CONTINUED RESEARCH ON SELECTED PARAMETERS TO MINIMIZE COMMUNITY ANNOYANCE FROM AIRPORT NOISE

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ABSTRACT

Results from continued research on selected parameters to minimize community annoyance from airport noise are reported. First, a review of the initial work on this problem is presented. Then the research focus is expanded by considering multiobjective optimization approaches for this problem. A multiobjective optimization algorithm review from the open literature is presented. This is followed by the multiobjective mathematical formulation for the problem of interest. A discussion of the appropriate solution algorithm for the multiobjective formulation is conducted.

Alternate formulations and associated solution algorithms are discussed and evaluated for this airport noise problem. Selected solution algorithms that have been implemented are then used to produce computational results for example airports. These computations involved finding the optimal operating scenario for a moderate size airport and a series of sensitivity analyses for a smaller example airport.

CONTINUED RESEARCH ON SEL'ˈTED PARAMETERS TO MINIMIZE COMMUNITY ANNOYANC' FROM AIRPORT NOISE

1. INTRODUCTION

This report describes the continued research on selected control parameters for minimizing community annoyance with airport noise. This research provides a means of assessing the relative importance of problem parameters and their impact in terms of annoyance on communities subjected to airport noise.

The second section of this report briefly reviews the work performed on this problem during the initial one year's investigation. Section three provides a literature review for Multiobjective Optimization techniques. Alternate solution algorithms and formulations are discussed in the fourth section of this report. Computational Results for a moderate size airport are reported upon in section five once the mathematical formulation is made available. The concluding section of this report presents a series of sensitivity analyses conducted during the course of this study.

2. INITIAL STUDY ON SELECTED PARAMETERS TO MINIMIZE COMMUNITY ANNOYANCE FROM AIRPORT NOISE

The initial research on the airport noise problem first concerns itself with the development of mathematical models that functionally represent the problem of interest. These models may be classified as corresponding to nonlinear integer mathematical programming problems. The objective function in every case represents the minimization of a noise annoyance measure represented as a function of airport activity. The controls (decision variables) manipulated to achieve this objective were aircraft fleet mix composition for the airport along with runway assignment as a function of the time of day. Considered as constraints were:

1. The demand for flight services
2. The fleet mix composition flexibility
3. A limitation on daily flights
4. Restrictions on the use of specified runways as a function of type of aircraft and/or time of day

The constraints as formulated were linear in terms of the decision variables.

The solution algorithms investigated as methods to obtain solutions for the mathematical models developed were separable programming, relaxation and partitioning procedures. With relaxation and partitioning one solves special structured problems, for which efficient solution procedures exist, in an iterative fashion. Separable programming involves successive linear approximation to obtain solutions to the nonlinear problem of interest. These solution algorithms were applied upon several applications and optimal (i.e., minimum annoyance) operating scenarios were obtained. It was determined that the relaxation technique that involved the solution of successive linear programs was the most efficient algorithm for the type of airport problems faced.

3. MULTIPLE OBJECTIVE FUNCTIONS

It is recognized that working singularily towards the objective of minimizing annoyance from airport noise can degrade other potential goals such as fuel consumption and aircraft time delay. Because of this the additional objectives of fuel consumption and time delay of aircraft are now considered.

3.1 MULTIPLE OBJECTIVE FUNCTION FORMULATION

There are various types of air terminal time delay. Reference (11) identifies four types of such delays. Namely they were identified as holding, path-stretching, early descent/speed control and procedural routing. The delay of interest in this research is that of procedural routing. Probably the most common type of procedural routing is designed to comply with local noise abatement policy. It has been shown in (11) that the additional fuel consumption due to procedural routing is a linear function of the associated time delay. Time delay can be determined for any aircraft knowing its speed and the trajectory selected as well as the minimum time path. Notation:

u_{idj}	the time delay for the $i^{\underline{th}}$ type aircraft arriving from direction d and utilizing trajectory j.
v_{ikdj}	the time delay for the $i^{\underline{th}}$ type aircraft of stagelength k departing for direction d and utilizing trajectory j.
e_{idj}	the additional fuel consumption for the $i^{\underline{th}}$ type aircraft arriving from direction d and using trajectory j.
f_{ikdj}	the additional fuel consumption for the $i^{\underline{th}}$ type aircraft of stagelength k departing for direction d and utilizing trajectory j.
T	the total amount of time delay allowable.
F	the maximum amount of additional fuel consumption allowed.
ND	the number of directions considered.
$X_{ikdj\ell}$	number of departures of type i aircraft with stagelength k going in direction d utilizing trajectory j during period ℓ.
$Y_{idj\ell}$	number of type i arrivals from direction d utilizing trajectory j during period ℓ.

The alternative objective of minimizing the total time delay may be forulated as

$$\sum_{i=1}^{NI} \sum_{j=1}^{NJ} \sum_{d=1}^{ND} \sum_{\ell=1}^{NL} \sum_{k=1}^{NK} (v_{ikdj} \quad X_{ikdj\ell} + u_{idj} \quad Y_{idj\ell}) \equiv 0_2 \qquad (3\text{-}1)$$

The objective of minimizing the total amount of additional fuel consumed may be represented by

$$\sum_{i=1}^{NI} \sum_{j=1}^{NJ} \sum_{d=1}^{ND} \sum_{\ell=1}^{NL} \sum_{k=1}^{NK} (f_{ikdj} \quad X_{ikdj\ell} + e_{idj} \quad Y_{idj\ell}) \equiv 0_3 \qquad (3.2)$$

The additional constraints considered were,

$$0_2 \leq T \qquad (3.3)$$

$$0_3 \leq F \qquad (3.4)$$

Constraint (3.3) restricts the amount of time delay allowed where as (3.4) specifies an upper limit on the additional fuel which may be consumed.

To complete the multiple objective formulation equations the original model (Appendix A) would be changed to account for the extra dimension (direction) on the decision variables.

A multiobjective solution algorithm has of yet not been implemented since the noise objective function was of primary concern to the users of this research. An alternative to a multiobjective solution algorithm would be to treat the objectives of time delay and additional fuel consumption with constraint equations (3.3) and (3.4) and then perform extensive sensitivity analyses by varying the right-hand-side of these equations. A fuel minimization computer run was accomplished and the results reported on in Appendix E.

3.2 LITERATURE REVIEW

The consideration of the alternative objective functions of minimizing fuel consumption and minimizing passenger and or freight delay leads one to consider the potential of vector optimization solution procedures. The following literature review is meant to provide background material for the consideration of multiple objective optimization.

Hwang (12) has classified multiple objective decision methods (MODM) into four categories based on the availability of preference information from a decision maker. They are,

1) no articulation of preference information
2) "a priori" articulation of preference information
3) progressive articulation of preference information (iterative methods)
4) "a posteriori" articulation of preference information (generating techniques)

The last three categories are of primary interest and will be discussed in greater detail.

3.2.1 "A Priori" Articulation of Preference Information

These methods depend on the ordering or prioritizing of the objectives of interest. This helps to reduce the size of the solution set which must be considered. The primary difference between the various methods that belong to this MODM category is in the nature of the preference information required. Several classes of methods that belong to this category are now discussed.

Utility Function Method

This method is based upon obtaining cardinal information from the decision maker in the form of specific objective preference levels. A satisfactory solution can be obtained if the proper utility function is available and is used in conjunction with the decision maker's preferences. The prime difficulties in using such methods are the specification of the appropriate utility function and the computational burden that increases exponentially with the number of objectives considered.

Goal Programming

Goal programming methods seek to minimize the absolute deviations of the objectives from predetermined targets or goals. Originally proposed by Charnes and Cooper (3) many different approaches have been developed. Lee (13), Ignizio (12) and Cohen (4) describe many such methods.

The various approaches differ in the formulation of the surrogate objective function. Advantages of Goal Programming is that it is relatively efficient computationally and requires less effort by the decision maker in specifying preference information. The major disadvantage of Goal Programming is that any solution determined is a function of the goals set by the decision maker. Unless the decision maker is careful in establishing a priority structure one can arrive at an infeasible or dominated solution.

Electre Method

The Electre Method, proposed by Roy (18) is based upon an "outranking" relationship for non-inferior solutions. A partial ordering of the solutions is obtained based upon two conditions; a concordance condition and a discordance condition. The function of the concordance condition is to indicate a tolerance for errors in comparing candidate solutions. The discordance condition is used to indicate which solutions need to be compared.

3.2.2 Iterative Methods

All of these methods basically utilize two steps in obtaining solutions. First a technique is employed to generate a solution. Secondly, methods are used to elicit the decision makers reaction to the given solution and then use this reaction to modify the problem by a modified priority structure. Two such methods are now reviewed.

Surrogate Worth Trade-off Method (SWT)

This method proposed by Haines, Hall and Freedman (10) consists of (1) generation of candidate (nondominated) solutions which are used to form trade-off functions and (2) the search for a preferred solution using a surrogate worth function. The trade-off function is formed from the relative trade-offs of marginal increases or decreases from any two objective functions. The surrogate worth function is the decision maker's assessment of how much one prefers trading one objective for another. The trade-off functions are generated for any two objectives assuming that all remaining objectives are fixed. This means it provides limited information to the decision maker about the nondominated solution set.

Step Transform Evaluation Method (STEM)

The use of Linear Programming (LP) with Multiple Objective Functions Step Method has been proposed by Benayoun and others (1). This method involves,

1) construction of a pay-off table after optimizing with p objective functions separately

2) solving an LP which is "nearest" in the MINIMAX case to an ideal solution.

3) specifying a trade-off relationship by the decision maker.

Other interactive methods are described by Geoffrion (9), Zionts and Wallinius (20).

3.2.3 Generating Techniques

Generating Techniques involve the generation of the nondominated solution set. These methods seek to develop an exact or approximate representation of the entire solution set. Because of this they are computationally intensive and feasible only for problems with a small number of objectives.

