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Transfer-Matrices for Series-Type Microwave Antenna Circuits

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TRANSFER MATRICES FOR SERIES-TYPE MICROWAVE ANTENNA CIRCUITS

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April 1981

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ABSTRACT

This document develops transfer matrices which permit analysis and computer evaluation of certain series-type microwave antenna circuits associated with an L-Band microwave radiometer (LBMR) under investigation at Goddard Space Flight Center. This radiometer is one of several diverse instrument designs to be used for the determination of soil moisture, sea state, salinity, and temperature data. Four-port matrix notation is used throughout for the evaluation of LBMR circuits with mismatched couplers and lossy transmission lines. Matrix parameters in examples are predicated on an impedance analysis and an assumption of an array aperture distribution. The notation presented here is easily adapted to longer and more varied chains of matrices, and to matrices of larger dimension.

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LIST OF ILLUSTRATIONS

GLOSSARY OF NOTATION

g_i	Elements of a column matrix
h_i	Elements of a column matrix
ψ_i	Phase shift (radians)
$[T]$	Transfer matrix
R,L	Right, left (in context)
c_1, c_2	Coupler voltage coefficients
ℓ	Transmission line length
Z_0	Characteristic impedance of line
V, I	Standing-wave voltage and current
γ	Propagation constant
α	Attenuation constant, nepers per unit length, proportional (in context)
β	Phase constant, radians per unit length
A,B	Incident and reflected wave amplitudes
Γ_r	Complex reflection coefficient (generic)
Γ_t	Complex transmission coefficient (generic)
Γ_0	Complex reflection coefficient at termination
Z_L	Termination impedance
V_{in}, V_L	Standing wave voltages at input and output of line
α_r	Attenuation constant (mismatched line)
K_i	Array element current
P_i	Array element power
E	Electrical far-field
P_{rad}	Radiated power (density)
r	Radius
ζ_0	Intrinsic impedance of free space

k_0	Wave number
$g(\theta,\phi)$	Complex radiation pattern
h_1	Traveling wave voltage
Z_{NET}	Input impedance to LBMR net
Z_s	Generator impedance
R, G	Resistance, conductance
δP_i	Increment of P_i
T	Matrix superscript for transpose
*	Matrix superscript for complex conjugate
[I]	Identity matrix
dB	Decibel
L, C	Inductance, capacitance

TRANSFER MATRICES FOR SERIES-TYPE MICROWAVE ANTENNA CIRCUITS

INTRODUCTION

This document records some of the preliminary work which has been done to analyze the series-type circuit associated with the L-Band Microwave Radiometer (LBMR) design at Goddard Space Flight Center. The matrix formulation presented here is not restricted to the LBMR, and may be utilized in a general way. Furthermore, additional parallel paths may be obtained by increasing the dimensions of the matrices. A relevant example may be found in Ref. 1.

The transfer matrices in this document are defined over a field of complex numbers. They are a direct consequence of the definition of a scattering matrix of some microwave component or junction. See Ref. 2. The latter may be postulated or may be obtained by means of a direct physical measurement on a "black-box."

A general or bilateral transfer matrix may be regarded, after appropriate rearrangement and partitioning, as being composed of two null matrices and two unilateral ("transmission" and "reception") matrices of equal dimension. This partitioning has been exploited throughout the present document; only unilateral "transmission" matrices are used. In this manner 75-percent of the matrix-element bookkeeping is eliminated. The notational conventions, conversion from scattering to transfer matrix, and dimensional reduction are detailed in the beginning of this document. The development is then carried to the design stage for the overall problem.

TRANSFER MATRIX CONVENTIONS AND REDUCTION

The conventions used in this document are those of Ref. 1, and are made clear by means of a simple example: two unequal lossless line-lengths in parallel.

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} = \begin{bmatrix} e^{-j\psi_1} & 0 & 0 & 0 \\ 0 & e^{j\psi_1} & 0 & 0 \\ 0 & 0 & e^{-j\psi_2} & 0 \\ 0 & 0 & 0 & e^{j\psi_2} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} \tag{1}$$

An output column matrix ($\bar{g}$) results from the product of a transfer matrix $[T]$ with an input column matrix ($\bar{h}$).

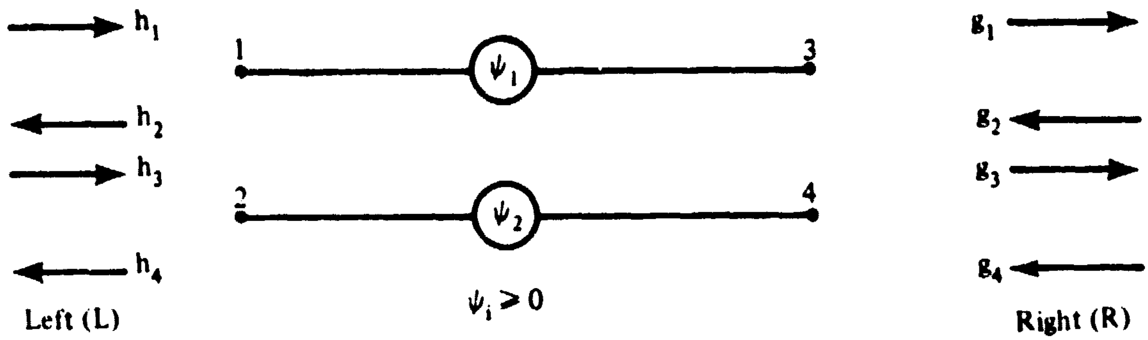

Figure 1. Transfer-Matrix Conventions.

