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NOMENCLATURE

$C^n(S)$	n times continuously differentiable on S
p	pressure
U	component velocity in x-direction
v	component velocity in y-direction
$\Delta[\ ]$	$\partial^2[\ ]/\partial x^2 + \partial^2[\ ]/\partial^2[\ ]/\partial y^2$
$\Delta\Delta[\ ]$	$\partial^4[\ ]/\partial x^2 + 2\dfrac{\partial^2\partial^2[\ ]}{\partial x^2\partial y^2} + \partial^4[\ ]/\partial y^4$
∂S	boundary of domain S
$[\ ]_\alpha$	$\dfrac{\partial}{\partial\alpha}[\ ]$
$\Rightarrow$	implies

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SUMMARY

For more than the last 20 years there has been a concerted effort to solve the stationary Navier-Stokes equations; however, this has only been successful for a few special cases of primarily academic interest. An alternative approach has been to solve the equations numerically, and then compare the results with experiment. On occasion, such comparisons are in good agreement. However, such results are of dubious value since one has no a-priori way of knowing the relevance of such results until they are explicitly compared against experiment. Therefore, it would seem reasonable to conclude that the present approaches to solving the Navier-Stokes equations are of limited value. Accordingly, it is the purpose of this paper to show that there does, indeed, exist an equivalent representation of the problem that has significant potential in solving such problems. This is due to the fact that this equivalent representation of the problem consists of a sequence of Fredholm Integral Equations of the second kind, and the solving of this type of problem is very well developed. In addition, for the problem in this form, there is an excellent chance to also determine explicit error estimates, since one would now be dealing with bounded linear operators, rather than unbounded.

INTRODUCTION

Ideally, one would like to obtain a numerical solution to the Stationary Navier-Stokes problem that is within some prescribed degree of accuracy of the true solution. Unfortunately, it is not possible to accomplish this objective by just applying some existing numerical technique directly to the Stationary Navier-Stokes equations, since error analysis for nonlinear equations is, for all practical purposes, nonexistent. However, since it is possible to replace the Stationary Navier-Stokes problem by an equivalent sequence of linear partial differential equations (ref. 1), providing some rather general conditions can be met, it is at least theoretically possible to obtain error estimates due to the vast wealth of knowledge known about linear equations. However, even though attaining such error estimates is within the realm of possibility, one should not be misled into thinking that such a task is a small undertaking, for this is most certainly not the case. However, the chances of someone accomplishing this task are significantly greater if the basic sequential problem is transformed into an equivalent form that is more suitable to error analysis. Accordingly, in this paper it will be proven that the said sequential problem can be transformed into a form that has significantly greater changes of yielding explicit error estimates.

GENERAL DEVELOPMENT

The Navier-Stokes equations are given by

$$\left.\begin{aligned} uu_x + vu_y + p_x - \nu\,\Delta u + f_1(x,y) &= 0 \\ uv_x + vv_y + p_y - \nu\,\Delta v + f_2(x,y) &= 0 \end{aligned}\right\} \text{ in } S \tag{1}$$

with boundary conditions

$$u(\partial S) = -b_2(\partial S)\ , \quad v(\partial S) = b_1\ ,$$

where S is a two dimensional Green's domain with surface ∂S,

$$f_1(x,y) \in C^1(S)\ , \quad \text{and} \quad f_2(x,y) \in C^1(S)\ ,$$

or equivalently,

$$\nu\Delta\Delta\psi + \psi_y\,\Delta\psi_x - \psi_x\,\Delta\psi_y + f_{1_y} - f_{2_x} = 0 \quad \text{in} \quad S \tag{2}$$

$$\psi_x(\partial S) = b_1\ , \quad \psi_y(\partial S) = b_2$$

Let

$$\nu\Delta\Delta\psi + \psi_y\,\Delta\psi_x - \psi_x\,\Delta\psi_y + f_{1_y} - f_{2_x}$$

be denoted by $P(\psi)$. Hence (2) can be expressed by

$$P(\psi) = 0\ , \quad \psi_x(\partial S) = b_1\ , \quad \psi_y(\partial S) = b_2$$

where P can be interpreted as a mapping from $C^4(S)$ into $C^0(S)$. As demonstrated in reference 1, solving (2) is equivalent to solving the sequence of equations

$$\left.\begin{aligned} &P(\psi_n) + P'(\psi_n)(\psi_{n+1} - \psi_n) = 0 \\ &\left.\frac{\partial\psi_n}{\partial x}\right|_{\partial S} = b_1\ , \quad \left.\frac{\partial\psi_n}{\partial y}\right|_{\partial S} = b_2 \qquad \text{for } n = 0,1,2,\ .\ .\ .\ , \end{aligned}\right\} \tag{3}$$

providing certain conditions are satisfied (ref. 1), (i.e., primarily that ψ_0, the initial guess is reasonably good), for further insight into the significant latitude on ψ_0, the reader should consult reference 2.

