NASA CR-165,747

NASA Contractor Report 165747

NASA-CR-165747
19810021847

WEIGHTED-MEAN SCHEME FOR SOLVING
INCOMPRESSIBLE VISCOUS FLOW

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Contract NAS1-15604
May 1981

NASA
National Aeronautics and
Space Administration

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FIGURE CAPTIONS

SECTION 1 - INTRODUCTION

The turbulent bursting phenomenon is recognized as a dominant feature of time-dependent turbulent boundary-layer flow. The nature of the boundary layer formed on a flat plate located beneath a convecting rectilinear vortex embedded in a uniform flow was investigated by Doligalski and Walker (1). They obtained numerical solutions for the temporal development of the boundary layer induced by the motion of a rectilinear vortex. The boundary layer was expected to erupt from the wall into the inviscid flow.

In the present study the flat plate is replaced by a porous wall where the velocity at the wall obeys Darcy's law. The point of inflexion of the boundary layer velocity profile is controlled by varying the strength and/or the location of the vortex. Our interest is focused on how the inflected velocity profile responds to wall mass transfer induced by the motion of the vortex.

The governing equations for a two-dimensional incompressible viscous flow are the vorticity transport equation and the stream function equation. These equations can be solved simultaneously at each time step by means of a finite difference scheme. We propose to use the weighted-mean scheme of Fiadeiro and Veronis (2) for the vorticity transport equation. This scheme is first applied to a series of test problems to determine its accuracy, stability and efficiency. For the stream function equation, we use the usual central difference scheme.

SECTION 2 - THE WEIGHTED-MEAN SCHEME

The advection-diffusion equation is written in conservative form:

$$\frac{\partial \omega}{\partial t} + \frac{\partial (u\omega)}{\partial x} + \frac{\partial (v\omega)}{\partial y} = \nu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) \tag{1}$$

where ν is the diffusivity constant

u and v are the velocity components in the x and y directions, respectively and

ω is the vorticity

For cases when the vorticity field is subject to strong advection, centered-finite difference procedures are inefficient; consequently, the weighted-mean scheme is adopted for the present investigation. An explicit (leap-frog) centered-time difference in equation (1) is used except for the central term (ω_{ij}^{n}) that results from the spatial differencing evaluated as $(\omega_{ij}^{n+1} + \omega_{ij}^{n-1})/2$ where the superscript denotes the time step. When the weighted-mean scheme is applied to the spatial derivatives the equation for the calculation of ω_{ij}^{n+1} given values at ω_{ij}^{n} and ω_{ij}^{n-1} is

$$\omega_{ij}^{n+1} = \frac{2\Delta t}{1+\alpha_o \Delta t} (\alpha_{i-1} \omega_{i-1,j}^{n} + \alpha_{i+1} \omega_{i+1,j}^{n} + \alpha_{j-1} \omega_{i,j-1}^{n} + \alpha_{j+1} \omega_{i,j+1}^{n}) + \frac{1-\alpha_o \Delta t}{1+\alpha_o \Delta t} \omega_{ij}^{n-1} \tag{2}$$

where Δt is the time increment

h is the spatial increment

With all α's evaluated at time step n, we have:

$$\alpha_{i+1} = \frac{u_{i+\frac{1}{2}}}{2h} \left[\coth \left(\frac{h\, u_{i+\frac{1}{2}}}{2\nu}\right) - 1\right]$$

$$\alpha_{i-1} = \frac{u_{i-\frac{1}{2}}}{2h} \left[\coth \left(\frac{h\, u_{i-\frac{1}{2}}}{2\nu}\right) + 1\right]$$

$$\alpha_{j+1} = \frac{v_{j+\frac{1}{2}}}{2h} \left[\coth \left(\frac{h\, v_{j+\frac{1}{2}}}{2\nu}\right) - 1\right] \tag{3}$$

$$\alpha_{j-1} = \frac{v_{j-\frac{1}{2}}}{2h} \left[\coth \left(\frac{h\, v_{j-\frac{1}{2}}}{2\nu}\right) + 1\right]$$

$$\alpha_o = \alpha_{i-1} + \alpha_{i+1} + \alpha_{j-1} + \alpha_{j+1}$$

The error of approximation is $O(\Delta t^2, h^2)$. The formulation uses only a five point operator for two-dimensional flow and is antisymmetric in relation to the velocity field. When the component change sign, the coefficients upstream and downstream of the point are automatically reversed, a feature particularly useful in computer programming because the sign of the velocity components is not generally known a priori.