The Parametric method proposed by Gal and Nedoma (8) and Zadeh (19) and the Epsilon-Constraint method due to Marglin (14) are examples of Generating Techniques that have been applied to multiple objective decision problems.

Another approach sometimes used is to relegate all objectives except the one considered most important to constraint equation status. The problem is first solved with the single selected objective function. Post optimality analysis is then employed by varying the right-hand-sides of the constraints that originally corresponded to alternate objective functions. Thus, an entire family of solutions may be generated as a function of the alternate objective functions. It is this approach that is presently being implemented to deal with the alternate objectives of additional fuel minimization and passenger/freight delay.

In order to select a solution strategy for a given multi-objective problem it is necessary to compare the effectiveness of potential solution techniques for the specific problem of interest. Normally one would evaluate candidate solution algorithms on the basis of criteria such as

1) computational burden
2) quantification of tradeoffs
3) availability of decision making information

Such research has been initiated and will continue over the coming year.

4. ALTERNATIVE FORMULATIONS AND SOLUTION ALGORITHMS

The research into the design and development of efficient solution algorithms for minimizing community annoyance from airport noise has been continued. The reason for this is that the simplest and most efficient solution algorithm possible is desirable. However, such an algorithm should produce superior results for the problem of interest.

The solution algorithm implemented for the airport noise problem (see Appendix B for a description) does not guarantee a global optimum. However it does appear capable of always generating very good solutions for the airports studied so far. In some instances the use of the selected solution algorithm would be quite costly and time consuming. Because of this several alternatives to the implemented solution algorithm are proposed, discussed and limited computational experience reported.

For airports with a large fleet of many different servicing aircraft that utilize many different departure and arrival trajectories the number of decision variables and constraint equations may become prohibatively large. In such cases a simplified p oblem may be solved. The simplification is that the distinction bet een aircraft types is dropped and all operations are for a "typical" aircraft. The corresponding mathematical model is of the same form as the original model (see Appendix A) except that the type of aircraft designation is removed. Therefore the decision variables are

$X_{j\ell}$ = number of departures utilizing trajectory j during time period ℓ.

$Y_{j\ell}$ = number of arrivals utilizing trajectory j during time period ℓ.

For comparison purposes an airport with the following characteristics would have 2510 decision variables for the original model whereas it would involve only 60 variables with the simplified model.

14	different	types of aircraft
15	"	trajectories
2	"	time periods
5	"	departure stage lengths

The use of this simplified model would become even more attractive if it imperative to consider nonlinear constraints or problems with more than two time periods. For example if constraint (A.3) were to be considered for an actual airport exercise we could not use the original solution algorithm since it relies on all constraints being linear. Another advantage associated with this simplified model is that nonlinear optimization solution algorithms might be successfully applied if the number of variables and constraints were not excessive.

During recent years the parametric transformation technique known as the Augmented Lagrangian (ALAG) Penalty Function Technique has gained recognition as one of the most effective type of methods for solving contrained minimization problems. In the opinion of many researchers in this field, the ALAG penalty function technique is the best method available for solving problems with nonlinear constraints in the absence of special struc- (2). The disadvantage of the method are negligible and the advantages are strong, especially the lack of numerical difficulties and the ease of using the unconstrained minimization routine. The method has global convergence at an ultimately superlinear rate, the computational effort per minimization falls off rapidly, initial starting point need not be feasible and the function is defined for all values of the parameters (2).

The ALAG penalty function technique is a balance between the classical penalty function technique and the Lagrangian primal-dual method which are both parametric transformation techniques. The design of this method was motivated by efforts to overcome the numerical instability of the penalty function technique near the solution (12), (18) and attempts to eliminate the "duality gap" in nonconvex programming (6). This ALAG penalty function nonlinear optimizing algorithm has been implemented to obtain solutions for the simplified model described previously. The documentation for this implemented solution algorithm is presented in Appendix C. Computational results for this algorithm are presented in Appendix D.

Another alternative solution algorithm would be to utilize a linearized version of the nonlinear objective function. This would amount to stopping after step 1 of the solution algorithm reported upon in the first years report (see Appendix B). The advantage of this approach is that one is solving Linear Programming problems for which very efficient solution techniques exist. The main disadvantage is that the surrogate objective function is just a linearized representation of the nonlinear objective function. Computational results, thus far, indicate that this surrogate objective function is an effective substitute for the nonlinear objective function. That is, it doesn't require near the computational effort and produces solutions nearly as good as the nonlinear algorithm.

Another useful formulation of the community response objective function would involve the goal of minimizing the maximum annoyance experienced by any population segment within the airport vicinity. The motivation for such a goal is that the annoyance due to airport noise is shared in a somewhat uniform manner by the effected population segments.

Several possibilities that have been considered are minimize the maximum,

1. noise impact index contribution from the various population segments
2. noise exposure forecast (NEF) or day-night level (L_{dn}) for the population segments
3. population weighted NEF or population weighted L_{dn}

The formulation of such goals can be accomplished by letting

n = the number of population segments

RI_k = response index for population segment k.

then

$$\text{minimize } \{\text{maximum}[RI_1, RI_2, \ldots, RI_n]\}$$

which may be written as

$$\begin{array}{ll} \text{minimize} & Z \\ \text{subject to:} & Z \geq RI_k \quad k : 1,2 \ldots, n \\ \text{where} & Z = \text{maximum } [RI_1, RI_2, \ldots, RI_n] \end{array}$$

An objective function utilizing a population weighted NEF has been formulated and run for the example airport number two described in (7). The computational results are presented in Tables F10 and F11 of Appendix F.

5. APPLICATION AIRPORT FOR SOLUTION PROCEDURE

This section presents an example, medium sized airport to which the suggested solution procedure has been applied. The analysis contained herein is limited to commercial airline traffic for the particular airport of interest. A series of tables and figures are first presented defining parameter values needed in constructing the appropriate mathematical formulation. The profiles utilized are the standard ones described previously in (7). The computational results obtained from application of the solution algorithm to this example airport are displayed in Table 5.7.

TABLE 5.1

Demand for Incoming Flights

Runway Tracks	Day	Night
(15)24	6	1
(13)12 R	90	10
(14)30 L	98	11

TABLE 5.2

Take-off Flight Demands

Runway Tracks		Day Stage				Night Stage			
		1	2	3	4	1	2	3	4
(6)	6	2	2	0	0	1	0	0	0
(11,12)	24	2	2	0	0	1	0	0	0
(1,2,3,4,5)	12R	46	25	8	6	6	2	0	2
(7,8,9,10)	30L	60	32	10	8	7	2	1	2

