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Summary

This report describes a novel numerical procedure for the iterative solution of inviscid flow problems and demonstrates its utility for the calculation of steady subsonic and transonic flow fields. The method is more general than previously developed iterative methods in that no assumptions concerning the existence of either a velocity potential function or a stream function are required.

Application of the surrogate-equation technique defined herein allows the formulation of stable, fully conservative, type-dependent finite difference equations for use in obtaining numerical solutions to systems of first-order partial differential equations, such as the steady-state Euler equations. Included among the results presented are steady, two-dimensional solutions to the Euler equations for both subsonic, rotational flow and supersonic flow and to the small-disturbance equations for transonic flow. A computational efficiency in excess of that obtained by means of the standard perturbation-potential approach is indicated for the small-disturbance equations. Possible improvements to, and extensions of, the method are discussed.

Introduction

Motivating Factors

The present study is concerned with the numerical solution of steady inviscid flow problems. Many important physical situations encountered in modern engineering and applied science can be accurately modeled within the constraints of steady inviscid flow theory. Timely substantiation of this claim is provided by the generally good results presently being obtained from the use of such mathematical models for the design and analysis of transonic airfoils.

Most work, both past and present, has, however, dealt with that subset of inviscid flows that are irrotational and hence for which a velocity potential function exists. Although many flows of interest can be successfully modeled within this additional restriction, many others, constituting in all probability a larger class, cannot. Certainly all those flows of practical interest where significant gradients of entropy or total enthalpy can occur require a more general model than one based on potential flow theory. Significant gradients are virtually certain to occur in many internal flows, in particular in those through modern turbomachinery, and can also occur in external flows, especially when the flow over a number of interacting components is considered.

Of course the hyperbolic partial differential equations describing supersonic steady inviscid flow problems can be solved, for both potential and more general flow situations, by means of existing mathematical and numerical techniques. Consequently such flows are not the object of the present study. Rather it is the subsonic and transonic flow problems, described respectively by elliptic and mixed elliptic-hyperbolic equations, with which the research described herein deals.

At this juncture the issue of computational efficiency makes its importance felt. Subsonic and transonic inviscid flow problems can be solved by computing a temporal asymptote to the unsteady equations of motion. However, the computation times can be quite long. In contrast to this approach the present research describes a method for the direct solution of the steady equations. By proceeding in such a fashion the resolution of the transient physical states between the initial state and the desired final state is avoided. Thus a means for the more efficient solution of subsonic and transonic steady inviscid flow problems is provided. As described subsequently in this report the method is based on the creation of a higher order system that serves as a surrogate for the first-order partial differential equations of inviscid flow theory.

Literature Review

When written in primitive variable form the systems of partial differential equations used to describe the steady motions of an inviscid fluid are of first order and of mixed elliptic-hyperbolic type. Common examples of such systems are the transonic small-disturbance equations, the potential flow equations, and the Euler equations.

Because of the difficulties associated with both the formulation of robust finite difference analogs for such equations and the construction of stable iterative procedures for their numerical solution (refs. 1 and 2), these partial differential systems are not usually solved in their steady, primitive variable form. Rather, as is well known, the transonic small-disturbance equations and the potential flow equations are transformed into scalar, second-order,

partial differential equations by introducing either a velocity potential function (refs. 3 and 4) or a stream function (refs. 5 and 6). The steady Euler equations, on the other hand, are replaced by their unsteady versions, for which a temporally asymptotic steady solution is sought, either in real time (refs. 7 and 8) or in pseudo time (refs. 9 to 11).

Relatively few departures from these approaches are to be found in the literature. Steger and Lomax (ref. 12) developed an iterative procedure for solving a nonconservation form of the steady Euler equations for subcritical flow with small shear. Chattot (ref. 13) solved the transonic small-disturbance equations by differentiating them to obtain a second-order system, a special case of the approach discussed herein. He later adopted a variational formulation (refs. 14 and 15) and applied it to model problems representing the Euler equations. Ozer (ref. 16) developed a relaxation procedure for solving the equations of motion when they are reformulated in such a manner as to yield a second-order partial differential equation in the logarithm of the pressure, together with first-order equations for the remaining variables.

The work of these authors notwithstanding, contemporary numerical simulations of steady inviscid flow generally resort to either relaxation solutions of steady second-order equations with derived dependent variables or time-asymptotic solutions of unsteady first-order systems. In the former case, generality is lost; in the latter case the computational efficiency can be quite low.

Scope of Present Study

As is readily apparent in the foregoing literature survey there are two general approaches to the numerical solution of steady inviscid flow problems. One approach involves the time-accurate solution of the complete, unsteady Euler equations of motion. Taken in their time-dependent form the governing Euler equations are of hyperbolic type and their numerical solution is a relatively straightforward matter. Hence, one may attempt to obtain the solution to a steady flow problem as the temporal asymptote of the solution to an unsteady flow problem. This approach has been successfully employed by several researchers.

An alternative approach is to solve the Euler equations by a method that is not time accurate but that produces the desired steady-state result. Such methods are generally referred to as relaxation or iterative methods. There is substantial opinion and a considerable body of evidence that relaxation methods can provide a converged solution more quickly than can time-accurate methods and hence lead to more efficient use of computer resources. Because of the limited capacity of presently available computers and the complex nature of the phenomena under investigation, this issue of computational efficiency is of great importance. This is especially true if numerical solutions are to be used for design purposes, which typically require large numbers of cases to be computed.

At present, steady-state solutions of the Euler equations for subsonic and transonic flow problems are found primarily by means of the time-accurate approach. Since this approach can in the words of Lomax and Steger (ref. 2) be "disastrously slow," the development of efficient relaxation procedures could be quite beneficial. It is, however, no coincidence that with the few exceptions indicated previously little has been achieved toward creating a suitable relaxation procedure for the complete, steady Euler equations. The task of constructing stable iteration algorithms for equations such as the Euler equations, which involve only first derivatives of primitive variables, has received little attention in the literature. The questions as to whether, and under which circumstances, such procedures exist have not yet been satisfactorily answered.

The present work describes a new procedure, referred to as the surrogate-equation technique, that is designed to circumvent the difficulties associated with the nature of the steady Euler equations and to permit their solution by means of conventional iterative techniques. The numerical stability problems normally encountered when attempting to formulate finite difference equations for the steady-state, first-order Euler equations in regions where they exhibit elliptic behavior are avoided by introducing an alternative higher-order partial differential system for which proven numerical solution techniques are readily available.

For clarity the essential ideas comprising the surrogate-equation technique are introduced within the context of a quite simple and well-understood first-order system of partial differential equations, namely, the Cauchy-Riemann system. More interesting subsonic and transonic flow problems are subsequently discussed and solved.

By proceeding in this fashion a basis for the iterative solution of subsonic and transonic inviscid flow problems that lie beyond the restrictions of potential flow theory is developed. Although the efficient use of computer resources is a motivating force, no attempt is made here to proceed beyond the now standard, successive-line-relaxation solution procedure. The application of convergence acceleration techniques is reserved for future study. It is nevertheless interesting to note that when applied to the transonic small-disturbance equations, as

discussed in the section Transonic Flow, the surrogate-equation technique leads to an algorithm that, on the basis of the computational experimentation reported herein, appears to exceed the computational efficiency of the standard Murman and Cole algorithm by several multiples.

Cauchy-Riemann Problem

To illustrate the essential aspects of the surrogate-equation technique, we examine its application to a first-order system of partial differential equations of minimal complexity. Consider the equations describing in two dimensions the flow of an incompressible and irrotational fluid,

$$u_x + v_y = 0 \tag{1}$$

$$v_x - u_y = 0 \tag{2}$$

where u and v represent the components of velocity in the x and y directions, respectively. As is well known, these equations are simply the Cauchy-Riemann equations.

Potential Formulation

To solve equations (1) and (2), hydrodynamicists have long made use of the fact that a velocity potential, $\varphi = \varphi(x,y)$, can be introduced such that the condition of irrotationality (eq. (2)) is identically satisfied:

$$\varphi_x = u$$

$$\varphi_y = v$$

Substitution of the velocity potential into the incompressibility condition (eq. (1)) then leads to a Laplace equation in φ:

$$\varphi_{xx} + \varphi_{yy} = 0$$

A succinct and informative discussion of this development from the viewpoint of complex variable theory is given in reference 17.

Hence for a particular incompressible and irrotational flow problem the resulting boundary value problem for the first-order Cauchy-Riemann system can be reformulated as a boundary value problem for the Laplace equation in the single scalar dependent variable φ. As the theory of harmonic functions is a quite mature branch of mathematics, a large number of particular solutions are available for this problem. Furthermore, if circumstances should indicate the desirability of a numerical solution, the fact that most of the equations of mathematical physics are second-order partial differential equations means that a large variety of proven numerical methods exist and are at our disposal.

Surrogate-Equation Formulation

Consider now another approach to obtaining a solution to equations (1) and (2) that for convenience is referred to as the surrogate-equation technique. This technique is of general application and is not restricted to the class of irrotational, incompressible fluid flows that we are treating here only by way of an example.

By defining the two-component vectors f and g such that

$$f = \begin{bmatrix} u \\ v \end{bmatrix}$$

and

$$g = \begin{bmatrix} v \\ -u \end{bmatrix}$$

we can rewrite equations (1) and (2) as

$$f_x + g_y = 0 \tag{3}$$

Equation (3) can then in turn be reexpressed as $f_x + Af_y = 0$, where the Jacobian matrix A is defined such that $A = \partial g / \partial f$. Since A is a constant matrix, we can write $f_x + (Af)_y = 0$ or more conveniently in differential operator notation,

$$\left[\frac{\partial}{\partial x}(I\ ) + \frac{\partial}{\partial y}(A\ ) \right] f = 0 \tag{4}$$

where I represents the two-dimensional identity matrix.

If we now operate on equation (4) with the differential operator

$$\frac{\partial}{\partial x}(M\ ) + \frac{\partial}{\partial y}(N\ ) \tag{5}$$

with $M = I$ and $N = -A$ (or $N = A^T$), we obtain a second-order partial differential equation for f that has many pleasing properties. The form of M and N might be suggested by analogy with holomorphic function theory (ref. 18). In any case the present exposition undertakes to illustrate the utility of such a choice.

Clearly the equation resulting from an application of operator (3) to equation (4) is

$$\left[\frac{\partial^2}{\partial x^2}(I\ ) - \frac{\partial^2}{\partial y^2}(A^2\ )\right] f = 0 \tag{6}$$

In the case under consideration, where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and hence $A^2 = -I$, equation (6) reduces quite simply to

$$\left[I\frac{\partial^2}{\partial x^2} + I\frac{\partial^2}{\partial y^2}\right] f = 0 \tag{7}$$

This is the two-dimensional Laplace equation for the vector dependent variable f. This should come as no surprise since the generality of the surrogate-equation technique should not prevent the specific nature of the particular partial differential equations to which it is applied from manifesting itself. Indeed, in this case the appearance of the Laplace equation is a consequence of the fact that holomorphic functions are harmonic.

Problem Specification

At this point it is instructive to consider an application of the surrogate-equation technique to a particular boundary value problem for the Cauchy-Riemann system of equations (1) and (2). We choose the closed rectangular domain D in the $x-y$ plane such that

$$D = \{(x,y)\,|\,0 \le x \le l_x,\ 0 \le y \le l_y\}$$

We require that the Cauchy-Riemann system be satisfied on D, subject to the boundary conditions

$$\begin{aligned} u(0,y) &= q_1(y) \\ u(l_x,y) &= q_2(y) \\ v(x,0) &= q_3(x) \\ v(x,l_y) &= q_4(x) \end{aligned} \tag{8}$$

Following the classical approach, this Dirichlet problem for the Cauchy-Riemann system would be transformed into a Neumann problem for the potential equation $\varphi_{xx} + \varphi_{yy} = 0$, subject to the boundary conditions

$$\begin{aligned} \varphi_x(0,y) &= q_1(y) \\ \varphi_x(l_x,y) &= q_2(y) \\ \varphi_y(x,0) &= q_3(x) \\ \varphi_y(x,l_y) &= q_4(x) \end{aligned}$$

and also to the additional constraint that the line integral of q_n $(n = 1, \ldots, 4)$ around the boundary of D must vanish (ref. 19). The solution to this problem is unique to within an arbitrary constant and, given a solution for φ, the unknowns u and v can be found by differentiating φ with respect to x and y, respectively.

An application of the surrogate-equation technique, on the other hand, requires the solution of a mixed boundary value problem on D. As we now deal with the Laplace equation (7) in the two-component vector dependent variable f, we require that one condition on f be specified at each point of the boundary of domain D. Half of the required conditions can be immediately obtained from the boundary conditions (8), which were applied to the original Cauchy-Riemann formulation. If we let f_1 and f_2 denote the first and second components of f, respectively, these conditions can be written as:

$$\begin{aligned} f_1(0,y) &= q_1(y) \\ f_1(l_x,y) &= q_2(y) \\ f_2(x,0) &= q_3(x) \\ f_2(x,l_y) &= q_4(x) \end{aligned}$$

We are left with the task of formulating the remaining four conditions. Since the ultimate objective is to obtain a solution to the Cauchy-Riemann system on D, it is natural to invoke equations (1) and (2), as required, to obtain the additional four conditions on the components of f. Note that, if we proceed in this manner, no additional information that might have an overconstraining effect is introduced into the problem. In particular, the use of the Cauchy-Riemann conditions merely restates the fact that u and v are harmonic conjugates. Hence, in the case at hand, the remaining conditions are found to be:

$$\begin{aligned} \frac{\partial}{\partial x} f_2\Big|_{(0,y)} &= \frac{\partial}{\partial y} q_1 \\ \frac{\partial}{\partial x} f_2\Big|_{(l_x,y)} &= \frac{\partial}{\partial y} q_2 \\ \frac{\partial}{\partial y} f_1\Big|_{(x,0)} &= \frac{\partial}{\partial x} q_3 \\ \frac{\partial}{\partial y} f_1\Big|_{(x,l_y)} &= \frac{\partial}{\partial x} q_4 \end{aligned}$$

If we now solve the resultant mixed boundary value problem, we will find the two conjugate harmonic functions u and v that satisfy the boundary conditions (8) imposed on the original Cauchy-Riemann system. Hence, we will have found a solution to the original Cauchy-Riemann problem by solving a second-order partial differential equation and without having made use of either the irrotationality or incompressibility conditions as field equations in our development. This is precisely the objective of the surrogate-equation technique: to find a second-order partial differential system that can serve during the solution procedure as a surrogate for the original first-order system while neither broadening nor restricting its set of admissible solutions.

For convenience the three boundary value problems discussed thus far are schematically illustrated in figure 1.

Computational Results

Given the potential and surrogate-equation formulations of the Cauchy-Riemann problem, as described in preceding sections, it is a quite straightforward matter to compute approximate numerical solutions to both of these boundary value problems.

