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FLOW RESEARCH COMPANY

A DIVISION OF FLOW INDUSTRIES, INC.

Flow Research Note No. 178

Numerical Solution of Transonic Full Stream Function Equations in Conservation Form

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Jerry, Your Comments are appreciated.
I am still working on the conservation results.

Mohamed

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By

Mohamed M. Hafez

October 1979

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Abstract

The stream function equation in conservation form is solved iteratively based on the artificial compressibility method. The density is not a unique function of the mass flux. To avoid the ambiguity near the sonic line, the density is updated in terms of the velocity, which is obtained through a simple integration of a first order equation step-by-step in the flow field. Iteration algorithms and finite difference approximations are discussed, and numerical results of both conservative and nonconservative calculations are presented.

*This work is sponsored by NASA Ames Contract No. NASA-10252.

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1. Introduction

The stream function formulation is suitable for design problems and internal flow as well as rotational flow calculations. The stream lines and potential lines (in the case of irrotational flows) provide an orthogonal grid with a natural body fitted coordinate.

There are, however, some difficulties in solving the stream function equation: (1) the density is not a unique function of the mass flux, and (2) stream function exits only for two-dimensional and axi-symmetric flows. Extension to three-dimensional flows in terms of more than one stream function is not simple to implement. Due to these stumbling blocks, the stream function has not been recently used.

In 1944 Emmons (Reference 1) solved the stream function equation by hand relaxation where the shock was fitted. In 1968 Sells (Reference 2) calculated only subcritical flows using line relaxations. Here transonic flows are calculated where shocks are captured, based on the artificial compressibility method (Reference 3), and the density is updated in terms of the velocity without ambiguity.

2. Stream Function Formulation

For irrotational flows, the governing equations are:

$$(\rho u)_x + (\rho v)_y = 0 \tag{1}$$

$$u_y - v_x = 0 \tag{2}$$

$$\rho = \left[1 - \frac{\gamma - 1}{2} M_\infty^2 (u^2 + v^2 - 1)\right]^{\frac{1}{\gamma - 1}} \quad . \tag{3}$$

A potential function exits and it is governed by:

$$(\rho \phi_x)_x + (\rho \phi_y)_y = 0 \tag{4}$$

$$\rho = (M_\infty^2 a^2)^{\frac{1}{\gamma - 1}} = \left[1 - \frac{\gamma - 1}{2} M_\infty^2 (\phi_x^2 + \phi_y^2 - 1)\right]^{\frac{1}{\gamma - 1}} \quad . \tag{5}$$

In terms of natural coordinates s , n Equation (4) becomes:

$$(1 - M^2)\phi_{ss} + \phi_{nn} = 0 \quad . \tag{6}$$

Similarly, a stream function exists such that:

$$\frac{-\psi_x}{\rho} = \phi_y = v \ , \ \frac{\psi_y}{\rho} = \phi_x = u \tag{7}$$

$$\left(\frac{-\psi_s}{\rho} = \phi_n \ , \ \frac{\psi_n}{\rho} = \phi_s \right) \tag{8}$$

Hence

$$\left(\frac{\psi_x}{\rho}\right)_x + \left(\frac{\psi_y}{\rho}\right)_y = 0 \tag{9}$$

$$\rho = \left[1 - \frac{\gamma - 1}{2} M_\infty^2 \left(\frac{\psi_x^2 + \psi_y^2}{\rho^2} - 1\right)\right]^{\frac{1}{\gamma - 1}} \tag{10}$$

or

$$(1 - M^2)\psi_{ss} + \psi_{nn} = 0 \quad . \tag{11}$$

While the density (the speed of sound) is uniquely determined in terms of the speed $q^2 = u^2 + v^2$, from Equation (3), Equation (10) indicates that there are two values of the density for certain mass flux (Sells, Reference 2, obtained rational approximations for the subsonic and supersonic branches).

We notice also that Equation (11) is a nonlinear mixed-type equation similar to Equation (6).

3. Artificial Viscosity

It is well known (Reference 3) that Equation (6) can be solved by adding an artificial viscosity term either explicitly or indirectly (through upwind differencing), namely,

$$(1 - M^2)\phi_{ss} + \phi_{nn} = -\varepsilon\phi_{sss} \tag{12}$$

The corresponding stream function equation is

$$(1 - M^2)\psi_{ss} + \psi_{nn} = -\varepsilon\psi_{sss} \tag{13}$$

Equation (13) is indeed equivalent to Equation (12) with the augmented artificial viscosity term.

Therefore, upwind differencing of Equation (11) in exactly the same way that Equation (6) is solved will yield nonconservative calculations. In the first case, however, the flow is irrotational and due to the nonconservative differencing, a proper mass balance across the shock is not achieved and the error can be represented as a source distribution at the shock. On the other hand, the mass is conserved automatically in the stream function formulation and due to the nonconservative differencing, a vorticity distribution at the shock is produced.

4. Shock Jump Conditions

Equation (1) admits a discontinuous solution such that

$$[\![\rho u]\!] - \left(\frac{dX}{dy}\right)_s [\![\rho v]\!] = 0 \qquad (14)$$

$$\left(\frac{dX}{dy}\right)_s [\![u]\!] + [\![v]\!] = 0 \qquad (15)$$

In terms of the velocity potential, Equations (14 and 15) read

$$[\![\rho\phi_x]\!] - \left(\frac{dX}{dy}\right)_s [\![\rho\phi_y]\!] = 0 \qquad (16)$$

$$[\![\phi]\!] = 0 \qquad (17)$$

and in terms of the stream function, they are:

$$[\![\psi]\!] = 0 \qquad (18)$$

$$\left(\frac{dX}{dy}\right)_s \left[\!\!\left[\frac{\psi_y}{\rho}\right]\!\!\right] - \left[\!\!\left[\frac{\psi_x}{\rho}\right]\!\!\right] = 0 \qquad (19)$$

Either the shock is fitted according to the above shock jump relations, or it is captured through a proper conservative finite differencing, such that in the limit of small mesh, the shock jump conditioning are satisfied.

5. Artificial Compressibility

In Reference (3), the full potential equation in conservation form is solved. An artificial viscosity is added through modifying the density, namely,

$$(\tilde{\rho}\phi_x)_x + (\tilde{\rho}\phi_y)_y = 0 \tag{20}$$

where

$$\tilde{\rho} = \rho - \mu\ \rho_s \Delta_s \tag{21}$$

$$\rho_s \Delta_s \cong \frac{u}{q}\ \rho_x\ \Delta_x + \frac{v}{q}\ \rho_y \Delta_y \tag{22}$$

Centered differences are used everywhere in Equation (20). Only ρ_s is evaluated using upwind differencing.

Similarly, the artificial viscosity can be added to the stream function equation as follows:

$$u = \frac{\psi_y}{\rho} + \mu\ \frac{u}{\rho}\ \rho_x \Delta_x \tag{23}$$

$$v = \frac{-\psi_x}{\rho} - \mu\ \frac{v}{\rho}\ \rho_y \Delta_y \tag{24}$$

Hence,

$$\left(\frac{\psi_x}{\bar{\rho}}\right)_x + \left(\frac{\psi_y}{\hat{\rho}}\right)_y = 0 \tag{25}$$

or

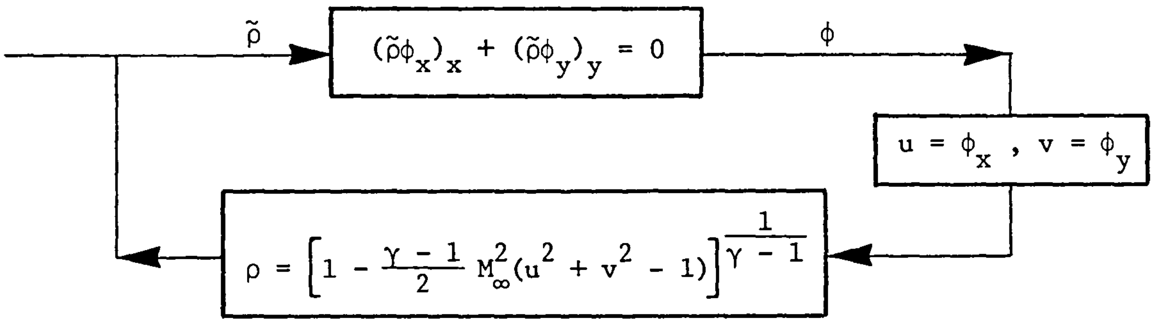
$$u = \psi_y / \tilde{\rho} \tag{26}$$

$$v = -\ \psi_x / \tilde{\rho} \tag{27}$$

and
$$\left(\frac{\psi_x}{\tilde{\rho}}\right)_x + \left(\frac{\psi_y}{\tilde{\rho}}\right)_y = 0 \tag{28}$$

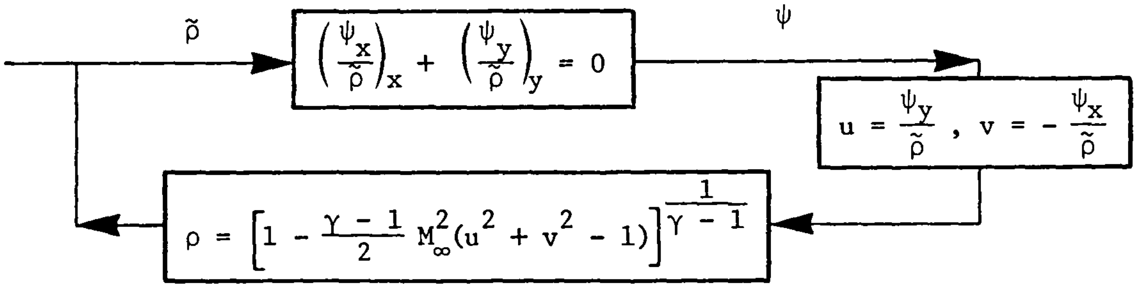
6. Iteration Algorithms

In the potential claculations, given the density from previous iterations, the velocity potential is calculated from Equation (20). Then the density is updated using Equation (5). The iteration loop is shown in Sketch 1.



Sketch 1

A similar iteration loop is implemented for stream function calculations as shown in Sketch 2. The algorithm converges only for subcritical flows.