TABLE 5.3

Available Aircraft for Arrivals

Type	Day	Night
DC-9-32	79	9
727-200	90	10
DC-8-55	23	3
-1011	12	1

TABLE 5.4

Available Aircraft for Departures

Type	Day	Night
DC-9-32	79	9
727-200	90	10
DC-8-55	23	3
L-1011	12	1

FIGURE 5.1

Ground Tracks for Example Airport

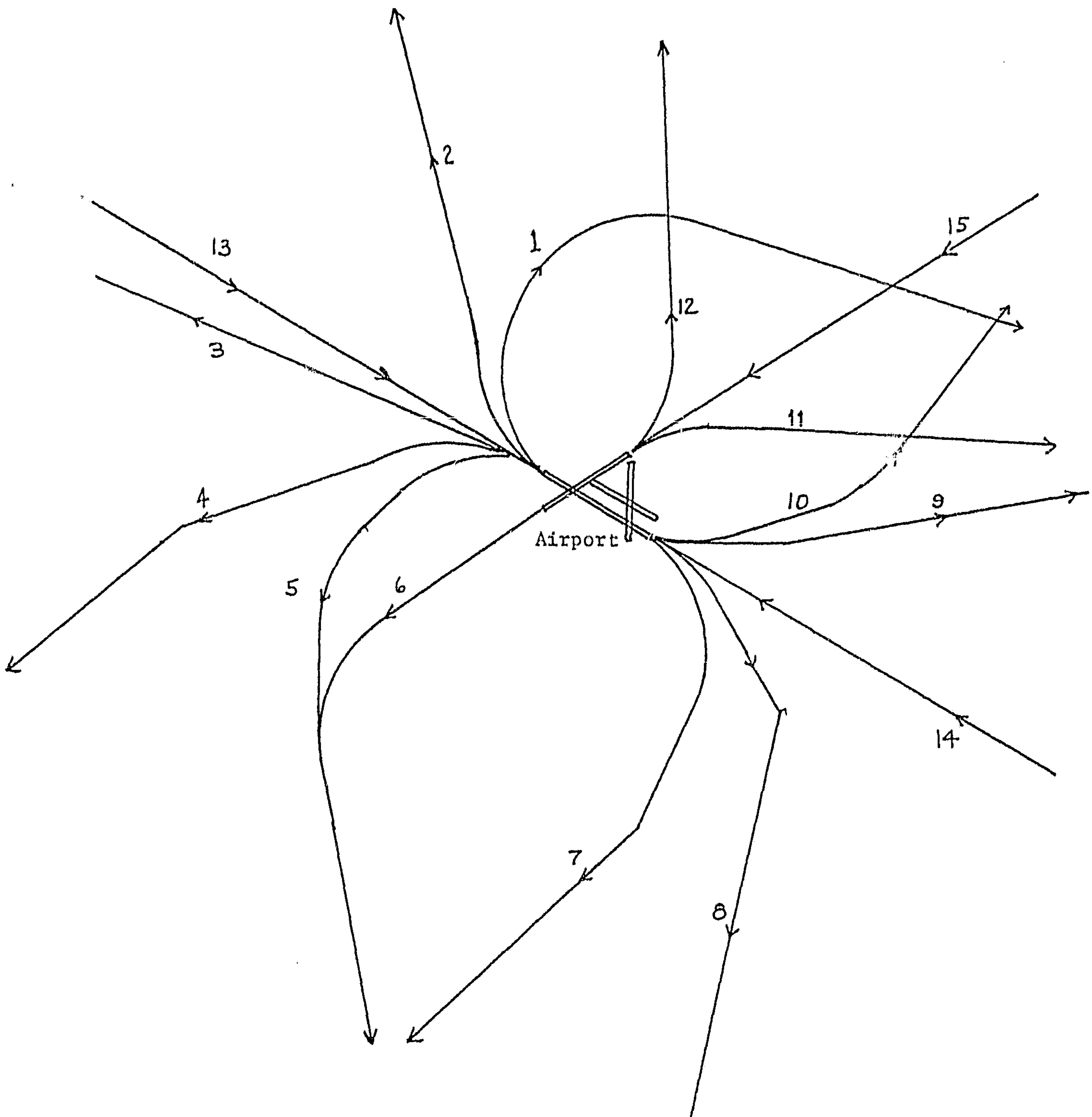

TABLE 5.5

Population of Areas in Vicinity of Example Airport

Area	Pop	Area	Pop
1	20078	34	879
2	16176	35	2127
3	8022	36	1161
4	8732	37	2591
5	11887	38	7447
6	12317	39	4294
7	4987	40	3585
8	5822	41	4401
9	7053	42	8060
10	19680	43	6075
11	9494	44	13940
12	3579	45	9578
13	5596	46	10334
14	946	47	6659
15	96	48	17991
16	8918	49	17268
17	5339	50	6714
18	9475	51	3048
19	532	52	8144
20	8874	53	13093
21	3799	54	5193
22	3996	55	5359
23	453	56	21192
24	1250	57	13785
25	1570	58	5640
26	4109	59	16827
27	5833	60	17408
28	6908	61	7977
29	10716	62	15239
30	11813	63	10034
31	2399	64	36311
32	15610	65	10852
33	10661		

FIGURE 5.2

Population Areas in the Vicinity of Example Airport

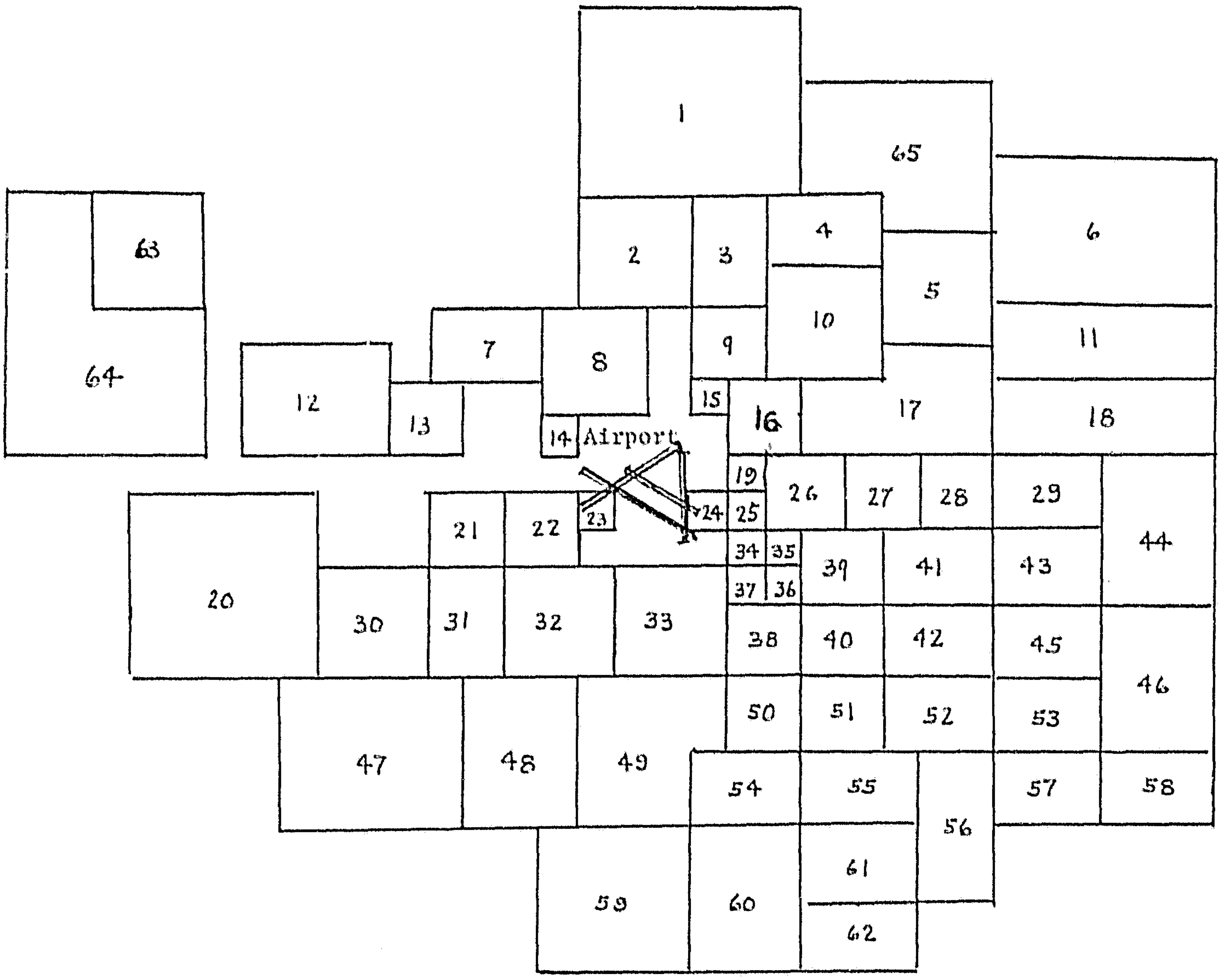

TABLE 5.6

Types of Aircrafts Considered

Aircraft Type	Number of Stagelengths*
DC-9-32	3
B727-200	4
DC-8-55	4
L-1011	4

* The stage length of a departing aircraft is a measure of the distance to the next destination. Stage lengths 1(0-500 nautical miles), 2(500-1000 nautical miles), 3(1000-1500 nautical miles) and 4(1500-2500 nautical miles)

TABLE 5.7

COMPUTATIONAL RESULTS FOR APPLICATION AIRPORT (DEPARTURES)

NII 0.17723

Number of Operations	Aircraft DC-9	727	DC-8	L-1011	Stage Length	Track Number	Time Day	Period Night
46		X			1	5	X	
10		X			2	5	X	
8		X			3	5	X	
6		X			4	5	X	
15			X		2	5	X	
4		X			1	5		X
2		X			2	5		X
2		X			4	5		X
5			X		1	5		X
2			X		1	6	X	
2			X		2	6	X	
60	X				1	8	X	
9	X				2	8	X	
10	X				3	8	X	
11		X			2	8	X	
8		X			4	8	X	
12				X	2	8	X	
7	X				1	8		X
2	X				2	8		X
1		X			3	8		X
1		X			4	8		X
1				X	4	8		X
2			X		1	12	X	
2			X		2	12	X	
1			X		1	12		X

TABLE 5.7 (continued)

COMPUTATIONAL RESULTS FOR APPLICATION AIRPORT (ARRIVALS)

NII 0.17723

Number of Operations	Aircraft DC-9	727	DC-8	L-1011	Stage Length	Track Number	Time Period Day	Night
73	X					13	X	
4		X				13	X	
13			X			13	X	
8	X					13		X
2			X			13		X
86		X				14	X	
12				X		14	X	
10		X				14		X
1				X		14		X
6	X					15	X	
1	X					15		X

6. SENSITIVITY ANALYSES

It should be recognized that the minimization solution to the noise problem under study is optimal only with respect to the specific model being used to represent the real problem, and such a solution becomes a reliable guide for action only after it has been verified as performing well for other reasonable representations of the problem as well. Furthermore, the model parameters (particularly the right-hand-side of constraint equations) are set as a result of operating policy and these decisions should be reviewed after seeing their consequences on what can be achieved.

For these reasons alone it is important to perform sensitivity analyses to investigate the effect on the optimal solution of the various problem parameters taking on a range of possible values. At times there will be some parameters that can be assigned any reasonable value without affecting the optimality of the problem of interest. However, there are usually other parameters which if perturbed in value would yield new optimal solutions. This is particularly serious if the change in a problem parameter results in a substantial change in the value of the objective function. Therefore, the basic goal of the on-going sensitivity analyses is to identify those particularly sensitive parameters, so that special care may be exercised in estimating them and in selecting solution algorithms that perform well for the most likely values of such parameters.

The sensitivity analyses conducted as a continuation of research into the noise problem are:

1. Variations in the demand for flight services
2. Fleet mix composition changes
3. Changes in operations by time period
4. Variations in objective function formulation

These sensitivity analyses considere and conducted for a particular airport are reported on in Appendix F. These results are summarized in Tables 6.1 and 6.2.

For comparison purposes the results from Tables E2-E5 should be grouped together. These Tables display the results for variations in demand for flight services. It is found that in a vast majority of the cases that

northwest (NW) through easterly (E)	flights	are	assigned	track	9
easterly through southwesterly (SW)	"	"	"	"	4
southwest through northwesterly	"	"	"	"	3

This is one indication that the assignment of aircraft to tracks is somewhat insensitive to variations in demand for flight services.

Tables E6-E9 results may also be grouped together for comparison purposes. These Tables corresponded to the use of only one type of aircraft for all commercial service. These cases display more variation in assignment, however, somewhat general observations can be made.