SECTION 3 - TEST PROBLEM 1

The one-dimensional advection-diffusion problem presented by Roache (3) to illustrate the weakness of the standard central difference relation is as follows:

$$\frac{1}{\nu}\frac{\partial\omega}{\partial t} + \frac{u}{\nu}\frac{\partial\omega}{\partial x} - \frac{\partial^2\omega}{\partial x^2} = 0 \qquad (3)$$

with boundary conditions

$\omega = 0$ at $x = 0$ $t \geq 0$

$\omega = 1$ at $x = 1$ $t \geq 0$

Regardless of the initial condition, the steady state solution for incompressible flow is

$$\omega = \frac{1 - e^{ux/\nu}}{1 - e^{u/\nu}}$$

The time derivative is written as first-order forward-difference approximation and the weighted-mean scheme is applied to the spatial derivatives.

Steady state results appear in Table 1 for weighted-mean scheme, upwind differencing scheme and central difference scheme for various values of cell Reynolds number

$$Rc = \frac{uh}{\nu}$$

The steady state form of equation (3) is

$$- u\,\frac{\delta\omega}{\delta x} + \nu\,\frac{\delta^2\omega}{\delta x^2} = 0 \qquad (4)$$

It can be shown that the solution of the finite difference analog of this equation agrees identically at the points x_i with the solution of the differential equation (4). We may observe this remarkable result in Table 1. Table 1 also reveals that the central difference scheme produces wiggles for values of Rc greater than 2.0. The upwind differencing scheme, although stable, is considerably in error at all values of Rc.

SECTION 4 - TEST PROBLEM 2

The problem is to determine the temperature field in a two-dimensional square, Bénard convection cell for a fluid heated uniformly from below and cooled from above. The length scale L and the maximum amplitude U of velocity are used in the non-dimensionalization.

The non-dimensional temperature equation has the form:

$$P \left[\frac{\partial (uT)}{\partial x} + \frac{\partial (vT)}{\partial y}\right] = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

where P = Péclet number = $\frac{UL}{K}$

K = thermal conductivity

The dimensionless velocity field is given by

$$u = -\sin x \cos y \qquad v = \cos x \sin y \qquad 0 \leq x \leq \pi$$

$$0 \leq y \leq \pi$$

Boundary conditions on T are

$T = 1$ at $y = 0$

$T = 0$ at $y = \pi$

$\frac{\partial T}{\partial x} = 0$ at $x = 0, \pi$

A relatively coarse grid system of 25 × 25 mesh points was used for different values of P. Although the scheme assures convergence to all iterative methods, we would like to compare the rates of convergences in Table 2. Line SOR combined with multigrid techniques are approximately four times faster than ADI method. Figures 1 and 2 show isotherms of the fluid for Péclet numbers 10 and 80. The intertwining tongue for the case $P = 80$ indicates the effect of strong convection. A final point that we would like to bring out for the strongly convective case ($P = 80$) is that distortions of isotherms were not introduced with relatively coarse grid (25×25).

SECTION 5 - TEST PROBLEM 3

We consider the flow in a driven square cavity. This problem is a standard test case for evaluating the accuracy, stability and efficiency of numerical schemes. For simplicity we choose a square with sides of length L = 1. The upper surface (fig. 3) moves to the right with a constant transverse velocity U = 1.

The flow of a viscous incompressible fluid in a square cavity is governed by the following coupled equations:

$$\frac{\partial \omega}{\partial t} = - \frac{\partial (u\omega)}{\partial x} - \frac{\partial (v\omega)}{\partial y} + \frac{1}{R} \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) \tag{5}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = - \omega \tag{6}$$

where ψ = stream function

ω = vorticity

R = Reynolds number = $\frac{UL}{\nu}$, ν = kinematic viscosity

$u = \frac{\partial \psi}{\partial y}$, $v = - \frac{\partial \psi}{\partial x}$

Boundary conditions:

upper surface $\psi_y = 1$, $\psi_x = \psi = 0$

bottom, left, right surfaces $\psi_x = \psi_y = \psi = 0$

The steady state solution of the system also may be reached by replacing equation (5) by its steady state form (the unsteady term is omitted)

$$- \frac{\partial (u\omega)}{\partial x} - \frac{\partial (v\omega)}{\partial y} + \frac{1}{R} \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) = 0$$

The vorticity transport equation (5) becomes equation (2) in finite difference approximations. The stream function equation (6) can be discretized using central difference formula. For concreteness, we will describe the computation cycle for the time-dependent approach.

The calculation starts at $t = 0$ where ψ and ω are known everywhere. From equation (5) we obtain the vorticity ω for all interiors points at $t = \Delta t$. The numerical solution of (6) gives ψ at $t = \Delta t$ and hence u and v. The last step of the computational cycle is to update the boundary values of ω using the most current values of ψ and ω at the interior points. This computational cycle is repeated until the steady state is reached to a specified convergence level. If we use the steady state form of equation (5), instead of "marching" the vorticity transport forward in time, new values of ω are obtained by an iterative process. The way in which the boundary values of the vorticity are approximated affects both the rate of convergence and the accuracy of the solutions. This fact has been noted by many authors. In the present case we use a second order form for evaluating wall vorticity given by

$$\omega/w = \frac{1}{2h^2} \left(8\ \psi/_{w+1} - \psi/_{w+2} - 7\ \psi/_{w}\right) - \frac{3}{h} \left.\frac{\partial\psi}{\partial n}\right|_{w} \qquad (7)$$

Studies indicated that instead of using formula (7) at time step n, the average $(\omega|^{n}_{wall} + \omega|^{n-1}_{wall})/2$ yielded a more stable process for the time-dependent and steady state approaches. Contour plots of stream function are shown in figure 4 and 5 for various grid sizes and Reynolds number. Numerical values also are listed in Table 3 for comparison.