Sketch 2

The iteration algorithm described in Sketch 2 is based on the following procedure:

$$\rho_{new} = \left[1 - \frac{\gamma - 1}{2} M_\infty^2 \left(\frac{\psi_x^2 + \psi_y^2}{\rho_{old}^2} - 1\right)\right]^{\frac{1}{\gamma - 1}} \tag{29}$$

Notice that ρ_{new} decreases if the mass flux $(\psi_x^2 + \psi_y^2)$ increases. This is true only for subsonic flows. Equation (10) can be rewritten in the form:

$$F(\rho) = \rho^{\gamma+1} - \left(1 + \frac{\gamma - 1}{2} M_\infty^2\right)\rho^2 + \frac{\gamma - 1}{2} M_\infty^2(\psi_x^2 + \psi_y^2) = 0 \tag{30}$$

A Newton iteration on the single nonlinear equation in ρ (Equation (30)) is given by:

$$J(\rho_{old}) \cdot (\rho_{new} - \rho_{old}) = - F(\rho_{old})$$

where

$$J(\rho) = (\gamma + 1)\rho^\gamma - 2\rho(1 + \frac{\gamma - 1}{2} M_\infty^2) \tag{31}$$

Notice that the Jacobian is negative or positive depending on whether ρ_{old} is larger than or smaller than ρ^* (ρ at $a = a^*$). For pure subsonic or supersonic flows, Equation (31) may be used. But for mixed flows, perturbing the sonic conditions yields two possible solutions, and there is no obvious way to exclude anyone of them based on local considerations, unless the sonic line is fitted.

7. Mixed Flow Calculations

To avoid the ambiguity in the density updating, a different approach is adopted to calculate the speed given the stream function without referring to the old density. Starting with Equation (2):

$$u_y - v_x = 0$$

or

$$u_y - \left(\frac{v}{u} \cdot u\right)_x = 0 \tag{32}$$

where

$$\frac{v}{u} = -\frac{\psi_x}{\psi_y}$$

Hence

$$u_y + \left(\frac{\psi_x}{\psi_y} \cdot u\right)_x = 0 \tag{33}$$

or

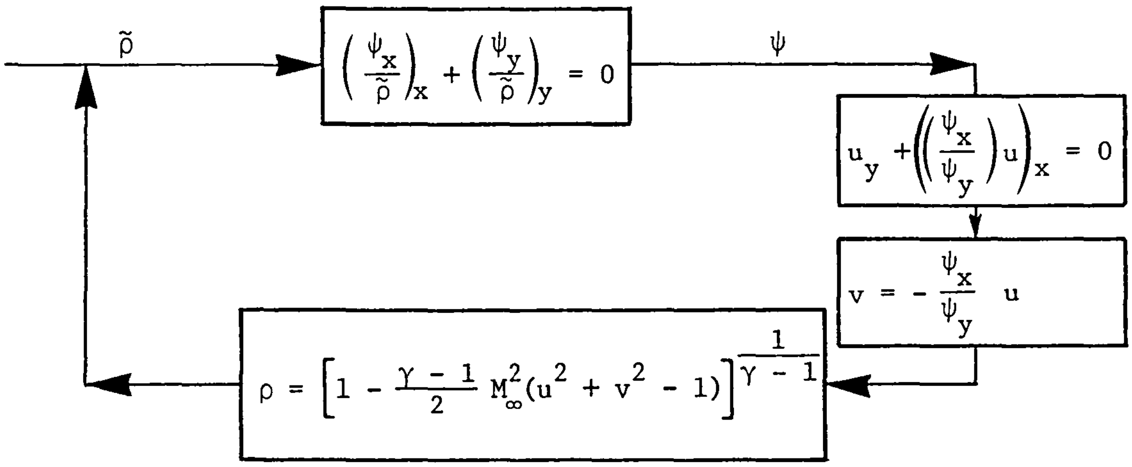
$$(\ln u)_y + \frac{\psi_x}{\psi_y} \cdot (\ln u)_x = -\left(\frac{\psi_x}{\psi_y}\right)_x \tag{34}$$

Knowing u on a curve crossing the characteristics of Equation (33), u can be calculated throughout the flow field step by step. The conditions along such a curve could be either pure subsonic or pure supersonic, where u can be obtained using a Newton iteration.

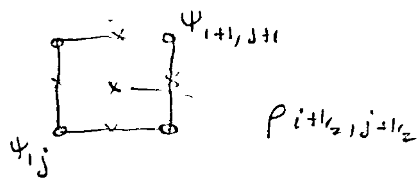
Once u is obtained, v is evaluated from Equation (35)

$$v = -\frac{\psi_x}{\psi_y} u \tag{35}$$

The iteration loop is shown in Sketch 3.



Sketch 3



let $\quad g = \dfrac{\psi_x}{\psi_y}$

$$u_y + (gu)_x = 0$$

$$(u_y)_{ij} = \frac{u_{i+1/2,\,j+1/2} - u_{i+1/2,\,j-1/2}}{2\Delta y} + \frac{u_{i-1/2,\,j+1/2} - u_{i-1/2,\,j-1/2}}{2\Delta y}$$

$$[(gu)_x]_{ij} = \frac{(gu)_{i+1/2,\,j} - (gu)_{i-1/2,\,j}}{\Delta x}$$

$$(gu)_{i+1/2,\,j} = \frac{1}{2}\left(g_{i+1/2,\,j+1/2}\, u_{i+1/2,\,j+1/2} + g_{i+1/2,\,j-1/2}\, u_{i+1/2,\,j-1/2}\right)$$

8. Relaxation Method

Horizontal line over relaxation (with additional ϕ_{xt} term, Reference 3) of the partial differential equation for ψ, given $\tilde{\rho}$, is found to be fast. To explain this, consider the transonic small disturbance system of equations:

$$v_y = (u - K)u_x - \varepsilon w_x$$

$$u_y = v_x$$

where does 3rd eq. come from.

$$\alpha w_y = - u_x + w \tag{36}$$

The characteristics of system (36) is given by

is $\lambda = \frac{dx}{dy}$ or $\frac{dy}{dx}$?

$$\lambda^3 - (u - k)\lambda - \varepsilon/\alpha = 0 \tag{37}$$

In the limit $\frac{\alpha}{\varepsilon} \to 0$, the system (36) is parabolic with $y = \text{constant}$ as a triple characteristic. Similarly, the streamlines are the characteristics of Equation (12).

Moreover, the iterative matrix of the horizontal over relaxation procedure seems to have a dominant eigenvalue allowing for even more acceleration.

9. Numerical Results

Transonic flows over NACA 0012 airfoil have been calculated based on the full potential equation as well as the stream function equation. In these calculations, linearized boundary conditions are used. The stream function equation is solved using the iteration procedure described in Sketch 3. Both conservative and nonconservative results are presented. The iteration history of both vertical and horizontal line relaxations are plotted. Results for different magnitudes of the artificial viscosity in the artificial density formulation are also shown.

Finally, we would like to mention that rotational flows can be easily calculated by modifying Equation (2) where the righthand side becomes the vorticity $\omega(\psi)$. The density formula is also slightly modified by $e^{-s(\psi)}$ where $s(\psi)$ is the entropy. Also, other iterative procedures (i.e., ADI, Fast Solver, MAD, etc.) are, of course, applicable.

Appendix
Transonic Small Disturbance Equation

Chin and Rizzetta (Reference 4) solved the small disturbance equation:

$$(1 - M^2)\psi_{xx} + \psi_{yy} = 0 \tag{A-1}$$

where $$1 - M^2 \cong 1 - M_\infty^2 - (\gamma + 1)M_\infty^2 u$$

Following Emmons (Reference 1), the small perturbation approximation of Equation (23), page 19,

$$\left(\frac{q}{q_i} = C(\xi)e^{-\int\frac{\psi\xi\xi}{\psi\eta}d\eta}\right)$$

is,
$$u_2 - u_1 = -\int_{y_1}^{y_2} \psi_{xx}\, dy \tag{A-2}$$

Equation (A-2) is consistent with the approximation

$$\phi_y = -\psi_x$$

hence $$\phi_{yx} = -\psi_{xx}$$

or

$$\phi_{xy} = - \psi_{xx} \tag{A-3}$$

Chin's calculations are nonconservative, not only because of Equation (A-1) but also because of Equation (A-2).

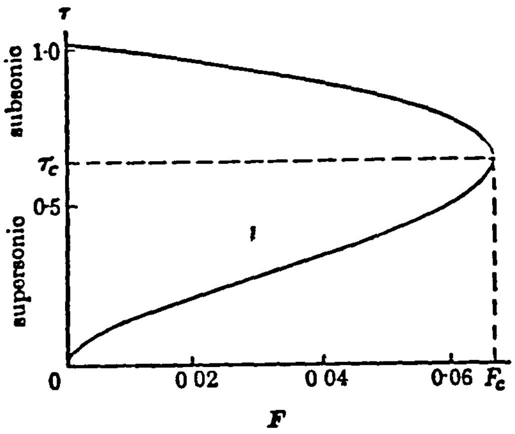
References

1. Emmons, H. W. (1944) "The Numerical Solution of Compressible Fluid Flow Problems," NACA TN 932.

2. Sells, C. C. L. (1968) "Plane Subcritical Flow Past a Lifting Airfoil," Proc. Roy. Soc. 308A, p. 377-401, London.

3. Hafez, M. M., South, C. J., Murman, E. M. (1979) "Artificial Compressibility Methods for Numerical Solutions of Transonic Full Potential Equation," AIAA Journal 17, No. 8, p. 838-844, August.

4. Chin, W. C. and Rizzetta, D. P. "Airfoil Design in Subcritical and Supercritical Flows," to be published in Journal of Applied Mechanics.



$$\tau = \frac{0{\cdot}6339395 + 1{\cdot}421451\,Q + 0{\cdot}5431537\,Q^2 - 0{\cdot}1998823\,Q^3}{1 + 1{\cdot}242281\,Q - 0{\cdot}1511548\,Q^2 - 0{\cdot}0553184\,Q^3}$$

$$\tau = \frac{0{\cdot}6339388 - 0{\cdot}8257193\,Q - 0{\cdot}8718085\,Q^2 + 1{\cdot}031123\,Q^3}{1 - 0{\cdot}3022746\,Q - 1{\cdot}449373\,Q^2 + 0{\cdot}3119677\,Q^3}.$$

$$\rho_s^{\gamma-1} = 1 + \tfrac{1}{2}M^2(\gamma - 1)$$

$$a_s^2 = \rho_s^{\gamma-1}/M^2$$

$$\frac{f^2}{\rho_s^2 a_s^2} = \frac{2}{\gamma - 1}\left[1 - \left(\frac{\rho}{\rho_s}\right)^{\gamma-1}\right]\left(\frac{\rho}{\rho_s}\right)^2.$$

$$\rho/\rho_s = \tau; \quad \tfrac{1}{2}(\gamma - 1)f^2/\rho_s^2 a_s^2 = F$$

$$F = \tau^2(1 - \tau^{\gamma-1})$$

$$F_c = \left(\frac{2}{\gamma+1}\right)^{2/(\gamma-1)}\left(\frac{\gamma-1}{\gamma+1}\right)$$

$$\tau_c = \left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)}.$$

$$Q^2 = (F_c - F)\frac{2}{(\gamma^2 - 1)\tau_c^{\gamma+1}}$$

$\gamma = 1{\cdot}4$,

$$Q = 1{\cdot}44[3(F_c - F)]^{\frac{1}{2}}.$$

Figure 1. Density - Mass Flow Relationship (Rational Approximation Formula After Sells, Reference 2).