For departures

NW	through	E	flights	are	assigned	track	9
E	"	SW	"	"	"	"	4
SW	"	NW	"	"	"	"	1

For approaches

NW	through	E	flights	are	assigned	track 9
E	"	SW	"	"	"	tracks 4 and 7

Even though greater variation is displayed in the results from Tables E6-E9 (due entirely to variations in approach assignments) it is conjectured that the assignments for this example are relatively insensitive to the type of aircraft considered. The reason for this is that there are a great

TABLE 6.1

SUMMARY OF SENSITIVITY ANALYSES RESULTS

FOR APPROACHES

Tracks	FROM TABLE								
	E2	E3	E4	E5	E6	E7	E8	E9	E10
1 (day)								54	
1 (night)								89	
2 (day)					54		11		24
2 (night)					11	22			
3 (day)	55	54	53	61		54	24		30
3 (night)	89	69	83		78	67	89		89
4 (day)	37	35	36	28			9	38	
4 (night)									
5 (day)									
5 (night)									
6 (day)									3
6 (night)									
7 (day)		3		2	38	38	14		35
7 (night)									
8 (day)									
8 (night)									
9 (day)	8	8	8	9	8	8	8	8	8
9 (night)	11	31	17		11	11	11	11	11

Entries in Table are percentage use
for each period.

TABLE 6.2

SUMMARY OF SENSITIVITY ANALYSES RESULTS

FOR DEPARTURES

Tracks	FROM TABLE								
	E2	E3	E4	E5	E6	E7	E8	E9	E10
1 (day)					21	21	21	21	15
1 (night									
2 (day)	13								
2 (night)									
3 (day)	8	22	18	20					6
3 (night)									
4 (day)	56	57	61	54	56	56	56	56	56
4 (night)	50	50	50		50	50	50	50	50
5 (day)									
5 (night)									
6 (day)		2.5		2					
6 (night)									
7 (day)									
7 (night)									
8 (day)		2.5		2					
8 (night)									
9 (day)	23	16	21	22	23	23	23	23	23
9 (night)	50	50	50		50	50	50	50	50

Entries in Table are percentage use
for each period.

many alternative solutions that give just as good or nearly as good an objective function value as the optimally selected solution.

Even though the results seem to indicate that the optimal solution to the airport noise problem is insensitive to the variations considered, for the one example airport, this should not lead one to conclude this would be true of all airports or all possible variations. The primary reason for the insensitivity displayed in the reported results is that the capacity constraints for the approach and departure trajectories were never binding in any of the cases considered. For a larger busier airport these capacity constraints might be binding for some flight service scenarios and one would expect greater variability in sensitivity analyses. If an annoyance metric were available that accounted for human annoyance as a function of the time frequency of flights and this metric was utilized to obtain noise optimal flight schedules then one would expect greater variation in sensitivity analyses. Tables E6 through E9 provide the corresponding computational results.

Table E10 describes the results obtained when the objective function was changed from the one originally given in equation (E1) to that of minimizing the maximum (minimax) annoyance for the various population segments considered. The use of such an objective formulation should result in a dispersion of annoyance more uniformly to the population areas surrounding the airport. The solution of a population weighted NEF minimax objective function is given in Table E11.

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APPENDIX A

AIRPORT NOISE MINIMIZATION MODEL

Reference (7)

A.1 Notation

CONSTANTS

A = area designation

i = type of aircraft designation

j = ground track designation

k = stage length designation

ℓ = time period designation

P = total affected population

R = runway number

NA = number of areas

ND = number of runways for departure aircraft

NI = number of types of aircraft

NJ = number of different ground tracks

NK = number of stage lengths

NL = number of time periods

NR = number of runways

NV = number of runways for arriving aircraft

N_d = limitation on number of operations that use ground track d

N_p = limitation on number of operations in time period P

N_r = limitation on number of operations for runway r

N_s = limitation on number of operations with stage length s

N_t = limitation on number of airport operations per day

P_A = population in area A

R_j = set of ground tracks associated with runway R

V_A = community response index critical value for area A

$NX_{i\ell}$ = number of type i aircraft available for take-offs during time period ℓ

$NY_{i\ell}$ = number of type i aircraft available for landing during time period ℓ

N_{tsdp} = number of type t aircraft with stage length s with ground track d that are required during time period p

NOISE PARAMETERS

EPNL = effective perceived noise level

$EPNL_{ijA}$ = effective perceived noise level for arriving aircraft of type i corresponding to ground track j experienced in area A

$EPNL_{ikJA}$ = effective perceived noise level for departing aircraft of type i with stage length k corresponding to ground track j experienced in area A

L_A = A-Weighted sound pressure level

L_{dn} = day-night level

L_{eq} = equivalent sound level

NEF = noise exposure forecast

RI_A = community response index for area A

$W(RI_A)$ = weighting factor as a function of community response index for area A

VARIABLES

$X_{ikj\ell}$ = number of departures of type i aircraft with stage length k utilizing trajectory j during period ℓ

$Y_{ij\ell}$ = number of arrivals of type i aircraft utilizing trajectory j during period ℓ

$$\text{minimize} \left[\sum_{A=1}^{NA} (P_A/P)(10) \log_{10} \left\{ \sum_{i=1}^{NI} \sum_{k=1}^{NK} \sum_{j=1}^{NJ} \left(10^{\frac{EPNL_{ijkA}}{10} - 8.8} \right. \right. \right.$$

$$(X_{ikj1} + 16.67X_{ikj2}) + 10^{\frac{EPNL_{ijA}}{10} - 8.8} \cdot$$

$$\left. \left. \left. (Y_{ij1} + 16.67Y_{ij2}) \right) - 1 \right\} \right] \tag{A.1}$$

This is the form of the objective function used in solving the example problems reported on in Section IV.

A.2 Constraints

Airport authorities seeking to reduce noise and still service passenger demand for their airports may impose various related operating constrants.

A.2.1 Flight Limitations

For any given airport flight limitations may be imposed for many different reasons and consist of many different constraints on aircraft operations. The flight limitations of primary interest in this research are those operational constraints that may be imposed in an attempt to improve the noise enviroment around an existing airport. The operational flight constraints formulated in this research are:

1. The annoyance as measured by a specified community response index may not exceed critical values for specified communities in the airport vicinity.
2. The number of certain types of aircraft that may operate into and out of a given airport cannot exceed some upper limit.
3. The number of certain types of operations such as the take-offs of long stage length aircraft may be limited.

4. Certain runways may only be used during certain periods or a limitation on the number of operations per period for any given runway may be specified.

5. Certain trajectories that correspond to specified ground tracks may only be used during certain periods or a limitation on the number of operations per period for any given ground track may be specified.

6. The total number of operations per time period may be constrained.

These flight limitations are now formulated mathematically:

1. The annoyance as measured by some community response index for a given area, A, must not exceed some critical value, V_A

$$RI_A \leq V_A \qquad A = 1, \ldots, NA \tag{A.2}$$

For example if NEF is selected as the community response index. then (3.3) would appear as a function of the decision variables as

$$10 \log_{10} \sum_{i=1}^{NI} \sum_{k=1}^{NK} \sum_{j=1}^{NJ} \text{antilog} (((EPNL_{ikja} + 10 \log_{10}(X_{ikj1} + 16.67X_{ikj2})$$

$$-8.8) + EPNL_{ijA} + 10 \log_{10}(Y_{ij1} + 16.67Y_{ij2}) - 88))/10) \leq V_A$$

$$A = 1, \ldots, NA \tag{A.3}$$

2. The number of aircraft type t allowed to operate in and out of a given airport is limited to N_t operations per day

$$\sum_{k=1}^{NK} \sum_{j=1}^{NJ} \sum_{\ell=1}^{NL} X_{tkj\ell} + \sum^{NJ} \sum^{NL} Y_{tj\ell} \leq N_t \quad t = 1, \ldots, NI \tag{A.4}$$

3. The number of take-offs with stage length s is limited to N_s operations per day

$$\sum_{i=1}^{NI} \sum_{j=1}^{NJ} \sum_{\ell=1}^{NL} X_{isj\ell} \leq N_s \quad s = 1, \ldots, NK \tag{A.5}$$

4. The number of operations for runway r is limited to Nr per day

$$\sum_{i=1}^{NT} \sum_{k=1}^{NK} \sum_{j\epsilon R_j} \sum_{\ell=1}^{NL} X_{ikj\ell} + \sum_{i=1}^{NI} \sum_{j\epsilon R_j} \sum_{\ell=1}^{NL} Y_{ij\ell} \leq N_r,$$

$$r = 1, \ldots, NR \tag{A.6}$$

5. The number of operations corresponding to ground track d is limited to N_d per day

$$\sum_i \sum_k \sum_\ell X_{ikd\ell} + \sum_{i=1}^{NI} \sum_{\ell=1}^{NL} Y_{id\ell} \leq N_d \quad d = 1, \ldots, NJ \tag{A.7}$$

6. The total number of operations in time period p must not exceed N_p

$$\sum_{i=1}^{NI} \sum_{k=1}^{NK} \sum_{j=1}^{NJ} X_{ikjp} + \sum_{i=1}^{NI} \sum_{j=1}^{NJ} Y_{ijp} \leq N_p \quad p=1, \ldots, NI \tag{A.8}$$

A.2.2 Aircraft Availability

Only a limited number of the various types of aircraft servicing an airport will be available during each time period for either landing or departure. This may be expressed analytically as

$$\sum_{j=1}^{NJ} Y_{ij\ell} \leq NY_{i\ell} \qquad i = 1, \ldots, NI; \quad \ell = 1, \ldots, NL \tag{A.9}$$

$$\sum_{k=1}^{NK} \sum_{j=1}^{NJ} X_{ikj\ell} \leq NX_{i\ell} \qquad i = 1, \ldots, NI; \quad \ell = 1, \ldots, NL \tag{A.10}$$

A.2.3 Passenger - Aircraft Demand

Passenger demand may be established for any given airport. Then passenger demand may be translated into aircraft demand. Such demands may take the form of requiring at least N_{tsdp} type t aircraft operations of stage length s, utilizing ground track d during time period p.

$$X_{tsdp} \geq N_{tsdp} \qquad \begin{array}{l} t = 1, \ldots, NI; \quad s = 1, \ldots, NK; \\ d = 1, \ldots, NJ; \quad p = 1, \ldots, NL \end{array} \tag{A.11}$$

This is one of the simplest ways in which passenger demand may be accounted for. More elaborate demand relationships could be derived and utilized if desired.