It follows that the reduced matrix for transmission from (L) to (R) is

$$\bar{g} = [T]\,\bar{h} = \begin{bmatrix} g_1 \\ g_3 \end{bmatrix} = \begin{bmatrix} e^{-j\psi_1} & 0 \\ 0 & e^{-j\psi_2} \end{bmatrix} \begin{bmatrix} h_1 \\ h_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{bmatrix} \begin{bmatrix} h_1 \\ h_3 \end{bmatrix} . \tag{2}$$

MATCHED DIRECTIONAL COUPLER

The scattering matrix (S) of the matched directional coupler of Ref. 2, p. 301, is rewritten, before reduction, as

$$[S] = \left[\begin{array}{cc:cc} 0 & 0 & C_1 & jC_2 \\ 0 & 0 & jC_2 & C_1 \\ \hdashline C_1 & jC_2 & 0 & 0 \\ jC_2 & C_1 & 0 & 0 \end{array}\right] \tag{3}$$

in view of the port designations of the previous discussion. By inspection, the reduced transfer matrix for "transmission" is

$$[T] = \begin{bmatrix} C_1 & jC_2 \\ jC_2 & C_1 \end{bmatrix} \tag{4}$$

and represents the four-port junction shown below.

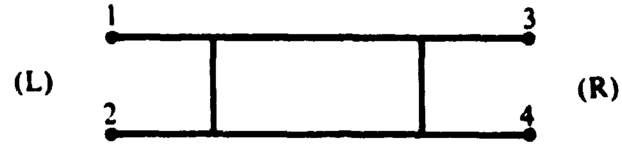

Figure 2. Matched Directional Coupler.

Since the matched-coupler matrix is both symmetrical and unitary,

$$C_1{}^2 + C_2{}^2 = 1 \ . \tag{5}$$

See Ref. 2, p. 301, and Appendix A of this document. Other [T] forms may be written for the matched coupler as there is arbitrariness in the phases of the signals at the output ports, depending on the choice of output reference planes.

LOSSY TRANSMISSION LINE

The transfer matrix for the lossy transmission line is predicated on equations found in Ref. 3, p. 13. Signal flow is assumed to be from left (L) to right (R) in Fig. 3, below.

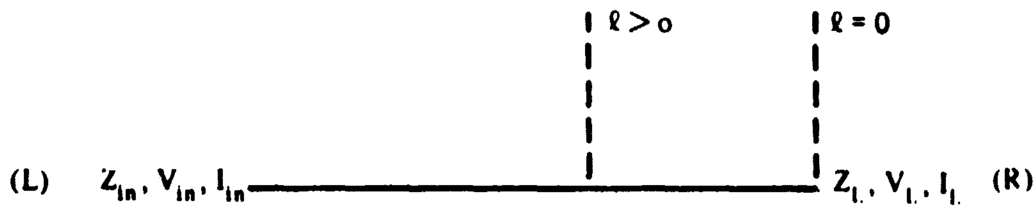

Figure 3. Lossy Transmission Line.

$$Z_z = \frac{V}{I} = Z_0 \, \frac{A\,e^{-\gamma z} + B\,e^{\gamma z}}{A\,e^{-\gamma z} - B\,e^{\gamma z}} \ , \tag{6}$$

where

$$z = -\ell \tag{7}$$

and

$$\gamma = \alpha + j\beta \ . \tag{8}$$

A voltage reflection coefficient at a generic point on the line is now defined, using

$$V_{inc} = A\,e^{\gamma \ell}$$

and

$$V_{ref.} = B\, e^{-\gamma \ell} \quad , \tag{9}$$

as

$$\Gamma_\ell = \frac{V_{ref.}}{V_{inc.}} = \frac{B}{A} e^{-2\gamma \ell} \quad . \tag{10}$$

At the special point $\ell = 0$, this reduces to

$$\Gamma_0 = \frac{B}{A} = \frac{Z_L - Z_0}{Z_L + Z_0} \quad . \tag{11}$$

The line voltages at $\ell \neq 0$ and $\ell = 0$ may then be written as

$$V_{in} = A\, e^{\gamma \ell} + A\left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) e^{-\gamma \ell} = \frac{2\, A\, (Z_L \cosh \gamma \ell + Z_0 \sinh \gamma \ell)}{(Z_L + Z_0)} \tag{12}$$

and

$$V_L = A\, e^{\gamma \ell} + B\, e^{-\gamma \ell} = A + B = A + A\left(\frac{Z_L - Z_0}{Z_L + Z_0}\right), \tag{13}$$

respectively. From these, after some algebra, the useful form

$$V_L = \left[\cosh \gamma \ell + \left(\frac{Z_0}{Z_L}\right) \sinh \gamma \ell\right]^{-1} V_{in} \tag{14}$$

is obtained.

An alternative to equation (14) is obtained more easily by writing the line voltages at $\ell \neq 0$ and $\ell = 0$ as

$$V_{in} = A\, e^{\gamma \ell} + \Gamma_0\, A\, e^{-\gamma \ell} \tag{15}$$

and

$$V_L = A + \Gamma_0 A = (1 + \Gamma_0)\, A \quad , \tag{16}$$

respectively. From these, the useful form

$$V_L = \frac{(1 + \Gamma_0)\, e^{-\gamma \ell}\, V_{in}}{(1 + \Gamma_0\, e^{-2\gamma \ell})} \tag{17}$$

is obtained. Equation (17) is now incorporated into a unilateral transfer matrix for a pair of lossy lines since two lines will be associated with the coupler output and since the entire series-type microwave circuit for the LBMR is cast in four-port notation.