Such illustrative computations are reported in detail in reference 20. There each problem is discretized by using second-order accurate central differencing, and the resulting systems of algebraic equations are solved by successive line overrelaxation (SLOR). The problem specification is completed by choosing the functions q_1 through q_4 for $0 \le x \le l_x$ and $0 \le y \le l_y$ such that

$$q_1(y) = \frac{0.5y}{l_y}$$

$$q_2(y) = 1.0$$

$$q_3(x) = \frac{0.5x(1.0 - 0.5x/l_x)}{l_y}$$

$$q_4(x) = \frac{0.5x(1.0 - 0.5x/l_x)}{l_y} - \frac{0.75\, l_y}{l_x}$$

The exact solution to this problem is

$$u(x,y) = \frac{0.5y(1.0 - x/l_x)}{l_y} + \frac{x}{l_x}$$

$$v(x,y) = \frac{0.5x(1.0 - 0.5x/l_x)}{l_y} - \frac{y(1.0 - 0.25y/l_y)}{l_x}$$

For both the computations with the potential formulation and those with the surrogate-equation formulation, the SLOR procedure was continued until the resultant approximations to the u and v velocity components differed from the exact solution by at most 1.0×10^{-6} at any point in the

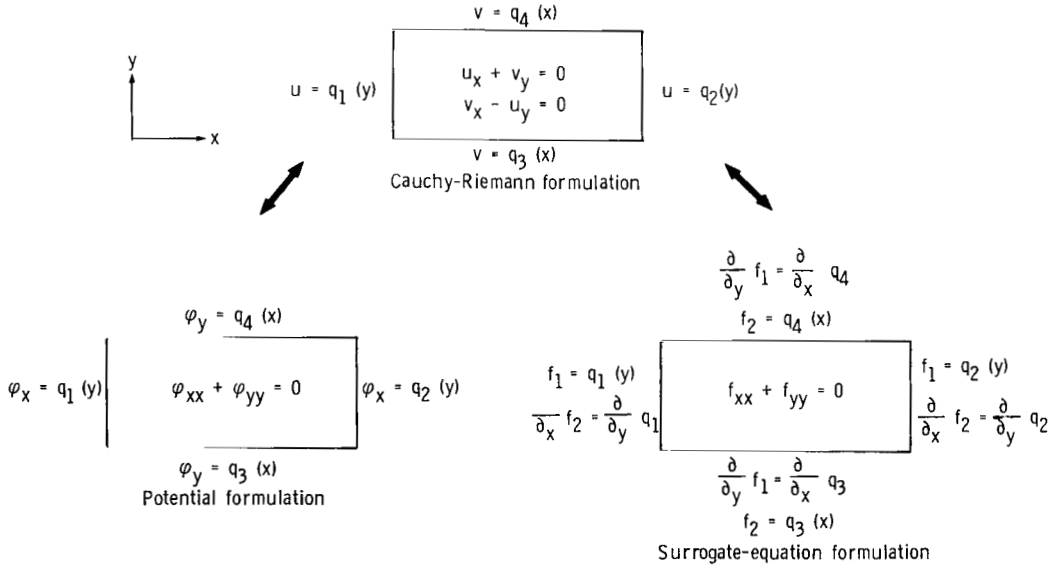


Figure 1. - Three equivalent formulations of the same boundary value problem.

computational domain. With the parameters l_x and l_y set at values of 2.0 and 1.0, respectively, computations were carried out on three meshes of successively doubling point density.

As expected, the numerical solution to the potential formulation behaved such that the u and v velocity components, as computed from the x and y derivatives of φ, respectively, converged to within the above specified tolerance of the exact solution in an orderly fashion. Hence the principal nontrivial result is that, as expected from the foregoing theoretical considerations, the numerical solution to the surrogate-equation formulation also behaved well, producing the desired approximate solution to the required degree of accuracy.

Summation

By using the Cauchy-Riemann equations describing the two-dimensional, incompressible, irrotational flow to provide a model problem, the essential features of the surrogate-equation technique have been illustrated. The objective of constructing a second-order partial differential system to serve during the solution procedure as a surrogate for the original first-order system while neither broadening nor restricting its set of admissible solutions appears on the basis of both theoretical considerations and the results of computational experimentation to have been achieved.

Euler Problem

Having illustrated the essential features of the surrogate-equation technique by means of the simple Cauchy-Riemann model problem, we now proceed to the consideration of more realistic and more complicated problems, where the utility and generality of the surrogate-equation technique can be more thoroughly displayed. This section treats the computational simulation of the steady flow of a perfect fluid under either purely subsonic or purely supersonic conditions. Transonic flows are discussed in the following section.

The conventional approach to the computational solution of steady, subsonic inviscid flow problems is to make use of either the potential or the stream function. By doing so, the Euler system of partial differential equations, which has only first-order derivatives of primitive variables, is replaced with a second-order partial differential equation in a derived dependent variable. Given this second-order equation, one can then draw on the large body of experience concerning the design of relaxation procedures for such equations in order to arrive at a solution algorithm. This convenience is compensated for by a loss of generality. The potential function formulation is limited in application, by definition, to irrotational flows. The stream function formulation is essentially two-dimensional and furthermore for transonic flows is hampered by density being a double-valued function of the mass flow parameter and by the saddle point that exists at the sonic line (refs. 1, 21, and 22).

An alternative to this approach is to seek a steady solution that is the temporal asymptote of solutions to the unsteady Euler equations. Assuming that such an asymptote exists, this method has the advantages both of avoiding the restrictions inherent in the potential and stream function formulations and of allowing one to deal with purely hyperbolic first-order partial differential equations. For such equations one may once again draw on a large body of experience when designing a solution procedure.

In the case of time-accurate, time-asymptotic solution of the unsteady Euler equations the principal disadvantage lies in the long computing times to be expected. As a result of efficiency comparisons of time-accurate methods with relaxation methods for certain model problems, Lomax and Steger (ref. 2) found ". . . that relaxation methods converge from one to two orders of magnitude faster than time-accurate ones." This led them to conclude that ". . . there is a real need to improve our relaxation techniques for many types of equations modeling steady-state fluid flows." More specifically, they found that, although ". . . some techniques for relaxing isentropic-flow equations in terms of the primitive variables have been developed" (refs. 1 and 23 to 25), ". . . a suitable relaxation procedure for the general Eulerian equations has not emerged, so far as the authors know" and further that "a major problem area where relaxation schemes have yet to be exploited is in the calculation of inviscid rotational and energy-input flows."

The absence of suitable relaxation procedures for the steady Euler equations can be accounted for in the observation that for regions of subsonic flow, where the equations exhibit elliptic behavior, the natural differencing techniques, when applied to these first-order equations, lead to unstable finite difference equations. At this writing, several attempts by various researchers at resolving this difficulty have, as discussed previously, provided interesting results and useful insights but have met with less than complete success.

The realization that the impasse in the development of relaxation methods for the steady Euler equations is caused by their first-order nature leads one quite naturally to consider an application

of the surrogate-equation technique. Since this technique provides a steady, second-order system of partial differential equations whose solution also satisfies the Euler equations, we can at once avoid the restrictions of the second-order potential and stream function formulations and use natural differencing techniques to create stable finite difference equations. Having thus devised a method for obtaining stable, discrete algebraic analogs to the steady Euler partial differential equations, we can then proceed to investigate various relaxation procedures with the aim of identifying those that provide a converged solution more efficiently than does the time-accurate solution of the unsteady equations. Note here that it is not the intention of the present work to perform such an investigation of the various possible relaxation procedures. Rather, these computations are of an illustrative nature. The goal then is to present the surrogate-equation technique and to show how its application to the equations of inviscid fluid flow leads to more powerful and general computational procedures than are presently available. In particular, we will presently illustrate its use for the computation of rotational subsonic flows by using the full Euler equations. In the next section the surrogate-equation technique is used to create stable, fully conservative, type-dependent finite difference equations for the numerical solution of an inviscid transonic flow problem.

The purely supersonic flow problem can be readily solved by any number of proven techniques. Consequently it is included here only to provide a test case for the surrogate-equation technique.

Equations of Motion

As is well known, the two-dimensional flow of a perfect fluid can be described by specifying four partial differential equations, together with the appropriate auxiliary relations and boundary conditions. These partial differential equations are known as the Euler equations and are statements concerning the conservation of fluid mass, momentum, and energy. For steady flow they can be written in vector form as

$$f_x + g_y = 0 \tag{9}$$

where x and y are the coordinates of a Cartesian reference frame,

$$f = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (E+p)u \end{bmatrix} \tag{10a}$$

and

$$g = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (E+p)v \end{bmatrix} \tag{10b}$$

Here the density, static pressure, and velocity components in the x and y directions and the total energy per unit volume are denoted by ρ, p, u, v and E, respectively. Furthermore the total energy per unit volume can be expressed as

$$E = \rho\left[e + \frac{1}{2}(u^2 + v^2)\right]$$

where the specific internal energy e is related to the pressure and density by the simple gas law

$$p = (\gamma - 1)\rho e$$

with γ denoting the ratio of specific heats.

For definitude we now assume that we wish to treat a flow that is approximately aligned with the x direction and hence wish to rewrite equations (9) solely in terms of f. To this end, note (ref. 26) that f and g are homogeneous functions of first degree in the components of the vector of conservative variables w, where w is defined such that

$$w = (\rho, \rho u, \rho v, E)^T$$

Hence it follows from Euler's theorem on homogeneous functions (ref. 27) that $f = Aw$ and $g = Bw$, where the Jacobian matrices A and B are defined such that $A \equiv \partial f/\partial w$ and $B \equiv \partial g/\partial w$ and their elements are given explicitly in appendix A. Thus, wherever A^{-1} exists, we can write

$$g = Tf$$

where $T \equiv BA^{-1}$. Since A is singular only when the absolute value of the u velocity component either vanishes or is equal to the local sonic velocity (appendix A), assuming A to be nonsingular introduces no essential limitations for the purely subsonic and supersonic flow cases presently under consideration. We can now rewrite equation (9) as

$$f_x + (Tf)_y = 0$$

This equation can in turn be simplified (appendix B) to yield

$$f_x + Tf_y = 0 \tag{11}$$

Thus far, we have done no more than to rewrite the Euler equations (9) in the form of equation (11), so that they are expressed in terms of the single vector of dependent variables f. Equation (11), together with the flow tangency condition and the appropriate upstream and downstream (or far field, if relevant) conditions, constitutes a completely specified partial differential problem.

Second-Order System

The Euler equations, as given in equation (11), can be rewritten by using differential operator notation to yield

$$\left(\frac{\partial}{\partial x} + T\frac{\partial}{\partial y}\right) f = 0 \tag{12}$$

Then, in a manner similar to that used previously for the Cauchy-Riemann equations, we can create the desired second-order system by operating on equation (12) with the differential operator

$$\frac{\partial}{\partial x} - T\frac{\partial}{\partial y} \tag{13}$$

to yield

$$\left[\frac{\partial^2}{\partial x^2} - T^2\frac{\partial^2}{\partial y^2} - (TT_y - T_x)\frac{\partial}{\partial y}\right] f = 0$$

This equation can be reexpressed (appendix B) as

$$\left[\frac{\partial^2}{\partial x^2} - T^2\frac{\partial^2}{\partial y^2} - (TT_y - T_y T)\frac{\partial}{\partial y}\right] f = 0$$

which can in turn be simplified to

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y}\left(T^2\frac{\partial}{\partial y}\right)\right] f = 0 \tag{14}$$

Equation (14) is the surrogate second-order equation that will be used here to obtain solutions to the Euler equations. Since this surrogate equation has been obtained from the original equation by a process of differentiation, we expect that its solution set will contain those solutions to the Euler equations that we seek. Any possible additional elements of the solution set to equation (14) that do not satisfy the Euler equations are eliminated by means of the boundary condition specification, which is discussed in the next subsection.

This surrogate second-order partial differential system possesses some interesting properties. By virtue of the form of the operator (13) the second-order system has no cross-derivative terms. This convenience results in a considerable simplification of its finite-difference equation analog. Furthermore the choice of the operator leads to a pleasing behavior of the characteristic directions associated with equation (14). We first recall that the characteristic directions of the Euler equations are determined by the eigenvalues of the matrix $A^{-1}B$. We then observe that the characteristic directions associated with the surrogate second-order system are determined by the square roots of the eigenvalues of the matrix T^2. However, the matrices $A^{-1}B$ and T are similar since

$$A^{-1}B = A^{-1}(T)A$$

and hence have the same eigenvalues. It then follows that the characteristic directions of the system of second-order partial differential equations described by equation (14) have slopes equal to $\pm\lambda$, where the λ are the eigenvalues of the matrix $A^{-1}B$. This means that, in applying the surrogate-equation technique to the Euler equations to obtain equation (14), we have retained the original Euler characteristic directions and added to them their reflections through the x axis. For subsonic flows this property may not be of great importance. However, for supersonic flow or for the extension of the method presented here to transonic flow the behavior of the characteristic directions gains considerably in significance. It is then quite useful to observe that, if the coordinate system is chosen such that the flow is aligned with the x direction, the standard successive-line-overrelaxation iterative procedure possesses the same symmetry with respect to the x axis as do the characteristic directions of the surrogate second-order system. From this point of view, one would then expect the introduction of the additional characteristic directions associated with the second-order system to have a minimal effect on the behavior of the SLOR solution procedure.

Boundary Conditions

Here we are concerned, in general, with two types of boundaries to the physical domain of interest. The

first type, which is referred to as a solid boundary, occurs at the interface of the fluid with some substantial obstacle, such as the wall of a passage or the surface of an airfoil. The second type, which is called a flow boundary, occurs when for practical reasons one arbitrarily prescribes a boundary in the fluid-flow field itself that is not a solid boundary but beyond which the flow simulation will not proceed.

We seek a solution to the Euler equations and, as is well known (ref. 28), it is a both necessary and sufficient solid boundary condition for these equations that the fluid-flow velocity vector be parallel to the wall slope at the point of contact. Hence we require that the surrogate second-order system also satisfy this wall tangency condition. Furthermore, to insure that we admit no solutions to the second-order system that do not satisfy the Euler equations, we require that precisely these Euler equations also serve as boundary conditions. Although no mathematical proof of the sufficiency of these additional conditions is presented herein, the computational results to be reported serve as a strong indication that this is in fact the case.

Since the flows to be treated subsequently are internal flows, the flow boundaries are of the inflow or outflow type rather than the far-field type associated with external flows. These far-field boundaries are normally treated by assuming that either "free stream" conditions can be used for the values of the unknown or that some far-field solution, obtained by other means, is available to determine their values.

The appropriate inflow and outflow boundary conditions for internal flows are dependent on the exact physical and mathematical nature of the problem to be solved. Quite often the inflow boundary is treated by specifying the values of all unknowns along its extent. Although such specification precludes any influence of the flow conditions downstream on those at the entrance, this nevertheless often results in a physically realistic boundary condition. For either supersonic or subsonic outflow a commonly used boundary treatment, again assuming that it is compatible with the physics and mathematics of the flow, is to assume that the values of the unknowns do not vary in the flow direction. Use is made of such flow boundary conditions in the calculations to be discussed subsequently. However, the simplified inflow and outflow conditions applied successfully to these model problems cannot in general be expected to yield good results in more complicated flow simulations.

Problem Specification

To test the surrogate-equation formulation described earlier, we consider the simulation of a number of two-dimensional internal flows. Since, as mentioned previously, very little appears to be known about the design of relaxation procedures for the Euler equations, the rational development of such a procedure requires that simple tests be made of the validity of the concepts involved.

