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EQUATIONS FOR THE ANGLES OF ARRIVAL AND DEPARTURE FOR MULTIVARIABLE ROOT LOCI USING FREQUENCY-DOMAIN METHODS

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ABSTRACT

Frequency-domain methods are used to study the angles of arrival and departure for multivariable root loci. Explicit equations are obtained. For a special class of poles and zeros, some simpler equations that are generalizations of the single input-single output equations are presented.

Keywords: root locus, angles of arrival and departure, poles, zeros, Smith-MacMillan form.

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1. Introduction

The study of the angles of arrival and departure for multivariable root loci began only recently. It has proceeded using two different approaches: the state-space approach used by Shaked, [3] which involves computationally arduous spectral decompositions of matrices; and the frequency-domain approach used by Postlethwaite [2], which involves the use of a Newton diagram to obtain a series approximation of the loci near the pole or zero of interest..

In this paper we take the frequency-domain methods of Postlethwaite and develop them further to obtain more detailed results. It is shown that, subject to certain conditions, multivariable root loci depart from poles and arrive at zeros in Butterworth patterns whose orders come from the McMillan indices of the transfer-function matrix G(s) at the pole or zero in question. Explicit equations for these angles, requiring only the evaluation of polynomials at the pole or zero in question, are obtained. We also define a special class of higher-order poles and zeros for which much simpler equations may be used. These simpler equations turn out to be generalizations of the single-input-single-output (SISO) root locus equations,

The problem considered is the standard root locus set-up, in which a system represented by the transfer-function matrix G(s) is placed in a feedback loop with a scalar gain k multiplying all channels. As k varies from zero to infinity, the closed-loop poles vary, and the plot of these variations in the complex plane is the root locus. G(s) is an mxm rational matrix function of the complex variables, and is assumed to have full rank and be strictly proper.

2. Background

The closed-loop poles are given by the solutions to

$$\Delta(g,s)=\det\left(gI-G(s)\right) = g^m - \mathrm{tr}G(s)\, g^{m-1} + \sum\begin{pmatrix}\text{principal minors} \\ \text{of order 2 of } G(s)\end{pmatrix} g^{m-2}$$

$$- \cdot \cdot \cdot + (-1)^m \det G(s) = 0 \qquad (1)$$

where $g = -1/k$. The multivariable root loci are thus branches of the algebraic function $s(g)$; however, for the purpose of determining the angles of arrival and departure we may regard the root loci as analytic functions of g. Multiplying (1) through by $A_m(s)$, the least common denominator of the non-zero principal minors of all orders of $G(s)$, we obtain

$$\Phi(g,s) = A_m(s)g^m + A_{m-1}(s)g^{m-1} + \ldots + A_1(s)g + A_o(s) = 0 \qquad (2)$$

where the $A_i(s)$ are all polynomials. The $A_i(s)$ will figure extensively in the results to follow. It may easily be shown (see Yagle [5],p.19) that, excepting single-point loci, the poles of $G(s)$ are the zeros of $A_m(s)$ and the finite zeros of $G(s)$ are the zeros of $A_o(s)$.

The Newton Polygon

The Newton polygon is a graphical device that can be used to find a series approximation to the function $\Phi(g,s)$ in the vicinity of a zero of the function. It is discussed in detail in Walker [4]; here we merely give instructions for constructing the polygon associated to the function $\Phi(g,s)$ around $s=0$.

(1) Write $A_i(s)=b_i s^{a_i}$ + (higher-order terms) for $i=0\ldots m$

(2) Set up u and v axes and plot the $m+1$ points $P_i=(u,v)=(i,a_i)$, $i=0,\ldots m$.

(3) Join P_o to P_m with a convex polygonal arc each of whose vertices is a P_i and such that no P_i lies below the arc.

(4) For each pair of points P_i, P_k forming the endpoints of a segment, compute the slope z of the segment and solve the equation $c^j b_j + c^k b_k = 0$ for c. If there is another point P_h on this segment, solve instead the equation $c^h b_h + c^j b_j + c^k b_k = 0$.