It is noted that both (V_{in}) and (V_L) above are, in general, standing-wave voltages on the transmission line. When $\Gamma_0 = 0$, only a single (traveling) wave is present.

$$[T] = \begin{bmatrix} \dfrac{(1+\Gamma_{01})\,e^{-\gamma_1 \ell_1}}{(1+\Gamma_{01}\,e^{-2\gamma_1 \ell_1})} & 0 \\ 0 & \dfrac{(1+\Gamma_{02})\,e^{-\gamma_2 \ell_2}}{(1+\Gamma_{02}\,e^{-\gamma_2 \ell_2})} \end{bmatrix} \tag{18}$$

where subscripts (1) and (2) refer to the upper and lower lines, respectively, and the (Z_L) used previously goes over to (Z_{L3}) and (Z_{L4}) for ports (3) and (4).

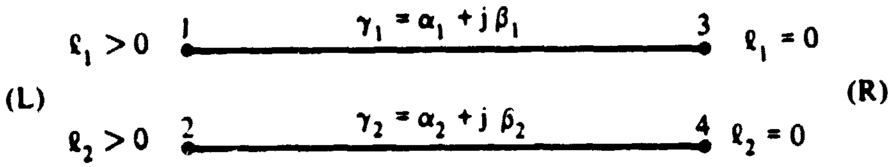

Figure 4. Lossy Transmission Lines.

The input impedance at any point (ℓ) on the transmission line, as given by equation (6), is now rewritten as

$$Z_z = Z_{in} = Z_0 \, \frac{Z_L + Z_0 \tanh \gamma\ell}{Z_0 + Z_L \tanh \gamma\ell} \quad . \tag{19}$$

See Appendix A of this document for conditions under which equation (18) represents a unitary matrix.

MISMATCHED DIRECTIONAL COUPLER

A unilateral or "transmission" transfer matrix for the mismatched directional coupler is now predicated on the reflection coefficients at coupler output ports (3) and (4). The conditions of Figure 5 are assumed; $Z_{L3} \neq Z_0$, and $Z_{L4} \neq Z_0$ in general. The unitary transfer matrix of the matched coupler is retained in the development. It is also assumed that there will be only a single input $h_1 \neq 0$ at port (1), and a termination $Z_{L2} = Z_0$ at port (2) of the coupler.

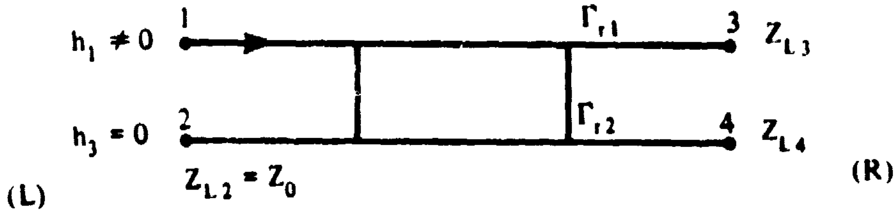

Figure 5. Mismatched Directional Coupler.

The voltages across (Z_{L3}) and (Z_{L4}) are the result of an incident and reflected wave superposition identical to that at the load-end of a mismatched transmission line ($\ell = 0$). Equation (16) suggests that an auxilliary matrix should follow the classical unitary transfer matrix of the coupler, and that it should have the form

$$[T] = \begin{bmatrix} (1 + \Gamma_{r1}) & 0 \\ 0 & (1 + \Gamma_{r2}) \end{bmatrix} \tag{20}$$

to effect a conversion from traveling to standing-wave voltage. This notation provides for independent complex-valued reflection coefficients at ports (3) and (4) which will in general differ from (Γ_{01}) and (Γ_{02}) associated with a pair of transmission lines to the right (R) of a coupler. Equation (20) may also be obtained by a fundamental argument based on energy conservation and may be displayed graphically on a Smith impedance chart.

In the event that $\alpha_1 = \alpha_2 = 0$,

$$|\Gamma_{r1}| = |\Gamma_{01}| \tag{21}$$

and

$$|\Gamma_{r2}| = |\Gamma_{02}| \quad . \tag{22}$$

but complex equality is obtained only when $\ell_1 = \ell_2$ equals some integer multiple of a half-wave length.

Since matrix products are not commutative the order of matrices for the case of mismatch must be as shown.

$$\begin{bmatrix} g_1 \\ g_3 \end{bmatrix} = \begin{bmatrix} (1+\Gamma_{r1}) & 0 \\ 0 & (1+\Gamma_{r2}) \end{bmatrix} \begin{bmatrix} C_1 & jC_2 \\ jC_2 & C_1 \end{bmatrix} \begin{bmatrix} h_1 \\ 0 \end{bmatrix} \quad . \tag{23}$$

That is

$$[T_{eq.}] = [T_n]\,[T_{n-1}] \ldots [T_1] \tag{24}$$

when the matrix subscripts increase in going from (L) to (R) in traversing a circuit.

It is noted that (23) serves for the unilateral (L) to (R) "transmission" under $h_3 = 0$ whereas matrix (4) allowed $h_3 \neq 0$. A brief discussion of the consequences of multiplying the two square matrices of (23) with a view toward obtaining an equivalent matrix under $h_3 = 0$ and an inquiry as to whether or not the result is unitary, may be found in Appendix A.

It is also noted that (h_1) above is not a standing-wave voltage, but (g_1) and (g_3) may be standing-wave voltages.