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SECTION 6 - COMMENTS ON THE VORTEX PROBLEM

The problem is shown schematically in figure 6. An isolated vortex embedded inside a boundary layer convects to the right over a porous wall. The velocity at the wall is given by Darcy's law:

$$V_w(x,t) = c(Pr- P_w(x_1t))$$

where c is a constant

Pr is a reference pressure

The wall mass transfer mechanism is driven by the wall pressure gradient induced by the motion of the vortex. Consequently, it is of interest to examine the effect of wall pressure fluctuations on velocity profile with an inflexion point inside the boundary layer. The point of inflexion initially may be created by varying the position and/or the strength of the vortex.

It is worthwhile to first consider a non-porous wall. The effect of two counter-rotating vortices of opposite and equal strength is equivalent to having an impermeable wall between them. In an inviscid flow, a dimensionless stream function of the vortices may be defined by

$$\psi_v = \frac{\Gamma}{2\pi} \ \ell n \ \frac{r_c^2 + (x-x_o)^2 + (y-y_o)^2}{r_c^2 + (x-x_o)^2 + (y+y_o)^2}$$

where Γ is the strength of the vortex

r_c is the core radius

(x_o,y_o) is the center of the vortex

Let the vortex be convected in an inviscid flow. The convection speed of the vortex is composed of 2 parts consisting of the local mean velocity and the self induced velocity due to the vortex image below the wall. Note that the inviscid solution predicts that the vortex will remain at constant height y_o. However, the inviscid solution is not valid for our realistic problem because it fails to satisfy the no-slip boundary condition on the wall.

We have also tested the possibility of satisfying the no-slip condition by means of a smoothing function, s. The modified stream function is defined as:

$$\psi_{modif} = s\ \psi_{orig}$$

and s vanishes strongly near the wall, to ensure zero derivatives of ψ_{modif} (zero velocity). Far from the wall, s is identically 1. However, this new stream function produced a reverse flow near the wall, introducing new complications in the flow pattern. It was therefore abandoned.

SECTION 7 - CONCLUDING REMARKS

The weighted-mean scheme has several significant advantages:

- it is locally a conservative scheme neither creating nor destroying the advected quantity artifically.
- it is stable for all grid spacings or cell Reynolds numbers. The scheme thus has the practical advantage of yielding solutions for relatively large values of grid size and is, therefore, economical with computer time. One can use this crude solution to construct a good initial guess solution for a finer grid.
- it becomes a central difference scheme for strongly diffusive cases and an upwind difference scheme for strongly advective cases. Furthermore, when the components of the velocity change sign, the coefficients upstream and downstream of the point are automatically reversed, a feature particularly useful in computer programming because the sign of the velocity components need not be known a priori.

The only major disadvantage is the loss in speed due to the computation of the hyperbolic tangent at each grid point in multidimensional problems.

For the vortex problem, the initial conditions need to correspond to some real initial situation since we are interested in the transient solution. Therefore one needs to satisfy initially the no-slip condition. One suggestion is that the vortex at time t equals zero, should be far away from the wall in the inviscid region so that the velocity induced by the vortex vanishes at the wall.

REFERENCES

1. Doligalski, T. L. and Walker, J. D. A., "Shear Layer Breakdown due to Vortex Motion," Proceedings of the AFOSR Workshop on Coherent Structure of Turbulent Boundary Layers, edited by C. Smith and D. Abbott, Lehigh Univ., Nov. 1978, pp. 288-339.
2. Fiadeiro, M. E., and Veronis, G., Tellus 29 (1977), 512.
3. Roache, P. J., "Computational Fluid Dynamics," Hermosa Publishers, New Mexico, 1972.
4. Burggraf, O. R., J. Fluid. Mech. 24 (1966), 113.
5. Bozeman, J. D. and Dalton, C., J. Computational Phys. 12 (1973), 348.
6. Hallasamy, M. and Krisna Prasad, K., J. Fluid. Mech. 79 (1977), 391.
7. Gosman A. D. et al., "Heat and Mass Transfer in Recirculating Flows," Academic Press, New York, 1969.
8. Lustman, L., Huynh, Q., and Hussaini, - "Multigrid Methods with Weighted-Mean Schemes," To be submitted at Symposium on Multigrid Methods, NASA-Ames Research Center, Oct. 21-22, 1981.

