate method for evaluaiton of the spectrum and directivity of the synchrotron radiation from an ensemble of electrons in a given magnetic field. In this paper we present the results of this method for the total emissivity at high harmonics. In §II we present the general formula and the details of the derivation of the emissivity for an arbitrary energy spectrum and pitch angle distribution of particles. The emissivity for some commonly encountered particle distributions are presented in §III where we also give some emperical relation valid for a large range of the parameters and compare the results from these analytic expressions with those obtained by detailed numerical calculations. A brief summary of the result is presented in §IV. Polarization and intensity of radiation in optically thick regions and at lower harmonics will be discussed in a future paper.

II. ANALYSIS AND GENERAL RESULTS

Consider particles with charge e, mass m and distribution $f(\mu,\gamma)$ where $fd\mu d\gamma$ is the number density of particles in the energy (in units of mc^2) interval γ to $\gamma + d\gamma$ and with pitch angle cosine between μ to $\mu + d\mu$. The specific emissivity $j_\nu(\theta)$ of the synchrotron radiation at frequency ν and at an angle θ with respect to the magnetic field of strength B is given by (cf. Bekefi 1966)

$$j_\nu(\theta) = \frac{2\pi e^2\nu_b}{c} \frac{\nu}{\nu_b \sin^2\theta} \int_\phi^\infty d\gamma \int_{-1}^{+1} d\mu f(\gamma,\mu)\nu\eta_\nu(\theta,\gamma,\mu), \tag{1}$$

where

$$\eta_\nu(\theta,\gamma,\mu) = \sum_{m=1}^{\infty} \left\{(\text{Cos}\theta-\beta\mu)^2 J_m^2(x) + (1-\beta\mu\text{Cos}\theta)^2[xJ_m'(x)/m]^2\right\} \delta(y), \tag{2}$$

and

$$y = m\nu_b/\gamma - \nu(1-\beta\mu\cos\theta), \qquad x = \nu\gamma\beta\sin\theta(1-\mu^2)^{\frac{1}{2}}\nu_b\,,$$

$$\nu_b = \frac{eB}{2\pi mc} = 2.8 \times 10^6(\text{B/gauss})\text{Hz}. \tag{3}$$

Here $J_m(x)$ is the Bessel function of order m and $\beta^2 = 1 - \gamma^{-2}$.

As mentioned above, we are interested in emission at frequencies ν such that optical depth $\tau_\nu \leq 1$. Under most astrophysical circumstances this occurs at frequencies much higher than the gyro frequency ν_b. We, therefore, seek solution to eq. (1) for $(\nu/\nu_b) >> 1$. In this case the delta function indicates that m is large and that the relevant harmonics are closely spaced so that the summation can be replaced by an integration. This integral can be carried out easily with the help of the delta function. This step then eliminates the summation, introduces a multiplying factor of γ/ν_b and replaces m by $\gamma\nu(1-\mu\beta\cos\theta)/\nu_b$.

Furthermore, since m is large and we are not in the extreme relativistic regime, the Bessel function and its deviation can be approximated as follows (cf. e.g., Abramowitz and Stegum 1970)

$$J_m(x) = (2\pi m)^{-\frac{1}{2}}(1-z^2)^{-\frac{1}{4}}\mathcal{Z}^m , \quad xJ_m'(x)/m = (1-z^2)^{\frac{1}{2}}J_m(mz),$$
$$\mathcal{Z} = \frac{ze^{(1-z^2)^{\frac{1}{2}}}}{1+(1-z^2)^{\frac{1}{2}}}, \quad z \equiv \frac{x}{m} = \frac{\beta\sin\theta\ (1-\mu^2)^{\frac{1}{2}}}{1-\mu\beta\cos\theta} . \tag{4}$$

This approximation is valid if $m(1-z^2)^{3/2} >> 1$. Note that $z \leq 1$ and that it has its largest value when $\theta = \frac{\pi}{2}$ and $\mu = 1.0$, i.e., $z \leq \beta$ or $1-z^2 > \gamma^{-2}$ so that the above approximation is valid for $\gamma^2 << \nu/\nu_b$. At higher energies $\gamma >> 1$ and $\nu/\nu_b >> 1$, one is in the ultrarelativistic regime where the spectral and angular distribution of the synchrotron radiation are well known (cf. e.g. Bekefi 1961, Ginzberg and Syrovatskii 1964). With the above approximations eq. (1) becomes

$$j_\nu(\theta) = \frac{e^2\nu_b}{c}\left(\frac{\nu}{\nu_b\sin^2\theta}\right)\int_1^\infty d\gamma\int_{-1}^{+1} d\mu f(\mu,\gamma)\ Y(\theta,\gamma,\mu)\mathcal{Z}^{2m}(\theta,\gamma,\mu)$$
$$Y(\mu,\gamma,\theta) = \frac{(\cos\theta-\mu\beta)^2 + (1-z^2)(1-\mu\beta\cos\theta)^2}{(1-z^2)^{\frac{1}{2}}\ (1-\mu\beta\cos\theta)} , \quad m = \frac{\nu\gamma}{\nu_b}(1-\beta\mu\cos\theta). \tag{5}$$

1) <u>Integration Over Pitch Angle</u>. In general, Y is a slowly varying function of μ while, because of the large exponent, $\mathcal{Z}^{2m}$ varies rapidly

from its maximum value at $\mu = \mu_{max}$ to zero at $\mu = \pm 1$. We, therefore, use the method of steepest descent to evaluate the integral over μ. This gives $m = (\nu\gamma/\nu_b)(1-\mu_{max}\beta\cos\theta)$ and

$$j_\nu(\theta) = \frac{e^2\nu_b}{c}\left(\frac{\nu}{\nu_b \sin^2\theta}\right)\int_1^\infty d\gamma\, f(\mu_{max},\gamma)\, Y(\theta,\gamma,\mu_{max}) \mathcal{Z}^{2m}(\theta,\gamma,\mu_{max})\left[\frac{2\pi}{d^2 \ln(\mathcal{Z}^{2m} f)/d\mu^2}\right]^{\frac{1}{2}} \tag{6}$$

where μ_{max} is the solution to the transcendental equation,

$$\mu = \beta\cos\theta(1-\varepsilon_1), \quad \varepsilon_1 = \frac{(1-\mu^2)}{(1-z^2)^{\frac{1}{2}}}\left[\ln \mathcal{Z} - \frac{\nu_b}{2\nu\gamma\beta\cos\theta}\;\frac{d\ln f}{d\mu}\right] . \tag{7}$$

For slowly varying pitch angle distributions $d\ln f(\mu,\gamma)/d\mu \ll \nu/\nu_b$ (see below for a more precise limit), the last term in the square brackets can be ignored. Then as we shall see below, ε_1 is of the order of $\nu_b/\nu \ll 1$ so that $\mu_{max} = \beta\cos\theta$ and eq. (6) becomes

$$j_\nu(\theta) = \frac{e^2\nu_b}{c}\left(\frac{\pi\nu}{\nu_b}\right)^{\frac{1}{2}}\int_1^\infty d\gamma f(\mu_{max},\gamma)\gamma^{-1}\left[(1+2\cot^2\theta/\gamma^2)(1-\beta^2\cos^2\theta)^{\frac{1}{4}}\right]\mathcal{Z}^{2m}_{max} \tag{8}$$

where

$$\mathcal{Z}_{max} = \frac{t\, e^{1/(1+t^2)^{\frac{1}{2}}}}{1+(1+t^2)^{\frac{1}{2}}}, \quad m = \frac{\nu}{\gamma\nu_b}(1+t^2), \qquad t \equiv \beta\gamma\sin\theta . \tag{9}$$

For $\theta = \frac{\pi}{2}$, $\left(\mu_{max} = \frac{\nu_b}{2\nu}\;\frac{d\ln f}{d\mu} = 0\right)$, the expression in the square bracket is equal to unity and

$$\mathcal{Z}^{2m}_{max}(\pi/2,\gamma,\mu_{max}) = e^{2\nu/\nu_b}\left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma\nu/\nu_b} \tag{10}$$

in agreement with the result obtained by Trubnikov (1958).