$$P(\psi_0)[\ ] = \nu\Delta\Delta[\ ] + \psi_{0_y}\,\Delta[\ ]_x + \Delta\psi_{0_x}[\ ]_y - \Delta\psi_{0_y}[\ ]_x - \psi_{0_x}\,\Delta[\ ]_y$$

Equivalent to equation (3) is the problem

$$P(\psi_n) + P'(\psi_n)\tilde{\psi}_n = 0\ ,\quad \left.\frac{\partial\tilde{\psi}_n}{\partial x}\right|_{\partial S} = 0\ ,\quad \frac{\partial\tilde{\psi}_n}{\partial y} = 0\ ,\quad n = 0,1,2,\ldots, \qquad (4)$$

by letting

$$\tilde{\psi}_n = \psi_{n+1} - \psi_n$$

Now consider (4) under a slight change in boundary conditions, in particular:

$$P(\psi_m) + P'(\psi_m)\tilde{\psi}_m = 0\ ,\quad \left.\tilde{\psi}_m\right|_{\partial S} = 0\ ,\quad \left.\frac{\partial\tilde{\psi}_m}{\partial n}\right|_{\partial S} = 0\ ,\quad m = 0,1,2,\ldots \qquad (5)$$

If $\tilde{\psi}_m^*$ is a solution to (5), it directly follows that $\tilde{\psi}_m^*$ is also a solution of (4), and vice versa (i.e., equation (4) $\Rightarrow \left.\tilde{\psi}_m\right|_{\partial S}$ = constant, without loss of generality assume constant = 0). Therefore, it is sufficient to focus our attention on equation (5), which in detail is given by

$$\nu\Delta\Delta\tilde{\psi}_m + \psi_{m_y}\,\Delta\tilde{\psi}_{m_x} + \Delta\psi_{m_x}\tilde{\psi}_{m_y} - \Delta\psi_{m_y}\tilde{\psi}_{m_x} - \psi_{m_x}\,\Delta\tilde{\psi}_{m_y} + P(\psi_m) = 0$$

With

$$\tilde{\psi}_m(\partial S) = 0\ ,\quad \tilde{\psi}_{m_n}(\partial S) = 0\ ,\quad m = 0,1,2,\ldots$$

Hence,

$$-\tilde{\psi}_m = \frac{1}{\nu}\int_S G(\psi_{m_y}\,\Delta\tilde{\psi}_{m_x} + \Delta\psi_{m_x}\tilde{\psi}_{m_y} - \Delta\psi_{m_y}\tilde{\psi}_{m_x} - \psi_{m_x}\Delta\tilde{\psi}_{m_y})ds + \frac{1}{\nu}\int_S PG\,ds \qquad (6)$$

where G is the Green's function of the biharmonic Equation [3]

$$\Delta\Delta\phi = 0 \text{ in } S, \text{ with } \phi(\partial S) = 0\ ,\quad \phi_n(\partial S) = 0$$

Lemma:

$$(6) \Leftrightarrow (5)$$

Proof. □

For convenience and clarity, equation (6) will be expressed in the form:

$$\tilde{\psi} = A(\psi_m)\tilde{\psi} + f(\psi_m) \tag{6}$$

where

$$A(\psi_m)[\ ] = \frac{1}{\nu}\int_S G(\psi_{m_y}\ \Delta[\ ]_x + \Delta\psi_{m_x}[\ ]_y - \Delta\psi_{m_y}[\ ]_x - \psi_{m_x}\ \Delta[\ ]_y)ds$$

and

$$f(\psi_m) = \frac{1}{\nu}\int_S P(\psi_m)G\ ds$$

Lemma: If the hypothesis of theorem 1 of reference 1 is satisfied, then there exists a solution $\tilde{\psi}_m^*$ of equation (6) for all m and

$$\psi^* = \lim_{n\to\infty}\sum_{i=0}^{n}\tilde{\psi}_i^* - \psi_0$$

where ψ^* is the unique of equation (2).