CONVERSION MATRIX

Equations (15) and (16) show that (V_{in}) and (V_L) are standing waves. Equation (23) shows that (h_1) is a single incident wave. If a self-consistent chain matrix is to be developed, an auxilliary matrix will be required to convert (V_L) across the input impedance (Z_{in}) to a coupler to an (h_1) implicitly across an impedance (Z_0). Since (Z_{in}) for the coupler at port (1) gives rise to a reflection coefficient,

$$\Gamma_{r_1}' = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} \quad , \tag{25}$$

the inverse of the process leading to equation (20) results in a matrix of the form

$$[T] = \begin{bmatrix} (1+\Gamma_{r1}')^{-1} & 0 \\ 0 & 0 \end{bmatrix} \tag{26}$$

which effects a conversion from standing to traveling wave voltage. It remains to develop an equation for (Z_{in}).

In analogy with transmission line theory, three waves superimpose at port (1) so that

$$Z_{in} = Z_0 \frac{A + B_1 + B_2}{A - B_1 - B_2} = \frac{1 + C_1{}^2 \Gamma_{r1} - C_2^2 \Gamma_{r2}}{1 - C_1{}^2 \Gamma_{r1} + C_2{}^2 \Gamma_{r2}} Z_0 \; . \tag{27}$$

The power into (Z_{in}) obviously equals the total power appearing at points (1), (2), (3), and (4) upon taking all internal coupler reflections into account. On the left side of the coupler, the relative power may be assessed using the equations

$$|C_1{}^2 \Gamma_{r1} - C_2{}^2 \Gamma_{r2}|^2 \; , \quad \text{port (1)} \tag{28A}$$

$$|j\, C_1\, C_2|^2\, |\Gamma_{r1} + \Gamma_{r2}|^2 \; , \quad \text{port (2)} \; . \tag{28B}$$

On the right side of the coupler relative power may be assessed using the equations

$$[|1 + \Gamma_{1r}|^2\, C_1{}^2]\, (R_e\, Z_{L3}/|Z_{L3}|^2) \; , \quad \text{port (3)} \tag{29A}$$

$$[|1 + \Gamma_{2r}|^2\, C_2{}^2]\, (R_e\, Z_{L4}/|Z_{L4}|^2) \; , \quad \text{port (4)} \; . \tag{29B}$$

Energy conservation is easily verified by the preceding, and absolute power division may also be determined.

ATTENUATION MATRIX

A four-port coupler with one input port terminated in its characteristic impedance is sometimes regarded as a three-port junction, leading to certain conformability problems regarding matrix multiplication. One convenient means of removing one of the coupler outputs on a chain-matrix calculation is to construct an attenuation matrix,

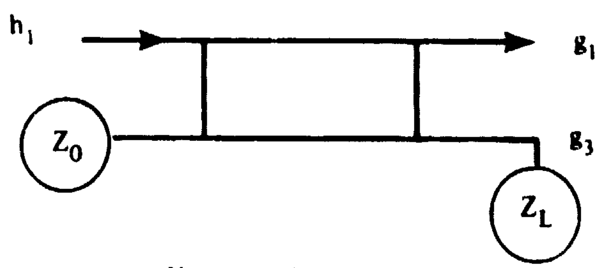

Figure 6. Physical Circuit.

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} . \tag{30}$$

which may be regarded as a specialization of a general attenuation matrix

$$T = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} , \tag{31}$$

with $\Gamma_{11} = 1$, and $\Gamma_{22} = \Gamma_{12} = \Gamma_{21} = 0$.

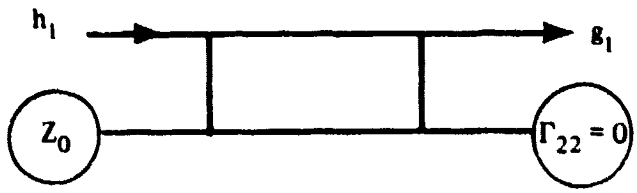

Figure 7. Equivalent Circuit.

Four-port notation may then be used throughout the analysis. The attenuation matrix may be introduced immediately after the coupler or some other component, such as a length of transmission line, may precede it. In any event, the order of equation (24) must be preserved since the matrix products are, in general, non-commutative.

PRACTICAL CIRCUITS (LBMR)

A restricted example is now given to illustrate the type of circuit associated with an LBMR antenna array. Only four radiators and three couplers are considered. See Fig. 8. All transmission lines are assumed to be lossy, and the loss parameter (α) may be different for each line of arbitrary length. Both antenna impedances and loss parameters may be functions of frequency. In the event that antenna impedances depart from the characteristic impedance (Z_0) of the lines, the loss parameter is redefined, in the presence of standing waves, as

$$\alpha_r = \alpha_m \frac{(1 + |\Gamma_0|^2)}{(1 - |\Gamma_0|^2)} \tag{32}$$

where $\alpha_m \equiv \alpha$ in Ref. 3, p. 31. Also see Appendix B of this document for a relationship between decibels and nepers when $\alpha \to \alpha_r$. In equation (32), the magnitude of the reflection coefficient is taken to be

$$|\Gamma_0| = \left| \frac{Z_L - Z_0}{Z_L + Z_0} \right| . \tag{33}$$

It is tacitly assumed here that all lines have the same characteristic impedance (Z_0) although the restriction may be lifted if necessary. A common notation is employed for all couplers, lines, etc. The values associated with the upper and lower feed system are designated (1) and (2), respectively. It follows that (α_1) and (ℓ_1) of one matrix may therefore be different from (α_1) and (ℓ_1) of some other matrix. The circuit is now represented by (n) matrices and the product

$$\bar{E} = [T_{eq}]\,\bar{h} = [T_n][T_{n-1}][T_{n-2}]\cdots[T_3][T_2][T_1]\,\bar{h} \ . \tag{34}$$

Here n = 15. Output may be obtained at any circuit interface by truncating the chain of matrices at the appropriate matrix $[T_{n-i}]$.