2) Integration Over the Energy. In general, the expression in the square bracket of eq. (8) varies slowly with γ. However, $\mathcal{Z}^{2m}$ increases rapidly with particle kinetic energy [as $(\gamma - 1)^{2m}$] when $\gamma - 1 \ll 1$ and approaches unity as $\gamma \to \infty$, while $f(\mu_{max},\gamma)$ decreases rapidly (as a power law, $f \propto (\gamma-1)^{-\delta}$; or exponentially, $f = e^{-\gamma/kT}$) with particle energy. Most of the contribution to the integral comes from the vicinity of the maximum of $f\mathcal{Z}^{2m}/\gamma$. Again, a good approximation to the integral is obtained by using the method of steepest descent which gives

$$j_\nu(\theta) = (\pi e^2\nu_b/c)(\nu/\nu_b)^{1/2}(1+2\cot^2\theta/\gamma_o^2)(1-\beta_o^2\cos^2\theta)^{1/4}f(\beta_o\cos\theta,\gamma_o)\mathcal{Z}^{2m}_{max}(t_o)X, \quad (11)$$

where $m = (\nu/\gamma_o\nu_b)(1+t_o^{\;2})$ and γ_o (or $t_o = \beta_o\gamma_o \sin\theta$) is the solution of the equation

$$t^{-2}(1+t^2)^{-1/2} + (1-\beta^2\cos^2\theta/t^2)\ln\mathcal{Z}_{max} = -[d\ln(f/\gamma)/d\gamma](\nu_b/2\nu\sin^2\theta) \equiv \frac{\varepsilon}{\sin^2\theta}. \quad (12)$$

In eq. (11)

$$X^{-2} \equiv -\left.\frac{\gamma_o^2 d\ln(\mathcal{Z}^{2m}_{max}(f/\gamma)}{2d\gamma^2}\right|_{\gamma=\gamma_o} = -\left.\frac{\gamma_o^2 d^2\ln(f/\gamma)}{2d\gamma^2}\right|_{\gamma=\gamma_o} - \left.\frac{d\ln(f/\gamma)}{d\ln\gamma}\right|_{\gamma=\gamma_o} W(t_o,\theta), \quad (13)$$

where

$$W = \frac{[t_o^4+t_o^2\,(3/2+\sin^2\theta/2)+\sin^2\theta]t_o^{-4}(1+t_o^2)^{-3/2} - (\cot^2\theta/\gamma_o^2)\ln\mathcal{Z}_{max}}{t_o^{-2}\,(1+t_o^2)^{-1/2} + (1-\beta_o^2\cos^2\theta/t_o^2)\ln\mathcal{Z}_{max}}. \quad (14)$$

3) Asympthotic Limits. These equations describe the synchrotron emissivity for a general electron spectrum and pitch angle distribution. Before we can express $j_\nu(\theta)$ explicitly in terms of ν and θ we must specify the distribution function $f(\mu,\gamma)$. Below we shall consider three kinds of distribution. Before doing so we first describe some general features and comment on the accuracy of these expressions by considering some asympthotic limits. This will allow us to simplify these complicated expressions considerably.

(a) High frequency, high energy limit. At high frequencies, at angles θ such that $\sin\theta$ is not negligible, and for distributions so that

$\varepsilon = \dfrac{\nu_b d\ln(f/\gamma)}{2\nu d\ln\gamma} << 1$, the right side of eq. (12) is much smaller than unity. This means that $t_o^2 >> 1$. In this limit the left hand side is approximately equal to $\dfrac{2}{3t^3}\left(1 + \dfrac{2-5\sin^2\theta}{10t^2}\right)$, so that to the first order in ε the solution of eqs. (12) and (14) becomes

$$t_o^3/\sin^3\theta = \beta_o^3\gamma_o^3 = \frac{2}{3\varepsilon\sin\theta} >> 1, \quad W = \frac{3}{2} \qquad (15)$$

Note that, in general, ε may be a function of γ (see examples given below), and that as θ decreases γ_o increases, which means that the largest contribution to the integral comes from higher energy particles. Eq. (15) is valid for

$$1 \geq \sin\theta >> \sin\theta_c \approx \varepsilon^{1/2} , \qquad (16)$$

however, even for $\theta = \theta_c$ (i.e. $t_o^2 \sim 1$) it is accurate to better than 20 percent.

(b) Small angle regime. For $\theta < \theta_c$ the right hand side of eq. (12) becomes larger than one so that t_o becomes less than unity, but γ_o is still much larger than one. In the extreme limit of $\theta \ll \theta_c$, we have $\varepsilon/\sin^2\theta \gg 1$ and $t^2 \ll 1$. In this case the left hand side of eq. (12) is approximately equal to $\frac{1}{t^2} \ln \frac{2}{t}$ so that for $\theta \ll \theta_c$ we have

$$t_o^2/\sin^2\theta = \beta_o^2\gamma_o^2 \simeq \varepsilon^{-1} \ln(\sin\theta_c/\sin\theta), \quad W \simeq 1 \, . \tag{17}$$

Note that this equation will not be of much practical use since for $\varepsilon \ll 1$ eq. (15) is valid at all angles except for emission along the field line, where the emissivity is negligible (emissivity is zero at $\theta = 0$ except at the gyro-frequency). Thus, except for the unrealistic (i.e. not attainable in astrophysical situation) case of uniform magnetic field within the observed beam, we can ignore emission at angles $\theta \leq \theta_c$ and use eq. (15) (cf. Epstein 1973).

(c) Non-relativistic limit and low harmonics. At lower harmonics and/or for very steep electron spectra (which means primarily non-relativistic particles) $\frac{d\ln(f/\gamma)}{d\gamma}$ may exceed ν/ν_b and the right side of eq. (12) becomes greater than one even at $\sin\theta \sim 1$. In the extreme case of $\varepsilon \gg 1$, it is easy to see that, independent of θ, the largest contribution to the emissivity comes from particles with non-relativistic energies. Thus, eqs. (14) and (12) reduce to

$$\beta_o^2\gamma_o^2 = 1/W = 1/\varepsilon \, , \quad \varepsilon \gg 1. \tag{18}$$

(d) Test of the accuracy of approximations used in previous section. In the next section we will discuss the accuracy of the above approximate expression for three different kinds of distribution $f(\mu,\gamma)$. Before doing so let us first examine the accuracy of the approximations used in the derivation of eqs. (11) to (14). In eq. (7) we assumed ϵ_1 to be much less than unity. Clearly this will not be the case at all energies and angles. Our results would, however, be valid so long as this inequality is satisfied for the pitch angles and energies of particles which have the largest contribution to the integral, namely, $\mu = \beta\cos\theta$ and $\gamma = \gamma_0$. For $\varepsilon << 1$ and γ_0 given by eqs. (15) and (17), the expression $(1-\mu^2)\ln\mathcal{Z}_{max} / (1-z^2)^{\frac{1}{2}}$ in eq. (7) is equal to $\varepsilon\sin/2$ and $\lesssim \varepsilon^2$, respectively. Ignoring this term then the approximation $\mu_{max} = \beta\cos\theta$ is justified[1] as long as

$$d\ln f/d\mu < 2\nu\cos\theta/\nu_b\sin^3\theta \quad . \tag{19}$$

[1]Note that at $\theta = \pi/2$ eq. (7) indicates that $\mu_{max} = \frac{\nu_b d\ln f/d\mu}{2\nu\gamma_0} \approx 0$ as long as $d\ln f/d\mu << \nu/\nu_b$. As θ decreases the inequality in (19) becomes less restrictive.

For $\varepsilon >> 1$ and γ_0 given by eq. (18) ε_1 is no longer negligible, but in this case $\mu_{max} \approx \beta\cos\theta\varepsilon_1 \approx \frac{\cos\theta}{\sqrt{\varepsilon}} \ln \frac{\sin\theta}{\sqrt{\varepsilon}} - \frac{\nu_b}{2\nu} d\ln f/d\mu \approx 0$ as long as $d\ln f/d\mu << 2\nu/\nu_b$. Thus, for particles with slowly varying pitch angle distribution, the relation $\mu_{max} = \beta\cos\theta$ is a good approximation throughout.