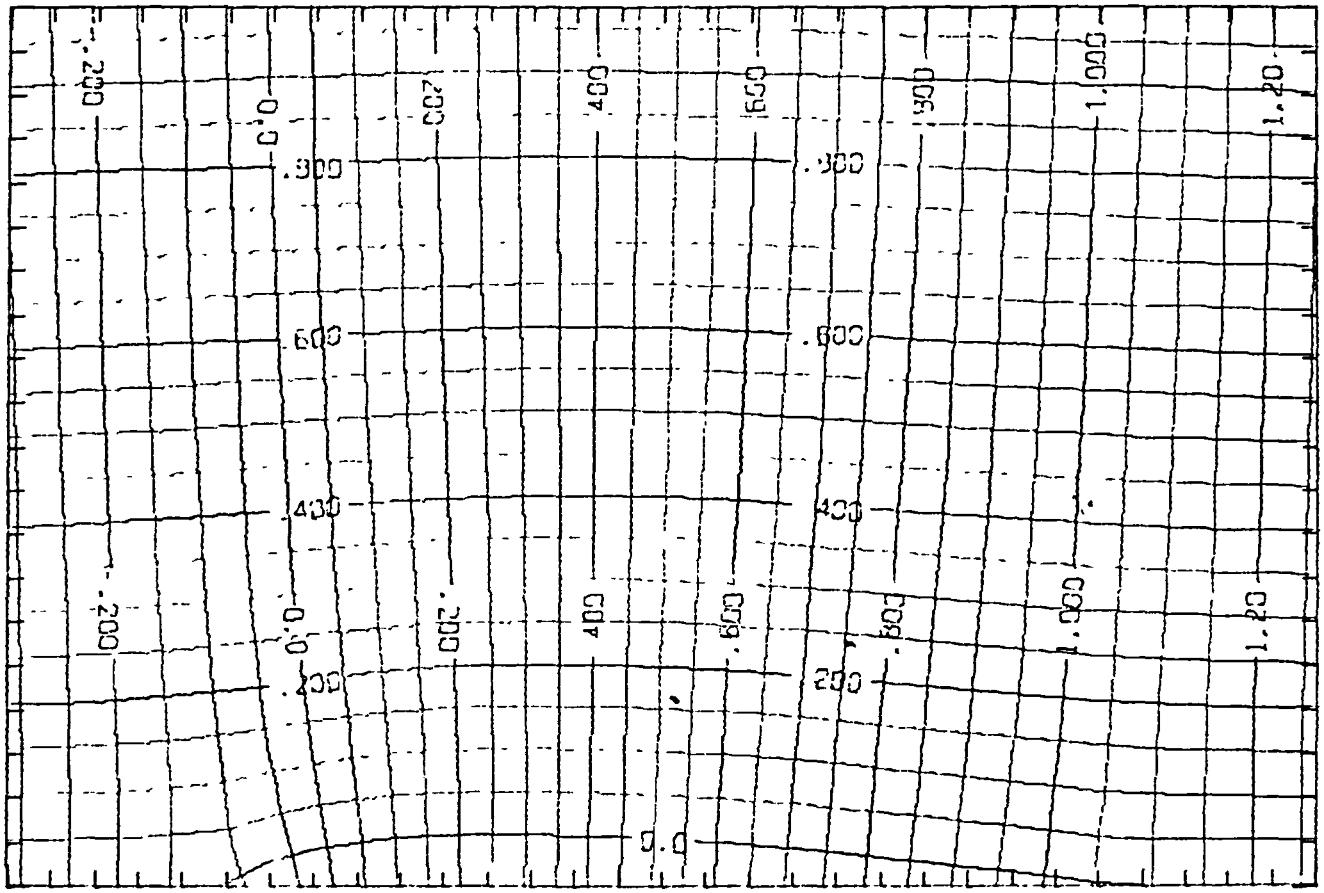


Figure 2. Potential and Stream Lines For Flows Around NACA 0012, $M_\infty = 0.85$.

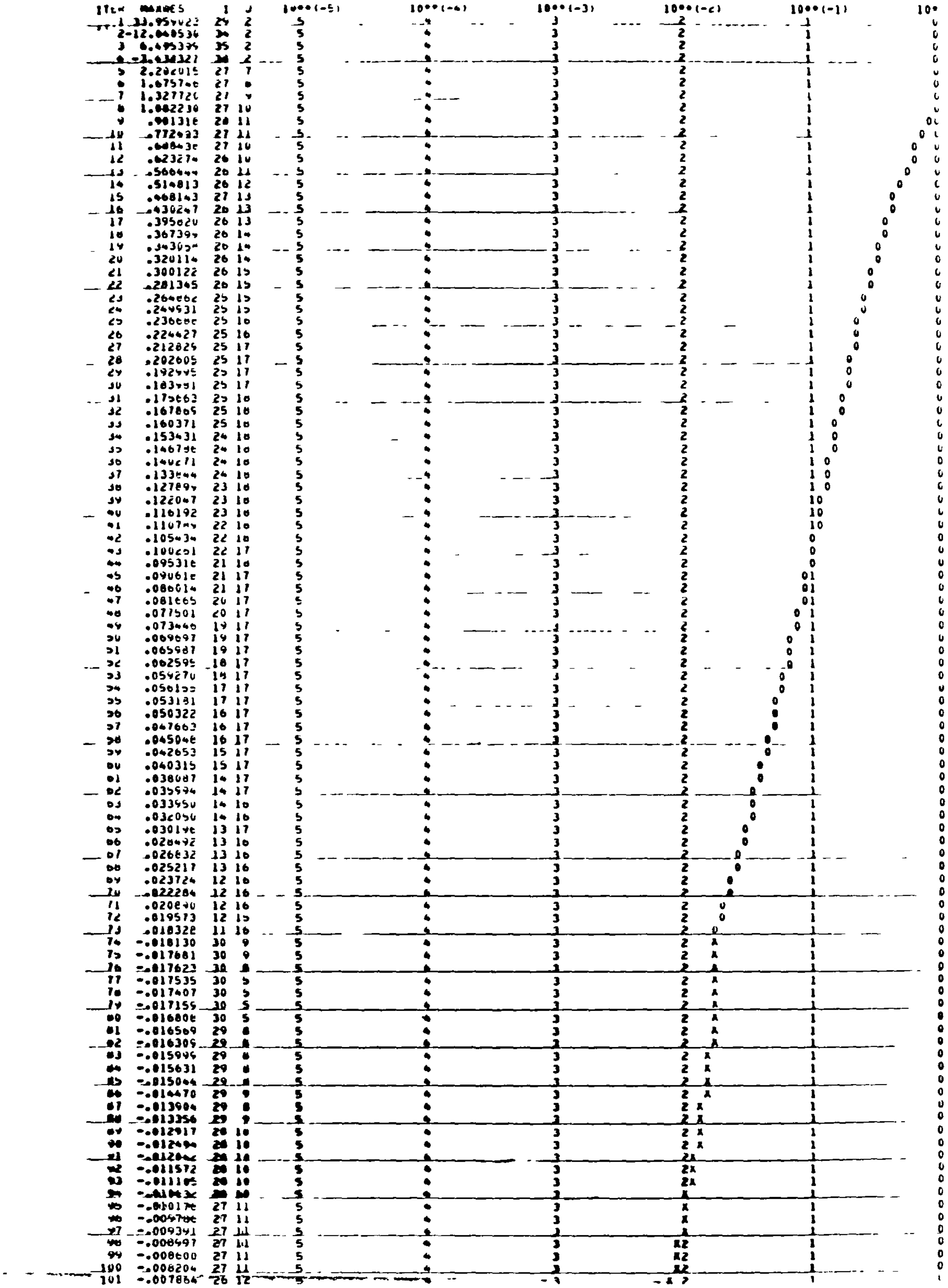


Figure 3-1. Iteration history of vertical line relaxation
$\omega = 1.8$, $M_\infty = 0.85$.