Table 1

	Rc = 1.0					
	Grid point					
	1	2	3	4	5	6
ES	0.0	0.0117	0.0433	0.1295	0.3636	1.0
WMS	0.0	0.0117	0.0433	0.1295	0.3636	1.0
UDS	0.0	0.0323	0.0968	0.2258	0.4839	1.0
CDS	0.0	0.0083	0.0331	0.1074	0.3306	1.0
	Rc = 2.0					
	Grid point					
	1	2	3	4	5	6
ES	0.0	0.0003	0.0024	0.0183	0.1353	1.0
WMS	0.0	0.0003	0.0024	0.0183	0.1353	1.0
UDS	0.0	0.0082	0.0331	0.1074	0.3306	1.0
CDS	0.0	0.0000	0.0000	0.0000	0.0000	1.0
	Rc = 4.0					
	Grid point					
	1	2	3	4	5	6
ES	0.0	0.0000	0.0000	0.0003	0.0183	1.0
WMS	0.0	0.0000	0.0000	0.0003	0.0183	1.0
UDS	0.0	0.0013	0.0077	0.0397	0.1997	1.0
CDS	0.0	0.0164	-.0328	0.1148	-0.3279	1.0

ES = exact solution

WMS = weighted-mean scheme

UDS = upwind differencing scheme

CDS = central difference scheme

Table 2

Bénard cell (25 × 25)				
P	SOR	ADI	MULTIGRID (8) in work unit (wu)	
			SOR	Line SOR
10	140	85	38	20
20	188	110	35	27
40	Δ	140	44	38
80	Δ	205	83	51
160	Δ	Δ	101	96

Δ The results for those specific Péclet numbers were not computed.

Table 3

R	Reference	Grid	Ψ_{vc}	Iterations
100	WMS	21 × 21	-0.094	55 (TD)
	WMS	41 41	-0.103	120 (ss)
	Burggraf (4)	41 41	-0.101	
	B + D (5)	51 51	-0.103	
	N + KP (6)	51 51	-0.102	
1000	WMS	61 61	-0.092	210 (ss)
	B + D (5)	51 51	-0.081	
	N + KP (6)	51 51	-0.097	
	Gosman et al. (7)	81 81	-0.099	