In connection with eq. (4) it was mentioned that the approximation used for the Bessell function and their derivatives is valid if $m(1-z^2)^{3/2} = \frac{\nu}{\nu_b\gamma_0}(1+t_0^2)^{-\frac{1}{2}} >> 1$. For $\varepsilon >> 1$ from eq. (18) we have $\gamma_0 \simeq 1$, $t_0^2 << 1$ so that this condition is valid if $\nu/\nu_b >> 1$, which is our basic assumption.

For $\epsilon << 1$ and $\sin\theta \gtrsim \sin\theta_c$ from eq. (15), we find that the above inequality reduces to

$$\left.\frac{d\ln(f/\gamma)/d\gamma}{(\epsilon\sin\theta)^{1/3}}\right|_{\gamma=\gamma_o} >> 1. \qquad (20)$$

which is satisfied for most practical spectra.

In the extreme limit of small angles $\theta << \theta_c$ the approximation for Bessell function may not be valid since in this case the right side of the inequality (19) is replaced by $\ln(\sin\theta_c/\sin\theta)$. However, this will happen at very small angles which, as mentioned in connection with eq. (17), is of no consequence except at ultrarelativistic energies (see Epstein 1973, and references cited therein).

Having justified all of our approximations, we can now substitute γ_o from eqs. (15) or (18) in eq. (11) to obtain to first order in ϵ or $1/\epsilon$

$$j_\nu(\theta) = (\pi e^2\nu_b/c)(\nu/\nu_b)^{\frac{1}{2}} f(\beta_o\cos\theta,\gamma_o) X \begin{cases} \sin^{\frac{1}{2}}\theta\exp\{d\ln(f/\gamma)/d\ln\gamma^2\big|_{\gamma_o}\}, & \epsilon<<1 \\ \dfrac{1+\cos^2\theta}{\sin^2\theta}(e\beta_o\sin\theta/2)^{2\nu/\nu_b}, & \epsilon>>1 \end{cases} . \qquad (21)$$

As evident from expressions (13) and (14), the quantity X is a complicated function of γ_o and $\sin\theta$. However, as we have seen above, $W(to,\theta)$ has a very simple form in the two limits. It turns out that, except for $\theta << \theta_c$, eq. (14) can be simplified considerably. Combining the two asymthotic forms of W given in eqs. (18) and (15), we find that we can replace eq. (14) by

$$W = 3/2 + 1/(\gamma_o^2 - 1) \qquad (22)$$

without loss of much accuracy. On Figure 1 we show the variation of the ratio of the actual value of W (from eq. 14) to its approximate value given by eq. (22) with γ_o and $\sin\theta$. This shows that, except for the uninteresting case of $\sin\theta << 1$, the above expression is an excellent approximation. We shall use eq. (22) instead of eq. (14) in the following section.

III. SOME SPECIAL EXAMPLES

In this section we derive the emissivity for three special particle distributions using the general eqs. (11), (12) and (21). Whenever possible we give a simple approximation to eq. (12).

1) Thermal spectrum. We first consider the emissivity of thermal, non-degenerate gas which was first investigated by Trubnikov (1961) and Drummond and Rosenbluth (1963)[2] and which has found application in fusion

[2]There was some disagreement initially between Trubnikov and Drummond and Rosenbluth (1960) which was settled in favor of Trubnikov. Our results confirm this.

plasma and in many astrophysical conditions such as solar flares, stellar coronae and accretion on white dwarfs (see e.g. Lamb and Masters 1979; Ramaty and Lingenfelter 1967; Ramaty and Petrosian 1972; Petrosian 1981).

If the particle distribution is isotropic and Maxwellian with temperature kT (in units of mc^2), then, in general

$$f(\mu,\gamma)/\gamma = Ce^{-\frac{\gamma}{kT}}(\gamma^2-1)^{\frac{1}{2}} , \tag{23a}$$

where for $kT \leq 1$

$$C \simeq \frac{n}{2}\left[2\pi(kT)^3\right]^{-\frac{1}{2}} e^{\frac{1}{kT}}\left(1 - \frac{3kT}{16} \ldots\right) \tag{23b}$$

and n is the number density of the particles. From this we find that

$$d\ln(f/\gamma)/d\gamma = -\frac{1}{kT}\left[1 - \frac{kT\gamma}{\gamma^2-1}\right], \quad \frac{d^2\ln(f/\gamma)}{d\gamma^2} = -\frac{\gamma^2+1}{(\gamma^2-1)^2} . \tag{24}$$

Our detailed calculation with eq. (12) (described below) shows that the second term in the square brackets is always less than $\nu_b/2\nu << 1$ unless $kT \geq 1$. However, if $kT >> 1$, then the method of steepest descent used in the derivation of eq. (11) does not give a good approximation to the integral over particle energies. This is not of much concern because in most astrophyscial conditions (except in the early phase of the universe) such high temperatures are not encountered. Even when such temperatures are encountered, other high energy processes involving the numerous electron-positron pairs become more important than synchrotron radiation. Thus, we shall restrict ourselves to $kT \leq 1$, and in what follows immediately we ignore the square brackets in eq. (24).

Let us first consider a Maxwellian gas with semi-relativistic temperature so that $\nu kT/\nu_b >> 1$. In this case ε in eq. (12) is small (except for the uninteresting case of $\sin\theta << \nu_b/\nu kT$) so that from eq. (15) we find

$$\beta_o \approx 1, \quad \gamma_o^3 \approx 4\nu kT/3\nu_b \sin\theta, \quad W = 3/2, \quad X = (2kT/3\gamma_o)^{\frac{1}{2}} \tag{25}$$

substitution of which in eq. (11) (keeping only the highest order term in γ_o), or in eq. (21) gives

$$j_\nu(\theta) \approx (2^{3/2}\pi e^2 \nu_b/3c)\, C(\nu kT/\nu_b) \exp\left\{ - \frac{\nu}{\nu_b} \left[\frac{4.5}{\sin\theta} \left(\frac{\nu_b}{\nu kT} \right) \right]^{1/3} \right\} . \tag{26}$$

This is in agreement with results we obtain from Trubnikov's expression in this limit. In particular, for $\theta = \pi/2-\alpha$ with $\alpha << 1$, the angular dependence reduces to $j_\nu(\theta) \propto \exp\left\{ - \frac{\nu\alpha^2}{\nu_b} \left[\frac{1}{48} \left(\frac{\nu b}{\nu kT} \right)^2 \right]^{1/3} \right\}$ as given by Trubnikov (see also Bekefi 1966, p. 205).

For a non-relativistic gas with $kT \ll 1$ so that $\nu kT/\nu_b$ is also much less than unity, the right hand side of eq. (12) is equal to $\nu_b/2\nu kT\sin\theta \gg 1$ so that its solution reduces to the simple expression in eq. (18):

$$\gamma_o \approx 1, \quad \beta_o^2 \approx 2\nu kT/\nu_b, \quad X = (2\nu/\nu_b)^{1/2} . \tag{27}$$

Substitution of this in eq. (11) or (21) gives

$$j_\nu(\theta) = (2\sqrt{2}\pi e^2\nu_b/c)C(\nu kT/\nu_b)^{3/2}\left(\frac{1+\cos^2\theta}{\sin^2\theta}\right)\exp\left\{-(\nu/\nu_b)\ln(2\nu_b/e\nu kT\sin^2\theta)\right\}. \tag{28}$$

We find that the expression given by Trubnikov reduces to eq. (28) in this limiting case also.

It is interesting that we find this agreement between the result here and those of Trubnikov even though the method of integration used here is quite different than that used by him and that eq. (12) bears no resemblance to his equation which plays the same role as our eq. (12). Presumably this agreement persists throughout. Note, however, that our result is more general and is applicable to a variety of spectral and pitch angle distributions (within the limitation specified in the previous section) while Trubnikov's result is applicable only to an isotropic Maxwellian gas with $kT \ll 1$.