Collectively Equations (A.2) through (A.11) define a mathematical model for this research. This model may be classified as a nonlinear integer mathematical programming model. Solution techniques for such a model are discussed in Section IV and Appendix B.

APPENDIX B

Original Solution Algorithm

Reference (7)

See Appendix A for the obj ective function form used.

Step 1

$$\text{minimize } Z^L = \sum_{A=1}^{NA} (P_A/P)(10)(S_A - 1) \tag{B.1}$$

subject to all linear constraints where,

$$S_A = \sum_{i=1}^{NI} \sum_{k=1}^{NK} \sum_{j=1}^{NJ} \{10^{\left(\frac{EPNL_{ikjA}}{10} - 8.8\right)} (X_{ikj1} + 16.67X_{ikj2})$$

$$+ 10^{\left(\frac{EPNL_{ijA}}{10} - 8.8\right)} (Y_{ij1} + 16.67Y_{ij2})\} \tag{B.2}$$

letting,

$$c_{ikjA} = 10^{\left(\frac{EPNL_{ikjA}}{10} - 8.8\right)} \text{ and } d_{ijA} = 10^{\left(\frac{EPNL_{ijA}}{10} - 8.8\right)}$$

$$S_A = \sum_{i=1}^{NI} \sum_{k=1}^{NK} \sum_{j=1}^{NJ} \{c_{ikjA}(X_{ikj1} + 16.67X_{ikj2})$$

$$+ d_{ijA} (Y_{ij1} + 16.67Y_{ij2})\} \tag{B.3}$$

This will provide a feasible solution to the problem but in no sense guarantees an optimum. Instead of the function given in Equation (4.1), the objective function should be

$$\text{minimize } Z^N = \sum_{A=1}^{NA} (P_A/P)(10)(\log_{10}(S_A) - 1) \tag{B.4}$$

However the surrogate objective function has provided very good solutions to the original problem for the few example problems solved.

The reason for such a surrogate objective function is that it is linear in the decision variables and subject to linear constraints, hence corresponds to a linear programming problem. With sophisticated computer implementation, linear programming solution techniques are capable of solving very large problems. For example, problems involving several thousand linear constraint equations and tens of thousands of variables are within the realm of possibility for the very efficient linear programming solution algorithms implemented on modern computers. However it should also be pointed out that to produce solutions to such large problems requires extensive efforts in data preparation, manipulation of the linear programming computer code, and interpretation of computational results.

Step 2

Obtain a truncated Taylor series expansion about the solution point from Step 1, say $\underline{S}^*$

$$u(\underline{S}) = Z^N(\underline{S}^*) + \nabla Z^N(\underline{S}^*)\ (\underline{S} - \underline{S}^*) \qquad \text{(B.5)}$$

Now minimize (4.5) subject to the original linear constraints. This corresponds to solving another linear programming problem. Denote the solution to this linear program (4.5) as $\underline{S}^1$. Since $u(\underline{S})$ is constructed from the gradient of Z^N at $\underline{S}^*$, an improved solution point can be secured only if $u(\underline{S}^1) < u(\underline{S}^*)$. This will not guarantee that $Z^N(\underline{S}^1) < Z^N(\underline{S}^*)$ unless $\underline{S}^1$ is in the immediate neighborhood of $\underline{S}^*$. However, given $u(\underline{S}^1) < u(\underline{S}^*)$ there must exist a point, say $\underline{S}^2$, on the line segment between $\underline{S}^*$ and $\underline{S}^1$ such that $Z^N(\underline{S}^2) < Z^N(\underline{S}^*)$. To determine $\underline{S}^2$ one solves

$$\underset{\alpha}{\text{minimize}}\ Z^N(\underline{S}^* + \alpha(\underline{S}^1 - \underline{S}^*$$

$$Z^N(\underline{S}^2) = Z^N(\underline{S}^* + \gamma(\underline{S}^1 - S^*)) = \underset{0\leq\alpha\leq 1}{\text{minimize}}\ Z^N(\underline{S}^* + \alpha(\underline{S}^1 - \underline{S}^*)) \qquad (B.6)$$

Set $\underline{S}^*$ equal to $\underline{S}^2$ and repeat Step 2 as many times as required to obtain the stopping condition, $u(\underline{S}^1) \geq u(\underline{S}^*)$. At this point no further improvement is possible.

APPENDIX C

COMPUTER PROGRAM: ALAG 1 DOCUMENTATION

LANGUAGE: FORTRAN

TECHNICAL REFERENCES: (6), (17)

ALAG 1

1. PURPOSE: To minimize a function $F(\bar{x}) = f(X_1, \ldots, X_n)$ subject to both equality and inequality constraints. Derivatives of all functions must be supplied in a user subroutine entitled ALAGB (see item 5). An initial estimate of the solution (not necessarily feasible) must be specified. This computer program is developed from algorithm of section .

2. USE: CALL ALAG1 (N,M,K,X,EPS, AKMIN, DFN, MAXFN, IPR1, IPR2, IW, MODE)

N An INTEGER set to the number of variables n ($N \geq 2$).

M An INTEGER set to the total number of constraints m ($M \geq 1$).

K An INTEGER set to the total number of equality constraints k.

X A REAL array of N elements in which the initial estimate of the solution must be set. ALAGI returns the solution $\bar{x}$ in X.

EPS A REAL array of N elements, in which the tolerances for the unconstrained minimizations must be set. EPS (I) should be set so that EPS (I)/X (I) = AKMIN, roughly speaking.

AKMIN A REAL number in which the relative error tolerance required in the constraint residuals must be set. ALAGI will exist when max$\{|c_i(x)|/$ scaling factor for $c_i\} \leq$ AKMIN for the active constraints $\{i\}$.

DFN A REAL number in which the likely reduction in $F(\underline{\bar{x}})$ must be set. This is done in the same way as for QNWTA - see the QNWTA description.

MAXFN An INTEGER in which the maximum number of calls of ALAGB on any one unconstrained minimization must be set.

IPR1 An INTEGER controlling the frequency of printing from ALAG1. Printing occurs every IPR1 iterations, except for details of increases to the c_i which are always printed. No printing at all occurs (except for error diagnostics) if IPR1 = 0.

IPR2 An INTEGER controlling the frequency of printing from QNWTA. IPR2 should be set as described in the QNWTA documentation.

IW An INTEGER giving the amount of storage available on COMMON/ALAGL/W(.). Set to 2500 unless wishing to change the restrictions (see Section 5).

MODE An INTEGER controlling the mode of operation of ALAG1. If any positive definite estimate if available of the hessian matrix of the penalty function, se' |MODE| = 2 or 3, otherwise set |MODE| = 1 (see QNWTA description). If estimates of the σ_i and Θ_i parameters are available (see item 8) set MODE < , otherwise set MODE > 0. A normal setting for a one-off job with no information available is MODE = 1.

3. LABELED COMMON AREAS:

Certain labeled COMMON areas must be declared and set on entry to ALAG1.

COMMON/ALGAGE/C(150) Set scale factors (>0) for the constraints in C(1), C(2), ...,C(M). Choose the magnitude of these scale factors to give an indication of the constraints evaluated about the initial approximation $\underset{\sim}{x}$. If any constraints are violated by an amount greater in modulus than that which is set, then the setting is increased accordingly. These scale factors are transferred to C(M+1), C(M+2),.....,C(@M) by ALAG1.

COMMON/ALAGF/GC(25,50) Set the derivatives of any linear constraints on entry rather than in ALAGB. This is the most efficient and the numbers are not disturbed. The manner of setting is described in item 4.

COMMON/ALAGG/T(150) If MODE < 0 is used, then set the parameters $\Theta_1, \Theta_2, \ldots, \Theta_m$ in T(1), T(2),...,T(M) and the parameters $\sigma_1, \sigma_2, \ldots, \sigma_m$ in T(M+1),T(M+2),...,T(2M) The meaning of these parameters may be found in section of this report.

COMMON/ALAGI/G2P(325) If $|MODE| = 2$ or 3 set the estimated hessian matrix of the penalty function in G2P(1),..., G2P(N·(N+1)/2). The manner of setting is that described in QNWTA under the heading MODE.

Local storage for ALAG1 is through labeled COMMON areas. These have been set on the assumption that $N \leq 25$ and $M \leq 50$. If it is desired to remove either or both of these restrictions, then it is necessary to increase the storage available in some or all of these areas. This can be done by defining the named COMMON areas in the users MAIN with the increased storage settings, in which case the extra storage will be effective throughout the whole program. The complete list of labeled COMMON used by ALAG1 and the corresponding values of N and M are as follows.

COMMON/ALAGC/F,M,K,IS,MK,NU		independent of N and M
"	D/G(50)	2N
"	E/C(150)	3M
"	F/GC(25,50)	N,M
"	G/T(150)	3M
"	H/GP(50)	μ (μ = max(M,N))
"	I(G2P(325)	N·(N+1)/2
"	J/V(50)	μ
"	K/WW(150)	3μ
"	L/W(2500)	μ^2
"	M/ZZ(100)	2μ
"	N/LT(100)	2M

4. ACCURACY: This iterative algorithm terminates normally when the following convergence condition is met: $\max \{|c_i(x)|/\text{scaling factor for } c_i\} \leq$ AKMIN for i an element of the set of active constraint indices. A diagnostic message for abnormal termination is printed when the program is unable to achieve the requested accuracy. This may be due to (i) a mistake in programming ALAGB, (ii) there is no feasible point (in which case $\sigma_i \to \infty$ and $c_i \to$ constant $\neq 0$), (iii) EPS has been set too large relative to AKMIN, (iv) the problem is too ill-conditioned.