We first compute the flow in the supersonic region of a two-dimensional hyperbolic nozzle. Any problems due to the introduction of additional characteristic directions by the second-order system should be revealed by such a case. Furthermore the necessary inflow boundary conditions and a series solution that is valid close to the sonic line are available from the work of Hall (ref. 29). A more complete specification of this test case is given in reference 30.

The next test is to compute the purely subsonic flow in a two-dimensional symmetric nozzle with sinusoidally shaped walls. This geometry is of interest because the subcritical flow through it should have two axes of symmetry: the nozzle centerline and the geometric throat. Further details concerning this case are given in reference 31.

We then consider the computational simulation of inviscid shear flows through curved passages. To this end, use is made of a class of inviscid, incompressible, rotational flows, presented by Shercliff (ref. 32), that can be described by the stream function

$$\psi = C \exp(ky) \cosh(lx)$$

where C, k, and l are constants and x and y are Cartesian coordinates.

As is explained in greater detail in reference 32, this stream function describes the flow of an incompressible fluid through a bend of angle $2 \arctan(l/k)$ that transitions between two asymptotic flows that are rectilinear shear flows. The bend is quite abrupt and the streamlines approach their asymptotes with exponential vigor. Also, as the streamtube cross section is greatest at the symmetry axis of the bend, these flows first decelerate and then reaccelerate as they complete their passage through the bend.

For precision, we have chosen here to examine computationally the compressible inviscid flow through finite sections of 90° Shercliff bends. The results can then be compared with one another and for reference with what is subsequently referred to as the augmented incompressible flow solution.

This augmented incompressible flow solution consists simply of velocity components obtained directly from the stream function that represents the

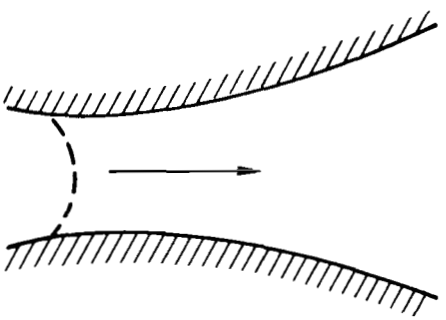
corresponding incompressible flow, together with density and static pressure values. These values are estimated by specifying density and static pressure profiles at the bend entrance and then obtaining their distributions throughout the flow field by making use of the constancy of both entropy and total enthalpy along streamlines while assuming that the perfect-gas law applies. In this way we obtain information suitable for specifying the entrance conditions for the compressible flow computations, for generating starting conditions for the iterative solution procedure, and for use as a baseline against which we can compare the results of our surrogate-equation compressible flow computations.

For variety a number of computations have been performed in a different sort of bend, referred to herein as a circular-arc bend. This is a bend that transitions between two rectilinear sections by utilizing a section whose walls are arcs of two concentric circles, with tangency being required at the joints. Further specifications for these cases are given in reference 20. For convenience, all four test cases are schematically illustrated in figure 2.

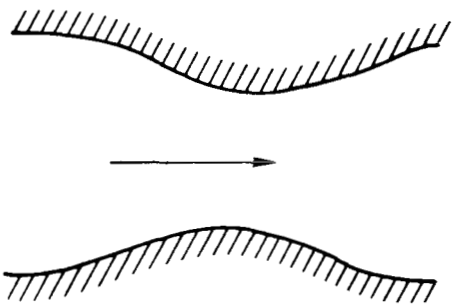
Computational Results

Extensive test computations have been carried out on the surrogate-equation representation of the Euler equations described previously. Of particular interest are the results obtained in the four test cases shown in figure 2. In all computations reported herein, a sheared coordinate system was chosen for the physical domain. Such a coordinate system although nonorthogonal is simple and convenient and, since the bounding walls are coordinate lines, it facilitates the precise application of wall boundary conditions.

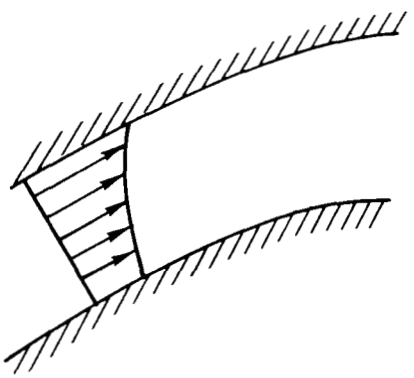
In each of the subsonic test cases the governing partial differential equations are elliptic and were discretized by means of second-order accurate central differences. The resulting algebraic equations were



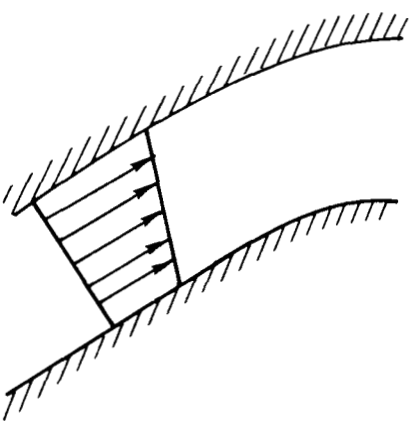
(a) Supersonic nozzle (see refs. 29 and 30 for details).



(b) Subsonic nozzle (see ref. 31 for details).



(c) Shercliff bend flow, decelerating flow with turning angles up to 90° (see refs. 20 and 32 for details).



(d) Circular-arc bend flow, constant-cross-section flow with turning angles up to 90° (see ref. 20 for details).

Figure 2. - Nomenclature for Euler equation test cases.

then solved as a coupled system by using an SLOR iteration scheme and a block-tridiagonal analog to the Thomas algorithm (ref. 33). Further details are contained in references 20 and 30.

The governing equations for the supersonic test case are of course hyperbolic. They were discretized by using three-point upwind differences in the flow direction and three-point central differences in the transverse direction. The resulting coupled system of algebraic equations was then solved by using an implicit marching scheme. More information concerning the details of both the discretization and the solution procedure are to be found in reference 30.

For all results presented in this section the following normalizations have been employed:

(1) For velocity components, c_*, the critical speed at the location of maximum entrance velocity

(2) For density, ρ_0, the density at this same location

(3) For static pressure, $\rho_0 c_*^2$

(4) For length, bend cross section at the axis of symmetry or the nozzle throat cross section

The results of the supersonic nozzle test are summarized in figures 3 through 6. There the velocity components, density, and static pressure calculated by using the surrogate-equation technique are compared with the predictions of the Hall solution. The agreement is excellent close to the sonic line, where the Hall solution is valid. As might be expected, the present solution deviates from the Hall

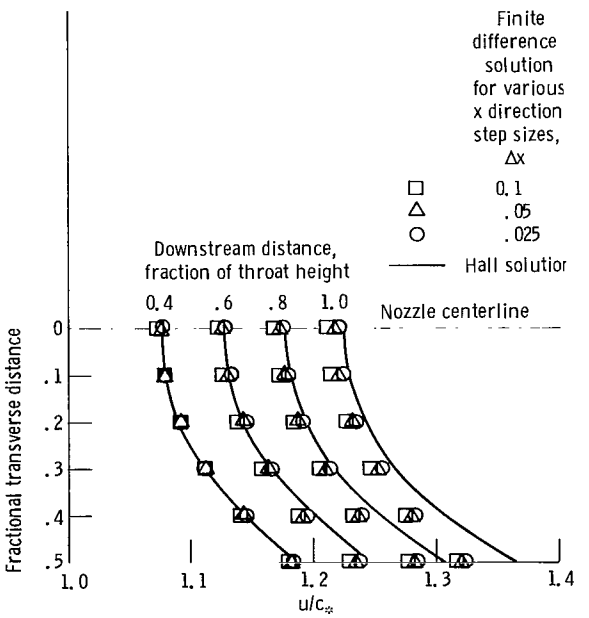


Figure 3. - Nondimensional u-velocity distribution - supersonic flow.

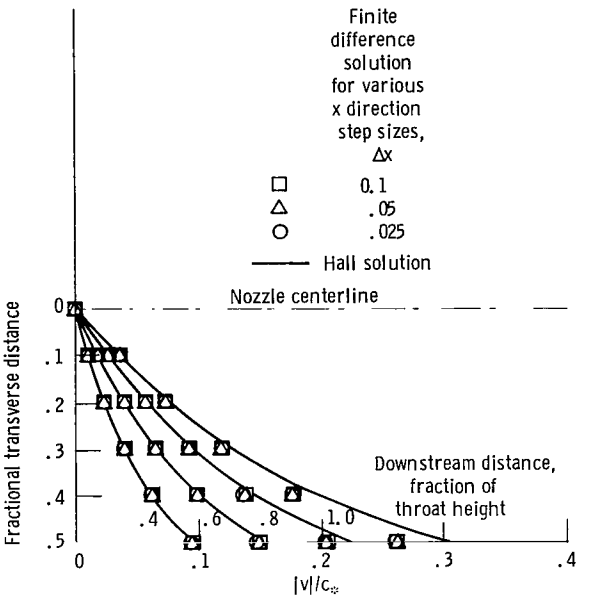


Figure 4. - Nondimensional v-velocity distribution - supersonic flow.

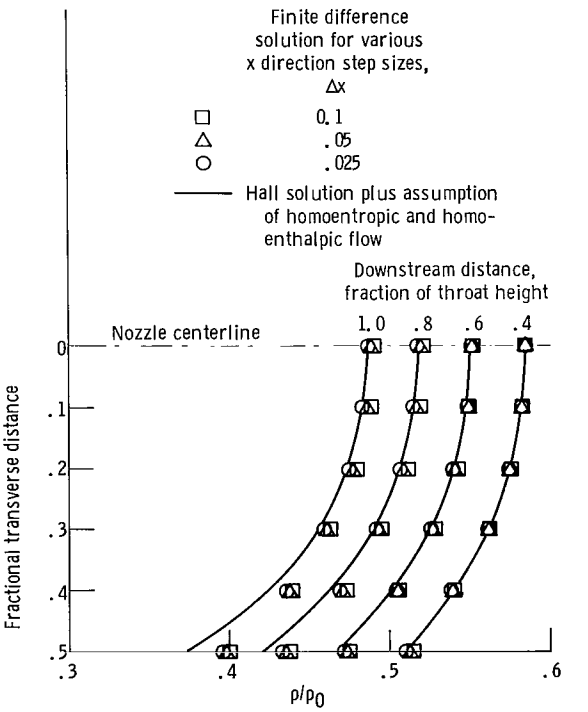


Figure 5. - Nondimensional density distribution - supersonic flow.

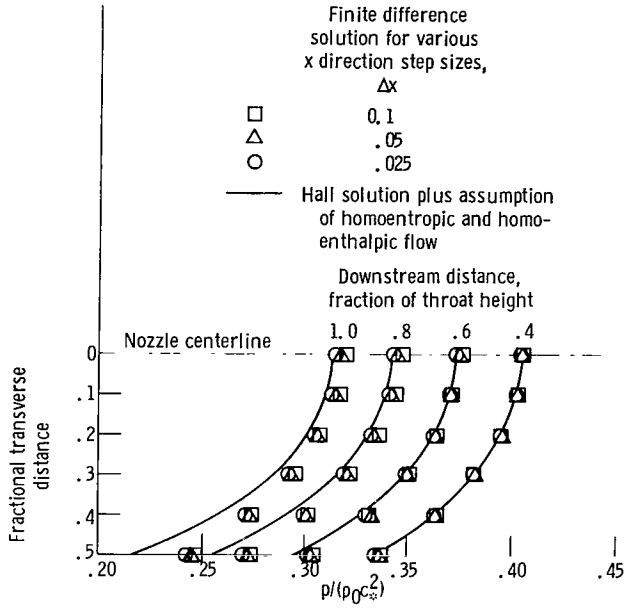


Figure 6. - Nondimensional static pressure distribution - supersonic flow.

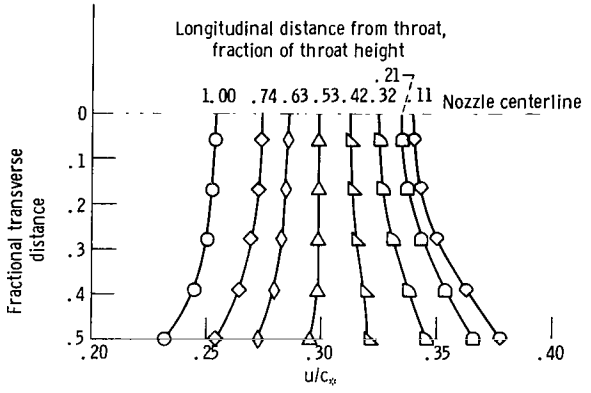


Figure 7. - Nondimensional u-velocity distribution - subsonic flow.

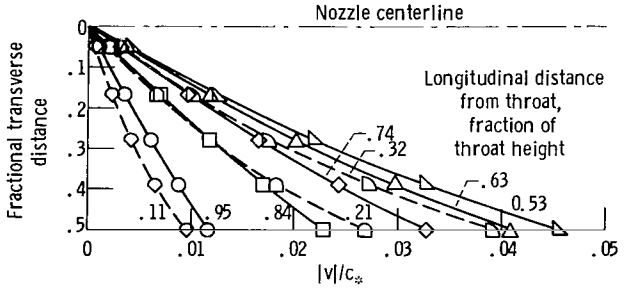


Figure 8. - Nondimensional v-velocity distribution - subsonic flow.

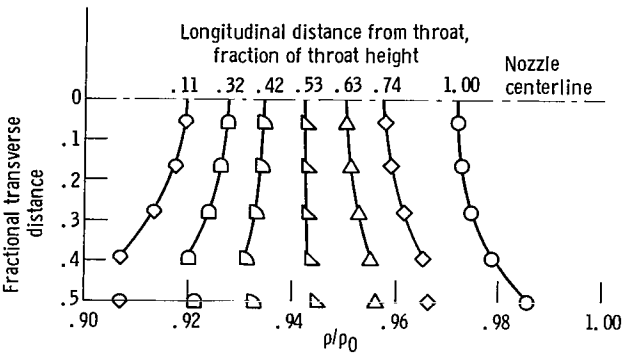


Figure 9. - Nondimensional density distribution - subsonic flow.

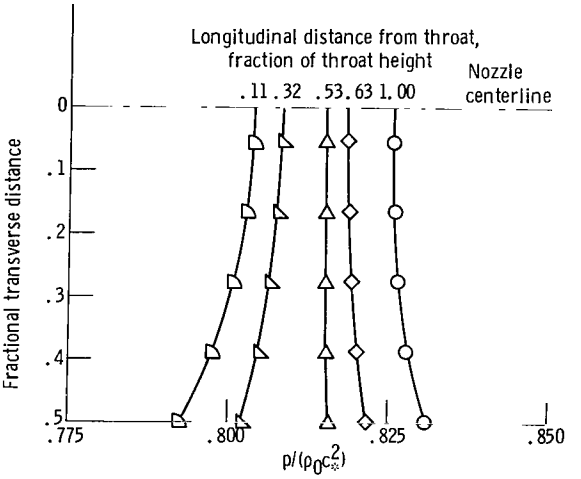


Figure 10. - Nondimensional static pressure distribution - subsonic flow.