(5) A series approximation to $\Phi(g,s)$ in the vicinity of s=0 is then $g = cs^{-z}$.

After obtaining this series approximation to a branch of the root locus, we may then compute its angle from Arg $\left[\frac{ds}{dg}\right]$. This is the method used by Postlethwaite [2]. Although construction of a Newton polygon is not necessary to employ any of the results of this paper, the Newton polygon is the basis for all of the proofs given in the Appendix.

3. Main Results

In this section we state the main result of this paper.

Definition. Let the Smith-McMillan form of G(s) be diag $\left(\frac{n_i(s)}{d_i(s)}\right)$, and let p by an nth-order pole. Let k_i be the largest integer such that $(s-p)^{k_j} \mid d_j(s)$, $j=1,\ldots m$. Then the $\{k_j\}$ are the structure indices associated with the pole p. Note that

(1) $\sum_{j=1}^{m} k_j = n$, (2) $k_1 \geq k_2 \geq \ldots \geq k_m$.

Structure indices for zeros are defined analogously. We may now state:

Theorem 1 The root loci departing from an nth-order pole depart generically in Butterworth patterns whose orders are the non-zero structure indices of G(s) at the pole. For a pole p with non-zero structure indices $[k_1,k_2,\ldots,k_r]$ the angles of departure are:

$$\Theta_{depart,i,j} = \frac{1}{k_i} \operatorname{Arg} \left[\left. \frac{\frac{d^{(n-k_1-k_2-\ldots-k_i)}}{ds^{(n-k_1-k_2-\ldots-k_i)}} A_{m-i}(s)}{\frac{d^{(n-k_1-\ldots-k_{i-1})}}{ds^{(n-k_1-\ldots-k_{i-1})}} A_{m-i+1}(s)} \right|_{s=p} \right]$$

$$+ \frac{j360°}{k_i} , \ j = 0,1,\ldots k_i-1 , \ i = 1,2,\ldots,r \qquad (3)$$

if and only if the following conditions are met:

(1) $k_1 \neq k_2 \neq \ldots \neq k_r$ (4a)

(2) $\det \begin{bmatrix} w_{11}(p) & \ldots & w_{1j}(p) \\ \vdots & & \vdots \\ w_{j1}(p) & \ldots & w_{jj}(p) \end{bmatrix} \neq 0 , \ j = 1,\ldots r$ (4b)

where $W(s) \triangleq V(s)U(s)$ and $U(s)$ are unimodular matrices that transform $G(s)$ in its Smith-McMillan form, i.e.,

$$G(s) = U(s) \operatorname{diag}\left(\frac{n_1(s)}{d_1(s)} \ \ldots \ \frac{n_m(s)}{d_m(s)} \right) V(s) \qquad (5)$$

For the angles of arrival we have:

<u>Theorem 2</u> The root loci arriving at an nth-order zero arrive generically in Butterworth patterns whose orders are the non-zero structure indices of $G(s)$ at the zero. For a zero z with non-zero structure indices $[k_{m-r}, \ldots k_{m-1}, k_m]$ the angles of arrival are:

$$\Theta_{arrival,i,j} = \frac{1}{k_{m-i}} \operatorname{Arg} \left[\left. \frac{\frac{d^{(n-k_m-k_{m-1}-\ldots-k_{m-i})}}{ds^{(n-k_m-k_{m-1}-\ldots-k_{m-i})}} A_{i+1}(s)}{\frac{d^{(n-k_m-\ldots-k_{m-i+1})}}{ds^{(n-k_m-\ldots-k_{m-i+1})}} A_i(s)} \right|_{s=z} \right]$$

$$+ \frac{j360°}{k_{m-i}} , \ j=0,1,\ldots k_{m-i}-1 , \ i=0,1,\ldots r \qquad (6)$$

if and only if the following conditions are met:

(1) $k_m \neq k_{m-1} \neq \ldots \neq k_{m-r}$ (7a)

(2) $\det \begin{bmatrix} w_{11}(z) & \ldots & w_{1j}(z) \\ \vdots & & \vdots \\ w_{j1}(z) & \ldots & w_{jj}(z) \end{bmatrix} \neq 0 \text{ , } j=m-1,m-2, \ldots m-r-1$ (7b)

where $k_{m+1} \triangleq 0$.