In the intermediate region, $\nu kT/\nu_b \simeq 1$, simple approximations such as those in eqs. (25) and (27) are not possible and we must solve eq. (12) numerically. In Figure 2 we show the variation of $(\gamma - 1)$ with $\nu kT/\nu_b$ for various values of $\sin\theta$ and for the limiting case of $kT \to 0$ (so that the square brackets in eq. (24) can be set equal to unity) and for $kT = 1$ (where we have included the square bracket in eq. (24) in our calculations). For $kT \to 0$ these curves show a smooth transition between the two limiting expressions (25) and (27) except for very small value of $\sin\theta$. For $kT \approx 1$, $\nu kT/\nu_b$ is always greater than unity and eq. (25) appears to be a good approximation throughout. From Figure 2 and eqs. (22) and (24) we find that the complicated eqs. (12) and be replaced by the following approximate equations without loss of much accuracy:

$$(\gamma_o^2 - 1) = \begin{cases} (2\nu kT/\nu_b)(1 + 4.5\nu kT\sin^2\theta/\nu_b)^{-1/3}, & kT << 1 \\ (4\nu kT/\nu_b \sin\theta)^{2/3} , & kT \approx 1 \end{cases} \tag{29}$$

$$X^2 = (2kT/\gamma_o^2)(\gamma_o^2 - 1)/(3\gamma_o^2 - 1) , \qquad kT \lesssim 1 .$$

These expressions asympthotically approach the expressions given in eqs. (25) and (27). In addition, as shown by the inset in Figure 2 where we plot the ratio of $\gamma_o^2 - 1$ from (29) to that obtained from exact expression (12), the above approximation agrees with the exact result to better than few percent for $\nu/\nu_b >> 1$ and $\sin\theta \approx 1$ and to within a factor of 2 throughout. The largest deviation occurs at $\nu kT/\nu_b \approx 1$ and $\sin\theta << 1$, where eq. (17) is the appropriate asymthotic limit. Although we can improve on this agreement by adding new terms to eq. (29), the resultant improvement is not worth the sacrifice in the simplicity of eq. (29).

To summarize, we can calculate the synchrotron emissivity from an isotropic Maxwellian gas at any $kT \leq 1$ using eqs.(11) and (29). Figures 3 compares the results obtained from these equations with results from detailed integration of eq. (1). As is evident,these simple analytic expressions give excellent results not only at $\nu/\nu_b >> 1$, but surprisingly at lower harmonics where the accuracy of the steepest descent method is lower.

2) Power law energy spectrum. Power law spectra are other commonly used spectra in astrophysical problems. Usually power law spectra are defined with a low energy cutoff. To avoid such discontinuities and to simplify our analysis, we assume a spectrum of the form

$$f(\mu,\gamma) = g(\mu)C\left(1 + \frac{\gamma-1}{\epsilon_c}\right)^{-\delta} \tag{30}$$

which converges at low energies;

$$C = n\,\frac{\delta-1}{\epsilon_c}\,, \qquad \int_{-1}^{+1} g(\mu)d\mu = 1. \tag{31}$$

In eq. (29) ϵ_c plays the role of the low energy cutoff. For energies much greater than ϵ_c the spectrum is a power law with index $-\delta$ but tends to a constant at lower energies. The particles can be classified as ultra-relativistic or non-relativistic if $\epsilon_c >> 1$ or $\epsilon_c << 1$. We are interested primarily in cases with $\epsilon_c \approx 1$. For $\epsilon_c >> 1$ the maximum in the function $(f\mathcal{Z}^{2m}/\gamma)$ used in derivation of eq. (11) becomes very broad so that the steepest descent method does not provide a good approximation to the integrals. In this case one must use the well known ultra-relativistic results.

For distributions which are not highly anisotropic (i.e., $d\ln g(\mu)/d\mu << \nu/\nu_b$) we can carry out calculation similar to that for a thermal gas. From eq. (30) we find

$$-\frac{d\ln(f/\gamma)}{d\ln\gamma} = 1 + \frac{\delta\gamma}{\gamma-1+\epsilon_c}\,, \qquad \frac{\gamma^2 d^2\ln(f/\gamma)}{d\gamma^2} = 1 + \frac{\delta\gamma^2}{(\gamma-1+\epsilon_c)^2}\,. \tag{32}$$

If $\nu_b\delta/\nu \ll 1$, then $\gamma_o \gg 1$ so that eqs. (13), (14), (15) and (22) give

$$\beta_o \approx 1, \qquad \gamma_o^2 = \frac{4\nu}{3\nu_b(1+\delta)\sin\theta} , \qquad X = (1+\delta)^{-1/2} , \tag{33}$$

substitution of which in eqs. (11) or (21) gives

$$j_\nu(\theta) = (\sqrt{3}\pi e^2\nu_b/2c)\, C\varepsilon_c g(\beta_o\cos\theta)\sin\theta\, e^{-\frac{(\delta+1)}{2}} \left(\frac{3\varepsilon_c^2\nu_b(\delta+1)\sin\theta}{4\nu}\right)^{\frac{(\delta-1)}{2}} . \tag{34}$$

This expression has exactly the same dependence on ν, ν_b and θ as the ultra-relativistic form (cf.e.g. Ginzberg and Syrovatskii 1964, p. 67). Even the numerical coefficients (which depend primarily on δ) agree to within a factor of 2, although the particle spectrum used in the ultra-relativistic calculations is different than that given by eq. (30).

In the other extreme limit, $\nu_b(1+\gamma)/\nu \gg 1$, which is not of much interest here (see below) unless $\varepsilon_c \ll 1$, from eq. (18) we find that $\gamma_o \approx 1$ and $\beta_o^2 \approx \frac{2\nu\varepsilon_c}{\nu_b\delta}$. This is identical to eq. (27) if we identify ε_c/δ as an equivalent kT. for the power law spectrum. In fact, substitution of this in eq. (11) or (21) gives an emissivity also identical to that in eq. (28) if we replace kT by ε_c/δ and set terms such as $e^{-\nu/\nu_b}(1 + \frac{\nu}{\delta\nu_b})^\delta \approx 1$.

In the intermediate ranges the expressions for γ_o or β_o are more complicated and depend on the value of ε_c. Unlike in the case of thermal spectrum, it is not possible to give a simple expression for $\gamma_o^2 - 1$ because here $\gamma_o^2 - 1$ varies rapidly between its two asympthotic values:

$$\gamma_o^2 - 1 = \frac{4\nu}{3\nu_b(1+\delta)\sin\theta} \qquad \nu_b(1+\delta)/\nu << 1 \qquad (35a)$$

$$\gamma_o^2 - 1 = \frac{2\nu\epsilon_c}{\nu_b(1+\delta/\epsilon_c)} \qquad \nu_b(1+\delta)/\nu >> 1 \qquad (35b)$$

in a manner similar to that found in degenerate Fermi distribution.

In Figures 4 and 5 we plot $(\gamma_o^2 - 1)3\nu_b/(1+\delta)\sin\theta/4\nu$ vs $\nu/\delta\nu_b$ for various values of ϵ_c, $\sin\theta$ and δ. We have attempted to fit the transition region with an emperical formula. We have not found a simple expression which could reproduce the transition region even within a factor of 2 at all θ and ν/ν_b. However, as evident from Figure 4 for $\epsilon_c \sim 1$, the high frequency form for $\gamma_o^2 - 1$ (eq. 35a) provides a good approximation at all frequencies.

On Figure 6 we compare the emissivity obtained using eqs. (11), (12), (22), (32) and (35a) with results from numerical integration of eqs. (1) and (2) for an isotropic power law distribution with $\delta = 3$, $\epsilon_c = 1$ and at $\theta = 45^o$. Just as for the previous case, here again we find excellent agreement between spectra down to the gyro-frequency. Even the absolute value of the emissivities at high ν agree within 30 percent. This slight inaccuracy is because the lefthand side of expression (20) is equal to 4, making this inequality not a strong one.

3) Non-isotropic pitch angle distribution. The two examples given above are valid if the inequality (19) is satisfied. This condition is violated (specially) at low frequencies) if the particles have a highly anisotropic pitch angle distribution. For example, if $f(\mu,\gamma) \propto (\mu-\mu_o)^n$ and if $n \gtrsim 2\nu/\nu_b$, then the method of steepest descent cannot be used in the derivation leading to eq. (6). Nevertheless, for a given n one can always find high enough frequencies where the above relations are valid. However, for a large n this could occur at such high harmonics where the emissivity is negligible.