Theorem: Under the condition of the above lemma, $\tilde{\psi}_m^*$ is a solution to the Fredholm Integral Equation of the second kind

$$\tilde{\psi}_m^*(x',y') + \int_S \tilde{\psi}_m^*(x,y)K_m(x',y',x,y)dx\ dy = f(\psi_m)(x',y') \tag{7}$$

where

$$K_m(x',y',x,y) = -\Delta G_y(x',y',x,y)\psi_{m_x}(x,y) + \Delta G_x\psi_{m_y} + 2\left[G_{xy}\left(-\psi_{m_{xx}} + \psi_{m_{yy}}\right)\right]$$
$$+ \psi_{m_{xy}}(G_{xx} - G_{yy})$$

Proof: Since $\tilde{\psi}_m^*$ will be a solution of (6) under the hypothesis, it is sufficient to show that a solution to (6) is alos a solution of (7). For clarity, denote (6) in the form

$$\psi^* - A(\psi_0)\psi^* = f(\psi_0)$$

or equivalently,

$$\psi^* - \frac{1}{\nu}\int_S G\left(\psi_{0_y}\ \Delta\psi_x^* + \Delta\psi_{0_x}\psi_y^* - \Delta\psi_{0_y}\psi_x^* - \psi_{0_x}\ \Delta\psi_y^*\right)ds = f(\psi_0)$$

By Green's Lemma

$$-\int_S G\psi_{0_y}\ \Delta\psi^*_x\ dx\ dy = \int_S \Delta\left(G\psi_{0_y}\right)\psi^*_x\ dx\ dy$$

$$= \int_S \Delta\left(G\psi_{0_y}\right)_x \psi^*\ ds - \left[\Delta\left(G\psi_{0_y}\right)\right]\psi^*\Big|_{\partial S}$$

$$= \int_S \Delta\left(G\psi_{0_y}\right)_x \psi^*\ ds$$

$$-\int_S G\ \Delta\psi_{0_x}\psi^*_y\ ds = +\int_S\left(G\ \Delta\psi_{0_x}\right)_y \psi^*\ ds$$

$$+\int_S G\ \Delta\psi_{0_y}\psi^*_x\ ds = -\int_S\left(G\ \Delta\psi_{0_y}\right)_x \psi^*\ ds$$

$$+\int_S G\psi_{0_x}\ \Delta\psi^*_y\ ds = -\int \Delta\left(G\psi_{0_x}\right)_y \psi^*\ ds$$

Therefore (6) reduces to

$$\psi^* + \frac{1}{\nu}\int_S\left\{\left[\Delta\left(G\psi_{0_y}\right)_x\right]\psi^* + G\left(\Delta\psi_{0_x}\right)_y\psi^* - \Delta\left(G\psi_{0_x}\right)_y\psi^* - \left(G\ \Delta\psi_{0_y}\right)_x\psi^*\right\}dS = f(\psi_0)$$

$$\therefore\ \psi^* + \frac{1}{\nu}\int_S \psi^*\left[\Delta\left(G\psi_{0_y}\right)_x + \left(G\ \Delta\psi_{0_x}\right)_y - \Delta\left(G\psi_{0_x}\right)_y - \left(G\ \Delta\psi_{0_y}\right)_x\right]dS = f(\psi_0)$$

$$\left[\Delta\left(G\psi_{0_y}\right)_x + \left(G\ \Delta\psi_{0_x}\right)_y - \Delta\left(G\psi_{0_x}\right)_y - \left(G\ \Delta\psi_{0_y}\right)_x\right]$$

$$= \Delta\left(G_x\psi_{0_y} + G\psi_{0_{xy}}\right) + G_y\ \Delta\psi_{0_x} + G\ \Delta\psi_{0_{xy}} - \Delta\left(G_y\psi_{0_x} + G\psi_{0_{xy}}\right) - G_x\ \Delta\psi_{0_y} - G\ \Delta\psi_{0_{xy}}$$

$$= \Delta\left(G_x\psi_{0_y} - G_y\psi_{0_x}\right) + \left(G_y\ \Delta\psi_{0_x} - G_x\ \Delta\psi_{0_y}\right)$$