WMS = Weighted-mean scheme

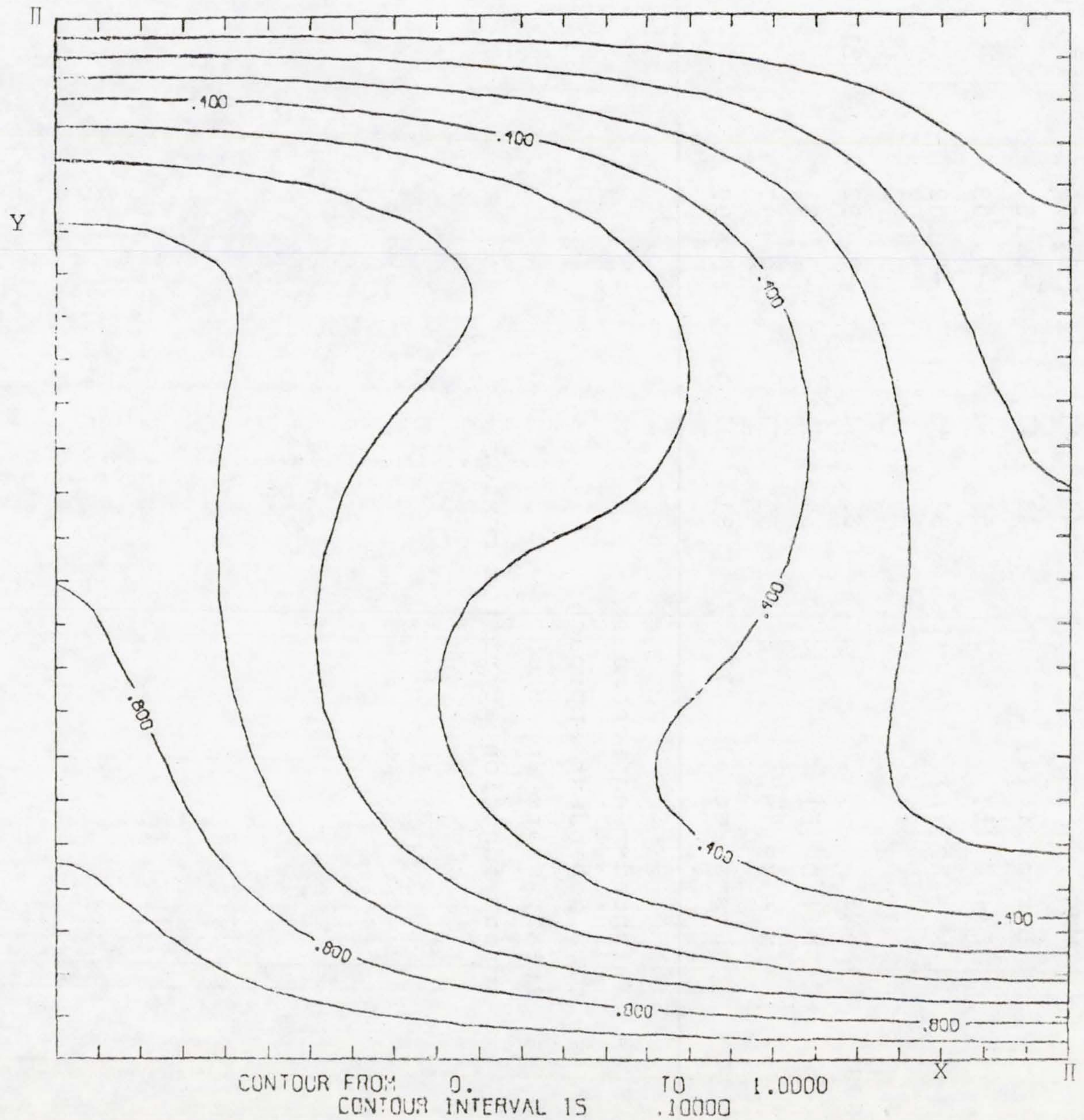
TD = time dependent approach