On the other hand, as mentioned above, because of self-absorption the flux of radiation at lower harmonics is negligible. Assuming that $\tau_\nu \gtrsim 1$ for $\nu \leq 5\nu_b$, then the inequality (19) would be violated if $n \geq 10$ for $\theta = \pi/2$ and for even greater n at $\theta < \pi/2$ (e.g., $n \gtrsim 20$ at $\theta = \pi/4$). It is difficult to envision astrophysical conditions where the pitch angle distribution of the particles would be so highly anisotropic except if this anisotropy is along the magnetic fidld lines, i.e. if particles form a tight beam along the field line. Even if particles are initially accelerated along the field lines, their interaction with the medium (through coulomb collisions, scattering by waves or by inhomogeneities in the magnetic field) could quickly disperse the beam in the pitch angle space so that the inequality (19) is satisfied. However, if collisions are infrequent and the magnetic field is very uniform, then the particle pitch angle could remain small.

To complete our discussion let us consider a highly anisotropic distribution in pitch angle of the form

$$f(\mu,\gamma) = (2/\alpha_o^2)e^{-\alpha_o^2/\alpha_o^2}\, f(\gamma), \qquad \alpha^2 = 1 - \mu^2, \qquad \alpha_o^2\nu/\nu_b << 1 \tag{36}$$

so that the inequality (19) is no longer satisfied. In this case the Taylor series expansion of eq. (5) gives

$$j_\nu(\theta) = \frac{e^2\nu_b n}{c}\left(\frac{\nu}{\nu_b}\right)\int_1^\infty d\gamma f(\gamma)\, Y(1,\gamma,\theta)\left[\frac{e\beta\sin\theta}{2(1-\beta\cos\theta)}\right]^{2m}\int_0^\infty e^{-\frac{\alpha^2}{\alpha_o^2}}\,\frac{2\alpha d\alpha}{\alpha_o^2}\,\alpha^{2m}\,e^{-\frac{\alpha^2}{\alpha_o^2}B}, \tag{37}$$

$$m = \frac{2\nu\gamma}{\nu_b}(1-\beta\cos\theta) \quad \text{and} \quad B = 1 + \frac{\nu\gamma\alpha_o^2}{\nu_b}\left[\frac{\beta^2\sin^2\theta}{4(1-\beta\cos\theta)} + \beta\cos\theta\ln\frac{2(1-\beta\cos\theta)}{\beta\alpha\sin^2\theta}\right]$$

As we shall see below, the largest contribution to the integral comes from electrons with γ of the order unity so that $\nu\gamma\alpha_o^2/\nu_b << 1$. Then we can neglect the slow logarithmic dependence of B in α and integrate eq. (37) to obtain to lowest order in $\nu\alpha_o^2/\nu_b$

$$j_\nu(\theta) = \frac{e^2\nu_b n}{c}\left(\frac{\nu}{\nu_b}\right)\int_1^\infty \gamma\,\sqrt{2\pi m}\left[\frac{e\beta^2\sin^2\theta\gamma\nu\alpha_o^2}{4\nu_b(1-\beta\cos\theta)}\right]^m f(\gamma)d\gamma \quad . \tag{38}$$

Now following our previous procedure, we find that the largest contribution to the integral comes from particles with

$$\beta_o = \cos\theta + O\left[1/\ln(\nu_b/\nu\alpha_o^2)\right] \quad \text{for} \quad \theta \neq \pi/2$$

$$\beta_o^2 = \frac{2}{\ln(\nu_b/\alpha_o^2\nu)} \qquad \text{for} \quad \theta = \pi/2 \ . \tag{39}$$

Substitution of these in eq. (38) gives

$$j_\nu(\theta) = \frac{e^2\nu_b n}{c}\,2\pi\left(\frac{\nu}{\nu_b}\right)\,f(\gamma_o)\,A, \tag{40}$$

when

$$A = \cot\,\exp\left\{-\frac{\nu\sin\theta}{\nu_b}\ln\frac{4\sin\theta\nu_b}{e\cos^2\theta\alpha_o^2\nu}\right\}\Big/\left[\ln\left(\frac{4\nu_b\sin\theta}{e\cos^2\theta\alpha_o^2}\right)\right]^{\frac{1}{2}}, \quad \theta \neq \pi/2 \ ,$$

$$A = \sqrt{2}\,\exp\left\{-\frac{\nu}{\nu_b}\ln\frac{2\nu_b}{e\alpha_o^2\nu}\right\}\Big/\ln(4\nu_b/e^2\nu\alpha_o^2) \quad , \qquad \theta = \pi/2 \ . \tag{41}$$

These are valid for semi-relativistic particles (i.e. for $\nu/\delta\nu_b$ or $\nu kT/\nu_b \geq 1$). The situation becomes more complicated for non-relativistic particles since there exists two small parameters $\alpha_o^2\nu/\nu_b$ and $\nu kT/\nu_b$ so that simple general expressions such as (40) are not possible.

IV. SUMMARY AND CONCLUSION

We have found a simple method of integrating the complex synchrotron emissivity over pitch angles and energies of ensemble of particles.

1) Eqs. (11) to (14) present the results in their most general form. These equations can be used for evaluation of the synchrotron emissivity from any particle distribution which falls fairly rapidly with energy and satisfies the inequality (19). The latter condition restricts the particle pitch angle distribution.

2) In most practical applications (high frequencies and at directions away from the direction of the field lines), eqs. (12) and (21) provide a simple expression for the emissivity.

3) Our result when applied to an isotropic thermal gas of temperature T agrees, in two asympthotic limits $\nu kT/\nu_b \ll 1$ and $\gg 1$, with the earlier results of Trubnikov (1961) even though the method used here is quite different from Trubnikov's, which is restricted to emissivity of a thermal gas. The simple emperical formula, eq. (29), connecting the two asympthotic limits is found to be sufficiently accurate for most practical application.

4) Our result for a power law particle spectrum when extended to high frequencies gives a result identical to that obtained from ultra-relativistic approximation as for its dependence on frequency, angle with respect to the magnetic field directions and on the parameters describing the particle distribution.

5) For isotropic pitch angle distributions, spectra from our simple analytic expressions agree with numerical results to a high degree of accuracy even at lower harmonics, making resort to such detailed numerical calculations unnecessary. The absolute value of the synchrotron emissivity also agrees with 20 to 30 percent at high frequency.

6) We also briefly describe a method for derivation of emissivity from particles with highly anisotropic pitch angle distribution. Polarization and emission at lower harmonics and optically thick regime will be discussed in a subsequent paper.

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FIGURE CAPTIONS

Figure 1. Ratio of exact value of W (eq. 15) to its approximate value (eq. 22) versus γ_o and θ. Note that the ratio deviates from unity significantly when $(\gamma_o - 1)$ is of order unity and only for $\sin\theta << 1$. For most practical purposes eq. (22) will be adequate.

Figure 2. Variation of $\gamma_o - 1$ (read γ_o instead of γ in this figure) with $\nu kT/\nu_b$ for an isotropic Maxwellian gas at temperature kT. For $\sin\theta$ of order unity $\gamma_o - 1$ varies smoothly and monotonically between the two asympthotic values. Deviations from this behavior occurs at very small values of $\sin\theta$. Note that at high frequencies γ_o is independent of value of kT and depends only on $\nu kT/\nu_b$. Note $kT = 0$ means $kT << 1$.

Inset. Ratio of the approximate value of $\gamma_o^2 - 1$ from eq. (29) to its exact value given by eq. (12) for $kT = 1$ and $kT \rightarrow 0$.

Figure 3. The synchrotron emissivity normalized to the Rayleigh-Jean value $(\phi_\nu \propto j_\nu/\nu^2)$ at $\theta = \pi/2$ for $kT \simeq 0.1$ (or 50 keV for an electron). Points from our analytic equations using approximations in eqs. (11) and (29). Solid line fron numerical integration of eq. (1) taken from Bekefi (1966) figure 6.10a, p. 203. Note the excellent agreement to much lower frequencies than expected from the analytic approximation.

Figure 4. Deviation of $\gamma_o^2 - 1$ (γ in the ordinate should be γ_o) from its high frequency asympthotic value versus $\nu/\delta\nu_b$ at $\theta = \pi/2$ for a power law spectrum. Note that for ϵ_c of order unity the asympthotic value provides an excellent approximation throughout. For $\epsilon_c << 1$, $\gamma_o^2 - 1$ changes rapidly between the high and low frequency asympthotic values (eq. 35), with the transition value of $\nu/\delta\nu_b$ being independent of δ and ϵ_c.