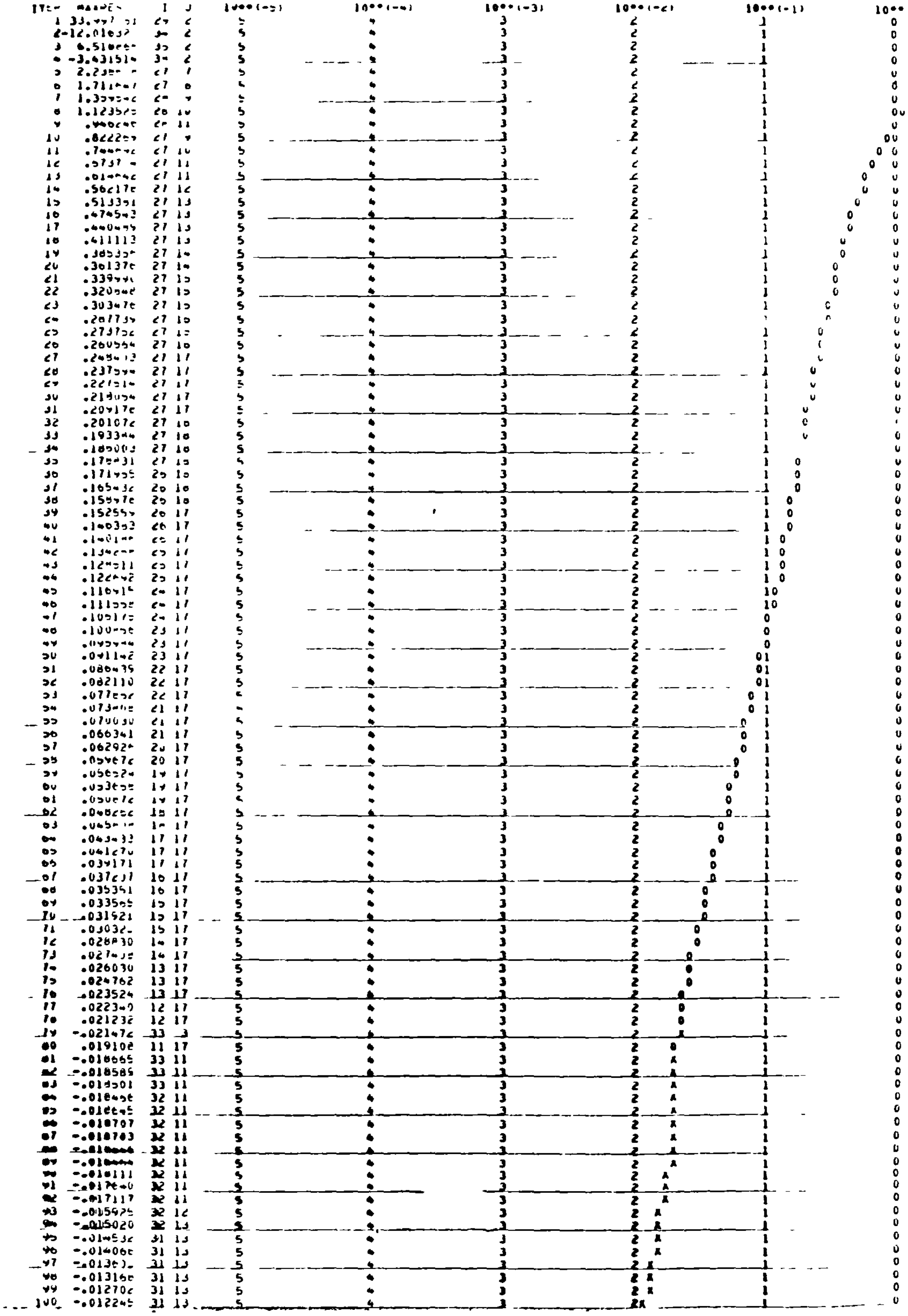


Figure 3-2. Iteration history of vertical line relaxation
$\omega = 1.8$, $M_\infty = 0.9$.

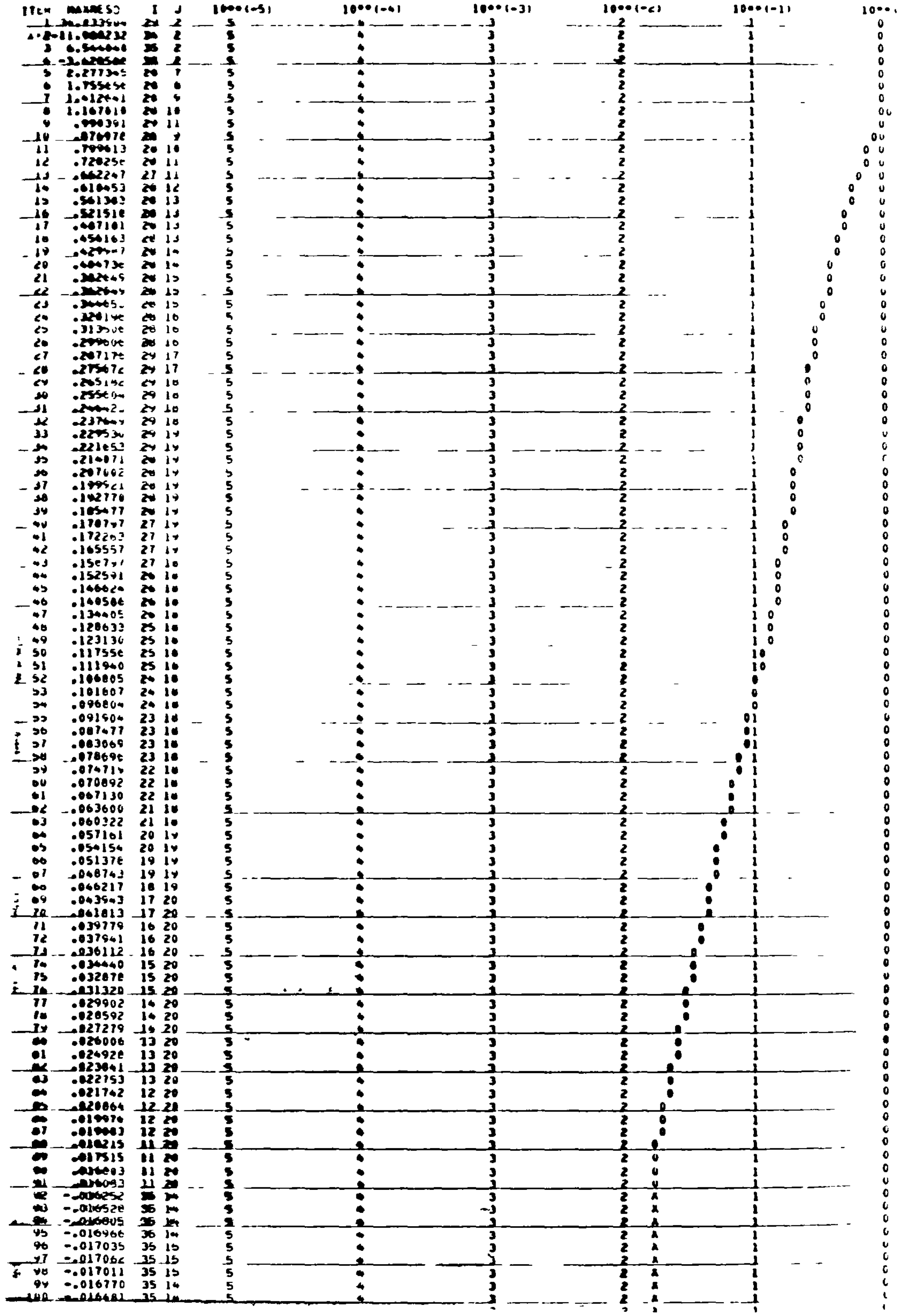


Figure 3-3. Iteration history of vertical line relaxation $\omega = 1.8$, $M_\infty = 0.95$.

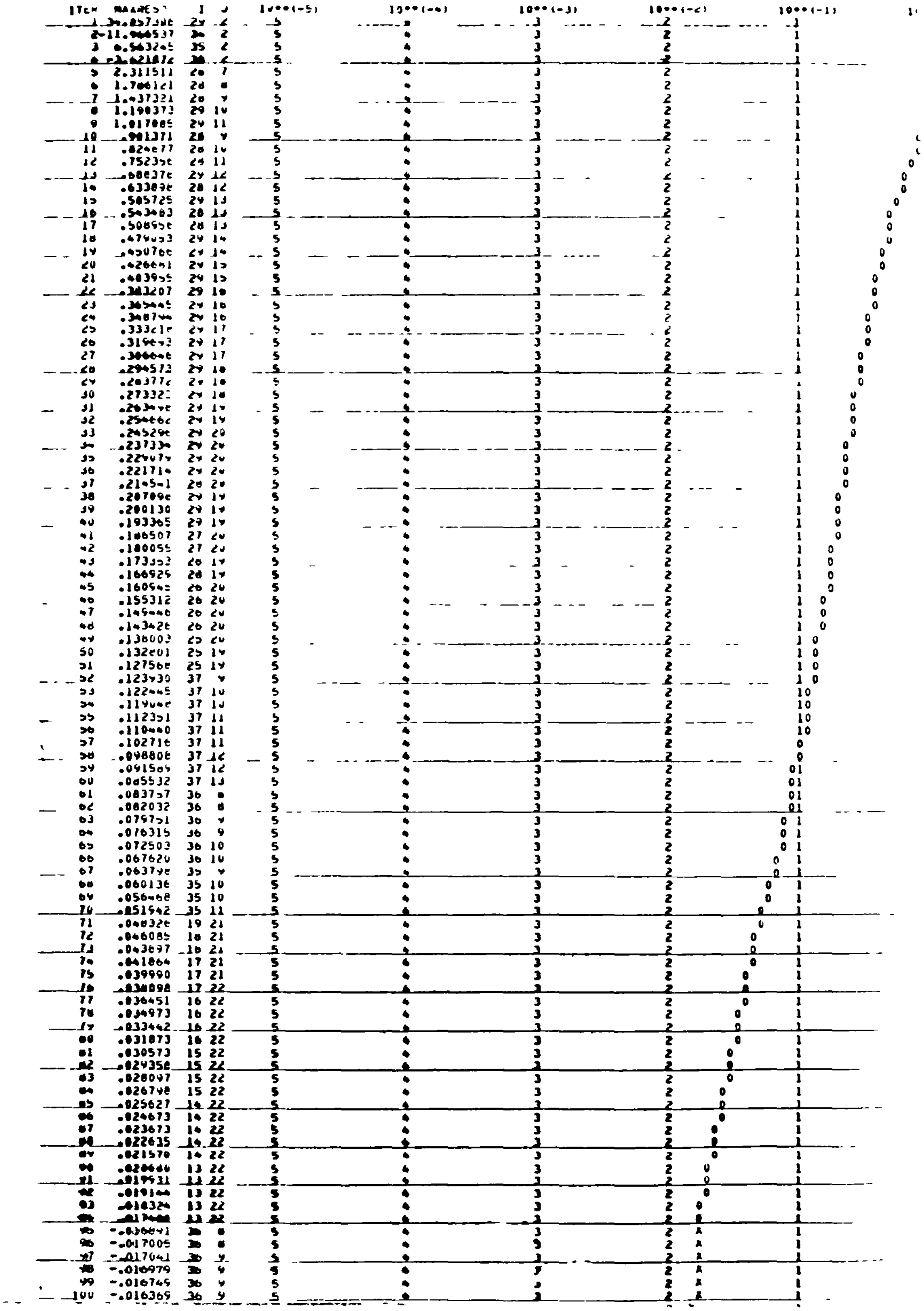


Figure 3-4. Iteration history of vertical line relaxation $\omega = 1.8$, $M_\infty = 0.98$.