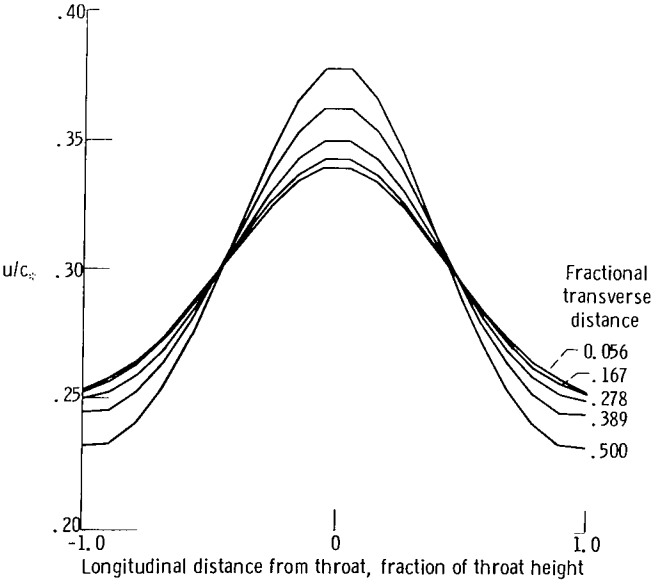


Figure 11. - Nondimensional u-velocity profiles - subsonic flow.

solution as the location becomes more remote from the sonic line and the nozzle axis of symmetry. No detrimental effects attributable to the additional charácteristic directions introduced by the surrogate-equation formulation are apparent in the computational results.

The subsonic nozzle test also produced encouraging results. Figures 7 through 10 show the computed distributions of velocity components, density, and static pressure; figure 11 presents the u velocity profiles obtained at several transverse locations.

To illustrate the utility of the surrogate-equation technique for obtaining solutions to the Euler equations in flow situations of contemporary interest, a computational study has been made of the effects of compressibility on the flow through a Shercliff 90° bend. The section of the bend used in the study, together with the computational grid employed, is illustrated in figure 12. The density and the static pressure were assumed to hold their normal atmospheric values at the bend entrance. The ratio of maximum to minimum entrance velocities was set equal to 1.5 for the entire study, while the minimum entrance velocity was increased from an initial value of 50 meters per second through 100, 150, 200, and 250 meters per second to a final value of 275 meters per second.

While the complete results are given in reference 20, a sampling of the results of the study is shown in figures 12 and 13. There, for the 275-meter-per-second case the augmented incompressible flow solution and the compressible flow solution obtained by means of the surrogate-equation technique are shown. The intervening cases, which are not illustrated here, show a gradual transition from essentially incompressible behavior to the highly compressible case of figure 13.

Results for a 30° circular-arc bend with a minimum entrance velocity of 50 meters per second, a ratio of maximum to minimum entrance velocities of 1.0, and atmospheric values of density and pressure at the upstream domain boundary are shown in figures 14 and 15. Figure 14 shows the solution obtained on the basic computational mesh; figure 15 shows the corresponding solution obtained on the refined mesh illustrated there. Only the relatively minor adjustments in the solution to be expected as a result of such a mesh refinement are in fact observed. It is, however, interesting to see that in both computations a slightly anomalous behavior is apparent in the density field. A more refined treatment of the wall boundary conditions must be considered as a strong candidate for the eventual elimination of this behavior.

During the course of the computations reported herein, some data regarding the efficiency, accuracy, and stability of the surrogate-equation algorithm for the Euler equations have also been collected.

The algorithm requires approximately 0.0026 central arithmetic unit (CAU) seconds per grid point per iteration when executed using double-precision arithmetic on the NASA Lewis Univac 1100/40 computer system.

For a simple model problem an experimental determination of the actual order of accuracy of the surrogate-equation algorithm was performed. For fixed flow conditions and a computational domain of fixed dimensions, computations were performed on three grids, with successive grids having the grid point spacings in each direction halved from their previous values to yield normalized mesh sizes of 1.0, 0.5 and 0.25, respectively. The results of this study are presented in figure 16, where the error, representing the difference between the exact and approximate computed solutions at a typical grid point, is plotted as a function of the normalized mesh size. By virtue of the logarithmic scales used in figure 16, the observed order of accuracy for the surrogate-equation algorithm used in this section can be easily estimated to be approximately 2.

As of this writing the observations made concerning the stability of the surrogate-equation algorithm are rather qualitative in nature. Formal stability bounds have yet to be determined and our computational experimentation, although quite extensive, does not suffice to estimate them. It does appear, however, that by using the surrogate-equation technique, we have, as a minimum, left the realm of unconditionally unstable finite difference analogs to the steady Euler equations and entered into the realm of conditionally stable, and apparently quite robust, ones. The conditional stability seems to be a consequence of the use of the first-order, steady Euler differential equations in the formulations of the finite difference equations used at solid boundaries. Given the rather perverse behavior of finite difference analogs to these equations in regions where they exhibit elliptic nature, it is quite plausible that their introduction would effect such a stability reduction. Hence an obvious area for further study is the more exact determination of, and subsequent improvement in, the stability properties of our algorithm.

Conservation Form

For the computations discussed in this section, we have applied the surrogate-equation technique to a nonconservation form, equation (11), of the Euler equations. Since the use of this nonconservation form presents no serious difficulties for the sort of computations discussed here, it was adopted because of its simplicity. Should one, however, wish to

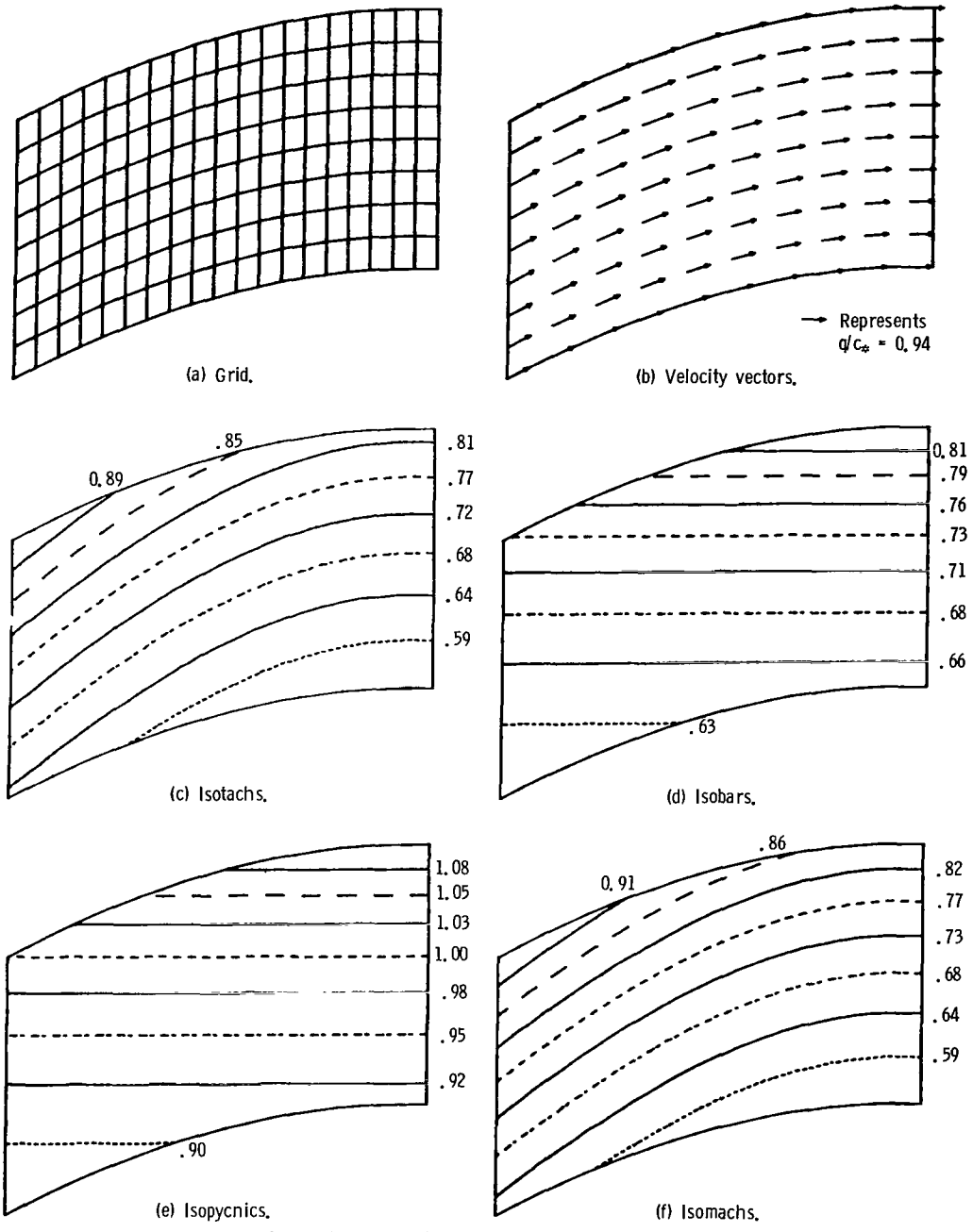


Figure 12. - Shercliff bend flow - initial guess.

employ the conservation-form analog to equation (14), it can be obtained quite simply as follows.

Given the Euler equations, written in conservation form as

$$f_x + (Tf)_y = 0 \tag{15}$$

we can rewrite them in operator notation as

$$\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y}(T\ )\right] f = 0$$

To create the surrogate second-order system, we employ the operator

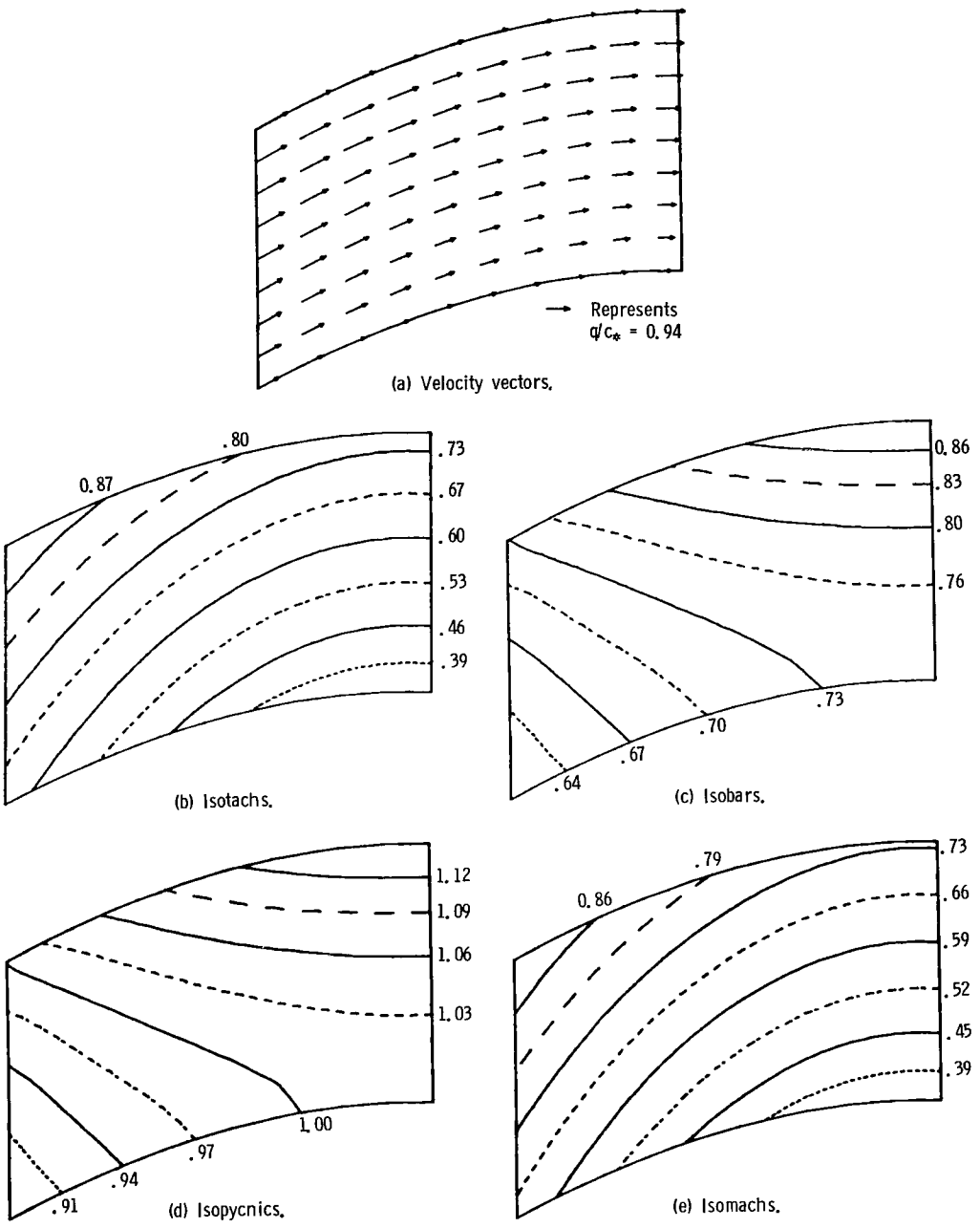


Figure 13. - Shercliff bend flow - converged solution.

$$\frac{\partial}{\partial x} - \frac{\partial}{\partial y}(T\ )$$

to yield

$$\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}(T\ )\right]\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial x}(T\ )\right]f=0$$

This equation can in turn be expanded and simplified to the form

$$\left\{\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y}\left[T\frac{\partial}{\partial y}(T\ )\right]\right\}f=0 \tag{16}$$