Figure 5. Same as Figure 4 but for various values of $\sin\theta$. Note the transition becomes sharper with decreasing values of $\sin\theta$ but occurs at the same value of $\nu/\nu_b\delta$ for all θ.

Figure 6. The synchrotron emissivity from an isotropic and power law particle distribution with $\delta = 3$ and $\epsilon_c \sim 1$ at $\theta = 45^o$. Points from our analytic eq. (11) using the approximation in eqs. (22) and (35a). The solid line from the numerical integration of eq. (1) taken from Ramaty and Petrosian (1972). Note the excellent agreement to much lower frequencies than expected from the analytic approximations.

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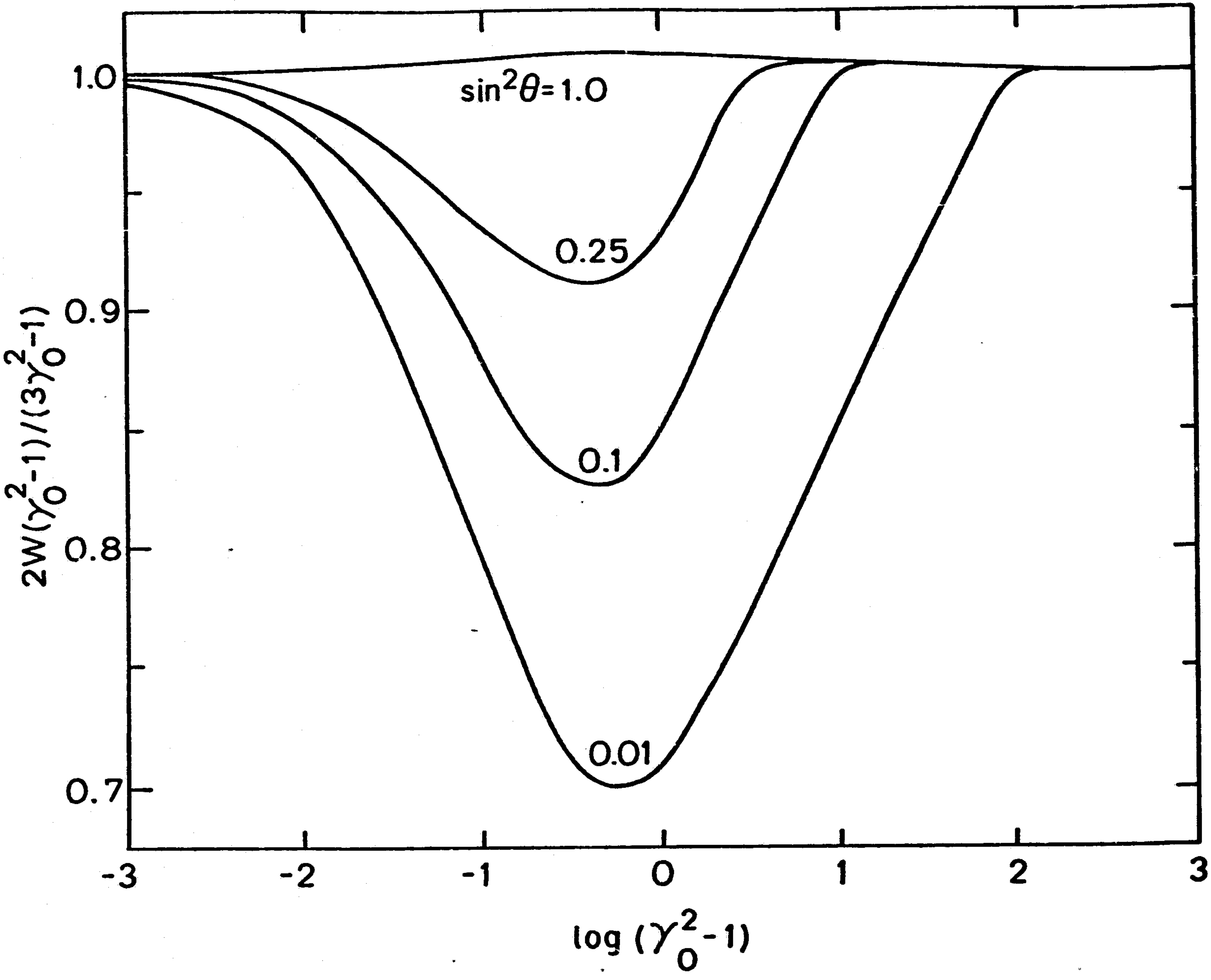


1.0
0.9
0.8
0.7
$2W(\gamma_0^2-1)/(3\gamma_0^2-1)$
$\sin^2\theta = 1.0$
0.25
0.1
0.01
-3
-2
-1
0
1
2
3
$\log(\gamma_0^2-1)$

Figure 1

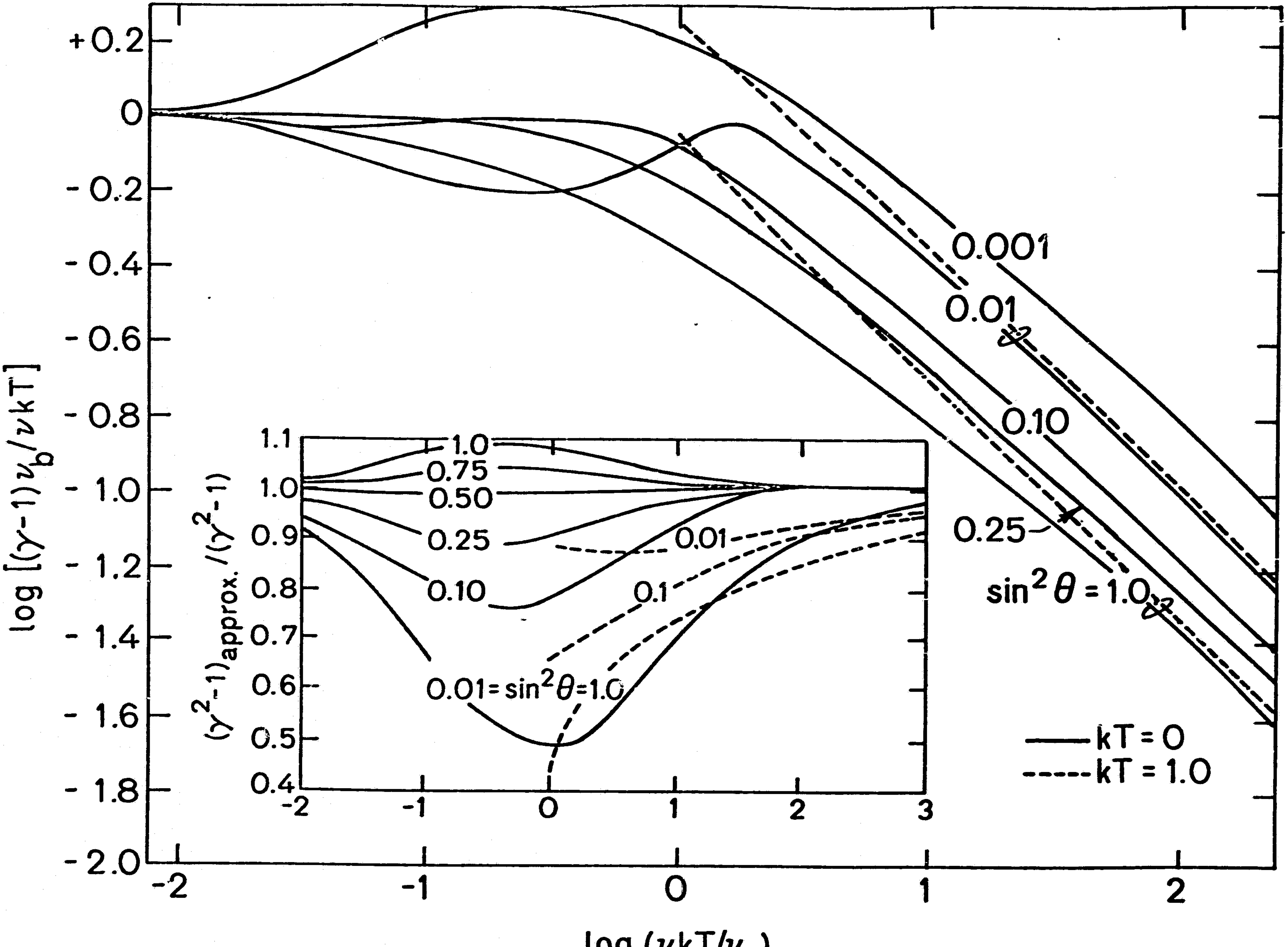

+0.2
0
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2
-1.4
-1.6
-1.8
-2.0
log [(γ-1)ν_b/νkT]
-2
-1
0
1
2
log (νkT/ν_b)
0.001
0.01
0.10
0.25
sin²θ = 1.0
kT = 0
kT = 1.0
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
(γ²-1)approx./(γ²-1)
3
0.75
0.50
0.1
0.01=sin²θ=1.0