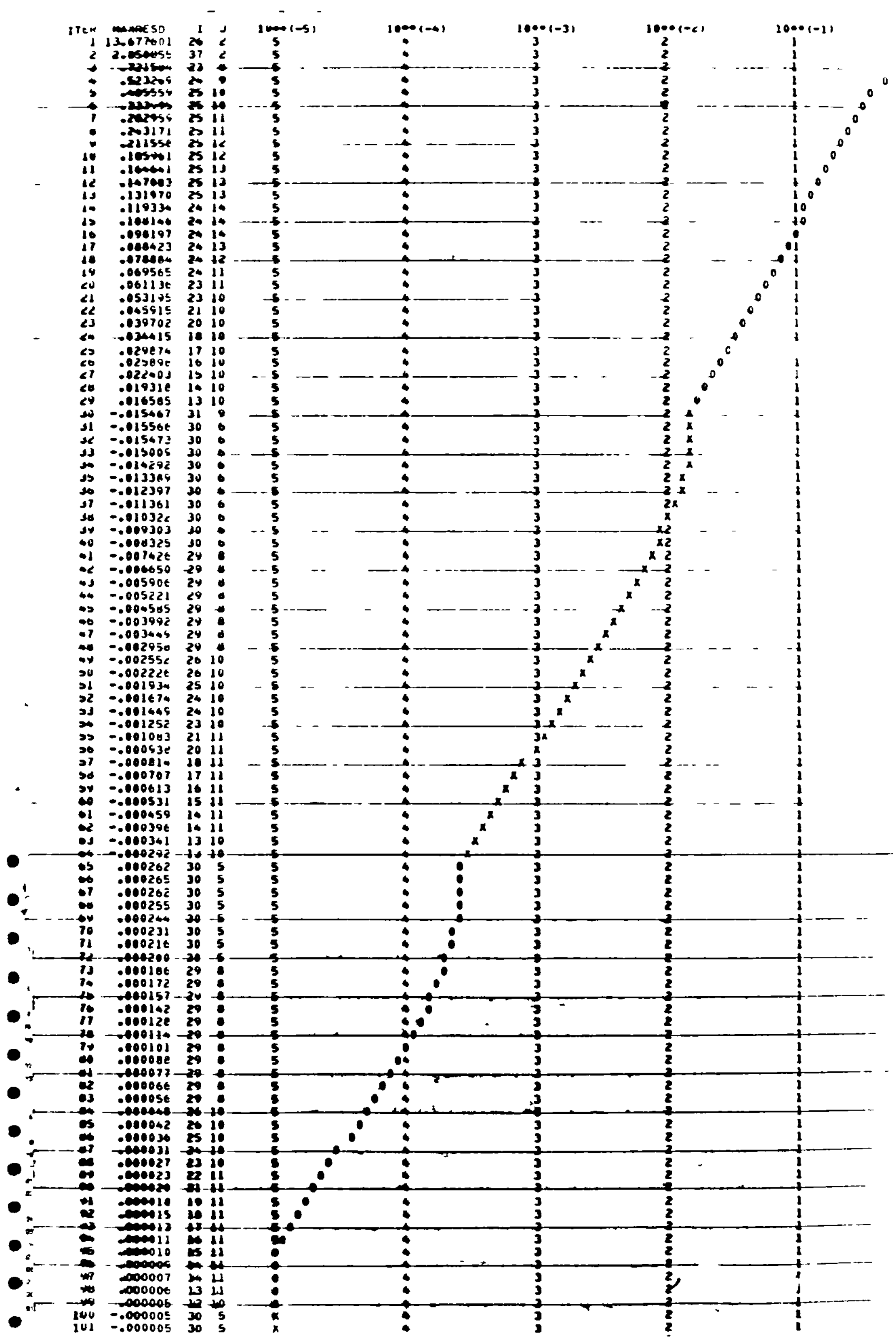


Figure 4-1. Iteration history of horizontal line relaxation
$\omega = 1.8$, $M_\infty = 0.85$.

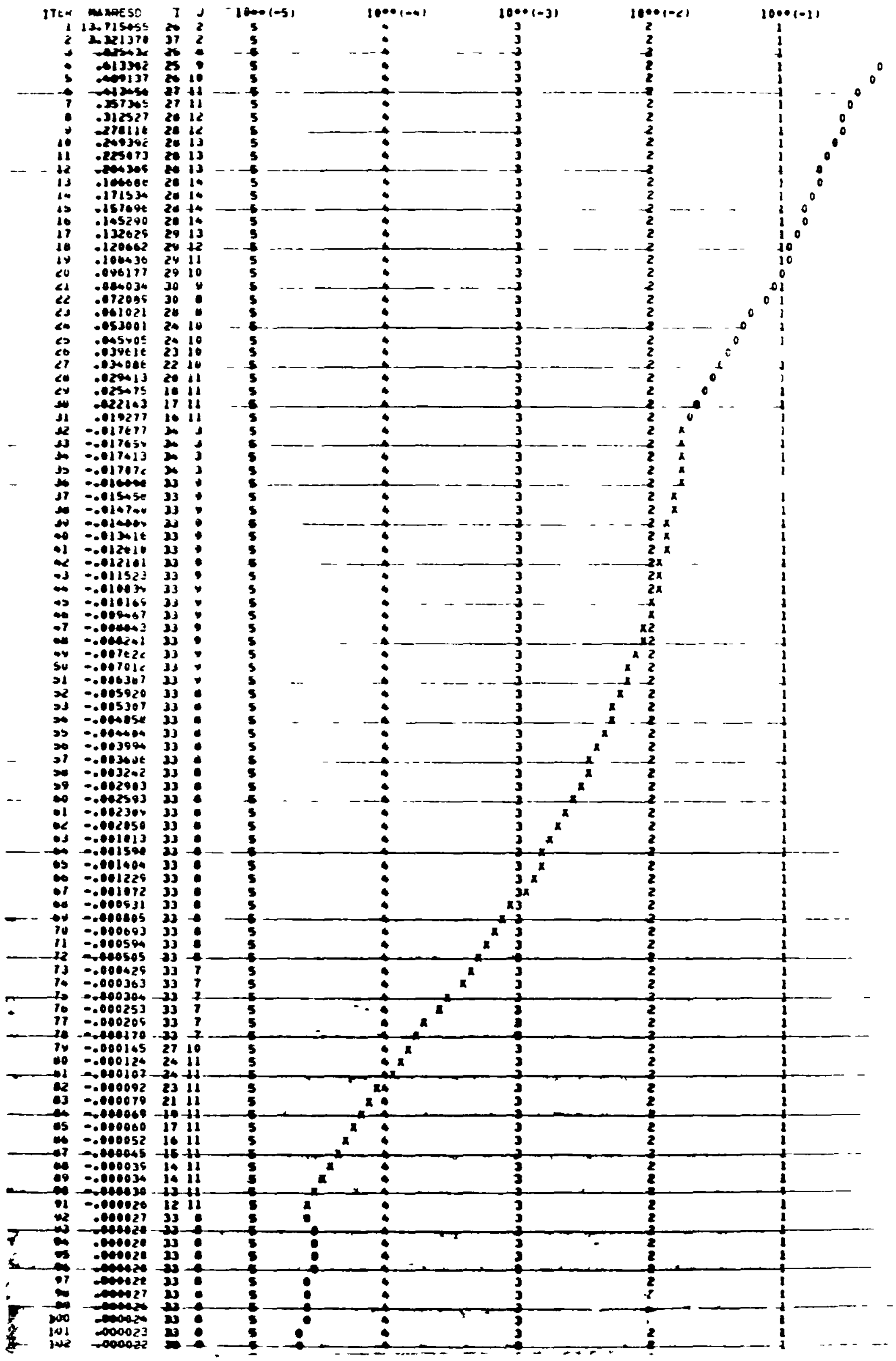


Figure 4-2. Iteration history of horizontal line relaxation
$\omega = 1.8$, $M_\infty = 0.9$.

ITER	MAXRESD	I	J
1	13.759605	26	2
2	3.087957	36	2
3	.879363	36	6
4	.710419	28	9
5	.585195	29	10
6	.511900	30	10
7	.458575	31	11
8	.414891	31	11
9	.370444	32	11
10	.344408	32	12
11	.316196	32	12
12	.290896	33	12
13	.269066	33	13
14	.249651	33	13
15	.231273	33	13
16	.215399	33	14
17	.199360	33	13
18	.183275	33	12
19	.166619	33	11
20	.149645	33	10
21	.132966	34	9
22	.116305	34	8
23	.099375	34	7
24	.084969	32	9
25	.073460	31	9
26	.064054	30	10
27	.055825	29	10
28	.048310	28	10
29	.041666	28	10
30	.036056	27	10
31	.031216	26	10
32	.027048	26	10
33	.023464	23	11
34	.020445	23	11
35	.017776	22	11
36	.015460	21	11
37	.013500	19	12
38	.011844	18	12
39	.010419	17	12
40	-.010279	37	5
41	-.010307	37	5
42	-.010136	37	5
43	-.009979	37	5
44	-.009790	37	5
45	-.009603	37	5
46	-.009159	37	4
47	-.008746	37	4
48	-.008236	37	4
49	-.007704	37	4
50	-.007243	37	4
51	-.006805	37	4
52	-.006393	37	4
53	-.005971	37	4
54	-.005577	37	4
55	-.005202	37	4
56	-.004845	37	4
57	-.004505	37	4
58	-.004182	37	4
59	-.003876	37	4
60	-.003586	37	4
61	-.003312	37	4
62	-.003054	37	4
63	-.002811	37	4
64	-.002585	37	4
65	-.002372	37	4
66	-.002174	37	4
67	-.001990	37	4
68	-.001818	37	4
69	-.001659	37	4
70	-.001513	37	4
71	-.001377	37	4
72	-.001252	37	4
73	-.001137	37	4
74	-.001031	37	4
75	-.000934	37	4
76	-.000846	37	4
77	-.000764	37	4
78	-.000690	37	4
79	-.000623	37	4
80	-.000562	37	4
81	-.000506	37	4
82	-.000455	37	4
83	-.000409	37	4
84	-.000368	37	4
85	-.000330	37	4
86	-.000296	37	4
87	-.000266	37	4
88	-.000238	37	4
89	-.000213	37	4
90	-.000191	37	4
91	-.000171	37	4
92	-.000153	37	4
93	-.000136	37	4
94	-.000122	37	4
95	-.000109	37	4
96	[illegible]	37	4
97	-.000087	37	4
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100	-.000062	37	4
101	-.000055	37	4
102	-.000049	37	4