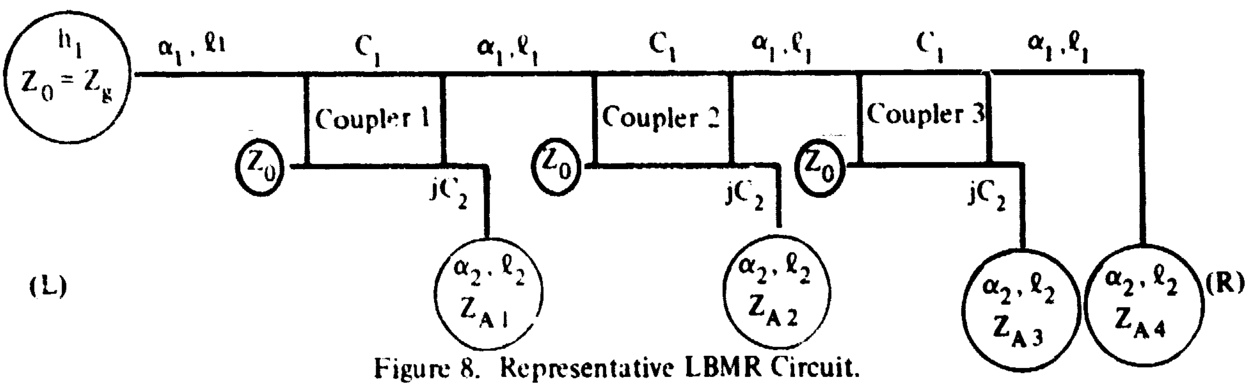

Figure 8. Representative LBMR Circuit.

The individual matrices are now written explicitly

$$[T_1] = \begin{bmatrix} \dfrac{(1+\Gamma_{01})\,e^{-\gamma_1 \ell_1}}{(1+\Gamma_{01}\,e^{-2\gamma_1 \ell_1})} & 0 \\ 0 & 0 \end{bmatrix} \tag{35}$$

$$[T_2] = \begin{bmatrix} (1+\Gamma_{r1}{}')^{-1} & 0 \\ 0 & 0 \end{bmatrix} \sim [T_7],\,[T_{12}] \tag{36}$$

$$[T_3] = \begin{bmatrix} C_1 & jC_2 \\ jC_2 & C_1 \end{bmatrix} \sim [T_8],\,[T_{13}] \tag{37}$$

$$[T_4] = \begin{bmatrix} (1+\Gamma_{r1}) & 0 \\ 0 & (1+\Gamma_{r2}) \end{bmatrix} \sim [T_9], [T_{14}] \tag{38}$$

$$[T_5] = \begin{bmatrix} \dfrac{(1+\Gamma_{01})e^{-\gamma_1 \ell_1}}{(1+\Gamma_{01}\, e^{-2\gamma_1 \ell_1})} & 0 \\ 0 & \dfrac{(1+\Gamma_{02})e^{-\gamma_2 \ell_2}}{(1+\Gamma_{02}\, e^{-\gamma_2 \ell_2})} \end{bmatrix} \sim [T_{10}], [T_{15}] \tag{39}$$

$$[T_6] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \sim [T_{11}] \;, \tag{40}$$

It is noted that $[T_2]$ is similar (~) to $[T_7]$, $[T_{12}]$, etc., and not identical since the matrix values are, in general, distinct.

PARAMETERS OF CIRCUITS

A procedure is now outlined for obtaining the parameters of a circuit such as the one shown in Fig. 8. The procedure begins at the extreme right (R) of the network, and works progressively toward the left (L), with an impedance analysis. A discussion of parameter determination unaviodably infringes on system design, which is beyond the scope of this document. Certain simplifying assumptions are made, however, it will be shown that these are realistic, not particularly restrictive, and may be lifted if desired. Further, it appears that a multi-stage procedure is expedient for sophisticated problems.

Initially, assume the following:

(1) All line characteristic impedances are equal to the value (Z_0), which is taken to be real. See Appendix C of this document.

(2) All load impedances are also equal to (Z_0).

(3) The loss parameters (α) are equal to zero.

(4) The current distribution is known for the array radiators.

(5) The phase distribution is known for the array radiators, and is taken to be a constant here for convenience.

(6) Coupler values (C_1) and (C_2) will be predicated on the current distribution magnitude, ignoring line loss (set $\alpha = 0$), standing wave loss (set $\alpha_r = 0$), and reflection loss (set $\Gamma_r = 0$).

Under the preceding assumptions, the individual matrices simplify to

$$[T_1] = \begin{bmatrix} e^{-j\beta\ell_1} & 0 \\ 0 & 0 \end{bmatrix} \tag{41}$$

$$[T_2] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \sim [T_7], [T_{12}] \tag{42}$$

$$[T_3] = \begin{bmatrix} C_1 & jC_2 \\ jC_2 & C_1 \end{bmatrix} \sim [T_8], [T_{13}] \tag{43}$$

$$[T_4] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim [T_9], [T_{14}] \tag{44}$$

$$[T_5] = \begin{bmatrix} e^{-j\beta\ell_1} & 0 \\ 0 & e^{-j\beta\ell_2} \end{bmatrix} \sim [T_{10}], [T_{15}] \tag{45}$$

$$[T_6] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \sim [T_{11}] \quad . \tag{46}$$

Suppose that a current distribution (K_i) is desired for (i) radiators, and that (i-1) couplers are to be used. See Figure 9.