Figure 2

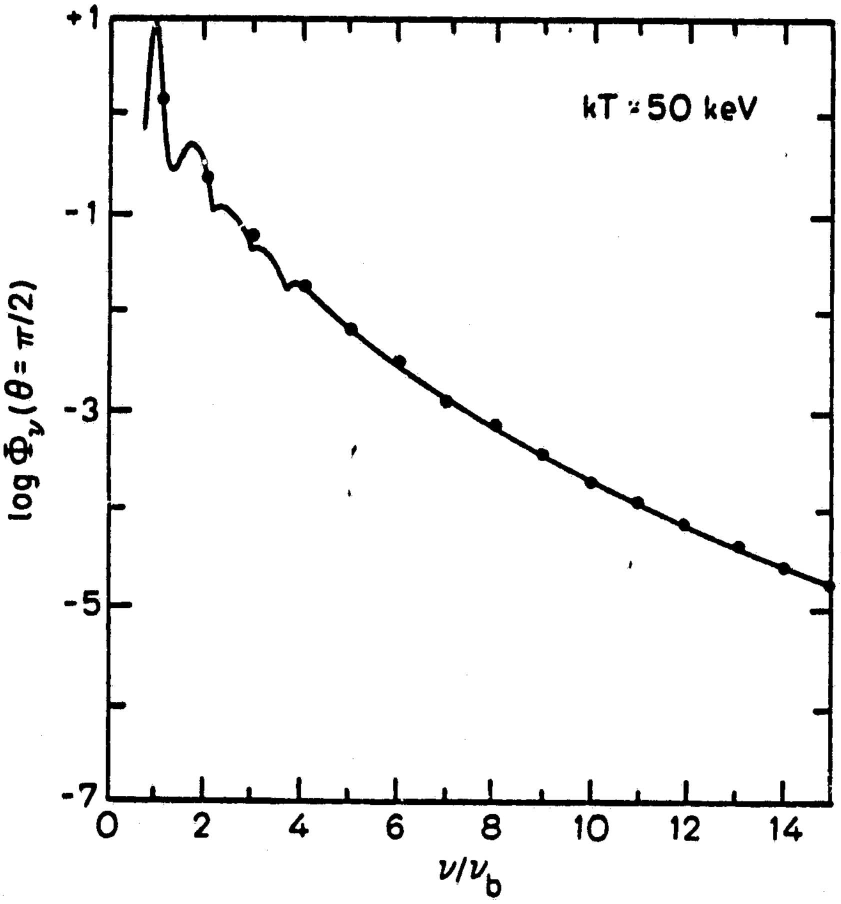
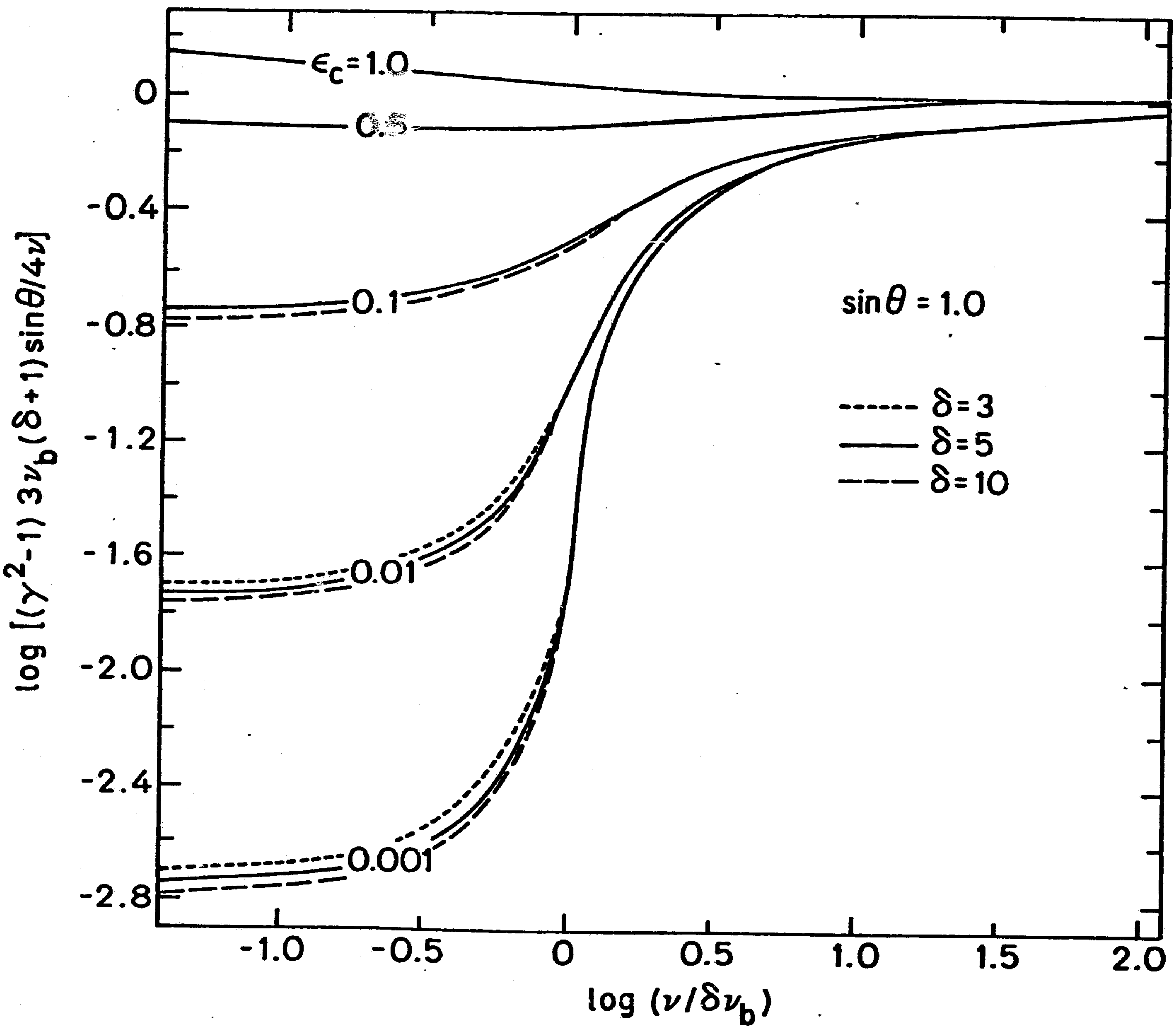
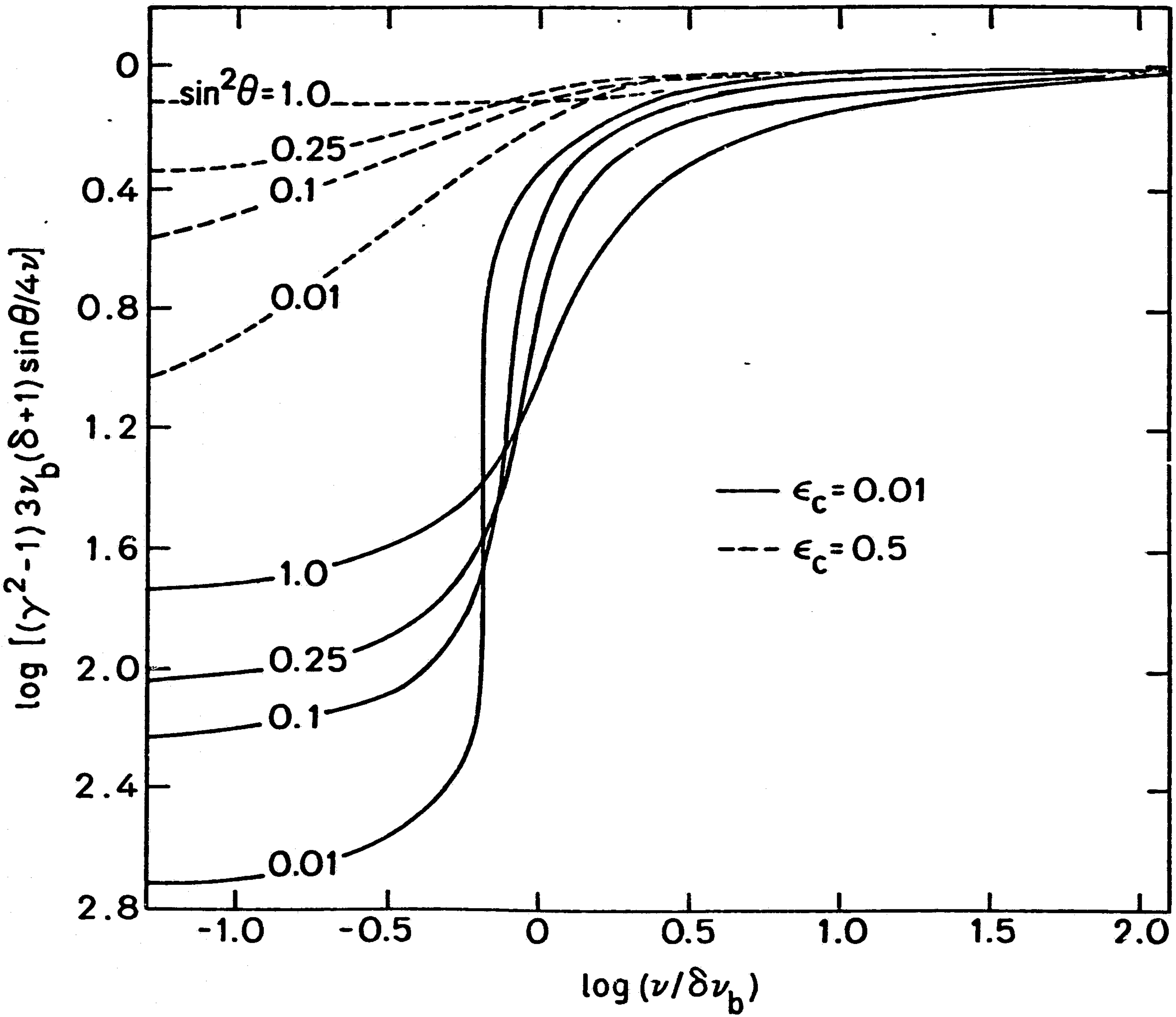