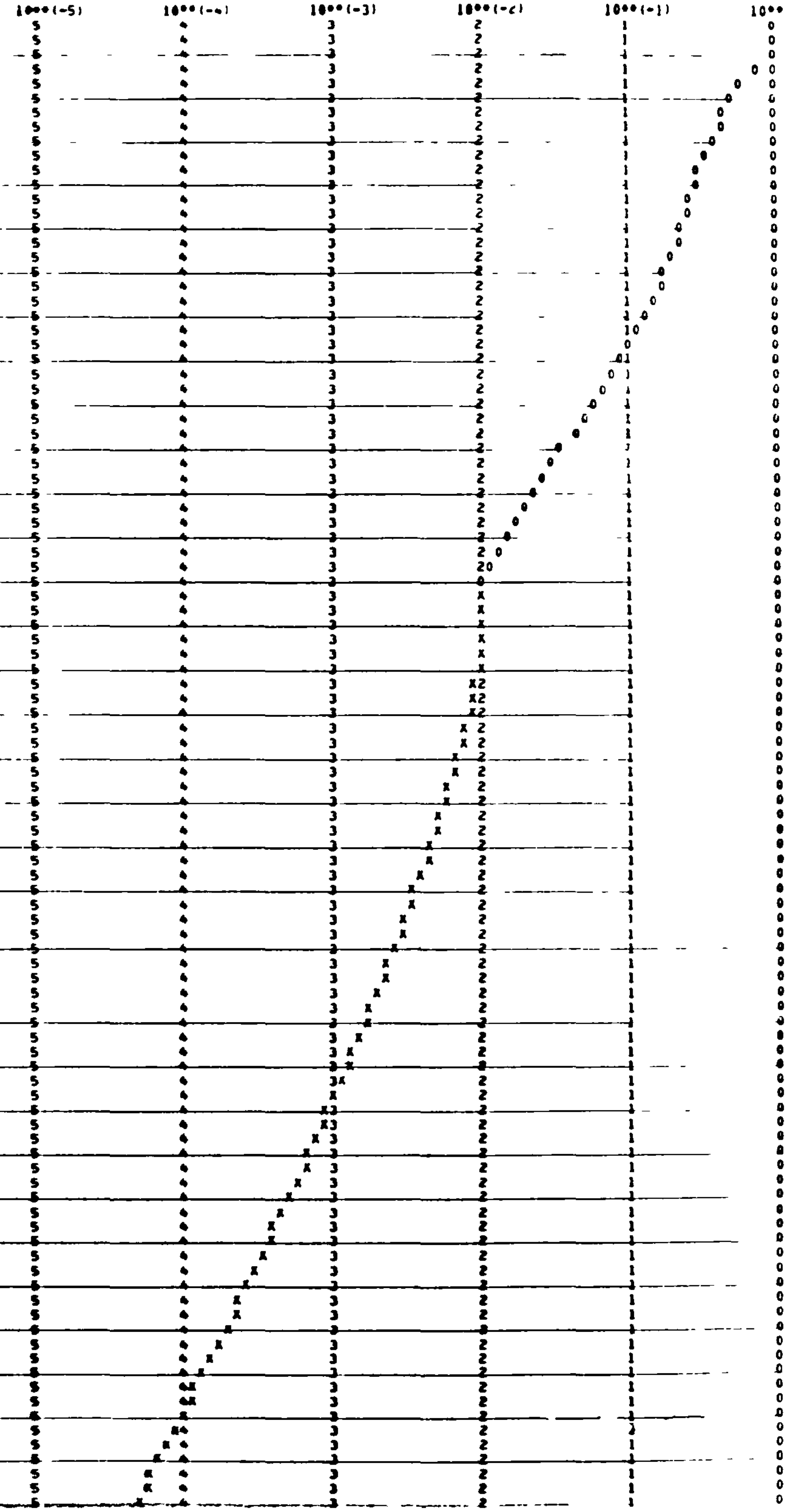


Figure 4-3. Iteration history of horizontal line relaxation $\omega = 1.8$, $M_\infty = 0.95$.

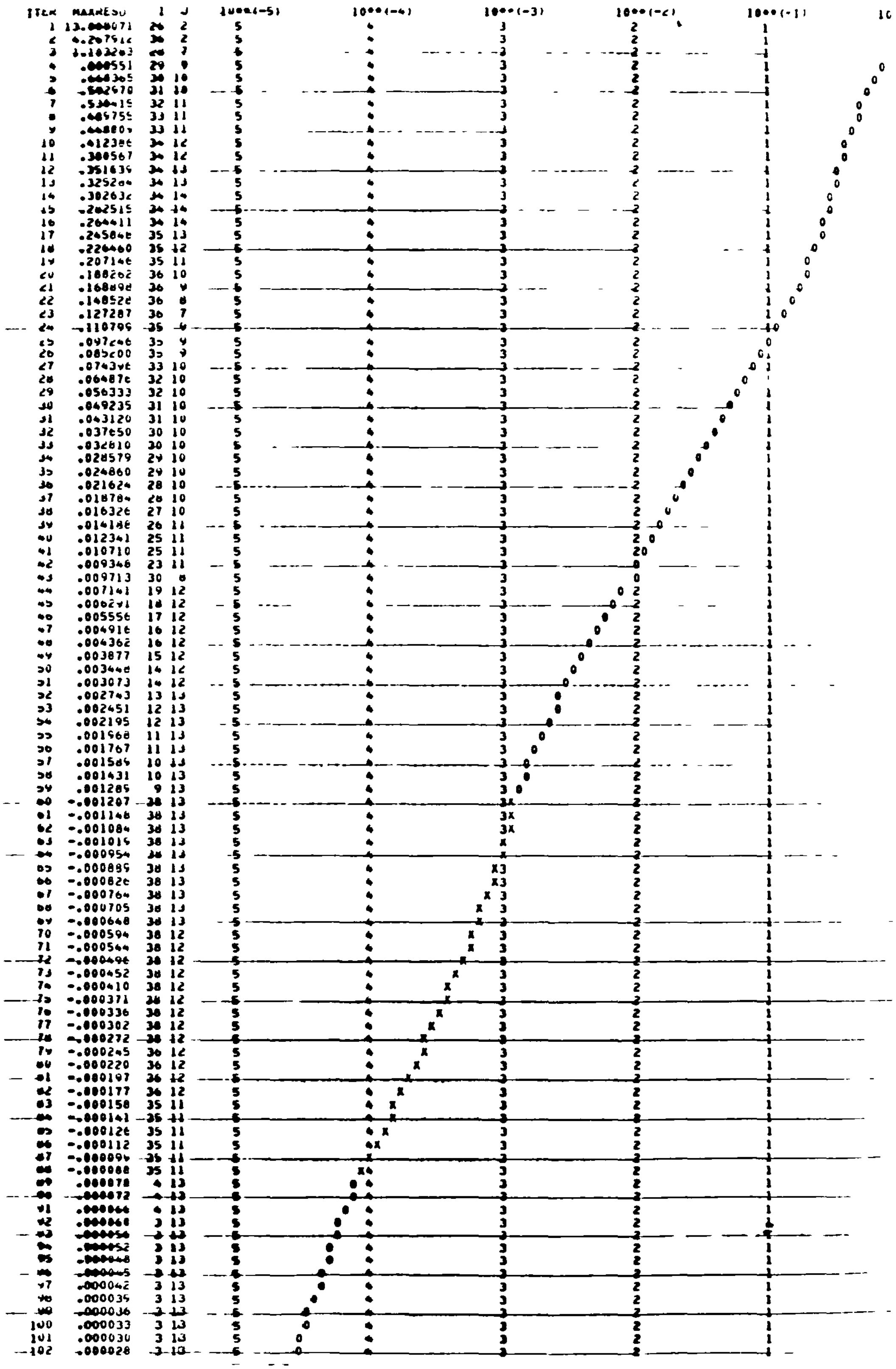


Figure 4-4. Iteration history of horizontal line relaxation
$\omega = 1.8$, $M_\infty = 0.98$.

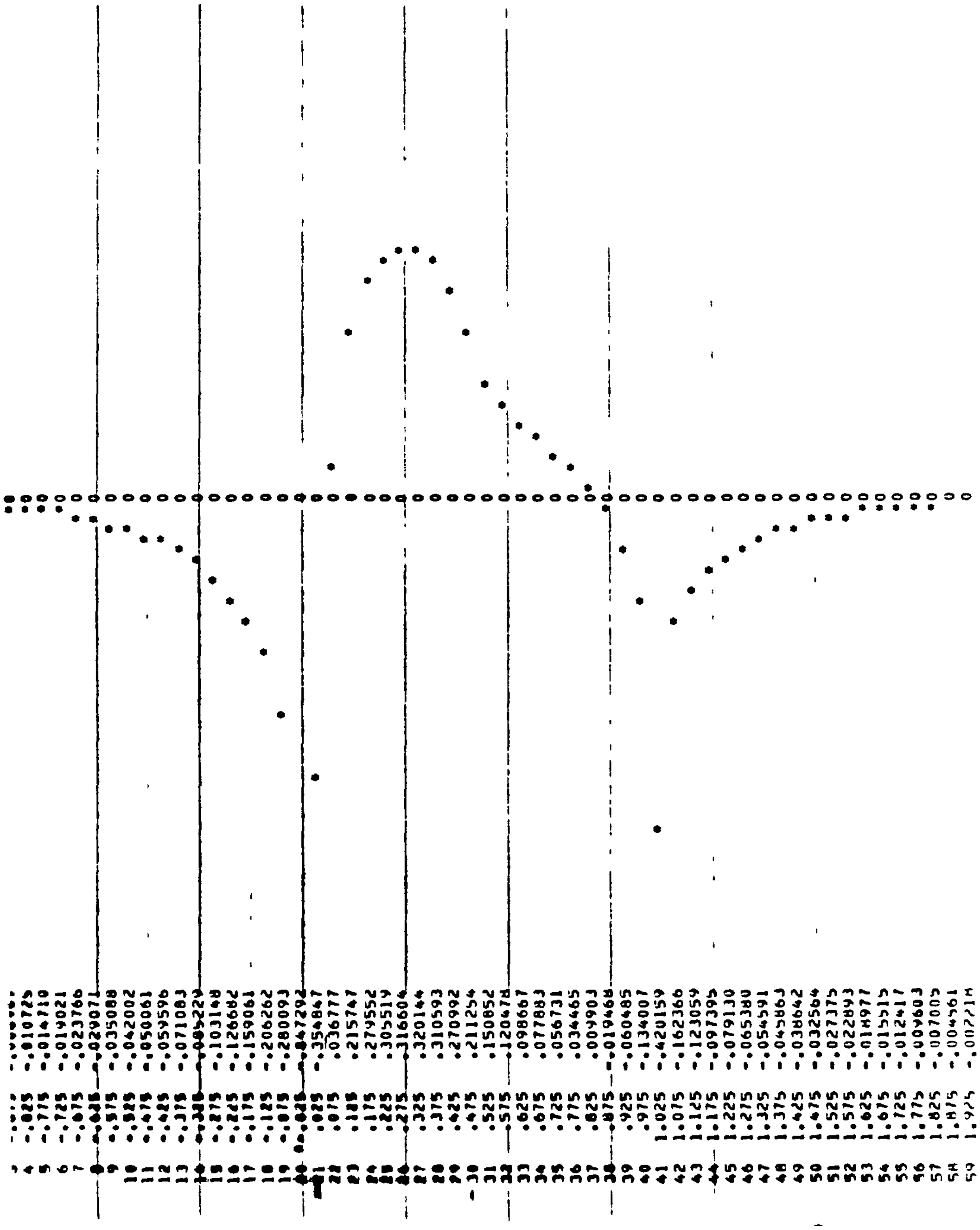


Figure 5-1. u versus x , M_∞ = 0.85 , viscosity factor = 1.3 .