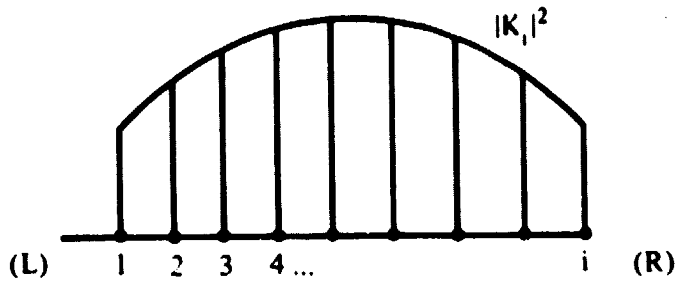

Figure 9. Power Distribution of Radiators.

Under impedance-match conditions ($Z_{A1} = Z_{A2} = \ldots = Z_{Ai} = Z_0$), the radiated power of each antenna is proportional to the square of the input current,

$$P_i \ \alpha \ |K_i|^2 = K_i^2 \qquad (41)$$

A rule can then be improvised to determine the coupler values for the initial set of assumptions,

$$C_1^2 \ \alpha \ \sum_{2}^{i} K_i^2 \qquad \text{(Coupler 1)} \qquad (42)$$

$$C_2^2 \ \alpha \ K_1^2 \qquad \text{(Coupler 1)} \qquad (43)$$

$$C_1^2 \ \alpha \ \sum_{3}^{i} K_i^2 \qquad \text{(Coupler 2)} \qquad (44)$$

$$C_2^2 \ \alpha \ K_2^2 \qquad \text{(Coupler 2)} \qquad (45)$$

$$C_1^2 \ \alpha \ K_i^2 \qquad \text{(Coupler i - 1)} \qquad (46)$$

$$C_2^2 \ \alpha \ K_{i-1}^2 \qquad \text{(Coupler i - 1)} \qquad (47)$$

As an example, when the current distribution is uniform, and

$$K_i^2 = \text{Constant} = 1 \, , \qquad (48)$$

and i = 4 as in Fig. 8, the coupler parameters are obtained via

$$C_1^2 = 3\, C_2^2 \qquad \text{(Coupler 1)} \qquad (49)$$

$$C_1{}^2 = 2\,C_2{}^2 \qquad \text{(Coupler 2)} \tag{50}$$

$$C_1{}^2 = C_2{}^2 \qquad \text{(Coupler 3)} \tag{51}$$

and

$$C_1{}^2 + C_2{}^2 = 1 \tag{5}$$

Since $\alpha = 0$, matrices $[T_1]$, $[T_5]$, $[T_{10}]$, $[T_{15}]$ behave as simple phase shifters and the line lengths (ℓ) are selected to obtain a zero phase gradient (or any other) across the array aperture by regarding the complex operator of the coupler matrix as a phase advance:

$$j = e^{j\pi/2} \quad . \tag{52}$$

Next, assume the following:

(1) All line impedances are equal to (Z_0), taken to be real.

(2) Load impedances $Z_{A1}, Z_{A2}, \ldots Z_{Ai}$ are complex values not necessarily related to (Z_0).

(3) The loss parameters (α) for each line are not equal to zero. For convenience assume (α) is the same for all lines.

(4) The square root of the radiated power will be regarded in the subsequent array analysis, and will replace the antenna current distribution used earlier. See Ref. 4, p. 142, which represents the electrical far-field as

$$E = \sqrt{\frac{P_{rad.}}{2\pi\,\zeta_0{}^{-1}}}\;\frac{e^{-jk_0r}}{r}\;g(\theta,\phi) \quad . \tag{53}$$

(5) The design power distribution will be retained, but will be modified to account for ohmic or joule power loss due to standing-waves on transmission lines with $\alpha \neq 0$.

(6) The phase distribution is known for the radiators and is taken to be a constant for convenience.

(7) Coupler values will be predicated on amended power distribution which anticipates line loss (α_r), but not on reflection loss (Γ_r). In practical situations where the Z_{Ai} may vary with frequency, reflection loss is ordinarily just accepted since it would be impossible to

design couplers to compensate over a band. In fact, the assumed coupler properties of equation (4) are a narrow-band approximation at best. Wide-band and single-frequency match are beyond the scope of this document.

Given the second set of assumptions, it is then possible to begin with the antenna impedances (Z_{A4}) and (Z_{A3}), evaluate the reflection coefficients (Γ_{01}) and (Γ_{02}) and mismatch line parameters (α_{r1}) and (α_{r2}), using equations (11), (33), and (32). From equation (19) the impedance values at ports (4) and (3) can be determined. These, in turn, lead to evaluation of reflection coefficients (Γ_{r1}) and (Γ_{r2}). From equations (27) and (25) the reflection coefficient (Γ'_{r1}) at port (1) is easily obtained using the (Z_{in}) resulting from triple-wave superposition. Since

$$\Gamma'_{r1} = \Gamma_{01} \tag{54}$$

for the transmission line matrix $[T_{10}]$ at coupler port (1), and since parameters γ_1, ℓ_1, γ_2, ℓ_2 for the line matrix are already known, all parameters of $[T_{15}]$ through $[T_{10}]$ are known with the exception of those in $[T_{13}]$.

From the relation

$$\mathrm{dB} = 10 \log_{10} \frac{(P_i + \delta P_i)}{(P_i)} = 8.686\, \alpha_r\, \ell \tag{55}$$

the power loss due to standing waves may be assessed, and the design power distribution value for the i^th^ feeder amended. The coefficients of the rightmost coupler (3), may then be set using logic similar to that leading to equations (46) and (47) once the losses of the lines (1) and (2) of matrix $[T_{15}]$ have been determined. Since equation (19) may be used again to determine impedances (Z_{L3}) and (Z_{L4}), with their associated relfection coefficients (Γ_{r1}) and (Γ_{r2}) at ports (3) and (4), respectively, of the next coupler (2), the indicated process may be continued until matrix $[T_1]$ is reached.