Figure 3



$\epsilon_c = 1.0$
0.5
0.1
0.01
0.001
$\sin\theta = 1.0$
$\delta = 3$
$\delta = 5$
$\delta = 10$
0
-0.4
-0.8
-1.2
-1.6
-2.0
-2.4
-2.8
-1.0
-0.5
0
0.5
1.0
1.5
2.0
$\log\,[(\gamma^2-1)\,3\nu_b(\delta+1)\sin\theta/4\nu]$
$\log\,(\nu/\delta\nu_b)$

Figure 4



sin²θ = 1.0
0.25
0.1
0.01
1.0
0.25
0.1
0.01
ε_c = 0.01
ε_c = 0.5
0
0.4
0.8
1.2
1.6
2.0
2.4
2.8
-1.0
-0.5
0
0.5
1.0
1.5
2.0
log [(γ²-1) 3ν_b(δ+1) sinθ/4ν]
log (ν/δν_b)

Figure 5

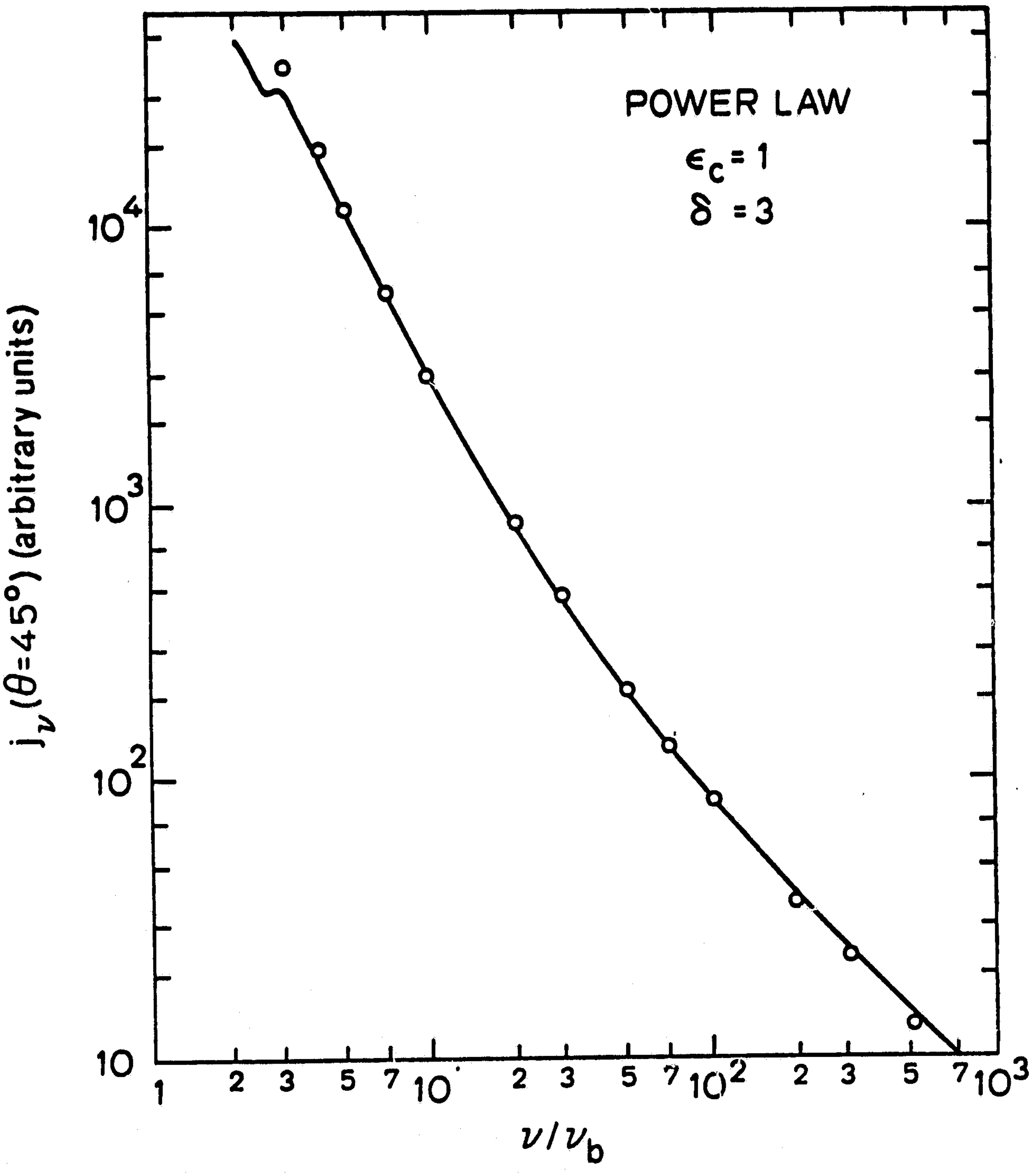

POWER LAW
$\epsilon_c = 1$
$\delta = 3$
$j_\nu(\theta = 45°)$ (arbitrary units)
10^4
10^3
10^2
10
1
2 3 5 7 10 2 3 5 7 10^2 2 3 5 7 10^3
ν/ν_b

Figure 6