ss = steady state approach

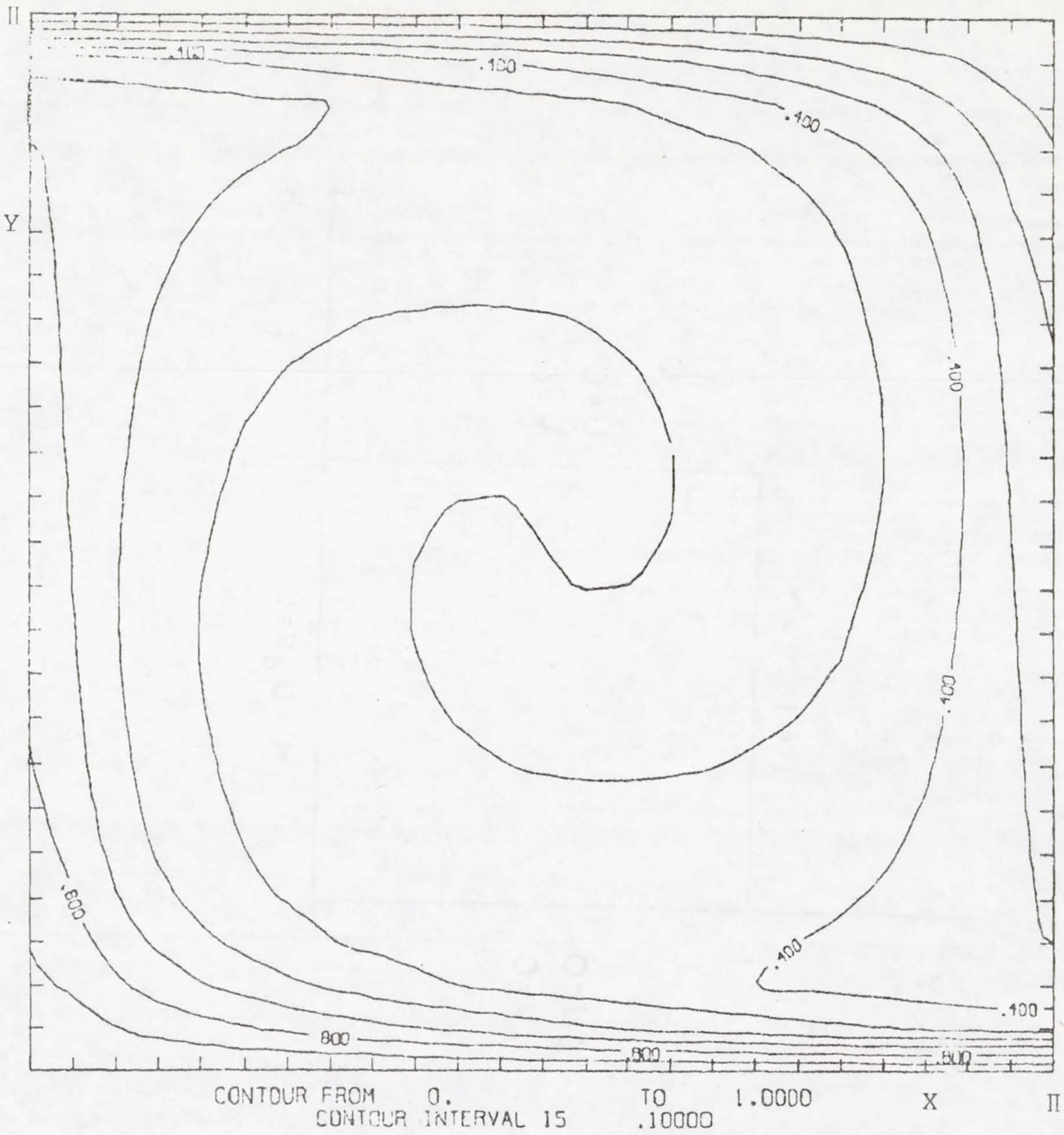
Ψ_{vc} = stream function at vortex center