The standing-wave voltage (V_{in}) at the left end of the transmission line described by matrix $[T_1]$ is given by

$$V_{in} = \left(\frac{h_1}{Z_0 + Z_{NE1}}\right) Z_{NE1} \tag{56}$$

where (Z_{NET}) is the input impedance to the entire network as seen to the right of the generator terminals. See Fig. 10.

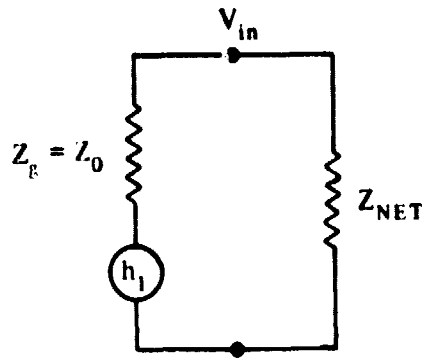

Figure 10. Equivalent Circuit.

The driving function for the entire problem is, therefore, a column matrix

$$\begin{bmatrix} V_{in} \\ 0 \end{bmatrix} \tag{57}$$

which operates on matrix $[T_1]$ in the non-commutative sequence

$$\begin{bmatrix} T_n \end{bmatrix} \cdots \begin{bmatrix} T_2 \end{bmatrix} \begin{bmatrix} T_1 \end{bmatrix} \begin{bmatrix} V_{in} \\ 0 \end{bmatrix} . \tag{58}$$

Returning to the discussion of coupler coefficients, suppose that an amended power distribution such as the one depicted by Fig. 11 has been developed by means of equation (55) and the original design distribution.

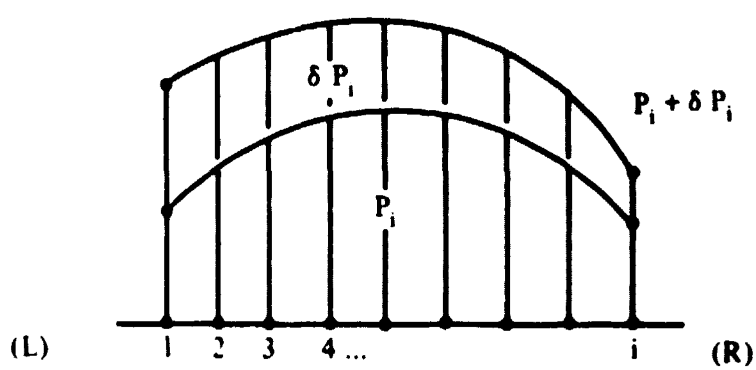

Figure 11. Amended Power Distribution.

The power $(P_i + \delta P_i)$ is associated with the i^{th} feeder, and the increment (δP_i) is intended to offset ohmic feeder loss in the presence of standing waves. Due to uncompensated reflection loss, the actual power arriving at the i^{th} radiator may be less than or equal to the design distribution value (P_i). Reflection losses may be at ports (1) and (2) of the couplers, depending on values (Z_{L3}) and (Z_{L4}).

Equations analogous to (46) and (47) can then be written as

$$\left| \frac{C_1 (1 + \Gamma_{r1})}{Z_{L_3}} \right|^2 R_{L_3} \ \alpha \ P_i + \delta P_i \tag{59}$$

$$\left| \frac{C_2 (1 + \Gamma_{r2})}{Z_{L_4}} \right|^2 R_{L_4} \ \alpha \ P_{i-1} + \delta P_{i-1} \tag{60}$$

The system of equations is solvable, using equation (5), and the summing process is carried out as before.

CONCLUSION

This document presents an approach to retaining transfer or chain-matrix methods in the presence of standing waves of voltage. An impedance analysis preceded the multiplication process to establish the matrix parameters. Current was not carried explicitly in the development.

The analysis of the mismatched coupler was predicated on the notion that its classical scattering-matrix properties are invariant with respect to impedance mismatch. On this premise, reflected signals were described in terms of complex reflection coefficients and superimposed in analogy with standard transmission line theory. Energy conservation was verified for the coupler. The analysis was restricted to a single input to the coupler. It was found that for the general mismatch case, two waves superimposed at each output port and three waves superimposed at the input port.

A brief discussion addressed the problem of determining the coupler coefficients for an assumed antenna array power distribution. The radiated design power from each antenna was used in the calculations since antenna input impedance was not necessarily identical among the array elements. Line losses were compensated in the determination of the coupler coefficients. Impedance mismatch losses over a frequency band were not compensated.

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APPENDIX A

UNITARY MATRICES

The term "unitary matrix" appears from time to time in the discussion on scattering and transfer matrices. In Ref. 2, p. 301, the author states that the coupler matrix is unitary. In an abstract mathematical sense any matrix which satisfies

$$[A]\,[A]^{*T} = [I] \tag{1-A}$$

is unitary when (*) implies "complex conjugate" and (T) implies "transpose." See Ref. 6, p. 229.

It can be seen that the coupler transfer matrix of equation (4) is unitary since

$$\begin{bmatrix} C_1 & jC_2 \\ jC_2 & C_1 \end{bmatrix} \begin{bmatrix} C_1 & jC_2 \\ jC_2 & C_1 \end{bmatrix}^{*T} = [I] \tag{2-A}$$

The notion of "reactive" or "lossless" is sometimes associated with the unitary matrix of a microwave junction, and reciprocity is evidently not a consideration. See Ref. 7, p. 273. It will be recalled that the objectives of the present document were restricted to "unilateral" transfer matrices, with the possible exception of energy conservation verification for the mismatched coupler.