OTHER ROUTINES: ALAG1 requires the use of ALAGB, ALAGZ, BQDMA, MULDA, MULDB, MULDE, and QNWTA

ALAGB: USER SUBROUTINE The user must define a subroutine headed by

```
SUBROUTINE ALAGB(N,M,X)
REAL X(1)
COMMON/ALAGC/F
COMMON/ALAGD/G(50)
COMMON/ALAGE/C(150)
COMMON/ALAGF/GC(25,50)
```

This subroutine takes the vector X and sets

(1) $F(\bar{x})$ in F; (2) $c_1(\bar{x}),\ldots, c_m(\bar{x})$ in C(1),...,C(M);

(3) $(\partial F/\partial X_1,\ldots,\partial F/\partial X_n)|_{\bar{x}}$ in G(1),...,G(N);

(4) $(\partial c_i/\partial X_1,\ldots,\partial c_i/\partial X_n)|_{\bar{x}}$ in GC(1,I),...

GC (N,I) for I = 1,...,M.

ALAGZ: This subroutine evaluates the augmented function comprised of the original objective function and penalty terms that is to be optimized.

SUBROUTINE ALAGZ (N, X, PHI, GPHI)

N and X as previously defined.

PHI is the value of the augmented function evaluated at X.

GPHI is the gradient of the augmented function evaluated at X.

BQDMA: The purpose of BQDMA is to find the values that minimize a quadratic of n variables subject to upper and lower bounds on some or all of the variables subject to upper and lower bounds on some or all of the variables.

The quadratic is defined by

$$Q(X) = 1/2\ X^t\ AX - B^t X$$

Subject to:

$$BL_i \leq X_i \leq BU_i \qquad i = 1,\ldots N.$$

SUBROUTINE BQDMA (N,A,IA,B,BL,BU,X,Q,LT,K,G)

N — an INTEGER which must be set by the user to the number of variables.

A — a REAL, two dimensional array, each dimension at least N; the elements in the upper triangle A(I,J) $I \leq J \leq N$ must be set by the user to the corresponding A_{ij} in (1), and will remain untouched by the subroutine. Elements A(I,J) $N \geq I > J$ are used as working space.

IA — an INTEGER giving the first dimension of A in the statement which assigns space to A.

B — a REAL array of at least N elements. The user must set b(I). B is not overwritten by BQDMA.

ALAGZ: This subroutine evaluates the augmented function comprised of the original objective function and penalty terms that is to be optimized.

SUBROUTINE ALAGZ (N, X, PHI, GPHI)

N and X as previously defined.

PHI is the value of the augmented function evaluated at X.

GPHI is the gradient of the augmented function evaluated at X.

BQDMA: The purpose of BQDMA is to find the values that minimize a quadratic of n variables subject to upper and lower bounds on some or all of the variables.

The quadratic is defined by

$$Q(X) = 1/2\ X^t\ AX - B^tX$$

Subject to:

$$BL_i \leq X_i \leq BU_i \qquad i = 1,\ldots,N.$$

SUBROUTINE BQDMA (N,A,IA,B,BL,BU,X,Q,LT,K,G)

N — an INTEGER which must be set by the user to the number of variables.

A — a REAL, two dimensional array, each dimension at least N: the elements in the upper triangle A(I,J) $I \leq J \leq N$ must be set by the user to the corresponding A_{ij} in (1), and will remain untouched by the subroutine. Elements A(I,J) $N \geq I > J$ are used as working space.

IA — an INTEGER giving the first dimension of A in the statement which assigns space to A.

B — a REAL array of at least N elements. The user must set B(I). B is not overwritten by BQDMA.

BL	a REAL array of at least N elements. The user must set BL(I) to the lower bound on the I^{th} variable. If the bound is non-existent, set it to a very small number like -1E75. BL is not overwritten by BQDMA.
BU	a REAL array of at least N elements. The user must set BU(I) to the upper bound on the I^{th} variable. If the bound is non-existent, set it to a very large number. BU is not overwritten by BQDMA.
X	a REAL array of at least N elements. BQDMA returns the solution in X(I).
Q	a REAL variable in which BQDMA returns the solution value of the quadratic.
LT	an INTEGER array of at least N elements, set by BQDMA to a permutation of the integers 1,2,...,N (see K and G below)
K	an INTEGER set by BQDMA to the number of free variables at the solution (those not on their bounds). These are the variables LT(1), LT(2),...,LT(K).
G	a REAL array of at least 3*N elements. G(1),....,G(N) are set by BQDMA to the gradient evaluated at the solution point. G is indirectly addressed so that G(I) contains the gradient with respect ot the LT(I) variable, whence G(1),...,G(K) will be found to be zero. G(N +1),...,G(3*N) are used by BQDMA as working space.

MULDA is a subroutine for use in problems which involve the addition or subtraction of rank one matrices $\sigma \underline{zz}^T$ to positive definite or semi-definite symmetric matrices A stored in factored form $A = LDL^T$, such that the resulting N xN matrix

$$A = A + \sigma \underline{zz}^T$$

is also known to be positive definite or semi-definite. Note that L is lower triangular with $\ell_{ii} = 1$, and D is diagonal with $d_i \geq 0$.

SUBROUTINE MULDA (A, N, Z, [illegible], W, IR, MK , EPS)

A — A REAL one dimensional array of N*(N+1)/2 elements in which the matrix $A = LDL^T$ must be given in factored form. The order in which elements of L and D are stored is $d_1, \ell_{21}, \ell_{31}, \ldots, \ell_{N1}, d_2, \ell_{32}, \ldots, \ell_{N2}, \ldots \ldots d_{N-1}, \ell_{N,N-1}, d_N$. The factors of the matrix $\tilde{A} = A + \sigma \overline{zz}^T$ will overwrite those of A on exit.

N — An INTEGER ($N \geq 1$) which must be set to the dimension of the problem.

Z — A REAL one dimensional array of N elements in which the vector $\overline{z}$ must be set. The array Z is overwritten by the routine.

SIG — A REAL variable in which the scalar σ must be set. SIG is not restricted to $\pm$., but if SIG<0 then it must be known from other considerations that A is positive definite or semi-definite, apart from the effects of round-off error.

W — A REAL array of N elements. If SIG>0 then W is not used, and the name of any one dimensional array can be inserted in the calling sequence. If SIG>0 then W is used as work space. In addition for SIG>0 it may be possible to save time by setting in W the vector v defined by $L\bar{v}=\bar{z}$. The ways in which this can occur are described under MK below.

IR — An INTEGER to be set that |IR| is the rank of A. If the rank of $\tilde{A}$ is expected to be different from that of A, set IR$\le$0. On exit from MULDA, IR($\ge$0) will contain the rank of $\tilde{A}$.

MK — An INTEGER to be set only when SIG<0, as follows. If the vector $\bar{v}$ defined by $L\bar{v}=\bar{z}$ has not been calculated previously, set MK=0. If MULDA has been used previously to calculate $A^{-1}\bar{z}$, then $\bar{v}$ is a by-product of this calculation and is stored in the W parameter of MULDE. In this case transfer v to the W parameter of MULDA and set MK=1. If $\bar{z}$ has been calculated as $\bar{z} = A\bar{u}$ for some arbitrary vector $\bar{u}$ using MULDD, then again $\bar{v}$ is a by-product of the calculation and is available in the W parameter of MULDD. In this case (or any other in which $\bar{v}$ is known) set $\bar{v}$ in the W parameter of MULDA and set MK=2.

EPS — A REAL variable to be set only when SIG<0 and $\tilde{A}$ is expected to have the same rank as A. In certain ill-conditioned cases a non-zero diagonal element of $\tilde{D}$ might become so small as to be indeterminate. Two courses of action are possible. One is to introduce a small perturbation in order that $\tilde{A}$ keeps the same rank as A. This is the normal course of action and is achieved by setting EPS equal to the relative machine precision ε. The other course of action is to let the rank of $\tilde{A}$ be one less than the rank of A. This is achieved by setting EPS equal zero.

MULDB - factorizes a positive definite symmetric matrix given in A. This matrix is then used in MULDA.

SUBROUTINE MULDB (A, N, IR)

A Must contain the elements of A in the order $a_{11},a_{21},\ldots,a_{N1},a_{22},a_{32},\ldots a_{N2},\ldots,a_{N-1,N-1},a_{NN}$; that is as successive columns of its lower triangle). One exit A will be overwritten by the factors L and D in the form described in MULDA.

N Order of the matrix A.

IR An INTEGER set by MULDB to the rank of the factorization. If the factorization has been performed successfully IR=N will be set. If IR<N then the factorization has failed because A is not positive definite (possibly due to round-off error). In this case the factors of a positive semi-definite matrix

of rank IR will be found in A. However the results of this calculation are unpredictable, and MULDB should not be used in an attempt to factorize positive semi-definite matrices.

MULDE calculates the vector $\bar{z}^* = A^{-1}\bar{z}$ where A is in factored form

SUBROUTINE MULDE (A, N, Z, W, IR)

A	Must be set in factored form.
N	Order of the matrix A.
Z	A REAL array of N elements to be set to the vector $\bar{z}$. On exit Z contains the vector $\bar{z}^* = A^{-1}\bar{z}$.
W	A REAL array of N elements which is set by MULDE to be vector $\bar{v}$ defined by $L\bar{v}=\bar{z}$. If this vector is not of interest, replace W by Z in the calling sequence to obviate the need to supply extra storage.
IR	An INTEGER which must be set to the rank of A.

QNTWA finds the minimum of a function $F(\bar{x})$ of several variables given that the gradient vector can be calculated. This routine is based upon a quasi-Newton method described by Fletcher in (F8).

SUBROUTINE QNWTA (FUNCT, N, X, F, G, H, W, DFN, EPS, MODE, MAXFN, IPRINT, IEXIT)

FUNCT	An IDENTIFIER of the users subroutine.
N	An Integer to be set to the number of variables ($N \geq 2$).
X	A REAL ARRAY of N elements in which the current estimate of solution is stored. An initial approximation

must be set in X on entry to QNWTA and the best estimate obtained will be returned on exit.

F — A REAL number in which the best value of $F(\bar{x})$ corresponding to X above will be returned.