Figure 1

P = 10



Π
Y
.400
.400
.400
.400
.400
.400
.800
.800
.800
.800
X
Π
CONTOUR FROM 0. TO 1.0000
CONTOUR INTERVAL IS .10000



Π
Y
.100
.100
.400
.400
.400
.400
.100
800
.800
.800
.800
CONTOUR FROM 0. TO 1.0000
CONTOUR INTERVAL IS .10000
X
Π

Figure 2
P = 80

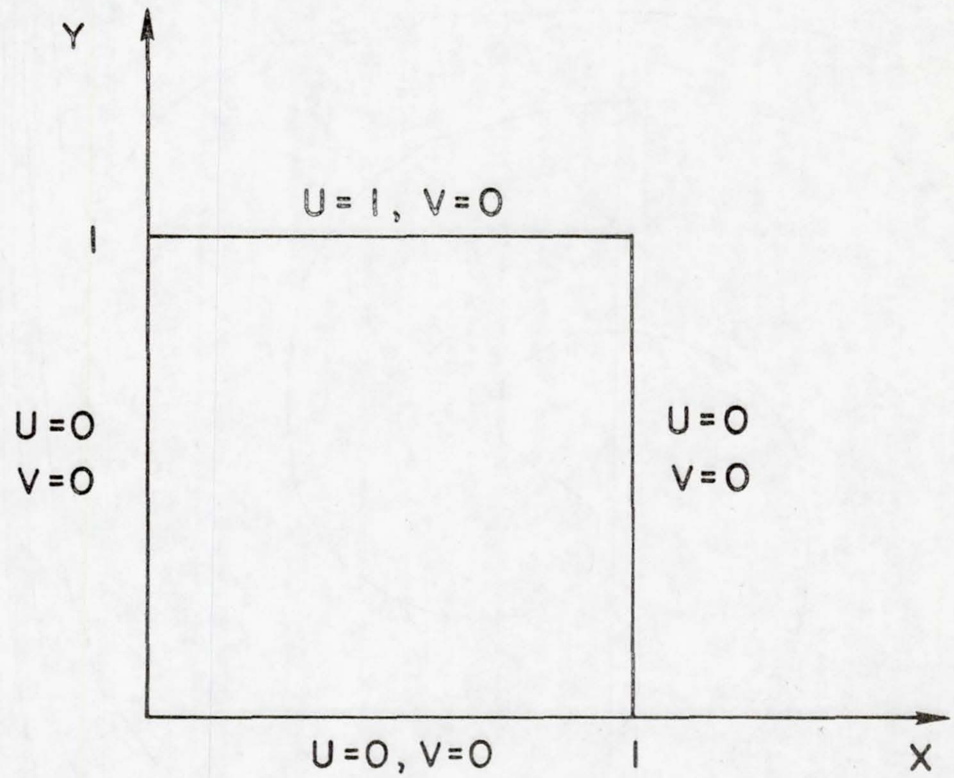
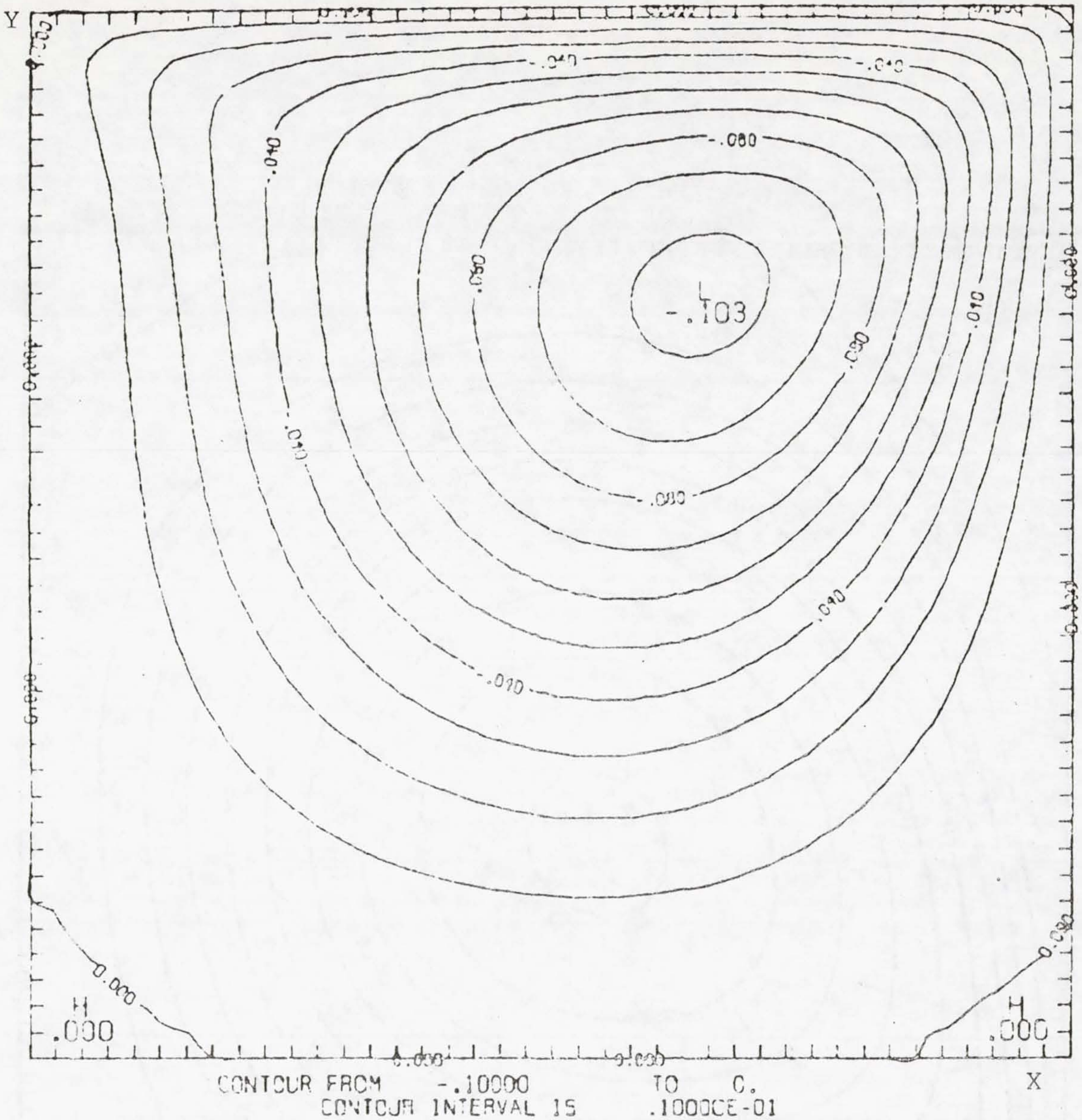