<u>Proofs</u>: See Appendix.

It should be noted that whether or not the conditions (4a) and (7a) hold, the orders of the Butterworth patterns are given by the structure indices as long as conditions (4b) and (7b) or similar conditions (see Appendix) hold. It is not clear that these conditions hold generically; however, it has been shown (Byrnes and Stevens [1]) that a necessary and sufficient condition for the Butterworth pattern orders to be given by the structure indices is for certain matrices arising in the block-diagonalization of G(s) to have simple null structure. This imples that the conditions (4b) and (7b) are generic.

The following example illustrates the application of Theorems 1 and 2.

<u>Example</u>: Compute the angles of departure for the root locus of

$$G(s) = \frac{1}{s^4 + 4s^3 + 8s^2 + 8s + 4} \begin{bmatrix} s^2 + 8s + 17 & s^3 + 10s^2 + 33s + 34 \\ s^3 + 9s^2 + 25s + 17 & 2s^4 + 21s^3 + 78s^2 + 117s + 68 \end{bmatrix}$$

It is straightforward to compute

$$\Phi(g,s) = (s^6 + 6s^5 + 18s^4 + 32s^3 + 36s^2 + 24s + 8)g^2$$
$$-(2s^6 + 25 s^5 + 125 s^4 + 325 s^3 + 493 s^2 + 420 s + 170)g$$
$$+(s^4 + 16s^3 + 98s^2 + 272s + 289) = 0$$

and to obtain (from $A_2(s)$) the third-order poles $-1 \pm j$. The structure indices for these poles are [2,1], and the condition (4b) is indeed satisfied. Thus loci depart from the pole -1+j in a second-order and a first-order Butterworth patterns with angles

$$\Theta_{depart,\,1} = \frac{1}{2}\text{Arg}\left[-\left.\frac{12s^5 + 125s^4 + 500s^3 + 975s^2 + 986s + 420}{120s^3 + 360s^2 + 432s + 192}\right|_{s=-1+j}\right]$$

$$+ \frac{n360°}{2}, \quad n=0,1$$

$$= 61.8°, \; 241.8°$$

$$\Theta_{depart,2} = \text{Arg}\left[-\left.\frac{s^4 + 16s^3 + 98s^2 + 98s^2 + 272s + 289}{12s^5 + 125s^4 + 500s^3 + 975s^2 + 986s + 420}\right|_{s=-1+j}\right]$$

$$= 33.7°.$$

By symmetry, the angles of departure from the pole -1-j are -61.8°, 118.2°, and -33.7°.

4. Simple Poles and Zeros

For a special class of poles and zeros the angles of arrival and departure may be computed more readily using the following theorems instead of Theorems 1 and 2.

Definition: An nth-order pole is said to be simple if its structure indices are [n,0, ... 0].

A similar definition applies for nth-order zeros. We now have:

Theorem 3 Let the Laurent expansion of G(s) at an nth-order pole p be

$$G(s) = \frac{1}{(s-p)^n} G_{-n} + \dots + \frac{1}{(s-p)} G_{-1} + G_o + \dots \qquad (8)$$

Then, the pole p is simple if and only if tr G_{-n} is non-zero, and the angles of departure from p are

$$\Theta_{depart} = \frac{1}{n}\text{Arg}\,[-\text{tr}\,G_{-n}] + \frac{j360°}{n}, \quad j=0,1,\ldots n-1$$

$$= \frac{1}{n}\text{Arg}\,[-(s-p)^n\,\text{tr}\,G(s)\big|_{s=p}] + \frac{j360°}{n}, \quad j=0,1,\ldots n-1, \qquad (9)$$

Theorem 4 Let z be an nth-order zero of $G(s)$, and let the Laurent expansion of $G^{-1}(s)$ at z be

$$G^{-1}(s) = \frac{1}{(s-z)^n} H_{-n} + \ldots + \frac{1}{(s-z)} H_{-1} + H_o + \ldots \qquad (10)$$

Then, if and only if $\operatorname{tr} H_{-n}$ is non-zero, the zero z is simple and the angles of arrival at z are

$$\Theta_{arrival} = \frac{1}{n} \operatorname{Arg} \left[\operatorname{tr} H_{-n}\right] + \frac{j360°}{n}, \; j=0,1, \ldots n-1$$

$$= \frac{1}{n} \operatorname{Arg} \left[(s-z)^n \operatorname{tr} G^{-1}(s)\Big|_{s=z}\right] + \frac{j360°}{n},$$

$$j=0,1, \ldots n-1. \qquad (11)$$

Proofs: See Appendix.