$$\Delta\left(G_x\psi_{0_y}\right) = \left(G_x\psi_{0_y}\right)_{xx} + \left(G_x\psi_{0_y}\right)_{yy}$$

$$= \left(G_{xx}\psi_{0_y} + G_x\psi_{0_y}\right)_x + \left(G_{xy}\psi_{0_y} + G_x\psi_{0_{yy}}\right)_y$$

$$= G_{xxx}\psi_{0_y} + G_x\psi_{0_{yyy}} + G_x\psi_{0_{xxy}} + G_{xyy}\psi_{0_y} + 2\left(G_{xx}\psi_{0_{xy}} + G_{xy}\psi_{0_{yy}}\right)$$

$$-\Delta\left(G_y\psi_{0_x}\right) = -\left(G_y\psi_{0_x}\right)_{xx} - \left(G_y\psi_{0_x}\right)_{yy}$$

$$= \left(-G_{xy}\psi_{0_x} - G_y\psi_{0_{xx}}\right)_x + \left(-G_{yy}\psi_{0_x} - G_y\psi_{0_{xy}}\right)_y$$

$$= -G_{xxy}\psi_{0_x} - G_y\psi_{0_{xyy}} - 2G_{xy}\psi_{0_{xx}} - G_y\psi_{0_{xxx}} - G_{yyy}\psi_{0_x} - 2G_{yy}\psi_{0_{xy}}$$

$$\Delta\left(G_x\phi_{0_y} - G_y\psi_{0_x}\right) = 2G_{xy}\left(\psi_{0_{yy}} - \psi_{0_{xx}}\right) + 2\psi_{0_{xy}}\left(G_{xx} - G_{yy}\right) + G_x\left(\psi_{0_y}\right) - G_y\ \Delta\psi_{0_x}$$

$$+ \psi_{0_y}\ \Delta G_x - \psi_{0_x}\ \Delta G_y$$

$$\therefore\ \Delta\left(G_x\psi_{0_y} - G_y\psi_{0_x}\right) + \left(G_y\ \Delta\psi_{0_x} - G_x\ \Delta\psi_{0_y}\right)$$

$$= 2\left[G_{xy}\left(\psi_{0_{yy}} - \psi_{0_{xx}}\right) + \psi_{0_{xy}}\left(G_{xx} - G_{yy}\right)\right] + \psi_{0_y}\ \Delta G_x - \psi_{0_x}\ \Delta G_y$$

Therefore, (6) reduces to

$$\psi^* + \frac{1}{\nu}\int_S \psi^*\left[\psi_{0_y}\ \Delta G_x - \psi_{0_x}\ \Delta G_y + 2\left(G_{xy}\left(\psi_{0_{yy}} - \psi_{0_{xx}}\right) + \psi_{0_{xy}}\left(G_{xx} - G_{yy}\right)\right)\right] ds = f(\psi_0)$$

Hence, (6) $\Leftrightarrow$ (7).

CONCLUSION

Therefore, under the conditions as cited, the sequence of solutions generated by (7) converge to the solution ψ^* of (1) in the following sense:

$$\sum_{i=0}^{n-1} \tilde{\psi}_i^* = \psi_n - \psi_0$$

$$\sum_0^{\infty} \tilde{\psi}_i^* = \psi^* - \psi_0$$

Therefore,

$$\psi^* = \psi_0 + \sum_0^{\infty} \tilde{\psi}_i^*$$

Hence the original problem, equation (1), which falls within the framework of nonlinear operator theory, an area that little is known, has been replaced by equation (5), which falls within the framework of bounded linear operator theory, an area for which there exists a vast wealth of information. Of course, it could be argued that it is theoretically possible that the sequential representation of the solution converges so slowly that the results are of questionable value. However, from all indications this will not be a problem, for as pointed out in reference 1, there are parameters at one's disposal in the method that can be adjusted to speed up the rate of convergence; in fact it was demonstrated in reference 2 that by judicious selection of the variable parameters available they were able to get numerically adequate convergence with just a few iterations.

REFERENCES

1. Gabrielsen, R. E.; and Karel, S.: A Family of Approximate Solutions and Explicit Estimates for the Nonlinear Stationary Navier-Stokes Problem. NASA TM X-62,497, January 1975.

2. Davis, Joseph E.; Gabrielsen, Ralph E.; and Mehta, Unmeel B.: A Solution to the Navier-Stokes Equations Based Upon the Newton Kantorovich Method. NASA TM-78,437.