Figure 3

R = 100

Grid 41 × 41

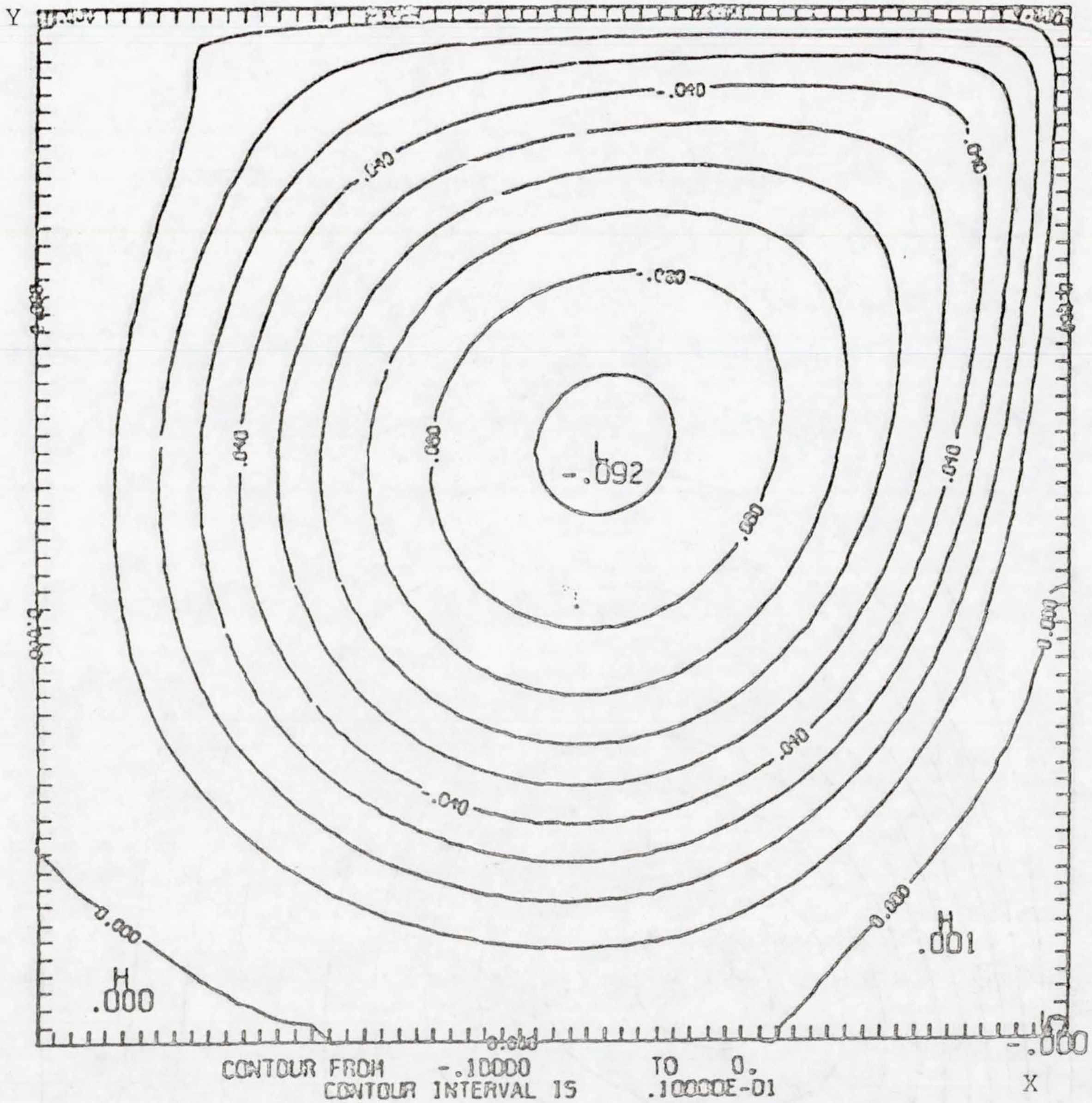
Figure 4

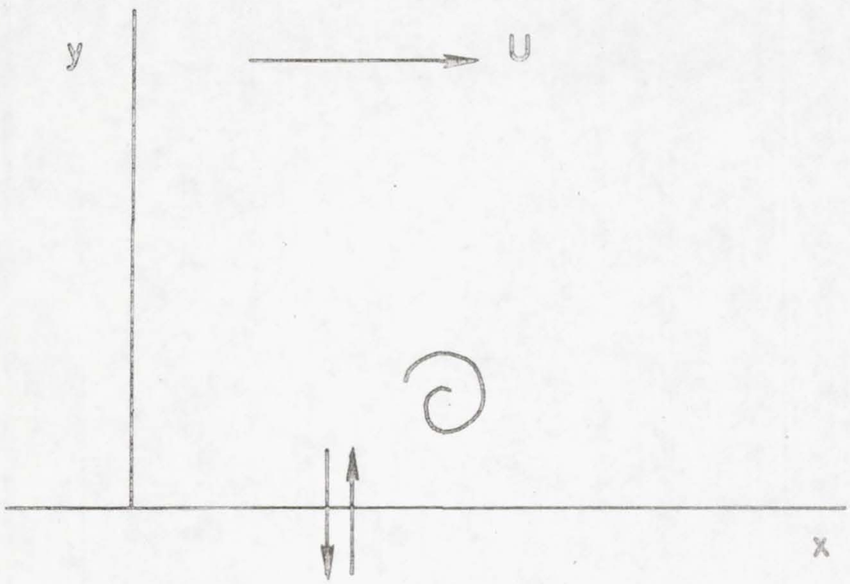


R = 1000

Grid 61 × 61

Figure 5





$$V_w(x,t) = C\,(P_r - P_w(x,t))$$

Figure 6

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA CR-165747		

4. Title and Subtitle	5. Report Date
WEIGHTED-MEAN SCHEME FOR SOLVING INCOMPRESSIBLE VISCOUS FLOW	May 1981
	6. Performing Organization Code
7. Author(s)	8. Performing Organization Report No.
Quyen Q. Huynh	R-SAL-05/81-01
9. Performing Organization Name and Address	10. Work Unit No.
Systems and Applied Sciences Corporation 17 Research Drive Hampton, VA 23665	11. Contract or Grant No. NAS1-15604
12. Sponsoring Agency Name and Address	13. Type of Report and Period Covered Contractor Report
National Aeronautics and Space Administration Washington, DC 20546	14. Sponsoring Agency Code

15. Supplementary Notes

Langley Technical Monitor: Dr. Julius E. Harris
Final Report

16. Abstract

The problem of how a boundary layer responds to the motion of a convected vortex on a porous wall is being investigated. The wall velocity is approximately given by Darcy's law. The vorticity-stream function approach was adopted for solving Navier-Stokes equations of two-dimensional incompressible viscous flows. The weighted-mean scheme was used for constructing finite difference approximations of spatial derivatives. Several test problems were solved and numerical results demonstrate clearly the accuracy, stability and efficiency of the scheme. The weighted-mean scheme then can be applied to the vortical flow problem.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Fluid Mechanics Viscous Flow Numerical Analysis	Unclassified - Unlimited Subject Category 34

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	27	A03

N-305