For the case of m=2, we may simplify (11) by noting that in this case we have

$$\operatorname{tr} G^{-1}(s) = \frac{\operatorname{tr} G(s)}{\det G(s)} \qquad (12)$$

so that (11) becomes

$$\Theta_{arrival} = \frac{1}{n} \operatorname{Arg} \left[(s-z)^n \frac{\operatorname{tr} G(s)}{\det G(s)}\Big|_{s=z}\right] + \frac{j360°}{n},$$

$$j=0,1,..n-1. \qquad (13)$$

Although Theorems 3 and 4 are easier to employ than Theorems 1 and 2, the most interesting thing about them is that they are striking generalizations of the SISO root locus equations for computing the angles of arrival and departure. The only difference is that the trace of the multivariable transfer function matrix is substituted for the SISO scalar transfer function. Note that this generalization is observed only for simple poles and zeros, since in the SISO case all higher-order poles and zeros are simple, while in the multivariable case only some are.

Further results are available for the case of first-order poles. It is easy to show (see [5],p.51) that the conditions for Theorems 1 and 3 are equivalent, i.e., $\operatorname{tr} G_{-1}$ is non-zero if and only if $w_{11}(p)$ is non-zero. If this condition does not hold, we may use the following theorem:

Theorem 5 Let the Laurent expansion of G(s) at a first-order pole p be (8). Then if

(1) $\text{tr } G_{-1} = 0$, (2) $\text{tr } (G_{-1}G_0) \neq 0$

the angles of departure from p is

$$\Theta_{depart} = \text{Arg}[\text{tr}(G_{-1}G_0)] \tag{14}$$

and the root locus branch departs as k^2 (a ½-order departure).

Proof: See [5], p.52.

5. Conclusion

The behavior of the anglos of arrival and departure for multivariable root loci has been studied from a frequency-domain point of view, and explicit equations for these angles have been obtained. Simpler equations are available for the case of "simple" higher-order poles and zeros, and it was noted that these equations are generalizations of the SISO root locus equations for angles of arrival and departure. All of these results depend on some genericity conditions. For first-order poles an equation for the angle of departure was given for a case wherein these conditions are violated. More work needs to be done in clarifying these assumptions, interpreting them, and obtaining equations for angles in cases where these assumptions are violated.

6. References

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Appendix

Proof of Theorem 1 We shall prove Theorem 1 by first constructing the Newton polygon, and then using it to obtain the angles of departure. A more detailed proof is given in [5].

From (1) and (5) we have

$$\Delta(g\ s) = \det\left(gI-G(s)\right)=\det\left(gI-U(s)\operatorname{diag}\left(\frac{n_i(s)}{d_i(s)}\right)V(s)\right)$$

$$= \det\left(gI - \operatorname{diag}\left(\frac{n_i(s)}{d_i(s)}\right) W(s)\right) \tag{15}$$

so that the coefficients of (2) may be written as

$$A_{m-h}(s) = A_m(s)(-1)^h \sum \begin{pmatrix}\text{principal minors of order} \\ h \text{ of } \operatorname{diag}\left(\frac{n_i(s)}{d_i(s)}\right) W(s)\end{pmatrix}$$

$$= A_m(s)(-1)^h \sum \left(\prod_{i_1 \dots i_h} \frac{n_j(s)}{d_j(s)}\right)\begin{pmatrix}\text{corresponding} \\ \text{principal} \\ \text{minor of } W(s)\end{pmatrix}, \quad h=1 \dots m \tag{16}$$

where we have used the Binet-Cauchy Theorem in (16)