Equation (16) is in conservation form and can be

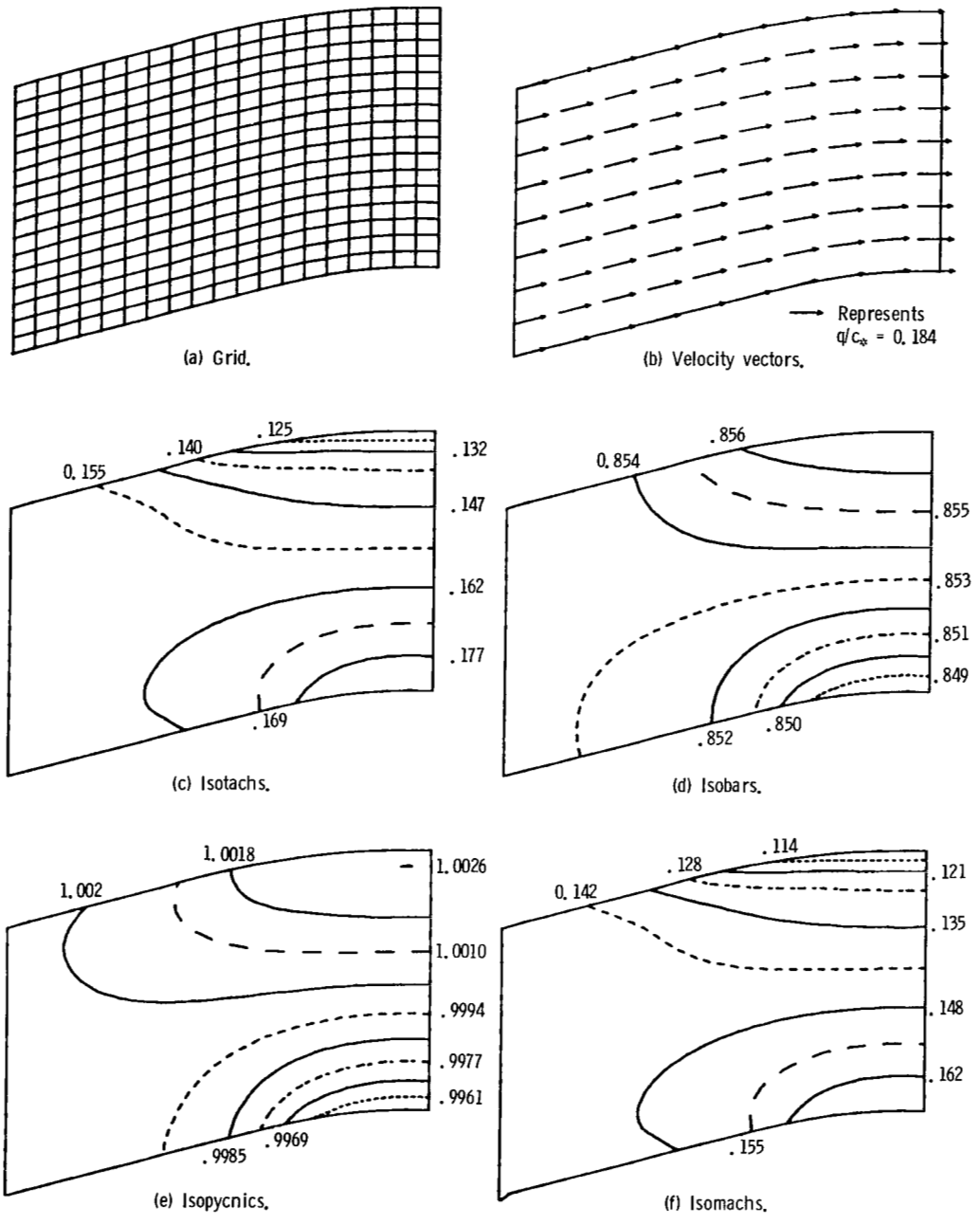


Figure 14. - Circular-arc bend flow - converged solution, basic mesh.

differenced by employing the usual techniques (i.e., second-order, accurate central differencing for purely subcritical flows) to yield fully conservative (refs. 13 and 34) finite difference equations.

Summation

In this section we have shown that the surrogate-equation technique can be used to perform computational simulations of the steady subsonic or supersonic flow of a perfect fluid. Solutions to the Euler equations are obtained without resort to either the potential or stream function formulations, and the inherent limitations of these formulations are not shared by the present approach. Furthermore the time-asymptotic solution of the unsteady Euler equations is also avoided, and convergence acceleration by either relaxation or some other non-

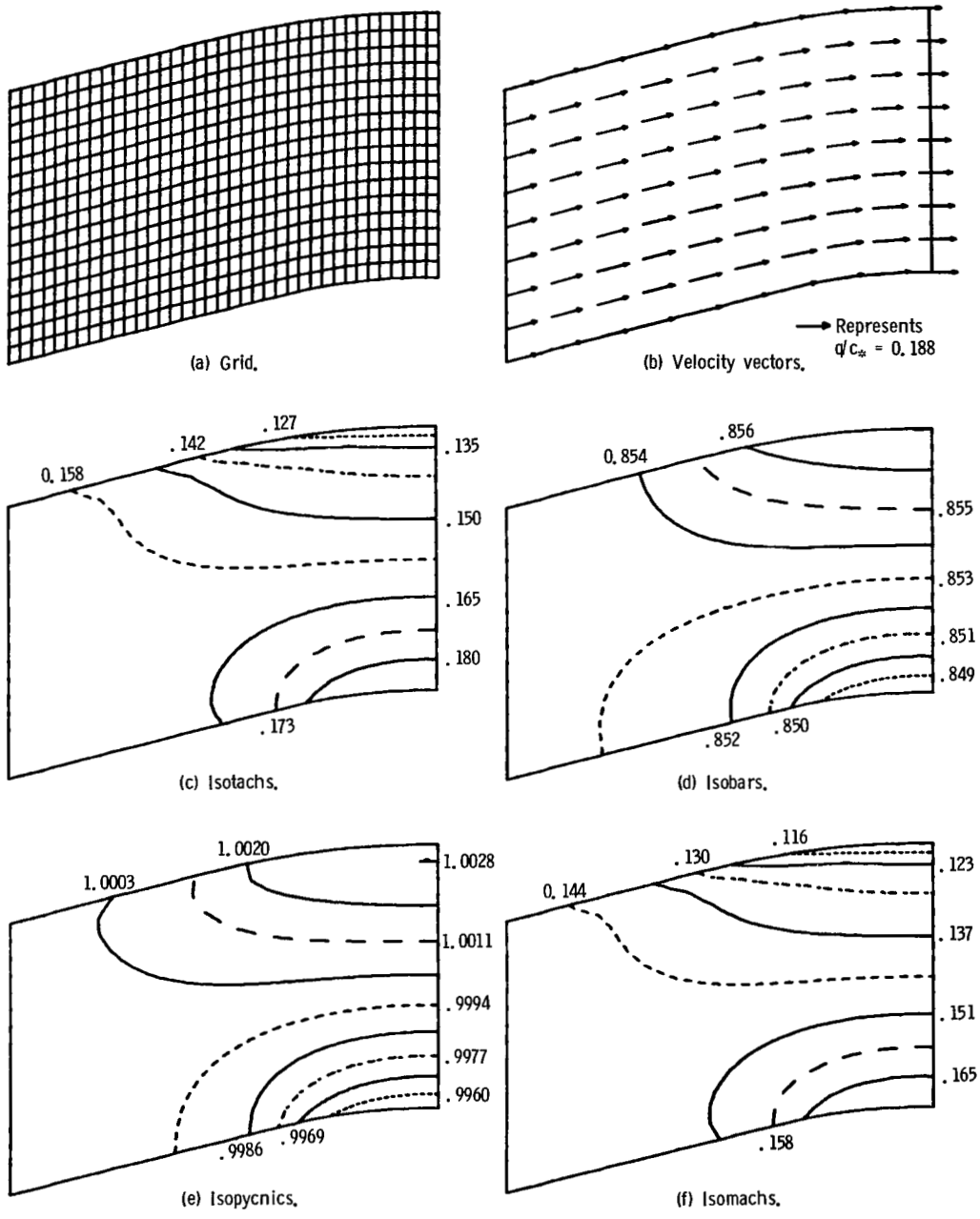


Figure 15. - Cicular-arc bend flow - converged solution, refined mesh.

time-accurate process is possible.

The algorithm presented here is reasonably efficient but possesses ample possibilities for improvement. It is approximately second-order accurate and is conditionally stable, with further study being necessary to determine precise stability bounds. Although the results presented here were based on a nonconservation form of the Euler equations, the development of an algorithm that uses a conservation form, as discussed in this section, presents no essential difficulties.

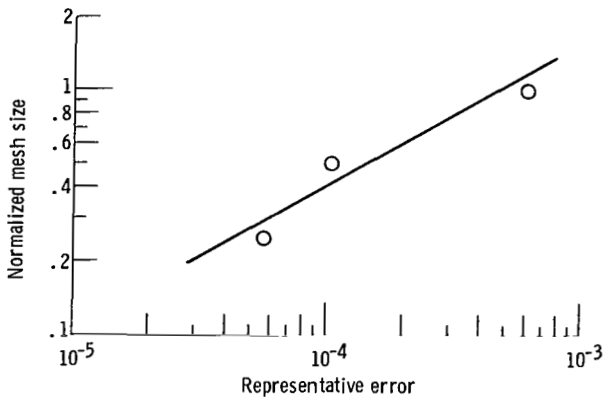


Figure 16. - Experimental accuracy determination for Euler problem.

Transonic Flow

Applications of the surrogate-equation technique are by no means limited to the classes of steady subsonic and supersonic flows from which the examples discussed in the previous section were drawn. In fact, as illustrated here, the technique can be used to good advantage for the computational simulation of steady transonic flows. Such flows are of considerable mathematical interest and have great technological importance.

As discussed previously the main body of methods for the computation of steady inviscid transonic flow that have been developed thus far either utilize the time-asymptotic solution of the unsteady Euler equations to obtain a steady solution or iteratively solve the steady potential or perturbation potential equation. In the former case computation times are long; in the latter case the generality of the model equations is sacrificed in the name of computational efficiency.

The transonic small-disturbance theory, upon which the perturbation-potential approach is based, is derived under the assumptions that body surface slopes are everywhere small (so that flow quantities are small perturbations about their free-stream values) and that the free-stream Mach number is near unity. In practical situations these assumptions are not always strictly met, but nonetheless many cases of engineering interest can be adequately treated. Where the assumptions of small-disturbance theory are grossly violated, resort is made to the full potential formulation.

The assumption of potential flow inherent in both of these approaches and the resultant restrictions to irrotational and isentropic conditions can prove quite troublesome under certain circumstances. Clearly shear flows cannot be treated, and hence a large class of technologically important flows lies beyond the reach of these methods.

There is a need for a technique for computationally simulating transonic flows that is, on the one hand, computationally efficient and, on the other hand, not subject to the restrictions of a potential flow formulation. In this section, we demonstrate the ability of the surrogate-equation technique to serve this need.

For simplicity and clarity the surrogate-equation technique is first applied within the context of transonic small-disturbance theory, where comparison can be made with the conventional perturbation-potential approach. In this manner the validation of the method for use in transonic flow is separated from the demonstration of its usefulness in rotational flow, which was presented in the preceding section. Its extension to the full steady Euler equations is discussed later in this section.

Small-Disturbance Approximation

The details of small-disturbance theory have been thoroughly discussed in the literature (refs. 35 to 39), and our remarks here are accordingly limited. In brief, if we assume that the flow of interest can be represented as a disturbance to a uniform flow, that in particular the disturbance velocity components are small with respect to the mean velocity, and that for transonic flow the Mach number of the undisturbed flow is close to unity, the exact Euler equations can be replaced by approximate transonic small-disturbance equations. These equations, although quite simple, retain the essential nonlinear, mixed elliptic-hyperbolic character of the exact equations. Furthermore weak solutions of the transonic small-disturbance equations that admit discontinuous jumps approximating shock waves can be obtained.

Although alternative forms of this equation are readily available and have in fact been employed successfully by other investigators, the following formulation is sufficient for our present purposes:

$$\left[1 - M_\infty^2 - (\gamma + 1)M_\infty^2 u\right]\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (17)$$

Here u and v represent the disturbance velocity components in the x and y directions, respectively, normalized by the uniform stream velocity, referred to as U_∞. Also, M_∞ is the Mach number of the undisturbed uniform stream and γ is the ratio of specific heats. Equation (17) can be rewritten in conservation form as

$$\frac{\partial \beta}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (18)$$

where

$$\beta \equiv \left(1 - M_\infty^2 - \frac{\gamma+1}{2} M_\infty^2 u\right) u$$

We can supplement this single equation in two unknowns by the irrotationality condition

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{19}$$

which is also a consequence of the small-disturbance assumption, to obtain a closed set of equations. Equations (18) and (19) then constitute a system of first-order, partial differential equations of mixed elliptic-hyperbolic type. They are elliptic or hyperbolic according to whether the u component of the disturbance velocity is smaller than or larger than the critical disturbance velocity u^*, which is defined as

$$u^* \equiv \frac{1 - M_\infty^2}{(\gamma+1) M_\infty^2}$$

Perturbation-Potential Equation

Although a computational procedure could be devised to calculate an approximate solution to the system of equations (18) and (19) directly, this is not normally attempted. This is so because equation (19) implies the existence of a scalar perturbation velocity potential φ such that $\varphi_x = u$ and $\varphi_y = v$. Hence equation (18) can be rewritten as

$$\left[\left(1 - M_\infty^2 - \frac{\gamma+1}{2} M_\infty^2 \varphi_x\right)\varphi_x\right]_x + \varphi_{yy} = 0 \tag{20}$$

while equation (19) reduces to the identity

$$\varphi_{yx} - \varphi_{xy} = 0$$

Equation (20) is a second-order partial differential equation of mixed elliptic-hyperbolic type in the scalar unknown φ. This appears to be advantageous on two counts. First, in solving equation (20) for φ and thence for u and v we can possibly obtain these velocity components with less computational effort than would be required by a direct solution of the system of equations (18) and (19). Second, since equation (20) is of second order, the formulation of stable, conservative finite difference equations for its discrete representation is greatly facilitated.

Although the transonic small-disturbance equations have been known for a considerable time, their essential nonlinearity has impeded progress on their analytical solution. On the other hand, their mixed elliptic-hyperbolic nature, together with the fact that the locations at which type changes occur cannot generally be prescribed apriori, confounded attempts at their approximate numerical solution. In 1969 Cole (ref. 35) summarized the status of transonic small-disturbance theory and set up the problem of plane mixed flow past an airfoil, including a discussion of the far field. Subsequently Murman and Cole (ref. 3) devised a numerical method for the computation of an approximate solution to this problem. The details of their basic method and its subsequent improvements and generalizations are given in the literature, particularly in references 3 and 40 through 44. A brief discussion of those aspects of the method that are germane to our present purpose follows.

Murman and Cole overcame the difficulties associated with the mixed elliptic-hyperbolic nature of the transonic perturbation-potential equation by introducing the idea of type-dependent differencing. By automatically adapting the finite difference equations at each grid point of the computational domain to suit the local nature of the flow, they were able to construct an iterative procedure for the solution of mixed flow problems that "captures" any shocks that may be present and represents them as steep gradients and that is computationally stable and in conservation form.

For the field equations and boundary conditions, second-order accurate central differencing is used for derivatives in the y direction and for derivatives in the x direction in regions where the flow is subsonic. Backward (or upwind) differencing of either first- or second-order accuracy is used in the x direction in regions where the flow is supersonic. At "sonic" and "shock" points special switching operators that preserve the conservative nature of the differencing scheme are employed. Hence domains of dependence are everywhere respected and intercellular fluxes are properly conserved. A further consequence of the conservation form of the partial differential and finite difference equations is that, should shocks be present, the proper (isentropic) jump conditions are attained.

The finite difference equations are written in implicit form, thereby avoiding the restriction of vanishingly small mesh width in the x direction upon approaching sonic velocity, which would be encountered with an explicit formulation. This system of algebraic equations is then solved iteratively by the method of successive line relaxation. In this fashion the approximate numerical solution is recomputed along lines transverse to the

flow direction as the computational domain is repeatedly traversed in the direction of the flow. In subsonic regions the solution is overrelaxed to accelerate convergence.

The boundary condition treatment is such that Neumann conditions on the perturbation potential are specified on the surface of an immersed body and applied, as is consistent with the small-disturbance approximation, along a coordinate line. At some finite distance from the body a domain outer boundary is chosen along which a far-field solution is used to provide the necessary boundary condition.