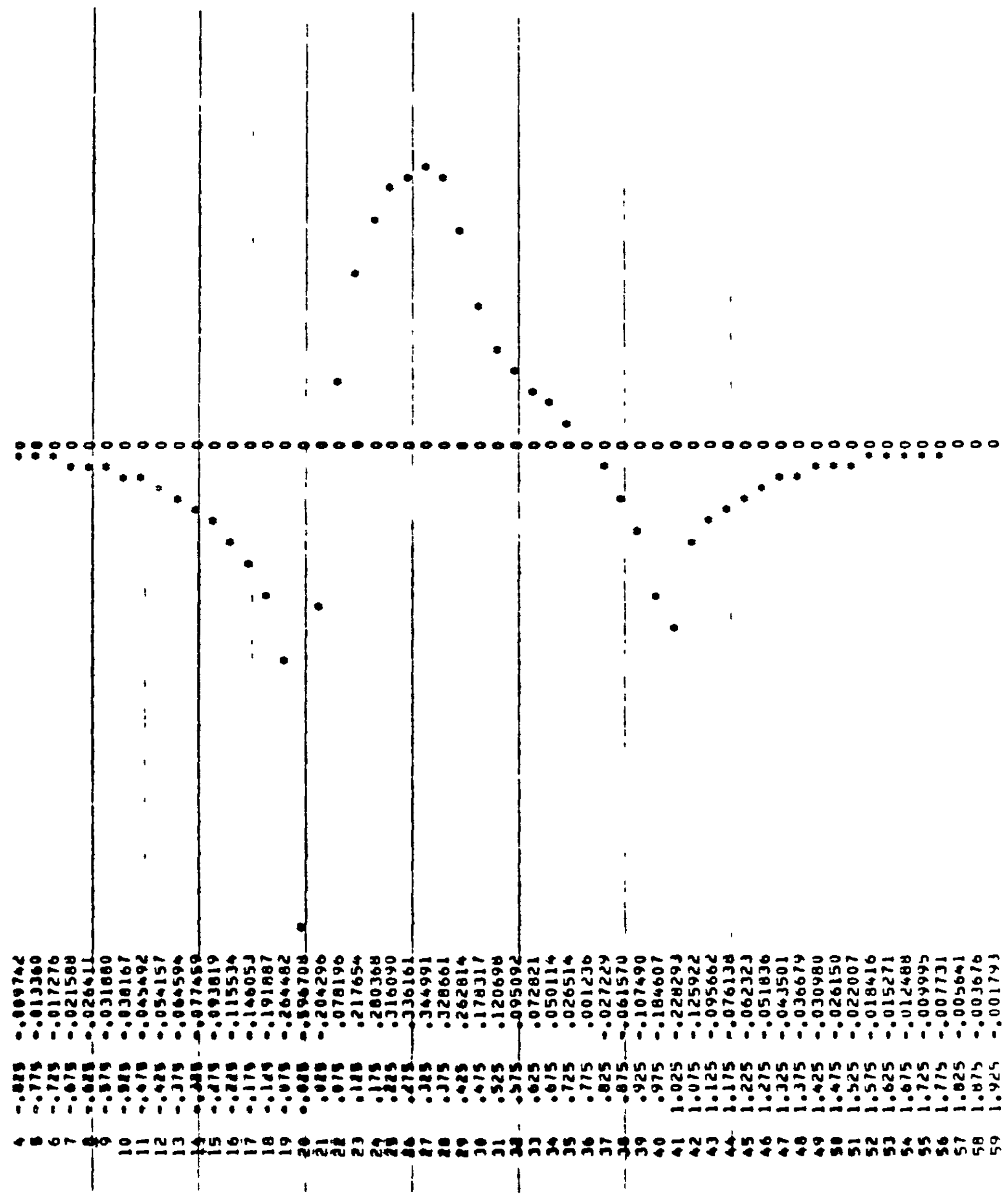


Figure 5-2. u versus x , $M_\infty = 0.85$, viscosity factor = 1.0 .

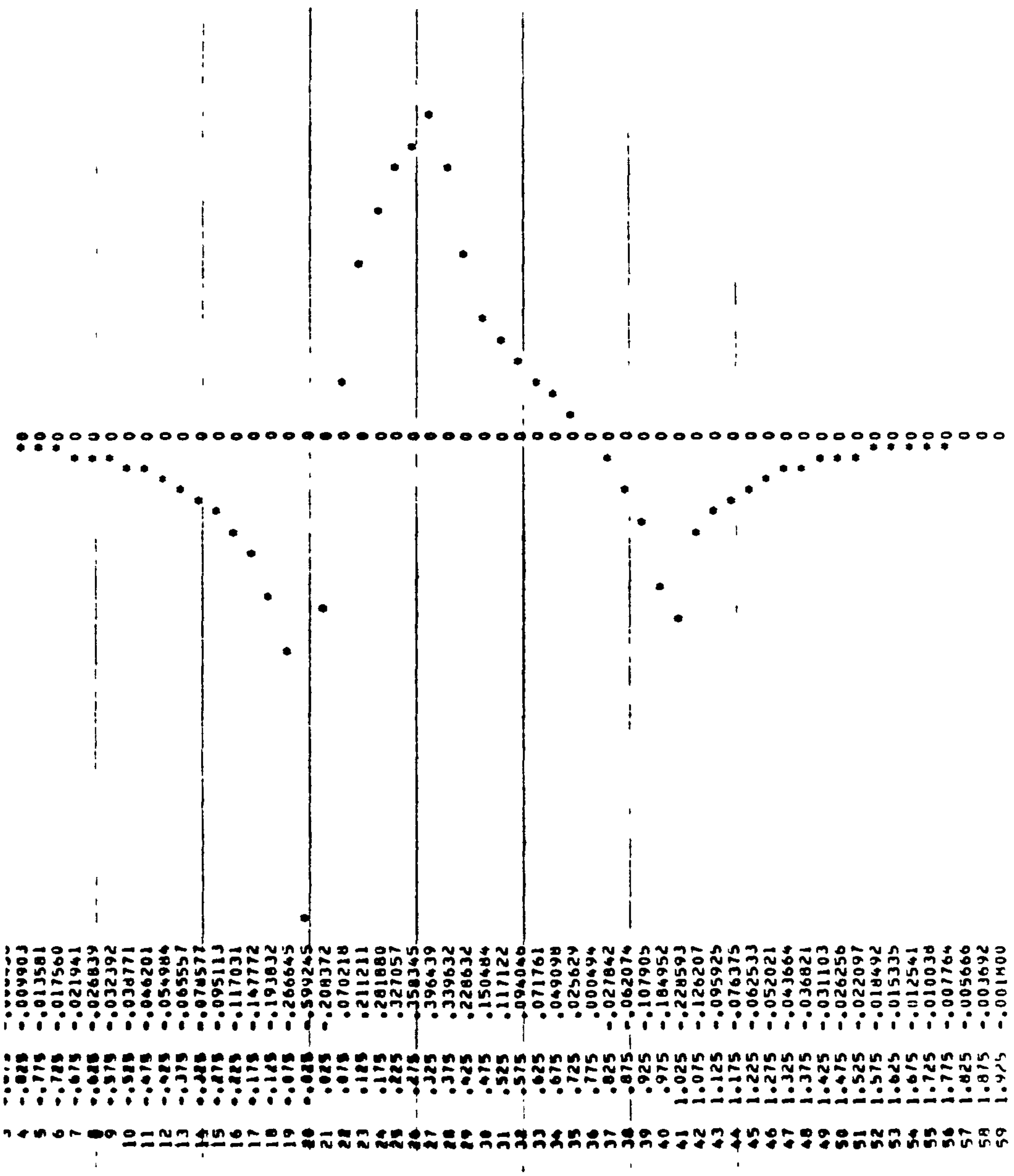


Figure 5-3. u versus x , $M_\infty = 0.85$, viscosity factor = 0.5 .