Now let p be an nth-order pole with non-zero structure indices $[k_1, k_2, \dots k_r]$, and assume that p is not a single-point locus. Then we may write

$$A_m(s) = (s-p)^n \tilde{A}_m(s)\ ,\ \tilde{A}_m(p) \neq 0 \tag{17}$$

$$d_i(s) = (s-p)^{k_i} \tilde{d}_i(s),\ \tilde{d}_i(p) \neq 0,\ i=1\dots m \tag{18}$$

and substituting these into (16) we get

$$A_{m-h}(s) = (s-p)^{(n-k_1 \dots -k_h)} \tilde{A}_m(s)(-1)^h \frac{n_1(s)\dots n_h(s)}{\tilde{d}_1(s)\dots\tilde{d}_h(s)} \cdot$$

$$\det\begin{bmatrix} w_{11}(s) & \dots & w_{1h}(s) \\ \vdots & & \vdots \\ w_{h1}(s) & \dots & w_{hh}(s)\end{bmatrix} + \begin{pmatrix}\text{higher-order terms} \\ \text{in } s-p\end{pmatrix}, \quad h=1 \dots m \tag{19}$$

(recall that $k_1 \geq k_2 \geq \ldots \geq k_m$). Since $n_i(s)$ and $d_i(s)$ are relatively prime, $n_i(p) \neq 0$ for $i=1,\ldots,r$. It should now be evident from (19) that the Newton polygon will take the form given in Fig. 1 if and only if the conditions (4a, 4b) are satisfied. Note, that if for example $k_1=k_2$, then (4b) is replaced by the condition

$$\frac{n_1(p)}{d_1(p)} w_{11}(p) + \frac{n_2(p)}{d_2(p)} w_{22}(p) \neq 0 \tag{20}$$

which we would still expect to be true in general.

From the Newton polygon we may approximate the root loci departing from p by the series

$$k \approx c_i(s-p)^{k_i}, \quad i=1\ldots r \tag{21}$$

where the coefficients c_i solve

$$(c_i)^{i-1} b_{i-1} + (c_i)^i b_i = 0 \tag{22}$$

and where b_i is defined from

$$A_{m-i}(s) = b_i(s-p)^{(n-k_1-\ldots-k_i)} + \begin{pmatrix}\text{higher-order} \\ \text{terms in } s-p\end{pmatrix} . \tag{23}$$

It should be evident from (21) that the departing loci are grouped into Butterworth patterns with orders $\{k_i, i=1..r\}$. The coefficients b_i may be obtained from (23) by taking repeated derivatives, and this yields (3), proving Theorem 1.

<u>Proof of Theorem 2</u> This is essentially the same as the proof of Theorem 1. The only difference is that we now use $i=m, m-1, \ldots m-r$ instead of $i=1, 2, \ldots r$.

Proof of Theorem 3 Taking the trace of (8) and using (17), we have

$$
\begin{aligned}
A_{m-1}(s) &= -A_m(s)\ \mathrm{tr}\ G(s) \\
&= -(s-p)^n\ \tilde{A}_m(s)\ \left(\frac{1}{(s-p)^n}\ \mathrm{tr}\ G_{-n} + \dots\right) \\
&= -\tilde{A}_m(s)\ \mathrm{tr}\ G_{-n} + \begin{pmatrix}\text{higher-order}\\ \text{terms in } s-p\end{pmatrix}
\end{aligned} \tag{24}
$$