As singularities can be present at the leading or trailing edges of an airfoil about which the flow field is to be computed, Murman and Cole chose to locate the boundary points of their computational domain one half of a cell width from the leading and trailing edges. Although this might seem to be a rather simplistic remedy, the nature of the singularities in question is such that this approach is reasonably good. Further details concerning the inner boundary condition specification are given in the original Murman and Cole article (ref. 3) and in particular in the work of Krupp (ref. 41).

Second-Order System

Preparatory to our ultimate goal of using the surrogate-equation technique to devise an iterative scheme for the Euler equations, we illustrate the basic process on the simpler, but nevertheless similar, transonic small-disturbance equations. In this way, we can develop the method, test it, and compare its performance with the Murman and Cole approach.

The transonic small-disturbance equations (18) and (19) can be written in vector form as

$$\left(A\frac{\partial}{\partial x}+B\frac{\partial}{\partial y}\right)f=0 \tag{21}$$

where

$$f=\begin{bmatrix}u\\v\end{bmatrix}$$

$$A=\begin{bmatrix}\frac{\partial\beta}{\partial u} & 0\\ 0 & 1\end{bmatrix}$$

$$B=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

Since β is not a homogeneous function of first degree in u, equation (21) is not equivalent to a conservation form of the transonic small-disturbance equations. However, as becomes apparent in the subsequent discussion, this is not an impediment to the formulation of a conservative second-order system. In any case this lack of homogeneity is not present in the full Euler equations.

To create a surrogate second-order equation for equation (21), we operate on it with a differential operator of the form

$$\frac{\partial}{\partial x}(M)-\frac{\partial}{\partial y}(N)$$

to yield

$$\left[\frac{\partial}{\partial x}\left(MA\frac{\partial}{\partial x}+MB\frac{\partial}{\partial y}\right)-\frac{\partial}{\partial y}\left(NA\frac{\partial}{\partial x}+NB\frac{\partial}{\partial y}\right)\right]f=0 \tag{22}$$

For any nonsingular choice of the matrix N, if M is chosen such that

$$M=NAB^{-1} \tag{23}$$

the characteristic directions of equation (22) are determined by the expression

$$\frac{\partial^2}{\partial x^2}-(MA)^{-1}NB\frac{\partial^2}{\partial y^2}$$

which reduces to

$$\frac{\partial^2}{\partial x^2}-(A^{-1}B)^2\frac{\partial^2}{\partial y^2}$$

Hence, here again, as was the case in the examples considered previously, the set of characteristic directions of the surrogate second-order system contains those of the original first-order system as a subset, with the added characteristics being the reflections of the original ones through the x axis. As mentioned previously this symmetry in the characteristic directions of the second-order system is of the same nature as the symmetry inherent in the successive-line-relaxation solution procedure to be employed here. Therefore one would expect the additional characteristic directions associated with the second-order system to have a minimal effect on

the solution procedure. Furthermore for the choice of the matrix M indicated in equation (23)

$$\left[\frac{\partial}{\partial x}\left(MB\frac{\partial}{\partial y}\right)-\frac{\partial}{\partial y}\left(NA\frac{\partial}{\partial x}\right)\right]f=0$$

so that the surrogate second-order system has no cross-derivative terms. This will serve to simplify its finite difference representation. The resulting second-order system can be written as

$$\left[\frac{\partial}{\partial x}\left(MA\frac{\partial}{\partial x}\right)-\frac{\partial}{\partial y}\left(NB\frac{\partial}{\partial y}\right)\right]f=0 \tag{24}$$

This equation is in conservation form and can be differenced to yield fully conservative finite difference equations.

We note in passing that, although the use of equation (23) results in several pleasant consequences, one may, under different circumstances, wish to consider other specifications of the matrix M.

At this point it is instructive to write out the scalar equations represented by equation (24) and examine them. It is a straightforward matter to perform the necessary algebra to obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial\beta}{\partial u}\frac{\partial u}{\partial x}\right)+\frac{\partial^2 u}{\partial y^2}=0 \tag{25}$$

and

$$\frac{\partial}{\partial x}\left(\frac{\partial\beta}{\partial u}\frac{\partial v}{\partial x}\right)+\frac{\partial^2 v}{\partial y^2}=0 \tag{26}$$

These equations are quite simple and present obvious differencing possibilities, as discussed subsequently.

Having formulated the surrogate second-order system for the transonic small-disturbance equations, we now proceed with a discussion of the boundary conditions necessary to completely specify the partial differential problem being considered for numerical solution. As we are numerically investigating a two-dimensional internal transonic flow, the boundary condition discussion is presented in such a context.

At the upstream and downstream flow boundaries, which are located far from any disturbance to the flow field and in regions where the velocity is uniform, we require that both the u and v disturbance velocity components vanish. At solid boundaries we require that the v component of the disturbance velocity be equal in magnitude to the local boundary slope. This is the usual solid-wall boundary condition of transonic small-disturbance theory. It then remains to specify conditions on the u disturbance velocity components at the solid boundaries. These are easily obtained from the original first-order system. One may, for instance, require that at one wall u be such that equation (18) is satisfied and at the opposite wall u be such that equation (19) is satisfied.

This then completes the specification of all necessary boundary conditions and furthermore does so in a manner designed on heuristic grounds, as discussed previously, to restrict the admissable solutions to our second-order system to be identical with those of the original first-order transonic small-disturbance equations.

Having discussed the formulation of the surrogate second-order system for the transonic small-disturbance equations, we now proceed to define the physical problem that will be used as a vehicle for testing the efficacy of the surrogate equation technique for inviscid transonic flow computation.

Problem Specification

Consider an inviscid flow in a two-dimensional channel with a uniform inlet velocity U_∞ and inlet Mach number M_∞. The upper surface of a bicircular arc airfoil is mounted on the lower channel wall. The channel height is one airfoil chord length, and the upstream and downstream flow boundaries are located one chord length upstream of the airfoil leading edge and one chord length downstream of the airfoil trailing edge, respectively. The airfoil half-thickness is equal to 10 percent of its chord length. Alternatively, this problem can be viewed as representing the flow past a 20-percent-thick bicircular-arc airfoil mounted at zero angle of attack on the centerline of a two-dimensional wind tunnel or as an unstaggered linear cascade with a gap-to-chord ratio of 2. The problem is schematically depicted in

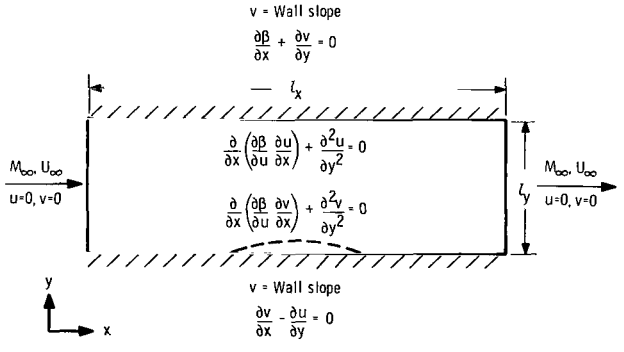


Figure 17. - Transonic flow problem.

figure 17, where the lengths l_x and l_y assume the values 3 and 1, respectively.

We will examine cases where the Mach number M_∞ is low enough so that, although local regions of supersonic flow may be present, the passage is not choked. Hence, the flow will be globally subsonic and in particular subsonic conditions will prevail at both the upstream and the downstream boundaries of the domain.

Discrete Formulation

Second-order systems like equations (25) and (26) can be quite easily discretized, in the spirit of Murman and Cole, for transonic flow computations. One such possible discretization is presented here.

We seek an approximate solution at a finite number of points distributed over the closed rectangular computational domain D, where

$$D=\{(x,y)\,|x_0 \le x \le x_0+l_x,\ y_0 \le y \le y_0+l_y\}$$

We call these points the grid (or mesh) points and define them to be the ordered pairs (x_i, y_j) such that

$$x_i=(x_0-\delta x)+i\delta x \qquad \text{where } i=1,\ldots,M$$

$$y_j=(y_0-\delta y)+j\delta y \qquad \text{where } j=1,\ldots,N$$

where δx and δy are the mesh spacings in the x and y directions, respectively. The value of some dependent variable a at the point (x_i, y_j) is denoted as $a_{i,j}$, that is,

$$a_{i,j} \equiv a(x_i,y_j)$$

Furthermore no notational distinction is made between the solution to the partial differential equations (25) and (26) and the approximate solution to the finite difference equations now to be constructed.

In regions of subsonic flow, where equations (25) and (26) exhibit elliptic behavior, second-order accurate central differencing can be employed in both the x and y directions to yield as finite difference representations

$$[r^2]a_{i,j-1}-\left\{2r^2+\left[\left(\frac{\partial\beta}{\partial u}\right)_{i+\frac{1}{2},j}+\left(\frac{\partial\beta}{\partial u}\right)_{i-\frac{1}{2},j}\right]\right\}a_{i,j}$$
$$+[r^2]a_{i,j+1}=-\left[\left(\frac{\partial\beta}{\partial u}\right)_{i+\frac{1}{2},j}\right]a_{i+1,j}$$
$$-\left[\left(\frac{\partial\beta}{\partial u}\right)_{i-\frac{1}{2},j}\right]a_{i-1,j} \qquad (27)$$

Here the mesh ratio r is defined such that $r\equiv\delta x/\delta y$, and a represents either u or v. Furthermore, to second-order accuracy, we may write

$$\left(\frac{\partial\beta}{\partial u}\right)_{i+\frac{1}{2},j}=\frac{1}{2}\left[\left(\frac{\partial\beta}{\partial u}\right)_{i+1,j}+\left(\frac{\partial\beta}{\partial u}\right)_{i,j}\right]$$

and

$$\left(\frac{\partial\beta}{\partial u}\right)_{i-\frac{1}{2},j}=\frac{1}{2}\left[\left(\frac{\partial\beta}{\partial u}\right)_{i,j}+\left(\frac{\partial\beta}{\partial u}\right)_{i-1,j}\right]$$

while $\partial\beta/\partial u$ can be expressed as

$$\frac{\partial\beta}{\partial u}=1-M_\infty^2-(\gamma+1)M_\infty^2 u$$

On the other hand, in regions of supersonic flow, where the governing equations exhibit hyperbolic behavior, we proceed differently. Although the second-order accurate differencing in the y direction is retained, in the x direction we switch to backward (or upwind) differencing, which is first-order accurate. Higher-order accurate differencing could be used; however, first-order accuracy may be preferable and is in any case sufficient for our purposes. This asymmetric differencing is designed to respect the domain of dependence of the partial differential equations in the hyperbolic region while yielding stable finite difference equations. In this fashion we obtain

$$[r^2]a_{i,j-1}-\left[2r^2-\left(\frac{\partial\beta}{\partial u}\right)_{i,j}\right]a_{i,j}+[r^2]a_{i,j+1}=$$
$$\left[\left(\frac{\partial\beta}{\partial u}\right)_{i,j}+\left(\frac{\partial\beta}{\partial u}\right)_{i-1,j}\right]a_{i-1,j}-\left[\left(\frac{\partial\beta}{\partial u}\right)_{i-1,j}\right]a_{i-2,j} \qquad (28)$$

Equations (27) and (28) determine the basic nature of the discrete approximation to the field equations.

As is now well-known (refs. 40 and 43), in transonic flow computations based on inviscid model equations it is not sufficient, in order to insure that the proper weak solution be captured, that the partial differential equations be written in conservation form and then differenced in a type-dependent fashion. One must, at points of transition between regions where either one or the other form of the difference equations is used, introduce special switching approximations. The function of these equations is to insure that the intercellular flux balances are not disturbed by the change from one form of difference operator to another. This in turn is intended to assure that the weak solution that is captured by the numerical procedure will depend only on the original partial differential equations and boundary conditions and not on local perturbations caused by flux imbalances.

The form of the switching equations required depends on the discretizations used in subsonic and supersonic regions and on whether the transition is from subsonic to supersonic flow, which is referred to as a sonic point, or from supersonic to subsonic flow, which is referred to as a shock point. As the names imply, sonic points occur at smooth transitions, while shock points occur at abrupt ones.

For the case at hand special attention need only be given to the treatment of x derivatives, since no change is made in the discretization in the y direction. Hence for the interior discretization we consider a term of the form

$$\frac{\partial}{\partial x}\left(\frac{\partial \beta}{\partial u}\frac{\partial a}{\partial x}\right)$$

where a represents either the u or the v perturbation velocity component. It is a straightforward matter to determine that intercellular fluxes will be conserved if we discretize this term such that at sonic points

$$\left[\frac{\partial}{\partial x}\left(\frac{\partial \beta}{\partial u}\frac{\partial a}{\partial x}\right)\right]_{i,j} = \left[\left(\frac{\partial \beta}{\partial u}\right)_{i,j} - \left(\frac{\partial \beta}{\partial u}\right)_{i-1,j}\right]\frac{a_{i,j}-a_{i-1,j}}{2\delta x^2}$$

and at shock points

$$\left[\frac{\partial}{\partial x}\left(\frac{\partial \beta}{\partial u}\frac{\partial a}{\partial x}\right)\right]_{i,j} = \left\{\frac{1}{2}\left[\left(\frac{\partial \beta}{\partial u}\right)_{i+1,j} + \left(\frac{\partial \beta}{\partial u}\right)_{i,j}\right] \times (a_{i+1,j}-a_{i,j}) - \left[\left(\frac{\partial \beta}{\partial u}\right)_{i-1,j}\right] \times (a_{i-1,j}-a_{i-2,j})\right\} \Big/ \delta x^2$$

For the boundary condition discretization we must construct sonic and shock point representations for terms of the form $\partial a/\partial x$ where a represents either v or β. Proceeding as in the previous case, we can readily determine that the required representations are such that at sonic points

$$\left(\frac{\partial a}{\partial x}\right)_{i,j} = \frac{a_{i,j}-a_{i-1,j}}{2\delta x}$$

and at shock points

$$\left(\frac{\partial a}{\partial x}\right)_{i,j} = \frac{a_{i+1,j}+a_{i,j}-2a_{i-1,j}}{2\delta x}$$

To decide in a consistent fashion which of these approximations is to be used at a given point (x_i, y_j) in the computational domain, we employ the following switching scheme:

If				Then
$\left(\frac{\partial \beta}{\partial u}\right)_{i-1,j} \geq 0$	and	$\left(\frac{\partial \beta}{\partial u}\right)_{i,j} \geq 0$		Subsonic
$\left(\frac{\partial \beta}{\partial u}\right)_{i-1,j} \geq 0$	and	$\left(\frac{\partial \beta}{\partial u}\right)_{i,j} < 0$		Sonic
$\left(\frac{\partial \beta}{\partial u}\right)_{i-1,j} < 0$	and	$\left(\frac{\partial \beta}{\partial u}\right)_{i,j} < 0$		Supersonic
$\left(\frac{\partial \beta}{\partial u}\right)_{i-1,j} < 0$	and	$\left(\frac{\partial \beta}{\partial u}\right)_{i,j} \geq 0$		Shock

Here $\partial\beta/\partial u$ changes sign from positive to negative as u exceeds u^*, the critical disturbance velocity.

Both in the present case and in the case of the perturbation-potential equation, the switching approximations that are obtained may not be locally consistent with the differential equations that they model. However, as we seek a weak solution to the conservation form of the equations, it is precisely

these switching approximations that allow the finite difference equations to be consistent in a global or integral sense.