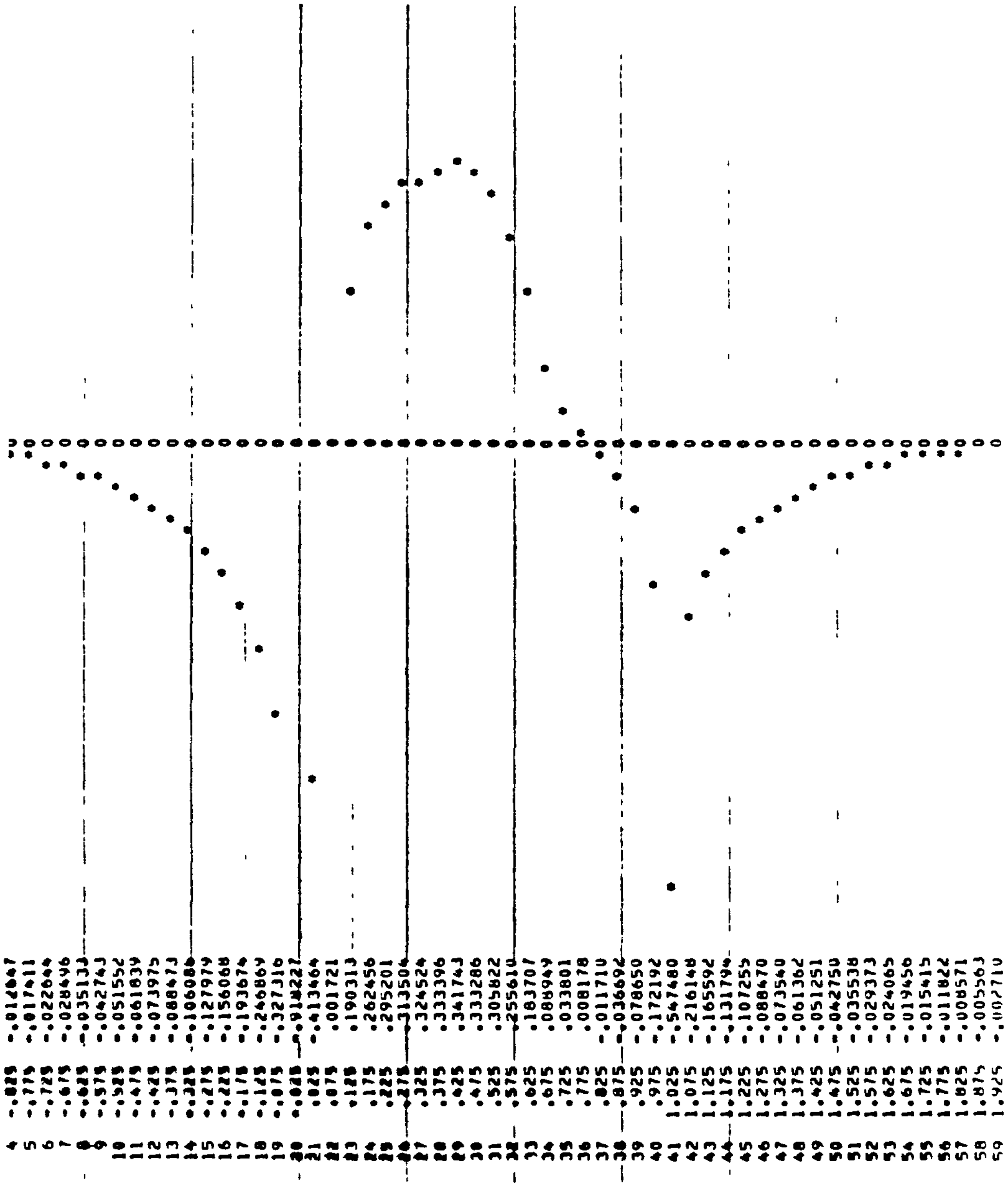
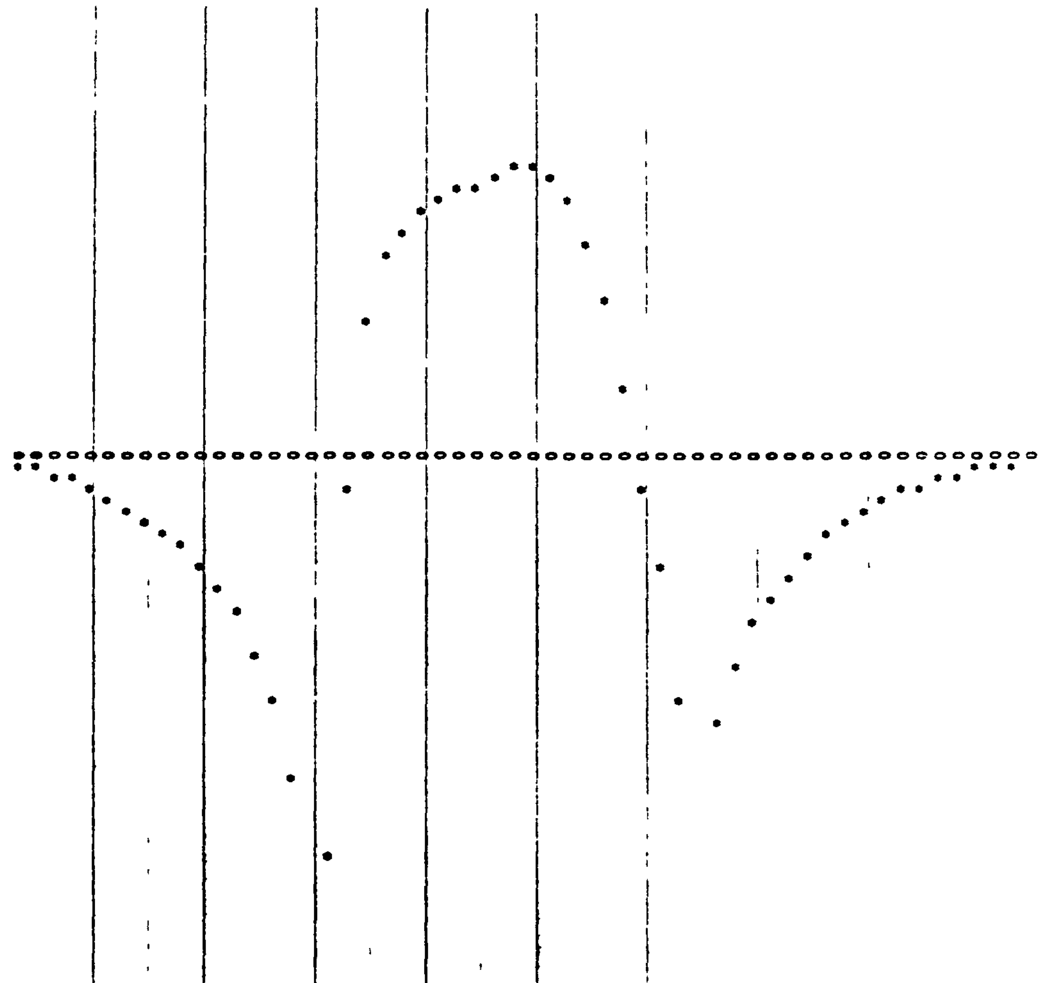


Figure 5-4. u versus x , $M_\infty = 0.9$.



3	[illegible]	[illegible]
4	-.825	-.013079
5	-.775	-.019343
6	-.725	-.025578
7	-.675	-.032797
8	-.625	-.041207
9	-.575	-.051026
10	-.525	-.062492
11	-.475	-.075887
12	-.425	-.091573
13	-.375	-.110052
14	-.325	-.132068
15	-.275	-.158787
16	-.225	-.192140
17	-.175	-.235481
18	-.125	-.294922
19	-.075	-.381903
20	-.025	-.999486
21	.025	-.482769
22	.075	-.044876
23	.125	.157112
24	.175	.236185
25	.225	.272940
26	.275	.294409
27	.325	.307078
28	.375	.316668
29	.425	.324518
30	.475	.333535
31	.525	.342246
32	.575	.344406
33	.625	.334470
34	.675	.305822
35	.725	.256086
36	.775	.186313
37	.825	.081065
38	.875	-.043372
39	.925	-.132888
40	.975	-.287651
41	1.025	-.771952
42	1.075	-.322340
43	1.125	-.254176
44	1.175	-.206598
45	1.225	-.170778
46	1.275	-.142392
47	1.325	-.119098
48	1.375	-.099554
49	1.425	-.082938
50	1.475	-.068716
51	1.525	-.056516
52	1.575	-.046056
53	1.625	-.037104
54	1.675	-.029454
55	1.725	-.022911
56	1.775	-.017282
57	1.825	-.012373
58	1.875	-.007981
59	1.925	-.003904

Figure 5-5. u versus x , $M_\infty = 0.95$.

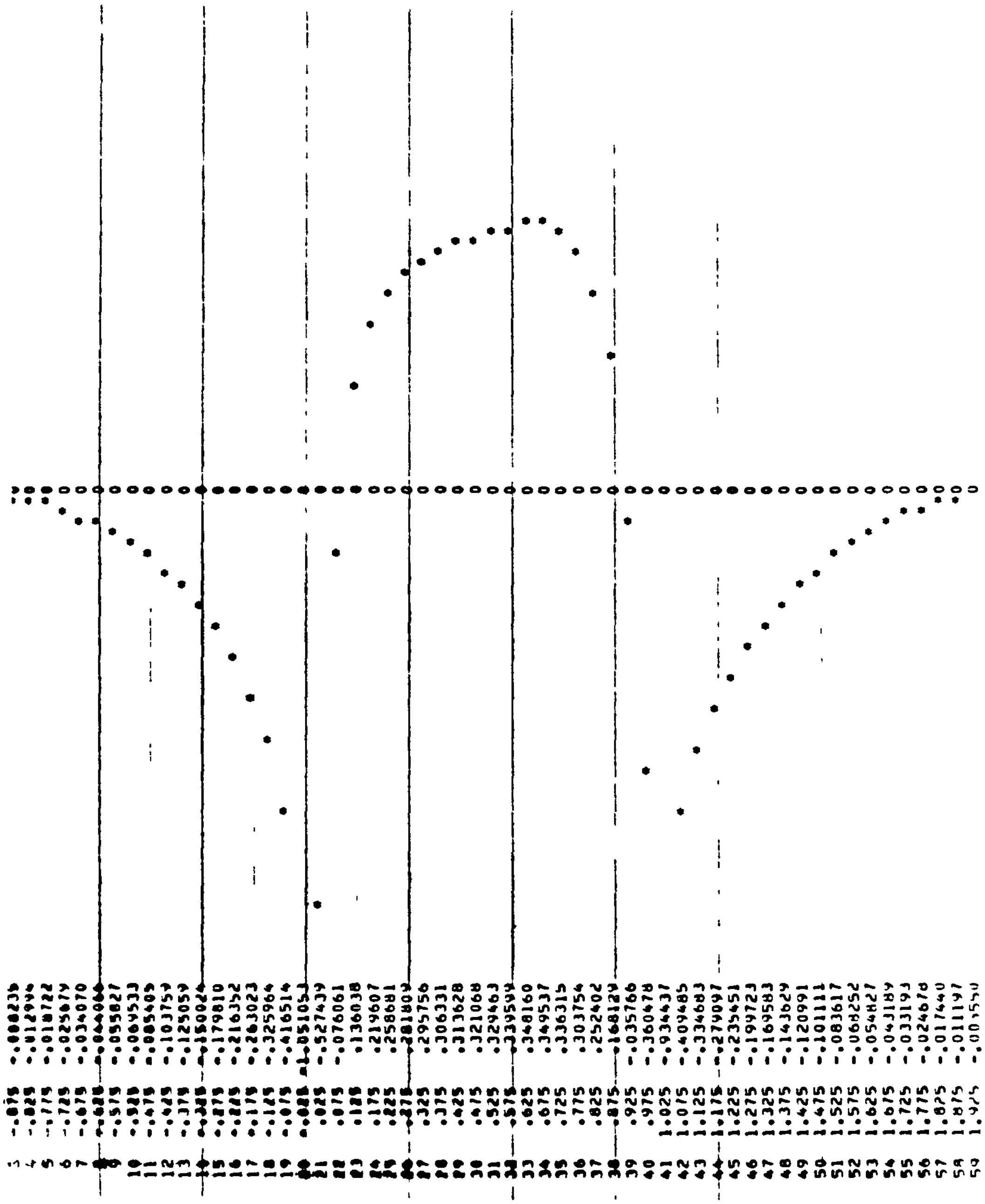


Figure 5-6. u versus x , $M_\infty = 0.98$.

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