G — A REAL ARRAY of N elements in which the gradient vector corresponding to X above will be returned. Not to be set on entry.

H — A REAL ARRAY of N*(N+1)/2 elements in which an estimate of the hessian matrix is stored. The matrix is represented in the product form LDL where L is a lower triangular matrix with unit diagonals and D is a diagonal matrix. The lower triangle of L is stored by columns in H excepting that the unit diagonal elements are replaced by the corresponding elements of D. The setting of H on entry is controlled by the parameter MODE.

W — A REAL ARRAY of 3*N elements used as working space.

DFN — A REAL number which must be set so as to give QNWTA an estimate of the likely reduction to be obtained in $F(\underline{\bar{x}})$. DFN is used only on the first iteration so an order of magnitude estimate will suffice. The information can be provided in different ways depending upon the sign of DFN which should be set in one of the following ways:

DFN>0 the setting of DFN itself will be taken as the likely reduction to be obtained in $F(\bar{x})$.

DFN=0 it will be assumed that an estimate of the minimum value of $F(\bar{x})$ has been set in argument F, and the likely reduction in $F(\bar{x})$ will be computed according to the initial function value.

DFN<0 a multiple |DFN| of the modulus of the initial function value will be taken as an estimate of the likely reduction.

EPS A REAL ARRAY of N elements to be set on entry to the accuracy required in each element of X.

MODE An INTEGER which controls the setting of the initial estimate of the hessian matrix in the parameter H. The following settings of MODE are permitted.

MODE=1 An estimate corresponding to a unit matrix is set in H by QNWTA.

MODE=2 QNWTA assumes that the hessian matrix itself has been set in H by columns of its lower triangle, and the conversion to LDL^T form is carried out by QNWTA. The hessian matrix must be positive definite.

MODE=3 QNWTA assumes that the hessian matrix has been set in H in product form. This is convenient when using the H matrix from one problem as an initial estimate for another, in which case the contents of H are passed on unchanged.

MAXFN An INTEGER set to the maximum number of calls of FUNCT permitted.

IMPRINT An INTEGER controlling printing. Printing occurs every |IPRINT| iterations and also on exit, in the form

Iteration No, No of calls of FUNCT, IEXIT (on exit only).

Function value

X(1),X(2),...,X(N) 8 to a line

G(1),G(2),...,G(N) 8 to a line

The values of X and G can be suppressed on intermediate iterations by setting IPRINT<0. All intermediate printing can be suppressed by setting IPRINT=MAXFN+1. All printing can be suppressed by setting IPRINT=0.

IEXIT An INTEGER giving the reason for exit from QNWTA. This will be set by QNWTA as follows:

IEXIT=0 (MODE=2 only). The estimate of the hessian matrix is not positive definite.

IEXIT=1 The normal exit in which |DX(1)|<EPS(I) for all I=1,2,...N, where DX(I) is the change in X on an iteration.

IEXIT=2 $G^T DX \geq 0$. Not possible without rounding error. Probable cause is that EPS is set too small for computer word length.

IEXIT=3 FUNCT called MAXFN times.

APPENDIX D

COMPUTATIONAL RESULTS FOR ALAG1 ALGORITHM

COMPUTATIONAL RESULTS FOR ALAGI OPTIMIZATION ALGORITHM

NII = 0.15276

Number of Operations	Arrival	Departure	Aircraft 727	Aircraft 737	Stage Length	Track Number	Time Period Day	Time Period Night
1	X		X			2	X	
3	X		X			3	X	
1	X		X			3		X
10	X			X		3	X	
5	X			X		3		X
2	X			X		5	X	
4	X		X			7	X	
12	X			X		7	X	
1	X			X		7		X
1	X		X			9	X	
1	X		X			9		X
4	X			X		9	X	
1	X			X		9		X
1		X	X		1	2		X
4		X	X		3	3	X	
10		X		X	2	3	X	
1		X		X	2	3		X
2		X		X	1	5	X	
2		X	X		1	7	X	
13		X		X	2	7	X	
1		X		X	3	6	X	
2		X	X		1	9	X	
5		X		X	2	9	X	

APPENDIX E

Sensitivity Analyses For Example Airport

The mathematical model employed for nearly all the sensitivity analyses is presented below. The only results that were not derived through the use of this model are those in Tables E10 and E11. These results are for a "minimax" objective function described in section four.

The mathematical model for this second example problem is

$$\text{minimize} \sum_{A=1}^{31} (P_A/P)(10)(\log_{10}(S_A) - 1) \tag{E.1}$$

subject to:

$$\sum_{i=1}^{NI} (X_{i111} + X_{i121} + X_{i131}) \geq 3 \tag{E.2}$$

$$\sum_{i=1}^{NI} (X_{i211} + X_{i231}) \geq 3 \tag{E.3}$$

$$\sum_{i=1}^{NI} (X_{i311} + X_{i321} + X_{i331}) \geq 2 \tag{E.4}$$

$$\sum_{i=1}^{NI} (X_{i141} + X_{i161} + X_{i171}) \geq 6 \tag{E.5}$$

$$\sum_{i=1}^{NI} (X_{i241} + X_{i251} + X_{i261} + X_{i271}) \geq 11 \tag{E.6}$$

$$\sum_{i=1}^{NI} (X_{i341} + X_{i351} + X_{i361} + X_{i371}) \geq 5 \tag{E.7}$$

$$\sum_{i=1}^{NI} (X_{i181} + X_{i191}) \geq 2 \tag{E.8}$$

$$\sum_{i=1}^{NI} (X_{i281} + X_{i291}) \geq 4 \tag{E.9}$$

$$\sum_{i=1}^{NI} (X_{i381} + X_{i391}) \geq 3 \qquad (E.10)$$

$$\sum_{i=1}^{NI} (X_{i242} + X_{i252} + X_{i262} + X_{i272}) \geq 1 \qquad (E.11)$$

$$\sum_{i=1}^{NI} (X_{i182} + X_{i192}) \geq 1 \qquad (E.12)$$

$$\sum_{i=1}^{NI} (Y_{i11} + Y_{i21} + Y_{i31}) \geq 20 \qquad (E.13)$$

$$\sum_{i=1}^{NI} (Y_{i41} + Y_{i51} + Y_{i61} + Y_{i71}) \geq 14 \qquad (E.14)$$

$$\sum_{i=1}^{NI} (Y_{i81} + Y_{i91}) \geq 3 \qquad (E.15)$$

$$\sum_{i=1}^{NI} (Y_{i12} + Y_{i22} + Y_{i32}) \geq 8 \qquad (E.16)$$

$$\sum_{i=1}^{NI} (Y_{i82} + Y_{i92}) \geq 1 \qquad (E.17)$$

$$\sum_{j=1}^{NJ} Y_{1j1} \leq 11 \qquad (E.18)$$

$$\sum_{j=1}^{NJ} Y_{2j1} \leq 28 \qquad (E.19)$$

$$\sum_{j=1}^{NJ} Y_{1j2} \leq 3 \qquad (E.20)$$

$$\sum_{j=1}^{NJ} Y_{2j2} \leq 7 \tag{E.21}$$

$$\sum_{k=1}^{NK} \sum_{j=1}^{NJ} X_{2kj1} \leq 9 \tag{E.22}$$

$$\sum_{k=1}^{NK} \sum_{j=1}^{NJ} X_{2kj1} \leq 32 \tag{E.23}$$

$$\sum_{k=1}^{NK} \sum_{j=1}^{NJ} X_{1kj2} \leq 2 \tag{E.24}$$

$$\sum_{k=1}^{NK} \sum_{j=1}^{NJ} X_{2kj2} \leq 1 \tag{E.25}$$

$$\sum_{i=1}^{NI} Y_{ij\ell} + \sum_{i=1}^{NI} \sum_{k=1}^{NK} X_{ikj\ell} \leq b_{j\ell} \quad \begin{array}{l} \ell = 1, \ldots, NL; \\ j = 1, \ldots, NJ \end{array} \tag{E.26}$$

Table E1 details the sensitivity analyses conducted for the application airport whose mathematical model was just presented.

TABLE E1

Demand Variations

Right Hand Side of Constraint	Case 1	Case 2	Case 3	Case 4
E2	5*	3	2*	3
E3	5*	3	2*	3
E4	3*	2	1*	2
E5	10*	6	4*	6
E6	16*	11	9*	12*
E7	8*	5	4*	5
E8	3*	2	1*	3*
E9	6*	4	3*	4
E10	5*	3	2*	3
E11	1	4*	1	0*
E12	1	4*	1	0*
E13	36*	20	14*	28*
E14	24*	14	9*	14
E15	5*	3	2*	4*
E16	8	9*	5*	0*
E17	1	4*	1	0*
E18	20*	11	9*	14*
E19	48*	28	21*	35*
E20	3	5*	2*	0*
E21	7	10*	5*	0*
E22	18*	9	7*	10*
E23	44*	32	24*	33*
E24	2	6*	2	2
E25	1	4*	1	1

* Designate changed values.

TABLE :2

COMPUTATIONAL RESULTS FOR DEMAND INCREASE FOR DAY FLIGHTS

NII = 0.15026

Number of Operations	Arrival	Departure	Aircraft 727	Aircraft 737	Stage Length	Track Number	Time Period Day	Time Period Night
11	X		X			3	X	
5	X		X			9	X	
25	X			X		3	X	
24	X			X		4	X	
1	X		X			3		X
1	X		X			9		X
7	X			X		3		X
5		X	X		1	2	X	
3		X	X		1	9	X	
3		X	X		2	2	X	
6		X	X		2	9	X	
10		X		X	1	4	X	
2		X		X	2	3	X	
16		X		X	2	4	X	
3		X		X	3	3	X	
8		X		X	3	4	X	
5		X		X	3	9	X	
1		X	X		1	9		X
1		X		X	2	4		X

The various cases correspond to the following variations.

Case 1 - Demand increase for day flights

Case 2 - " " "night "

Case 3 - Uniform decrease in demand

Case 4 - All flights are conducted during the day

The computational results are compiled in Tables E2 through E5.