It is interesting to test the transmission line transfer matrix of equation (18) to determine whether or not it represents a unitary matrix for certain special terminations after considering the general termination. Furthermore, it appears that the "unitary" test may include "reflectionless" as well as "lossless" for a given junction.

In Ref. 8, p. 7, it is stated that the operation of taking conjugates is distributive with respect to addition, subtraction, multiplication, and addition. Relying on this, the condition

$$\frac{1 + \Gamma_{01}\, e^{-j\beta_1 \ell_1}}{1 + \Gamma_{01}\, e^{-2j\beta_1 \ell_1}} \cdot \frac{1 + \Gamma_{01}^{\,*}\, e^{j\beta\ell_1}}{1 + \Gamma_{01}^{\,*}\, e^{2j\beta\ell_1}} = 1 \tag{3-A}$$

would have to be met under $\alpha_1 = 0$, and a similar expression for line (2) would result for $\alpha_2 = 0$.

An impedance match ($Z_L = Z_0$) results in a zero reflection coefficient ($\Gamma_{01} = 0$) so that equation (3-A) is satisfied upon introducing a resistor. Conversely, a perfect short-circuit ($Z_L = 0$) results in a non-zero reflection coefficient ($\Gamma_{01} = -1$) so that equation (3-A) is not satisfied upon introducing a lossless termination.

Matrix multiplication is valid under an associative law, raising the possibility of forming equivalent matrices, and inquiry about the unitary character of the result. Combining the two square matrices of equation (23) into an equivalent unilateral matrix is admissible, but the result will not be a unitary matrix. The introduction of ($\Gamma_{r1} \neq 0$) and ($\Gamma_{r2} \neq 0$) into equation (23) is an admission that energy will be reflected at the output end (R) of the coupler. The discussion on transmission lines and unitary matrices now becomes relevant here as reflection was associated with a non-unitary matrix ($Z_L = 0$, $\Gamma_r = -1$) previously.

Another example of a non-unitary equivalent matrix can be generated even when the reflection coefficients at the coupler output ports are equal to zero ($\Gamma_{r1} = \Gamma_{r2} = 0$). If an attenuation matrix such as that of equation (30) is placed between the generator and the coupler,

$$[T_{eq.}] = \begin{bmatrix} C_1 & jC_2 \\ jC_2 & C_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ jC_2 & 0 \end{bmatrix}, \qquad (4\text{-A})$$

which is not unitary due to power dissipation within the combination.

APPENDIX B

LOSSES IN MISMATCHED LINES

When $Z_L = Z_0$, equation (14) or (17) leads to

$$dB = 20 \log_{10} \left| \frac{V_S}{V_R} \right| = 20 \log_{10} \left| \frac{V_1}{V_2} \right| = 20 \log_{10} e^{\alpha \ell} \qquad (1\text{-B})$$

since

$$|e^{\gamma \ell}| = e^{\alpha \ell} \quad . \qquad (2\text{-B})$$

When $Z_L \neq Z_0$, and standing waves exist on the transmission line,

$$\alpha \rightarrow \alpha_r \quad , \qquad (3\text{-B})$$

as given by equation (32). Then

$$dB = 20 \log_{10} e^{\alpha \ell} \rightarrow 20 \log_{10} e^{\alpha_r \ell} \qquad (4\text{-B})$$

A convenient relationship may be derived relating decibels and nepers. In an abstract sense, if

$$\log_e N = M \quad , \qquad (5\text{-B})$$

and

$$\log_{10} N = P \quad , \qquad (6\text{-B})$$

then

$$e^M = 10^P \quad . \qquad (7\text{-B})$$

Taking the logarithm to the base ten of both sides,

$$\log_e N \log_{10} e = \log_{10} N \quad . \qquad (8\text{-B})$$

It follows that

$$dB = 20 \log e^{\alpha \ell} \approx 20 \, (.434) \log_e e^{\alpha \ell} = 8.686 \, \alpha \ell \rightarrow 8.686 \, \alpha_r \ell \qquad (9\text{-B})$$

when $Z_L \neq Z_0$.

APPENDIX C

CHARACTERISTIC IMPEDANCE OF LINES

The theory underlying transmission lines can be found in many standard texts. See Ref. 3 and 5. When series capacitance and shunt inductance are neglected, which is commonly done, the characteristic impedance is written as

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \tag{1-C}$$

When the series resistance (R) is small with respect to ($j\omega L$), and the shunt conductance (G) is small with respect to ($j\omega L$), the above is often simplified to

$$Z_0 \approx \sqrt{\frac{L}{C}} \quad , \tag{2-C}$$

which implies a purely resistive characteristic impedance when

$$R = G = 0 \ . \tag{3-C}$$

The propagation factor (γ) is usually written as

$$\gamma = [(R + j\omega L)(G + j\omega C)]^{1/2} \tag{4-C}$$

which simplifies to

$$\gamma = \alpha + j\beta = 0 + j\,\omega\,(LC)^{1/2} \tag{5-C}$$

when equation (3-C) holds. For lines with small losses, a binomial expansion (Ref. 3, p. 20) leads to

$$\alpha = \frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2}\sqrt{\frac{L}{C}} \text{ neper/meter.} \tag{6-C}$$

In practice, a purely resistive (Z_0) and a non-zero (α) are assumed even though this may appear paradoxical.