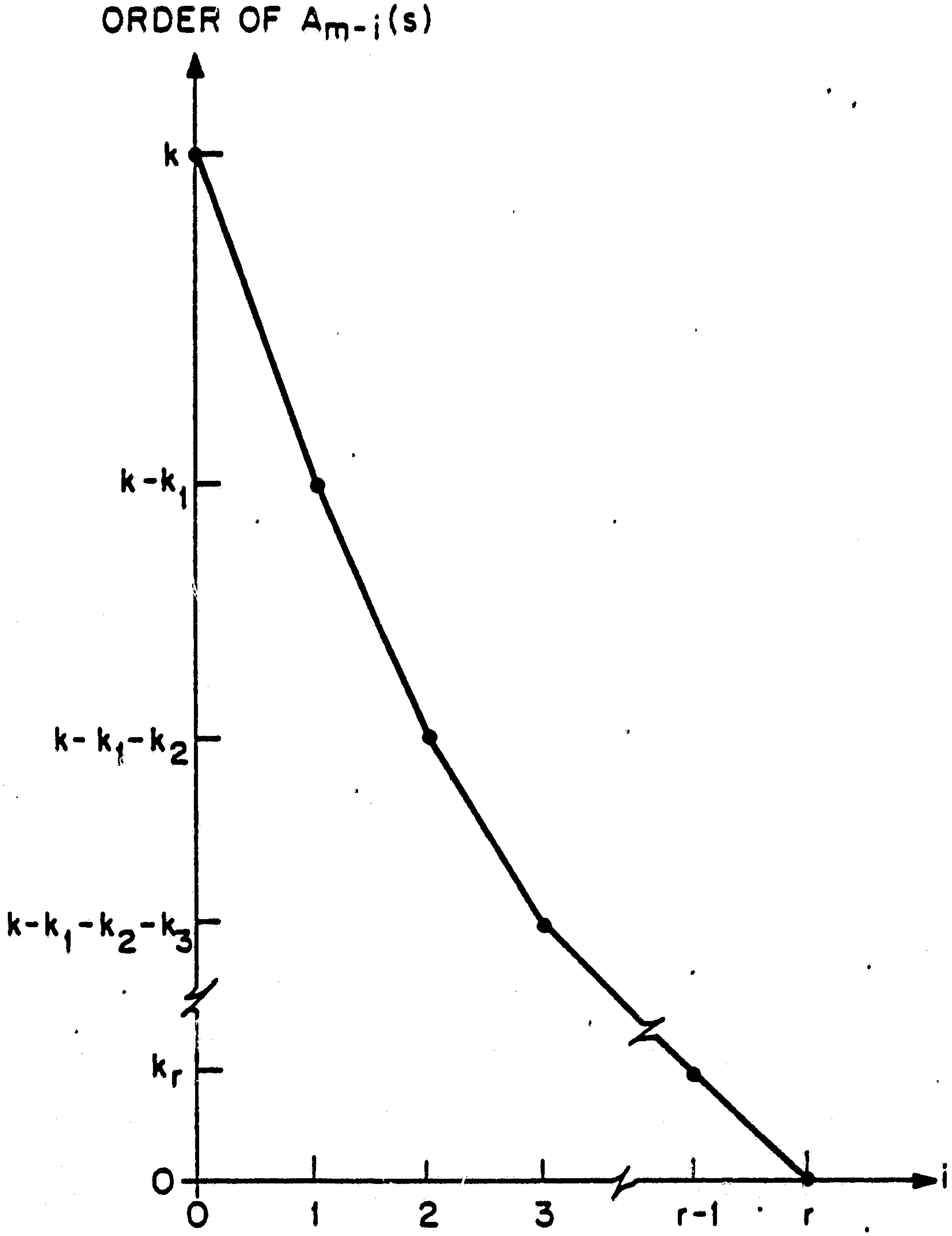
and if tr G_{-n} is non-zero Theorem 3 follows immediately from Theorem 1.

Proof of Theorem 4 The main diagonal elements of $G^{-1}(s)$ are the principal minors of order m-1 of G(s) divided by det G(s). So we have

$$
\begin{aligned}
A_1(s) &= A_m(s)\sum\begin{pmatrix}\text{principal minors of}\\ \text{order } m-1 \text{ of } G(s)\end{pmatrix}\ (-1)^{m-1} \\
&= A_m(s)\det G(s)\ \sum(\text{main diagonal elements of } G^{-1}(s))\ (-1)^{m-1} \\
&= -A_o(s)\ \mathrm{tr}\ G^{-1}(s)
\end{aligned} \tag{25}
$$

and the rest of the proof parallels the proof of Theorem 3.

Figure 1. Newton Polygon for the Angles of Departure in the Generic Case.



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