Another type of sensitivity analysis conducted for the given application was variations in the type of aircraft employed for all operations. The specific cases considered were:

Case 5 - all operations were by 737 aircraft

Case 6 - " " " " " " with SAM

Case 7 - " " " " 727 ".

Case 8 - " " " " " " with SAM

Tables E6 through E9 provide the corresponding computational results.

Table E10 describes the results obtained when the objective function was changed from the one originally given in equation (E1) to that of minimizing the maximum (minimax) annoyance for the various population segments considered. The use of such an objective formulation should result in a dispersion of annoyance more uniformly to the population areas surrounding the airport. The solution of a population weighted NEF minimax objective function is given in Table E11.

TABLE E3

COMPUTATIONAL RESULTS FOR DEMAND INCREASE FOR NIGHT FLIGHTS

NII = 0.17961

Number of Operations	Arrival	Departure	Aircraft 727	737	Stage Length	Track Number	Time Day	Period Night
5	x		x			3	x	
15	x			x		3	x	
9	x			x		3		x
13	x			x		4	x	
1	x		x			7	x	
3	x		x			9	x	
3	x		x			9		x
1	x			x		9		x
3		x		x	1	3	x	
3		x		x	2	3	x	
2		x		x	3	3	x	
6		x		x	1	4	x	
10		x		x	2	4	x	
5		x		x	3	4	x	
4		x		x	2	4		x
1		x			2	6	x	
1		x			2	8	x	
3		x			2	9	x	
3		x			3	9	x	
4		x			1	9		x

TABLE E4

COMPUTATIONAL RESULTS FOR UNIFORM DECREASE IN DEMAND

NII = 0.11016

Number of Operations	Arrival	Departure	Aircraft 727	737	Stage Length	Track Number	Time Day	Period Night
2	X		X			3	X	
2	X		X			9	X	
12	X			X		3	X	
9	X			X		4	X	
1	X		X			9		X
5	X			X		3		X
1		X	X		1	9	X	
3		X	X		2	9	X	
2		X		X	1	3	X	
4		X		X	1	4	X	
2		X		X	2	3	X	
9		X		X	2	4	X	
1		X		X	3	3	X	
4		X		X	3	4	X	
2		X		X	3	9	X	
1		X	X		1	9		X
1		X	X		2	4		X

TABLE E5

COMPUTATIONAL RESULTS FOR ALL DAY FLIGHTS

NII = 0.0816

Number of Operations	Arrival	Departure	Aircraft 727	Aircraft 737	Stage Length	Track Number	Time Period Day	Time Period Night
6	x		x			3	x	
1	x		x			7	x	
4	x		x			9	x	
22	x			x		3	x	
13	x			x		4	x	
3		x	x		1	9	x	
1		x	x		2	6	x	
1		x	x		2	8	x	
3		x	x		2	9	x	
3		x		x	1	3	x	
6		x		x	1	4	x	
3		x		x	2	3	x	
11		x		x	2	4	x	
2		x		x	3	3	x	
5		x		x	3	4	x	
3		x		x	3	9	x	

COMPUTATIONAL RESULTS FOR ALAG 1 OPTIMIZATION ALGORITHM

NII =

Number of Operations	Arrival	Departure	Aircraft 727	737	Stage Length	Track Number	Time Period Day	Night

TABLE E6

COMPUTATIONAL RESULTS FOR ALL 737 AIRCRAFT

NII = 0.09863

Number of Operations	Arrival	Departure	Aircraft 727	Aircraft 737	Stage Length	Track Number	Time Period Day	Time Period Night
20	X			X		2	X	
1	X			X		2		X
7	X			X		3		X
14	X			X		7	X	
3	X			X		9	X	
1	X			X		9		X
3		X		X	1	1	X	
3		X		X	2	1	X	
2		X		X	3	1	X	
6		X		X	1	4	X	
11		X		X	2	4	X	
5		X		X	2	4	X	
1		X		X	2	4		X
2		X		X	1	9	X	
4		X		X	2	9	X	
3		X		X	3	9	X	
1		X		X	1	9		X

TABLE E7

COMPUTATIONAL RESULTS FOR ALL 737/S.M AIRCRAFT

NII = 0.09981

Number of Operations	Arrival	Departure	Aircraft 727	Aircraft 737	Stage Length	Track Number	Time Period Day	Time Period Night
3	x			x		2		x
20	x			x		3	x	
6	x			x		3		x
14	x			x		7	x	
3	x			x		9	x	
1	x			x		9		x
3		x		x	1	1	x	
3		x		x	2	1	x	
2		x		x	3	1	x	
6		x		x	1	4	x	
11		x		x	2	4	x	
5		x		x	3	4	x	
1		x		x	2	4		x
2		x		x	1	9	x	
4		x		x	2	9	x	
3		x		x	3	9	x	
1		x		x	1	9		x

TABLE E8

COMPUTATIONAL RESULTS FOR ALL 727 AIRCRAFT

NII = 0.22506

Number of Operations	Arrival	Departure	Aircraft		Stage Length	Track Number	Time Period	
			727	737			Day	Night
11	x		x			2	x	
8	x		x			3		x
9	x		x			3	x	
9	x		x			4	x	
5	x		x			7	x	
3	x		x			9	x	
1	x		x			9		x
3		x	x		1	1	x	
5		x	x		2	1	x	
6		x	x		1	4	x	
16		x	x		2	4	x	
1		x	x		2	4		x
2		x	x		1	9	x	
7		x	x		2	9	x	
1		x	x		1	9		x

TABL E9

COMPUTATIONAL RESULTS FOR ALL 727/SA[illegible] AIRCRAFT

NII = 0.17922

Number of Operations	Arrival	Departure	Aircraft 727	737	Stage Length	Track Number	Time Period Day	Night
20	x		x			1	x	
8	x		x			1		x
14	x		x			4	x	
3	x		x			9	x	
1	x		x			9		x
3		x	x		1	1	x	
3		x	x		2	1	x	
2		x	x		3	1	x	
6		x	x		1	4	x	
11		x	x		2	4	x	
5		x	x		3	4	x	
1		x	x		2	4		x
2		x	x		1	9	x	
4		x	x		2	9	x	
3		x	x		3	9	x	
1		x	x		1	9		x

TABLE E10

COMPUTATIONAL RESULTS FOR MINIMIZE MAXIMUM ANNOYANCE

NII = 0.15968

Number of Operations	Arrival	Departure	Aircraft 727	Aircraft 737	Stage Length	Track Number	Time Day	Period Night
9	x			x		2	x	
11	x		x			3	x	
3	x		x			3		x
6	x			x		3		x
1	x			x		6	x	
13	x			x		7	x	
3	x			x		9	x	
1	x			x		9		x
3		x	x		1	1	x	
3		x	x		2	1	x	
2		x		x	3	3	x	
1		x	x		1	4	x	
5		x		x	1	4	x	
11		x		x	2	4	x	
5		x		x	3	4	x	
1		x	x		2	4		x
2		x		x	1	9	x	
4		x		x	2	9	x	
3		x		x	3	9	x	
1		x		x	1	9		x

TABLE E11

Area	Original NII Contribution $x10^2$	Objective Function NEF	Minimax NII Contribution $x\ 10^2$	Objective Function NEF
1	.049	8.57	-	-
2	.176	11.41	.295	14.86
3	.137	10.07	.676	21.16
4	.438	15.82	.567	17.63
5	.738	23.99	.896	25.61
6	1.038	27.05	1.071	27.33
7	.145	20.66	.498	31.36
8	.032	12.43	.084	19.25
9	.019	4.87	.061	12.20
10	.705	19.36	1.062	22.48
11	.581	20.90	1.004	25.28
12	.161	20.14	.263	23.98
13	.300	31.66	.374	33.98
14	.362	23.75	.147	16.94
15	.352	17.16	.670	21.91
16	.378	18.50	.697	23.14
17	.926	18.98	1.723	23.76
18	.338	23.58	.527	[illegible]
19	.140	39.19	.172	41.91
20	-	-	-	-
21	.542	24.40	.823	28.04
22	1.140	23.62	1.371	25.14
23	.867	26.67	.520	22.42
24	1.332	35.91	.836	31.18
25	.416	28.40	.209	22.54
26	-	-	-	-
27	-	-	-	-
28	.277	14.24	.592	19.59
29	.169	12.92	.379	18.52
30	.167	22.04	.119	19.49
31	1.054	33.79	.305	22.41

NII Totals $12.979x10^{-2}$ $15.968\ x\ 10^2$

TABLE E12

COMPUTATIONAL RESULTS FOR FUEL MINIMIZATION

NII = 0.17088

Number of Operations	Arrival	Departure	Aircraft 727	Aircraft 737	Stage Length	Track Number	Time Period Day	Time Period Night
2	x		x			1	x	
7	x			x		1	x	
3	x			x		1		x
1	x		x			2	x	
4	x			x		2	x	
1	x			x		2		x
1	x		x			5	x	
1	x		x			5		x
3	x			x		5	x	
3	x			x		6	x	
5	x		x			9	x	
1	x		x			9		x
11	x			x		9	x	
3	x			x		9		x

TABLE E12 (Continued)

COMPUTATIONAL RESULTS FOR FUEL MINIMIZATION

NII = 0.17088

Number of Operations	Arrival	Departure	Aircraft 727	Aircraft 737	Stage Length	Track Number	Time Period Day	Time Period Night
2		X	X		1	1	X	
1		X	X		2	1	X	
3		X		X	1	1	X	
7		X		X	2	1	X	
4		X		X	3	1	X	
1		X		X	2	4	X	
1		X	X		72	7	X	
1		X	X		1	7	X	
1		X		X	2	7	X	
1		X		X	3	7	X	
3		X	X		8	8	X	
3		X		X	1	8	X	
7		X		X	2	8	X	
5		X		X	3	8	X	
1		X		X	2	8		X