Solution Procedure

As in most of the previously considered examples the finite difference equations are solved by the method of successive line relaxation. We sweep repeatedly across the domain, moving in the flow direction, relaxing the solution on transverse (i = constant) lines in order to accelerate the convergence of the iterations. The boundary conditions are fixed during any given sweep of the computational domain and are then recomputed after the completion of each sweep.

The finite difference equations for the totality of the interior points along a given transverse mesh line can then be written in the form of a matrix equation:

$$M_i^n \Delta F_i^{n+1} = R_i$$

where M is a block-tridiagonal matrix consisting of 2×2 matrices,

$$\Delta F_i^{n+1} = \left[\Delta f_{i,2}^{n+1}, \Delta f_{i,3}^{n+1}, \ldots, \Delta f_{i,N-2}^{n+1}, \Delta f_{i,N-1}^{n+1} \right]^T$$

$$\Delta f_{i,j}^{n+1} \equiv f_{i,j}^{n+1} - f_{i,j}^n$$

$$f_{i,j} \equiv \begin{bmatrix} u_{i,j} \\ v_{i,j} \end{bmatrix}$$

and R_i is the resulting right-hand-side vector. In general a block-tridiagonal matrix elimination procedure would be used to solve such an equation. However, in the present case the equations for u and v are coupled only through their right-hand sides, and the matrix elements of the block-tridiagonal matrix M are therefore diagonal matrices. Hence we need only solve two simple tridiagonal equations: a reduction in the requisite computational effort. We accelerate the convergence of the iteration procedure such that, if we refer to the result of the tridiagonal elimination procedure as $\tilde{F}_i^{n+1}$, the relaxation process takes the form

$$F_i^{n+1} = \omega \tilde{F}_i^{n+1} + (1 - \omega) F_i^n$$

To compare the results obtained by using the surrogate-equation technique with those of the conventional perturbation-potential approach, two FORTRAN IV computer programs have been written for use on the NASA Lewis Univac 1100/40 computer system. Both have been written in the same spirit, and an equal degree of effort has been expended on making each program computationally efficient.

Computational Results

To generate information that can serve as a basis for evaluating the appropriateness and efficacy of the surrogate-equation approach to the numerical solution of transonic flow problems, two series of computations have been performed. For the same physical problem, computational domain, initial guess, and grid system, computations have been carried out for both subcritical and supercritical flow conditions with both the surrogate-equation program and the perturbation-potential program. Both because of the stringent convergence criterion employed here and for reasons of general prudence, all computations were done using double-precision arithmetic.

Four grid systems were employed during this study: extra coarse, coarse, medium, and fine. Essentially, each successive grid is constructed by halving the grid spacing of its predecessor. More detailed information on these grid systems is contained in table I. Note that we have sacrificed efficiency to simplicity by maintaining a uniform grid spacing everywhere rather than stretching the streamwise grid spacing upstream and downstream of the wall-mounted airfoil.

We consider one subcritical and one supercritical flow case, with upstream Mach numbers of 0.65 and 0.70, respectively. These two flow cases are described in table II. Note that the supercritical case does not represent a choked flow. A preliminary survey, using both the surrogate-equation program and the perturbation-potential program, indicated that choking occurs between the upstream Mach numbers 0.71 and 0.72. This is in reasonable agreement with the choking Mach number of 0.73 that is obtained by using an approximate relation derived by Spreiter, Smith, and Hyett (ref. 45) together with a constant for circular-arc profiles that has been estimated by Collins and Krupp (ref. 46) on the basis of experiments.

Before proceeding to the more interesting supercritical flow results, let us consider the

TABLE I.—DEFINITION OF MESHES FOR TRANSONIC FLOW COMPUTATIONS

Mesh designation	Number of points in x direction	Number of points in y direction	Total number of points	Mesh spacing in x direction, δ_x	Mesh spacing in y direction, δ_y
Extra coarse	30	6	180	0.1	0.2
Coarse	60	11	660	.05	.1
Medium	120	21	2520	.025	.05
Fine	240	41	9840	.0125	.025

TABLE II.—NOMENCLATURE FOR TRANSONIC FLOW TEST CASES

Flow designation	Upstream Mach number	Critical u disturbance velocity	Critical pressure coefficient
Subcritical flow	0.65	0.569527	−1.139053
Supercritical flow	.70	.433673	−.867347

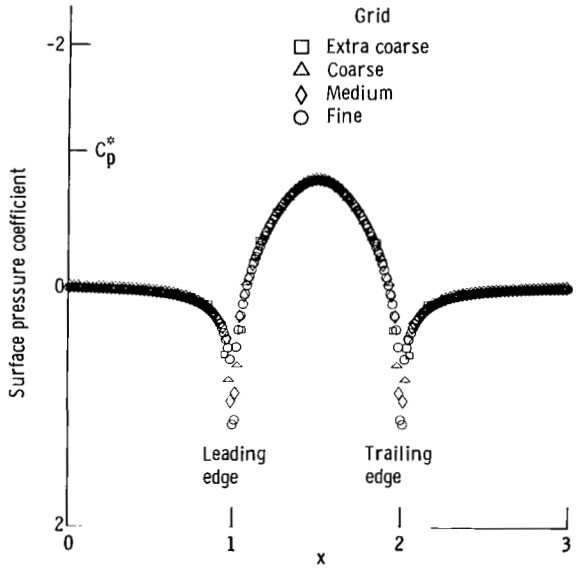


Figure 18. - Surface pressure coefficient distribution - surrogate-equation formulation, subcritical flow case.

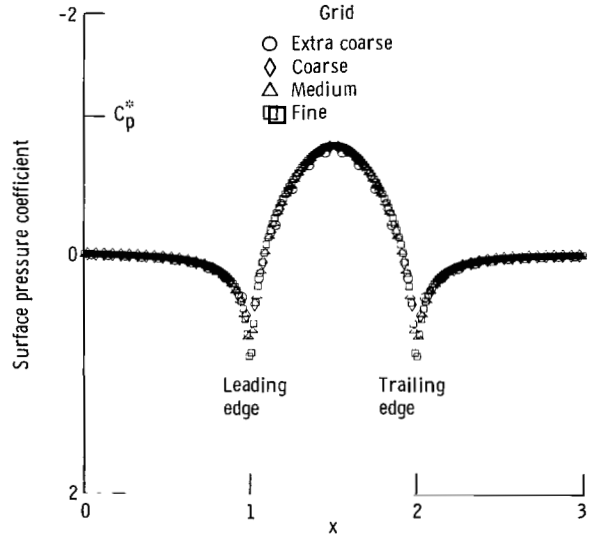


Figure 19. - Surface pressure coefficient distribution - perturbation-potential formulation, subcritical flow case.

subcritical flow solutions generated by the two programs. Figure 18 presents the pressure coefficient distributions on the streamline adjacent to the airfoil surface that result from computations on the extra-coarse, coarse, medium, and fine grids when using the surrogate-equation program. A similar sequence of results generated by the perturbation potential program is presented in figure 19. In each case the solution appears to converge to an asymptote as the grid is refined. Furthermore the expected fore-and-aft symmetry is present in both cases and stagnation points are well represented. Figure 20 shows the fine-grid pressure coefficient distributions for both the perturbation-potential and the surrogate-equation formulations. The results agree well with one another, an indication that both methods tend to the same asymptotic solution.

We now examine the supercritical flow case where a shock is present. In a fashion analogous to that of the subcritical flow case figures 21 and 22 present the

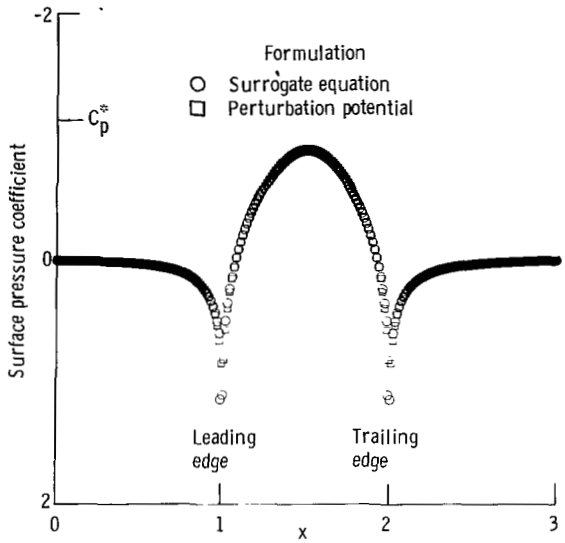


Figure 20. - Comparison of surface pressure coefficient distributions - subcritical flow case, fine grid.

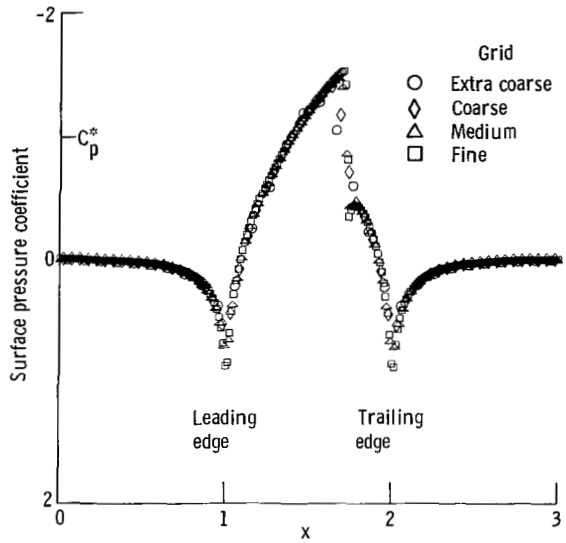


Figure 22. - Surface pressure coefficient distribution - perturbation-potential formulation, supercritical flow case.

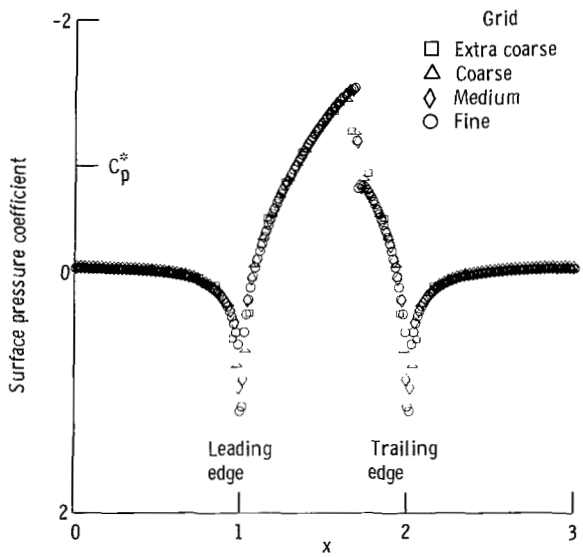


Figure 21. - Surface pressure coefficient distribution - surrogate-equation formulation, supercritical flow case.

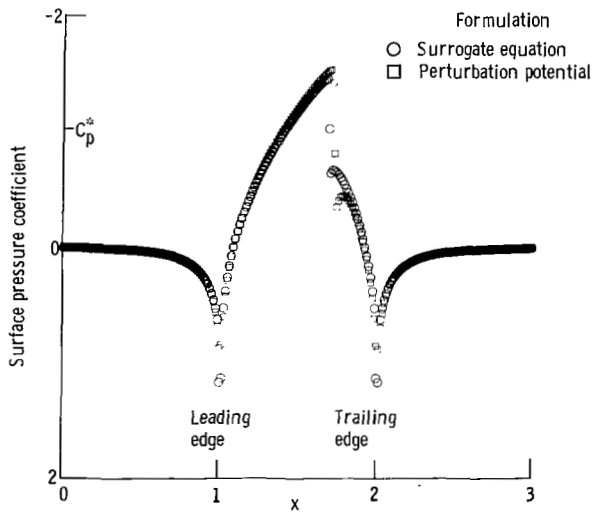


Figure 23. - Comparison of surface pressure coefficient distributions - supercritical flow case, fine grid.

pressure coefficient distributions on the streamline adjacent to the airfoil surface that result from computations on the aforementioned four-grid sequence obtained by using the surrogate-equation program and the perturbation-potential program, respectively. Here again, each solution appears to converge to an asymptote as the grid is refined.

The fine-grid pressure coefficient distributions obtained by means of the two approaches are compared in figure 23. The results agree well with one another. The shock location is reasonably well predicted and the shock strengths, although not identical, are in fair accord. There are several possible explanations for the observed underprediction of shock strength and slight difference in shock location obtained with the surrogate-equation algorithm. For example, the specific form chosen for the discretization at shock points can influence the behavior of the solution near the shock. In particular, although the shock point discretization is chosen such that a fully conservative difference scheme results, it is not consistent with the

governing partial differential equations to the same order as the subsonic and supersonic discretizations. This consistency problem also manifests itself at sonic points, but because of the smooth nature of the solution there it causes no extraordinary difficulties. Other factors contributing to the discrepancy in shock strengths may include differences in the boundary condition formulation and implementation in the two programs, differing dissipative properties due to different truncation errors creating different artificial viscosities, and the fact that the lack of homogeneity of β causes equation (21) to be not equivalent to a conservation form of the transonic small-disturbance equations. The prediction of shock strengths by means of the surrogate-equation algorithm is presently the object of further study and analysis.

It is interesting that post-shock reexpansion, first reported in the experiments of Ackeret, Feldmann, and Rott (ref. 47) and later discussed from a theoretical point of view by Zierep (ref. 48) and Oswatitsch and Zierep (ref. 49), is present in both solutions.

Hence, it appears that, when measured against the standard perturbation-potential approach, the surrogate-equation technique yields a viable method for obtaining numerical solutions to transonic flow problems. Recall that the main attraction of the surrogate-equation technique rests not on its value in providing an alternative method for the solution of the transonic small-disturbance equations but rather on its applicability to the solution of the full steady Euler equations. Nevertheless, a consideration of the computational efficiency of the perturbation-potential and surrogate-equation programs yields some interesting results.

In the absence of evidence to the contrary, one might well expect the perturbation-potential formulation to be superior to the surrogate-equation formulation when considered from the viewpoint of computational efficiency. Such an expectation would, no doubt, be based on the observation that less work is required to solve the one equation of the perturbation-potential formulation than is needed to solve the two equations that result from an application of the surrogate-equation technique. This is in fact the case. We have determined by computational experimentation that, in their present form, the perturbation-potential and surrogate-equation programs require 0.000175 and 0.000256 CAU second per iteration per grid point, respectively. Hence by this measure the surrogate-equation program requires approximately 1.46 times as much CAU time per iteration per grid point as the perturbation-potential program does.

To determine the computational effort each program requires to produce a converged solution, we must also take into account the number of iterations needed to reduce the residuals to the level at which we declare the solution to have been reached. To this end, a study involving both the subcritical and supercritical flow cases, all four computational grids, and both programs has been conducted. The results are shown in table III, where for each case the number of iterations required under optimum relaxation to reduce the absolute value of the maximum residual to 10^{-10} is indicated. This convergence criterion is rather severe, and a much milder one would suffice for engineering applications. The use of such a severe convergence criterion here can be viewed as a further test of the algorithms.

We note immediately from table III that the perturbation-potential program required, for the cases shown, on the average 4.36 times as many iterations in subcritical flow cases and 5.52 times as many iterations in supercritical flow cases as the surrogate-equation program. These results are illustrated graphically in figures 24 and 25 for the subcritical and supercritical flow cases, respectively. This marked superiority of the surrogate-equation program can be attributed to the fact that most of the

TABLE III.—CONVERGENCE BEHAVIOR OF TRANSONIC FLOW ALGORITHMS

	Number of iterations required for convergence							
	Subcritical flow				Supercritical flow			
	Extra-coarse mesh	Coarse mesh	Medium mesh	Fine mesh	Extra-coarse mesh	Coarse mesh	Medium mesh	Fine mesh
Perturbation-potential formulation	261	669	1879	6548	423	1098	3774	12 910
Surrogate-equation formulation	104	209	457	858	150	312	639	1 313

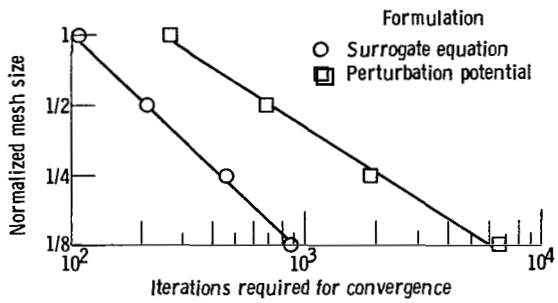


Figure 24. - Convergence behavior of transonic flow algorithms - subcritical flow case.

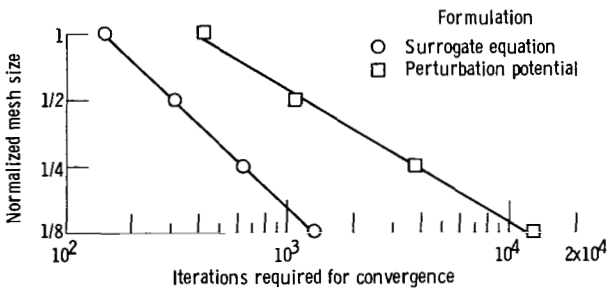


Figure 25. - Convergence behavior of transonic flow algorithms - supercritical flow case.

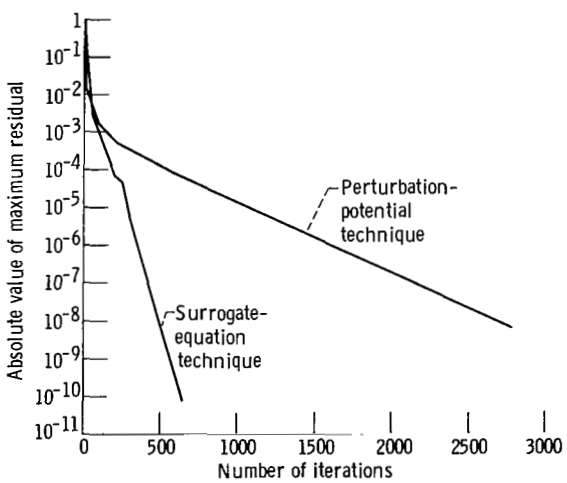


Figure 26. - Convergence rate comparison.

boundary conditions employed in this formulation are Dirichlet ones while all of those in the perturbation-potential program are Neumann ones. It further appears that the convergence of the surrogate-equation formulation is less affected by the presence of shocks than is that of the perturbation-potential formulation.

In figure 26 the behavior of the absolute value of the maximum residual is shown as a function of iteration level for both the perturbation-potential and surrogate-equation programs. For this comparison, we have chosen the supercritical flow case on the medium grid. From the data we estimate the spectral radii for the iteration matrices of the perturbation-potential and surrogate-equation programs to be 0.996 and 0.968, respectively. These in turn imply asymptotic convergence rates of 1.74×10^{-3} and 1.41×10^{-2}. Hence we estimate that the surrogate-equation program converges 8.1 times as fast as the perturbation-potential program.

By taking account of both the CAU time per iteration per grid point and the number of iterations necessary for convergence, we estimate that, for the cases considered here, the computational effort required by the surrogate-equation program varies from 0.15 to 0.58 of that required by the perturbation-potential program. Thus, even in this test case, chosen for simplicity and ease of comparison rather than to illustrate the power of the surrogate-equation technique, its application to the transonic small-disturbance problem results in a more efficient algorithm than is obtained by means of the conventional perturbation-potential formulation.

As we have no exact solution with which to compare the results, we can here only estimate the order of accuracy of the programs based on the computational results obtained on the sequence of four progressively finer grids described in table I. We do this by assuming the solution obtained on the fine grid to be exact and then calculating the absolute

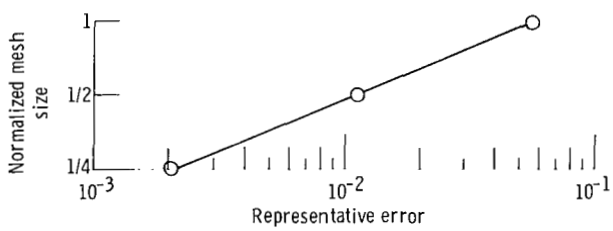


Figure 27. - Order of accuracy estimation - surrogate-equation formulation, subcritical flow case.

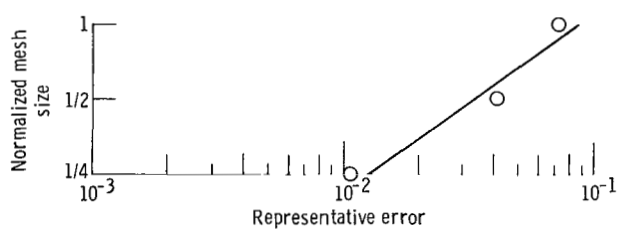


Figure 28. - Order-of-accuracy estimation - perturbation-potential formulation, subcritical flow case.

value of the difference between this solution and the solution on the various other grids at some representative location. This quantity is called the representative error and is plotted as a function of normalized mesh size for several cases in figures 27 through 30.

Comparison of figures 27 and 28 for subcritical flow shows that the surrogate-equation algorithm appears to be second-order accurate in the normalized mesh size but the perturbation-potential algorithm approaches first-order accuracy. This can be attributed to the fact that in the perturbation-potential formulation the flow variables from which the measure of error has been calculated must be obtained from the computed perturbation potential by differentiation.

Figures 29 and 30 illustrate the results obtained from applying this order-of-accuracy estimation to the supercritical flow test case. The surrogate-equation algorithm retains its second-order accuracy in the subsonic region while, here again, the perturbation-potential algorithm approaches first-order accuracy. We have also attempted to estimate the order of accuracy of the two programs at a point within the supersonic region. Because of the limited size of this region and the large difference in resolving power between the coarsest and finest meshes, this estimation proved very difficult to carry out and its results should be considered to be quite approximate. These remarks notwithstanding, the results of the order-of-accuracy estimations in the supersonic region are also shown in figures 29 and 30. For each algorithm the order of accuracy in the supersonic region is lower than that in the subsonic region. This is to be expected since the differencing used in the x direction at supersonic points is only of first-order accuracy.

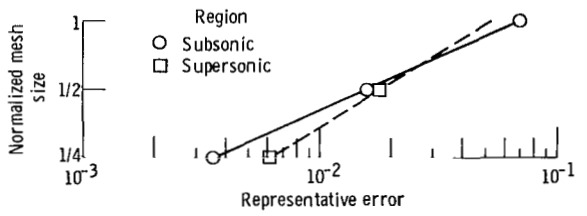


Figure 29. - Order-of-accuracy estimation - surrogate-equation formulation, supercritical flow case.

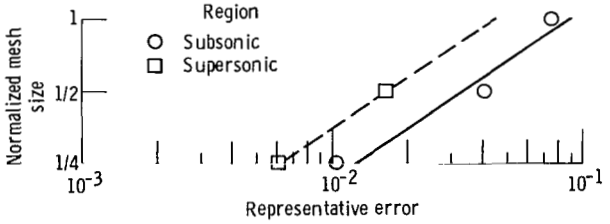


Figure 30. - Order-of-accuracy estimation - perturbation-potential formulation, supercritical flow case.

Murman (ref. 43) has stated that based on a linear stability analysis the Murman and Cole approach to solving the perturbation-potential equation, a version of which has been presented here, results in unconditionally stable finite difference equations. A similar result should hold for the surrogate-equation algorithm. Practical confirmation of the unconditional stability of the surrogate-equation algorithm was provided by the computations conducted during the course of the work reported herein.

Application to the Full Euler Equations

Having established that the surrogate equation technique can be successfully applied to the transonic small-disturbance equations, we now make use of the technique to develop a surrogate second-order system for the full, steady, two-dimensional Euler equations that will be valid for use in transonic flow computations. The resulting second-order system will be in conservation form, and fully conservative finite difference equations can be constructed for it in a fashion analogous to that previously discussed in the context of the small-disturbance equations.

The two-dimensional, steady Euler equations are normally written in conservation form as

$$f_x + g_y = 0 \tag{9}$$

where as noted previously

$$f = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (E+p)u \end{bmatrix}$$

$$g = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (E+p)v \end{bmatrix} \tag{10}$$

$$E = \rho\left[e + \frac{1}{2}(u^2 + v^2)\right]$$

and

$$e = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

Furthermore as discussed previously f and g are both homogeneous functions of first degree in the components of the vector w, where w is defined as $w=[\rho,\ \rho u,\ \rho v,\ E]^T$. It then follows from Euler's theorem on homogeneous functions (ref. 27) that $f=Aw$ and $g=Bw$, where A and B are the Jacobian matrices shown explicitly in appendix A. Hence we can reexpress the Euler equations in conservation form as

$$(Aw)_x+(Bw)_y=0$$

or in operator notation as

$$\left[\frac{\partial}{\partial x}(A)+\frac{\partial}{\partial y}(B)\right]w=0 \tag{29}$$

We now construct a surrogate second-order system by applying the differential operator

$$\frac{\partial}{\partial x}(M)-\frac{\partial}{\partial y}(N)$$

to equation (29) to obtain

$$\left\{\frac{\partial}{\partial x}\left[M\frac{\partial}{\partial x}(A)\right]+\frac{\partial}{\partial x}\left[M\frac{\partial}{\partial y}(B)\right]\right.$$

$$\left.-\frac{\partial}{\partial y}\left[N\frac{\partial}{\partial x}(A)\right]-\frac{\partial}{\partial y}\left[N\frac{\partial}{\partial y}(B)\right]\right\}w=0 \tag{30}$$

This is a general surrogate second-order system for equation (29). Its properties depend of course on the specific choices made for the 4×4 matrices M and N.

In particular, if we choose $M=NAB^{-1}$ and N as some nonsingular constant matrix, such as $N=I$, the identity matrix, then equation (30) reduces to the particularly simple form

$$\left\{\frac{\partial}{\partial x}\left[M\frac{\partial}{\partial x}(A)\right]-\frac{\partial}{\partial y}\left[N\frac{\partial}{\partial y}(B)\right]\right\}w=0$$

or more precisely

$$\left\{\frac{\partial}{\partial x}\left[AB^{-1}\frac{\partial}{\partial x}(A)\right]-\frac{\partial^2}{\partial y^2}(B)\right\}w=0 \tag{31}$$

Here again, as in the examples previously discussed, the set of characteristic directions of equation (31) contains those of the first-order system, equation (29), as a subset, with the additional characteristic directions being the reflections of the original ones through the x axis.

In the preceding formulation the inverse of the B matrix appears. Hence such a formulation would be inappropriate at points where the B matrix is singular. Such points occur where the v velocity component either vanishes or becomes sonic. We can avoid this undesirable behavior by making different choices for the matrices M and N in equation (30). If, for example, we choose $M=A$ and $N=B$, we obtain

$$\left\{\frac{\partial}{\partial x}\left[A\frac{\partial}{\partial x}(A)\right]+\frac{\partial}{\partial x}\left[A\frac{\partial}{\partial y}(B)\right]\right.$$

$$\left.-\frac{\partial}{\partial y}\left[B\frac{\partial}{\partial x}(A)\right]-\frac{\partial}{\partial y}\left[B\frac{\partial}{\partial y}(B)\right]\right\}w=0 \tag{32}$$

Although the cross-derivative terms do not cancel in equation (32), it does constitute a usable surrogate second-order system for equation (29) and the properties of the characteristic directions discussed previously are of course retained. Furthermore the formulation presented in equation (32) is uniformly valid for subsonic, transonic, and supersonic flow conditions.

Summation

We have demonstrated the capability of the surrogate-equation technique to generate second-order partial differential systems suitable for use in transonic flow computations. The fully conservative algorithm developed here to solve the surrogate second-order system for the transonic small-disturbance equations appears to be superior, when measured in terms of computational efficiency, to the standard Murman and Cole approach to solving the perturbation-potential equation.

The surrogate-equation technique shows promise toward providing a method for iteratively solving the full steady Euler equations in the transonic flow regime. Indeed, the results obtained thus far indicate that further research and algorithm development aimed at producing such a method are merited, and such work is presently in progress.

Finally, although the present discussion has been limited to two dimensions, the eventual extension of the surrogate-equation technique to three dimensions appears to be entirely feasible.

Summary of Results

We have been concerned during the course of this research with the development of a method to obtain approximate numerical solutions to fairly general classes of inviscid flow problems. We have chosen, on the one hand, to avoid the usual potential and stream function approaches with their inherent limitations and, on the other hand, to try to devise a method capable of greater computational efficiency than the time-asymptotic solution of the unsteady Euler equations.

The search for such a method has led to what we refer to as the surrogate-equation technique. This technique provides a means for obtaining, from a given first-order partial differential system, a replacement second-order system whose solution set contains the sought-after solution to the first-order system. The possibility of other, undesirable, solutions being admitted is eliminated by using the original first-order system to supply the additional boundary conditions required by the surrogate second-order system. Given the surrogate second-order system, it is a relatively straightforward matter to construct for it stable, fully conservative, type-dependent finite difference equations that can then be solved by using, for example, a successive-line-relaxation iterative procedure.

The essential aspects of the surrogate-equation technique were illustrated earlier, where it was applied to the Cauchy-Riemann equations. There numerical solutions of the surrogate second-order system were compared with those of the standard velocity potential formulation.

Problems of greater complexity, Euler problems, were chosen so as to more thoroughly display the utility and generality of the surrogate-equation technique. The full, steady Euler equations were used to model the rotational, inviscid subcritical flow through a two-dimensional bend. The relaxation solutions of the surrogate second-order system were in this case contrasted with analytic, incompressible flow solutions obtained by Shercliff.

The question of the applicability of the surrogate-equation technique to transonic flow computations was addressed next. A surrogate second-order system was constructed for the transonic small-disturbance equations. A solution algorithm based on the surrogate-equation formulation was then compared with the standard Murman and Cole type of algorithm for the perturbation-potential equation. The surrogate-equation algorithm produced good results and surpassed the computational efficiency of the perturbation-potential algorithm by several multiples. An indication was also given as to how the transonic results might be extended to the full, steady Euler equations.

The main conclusions are as follows:

1. It is possible to obtain an approximate numerical solution to a system of first-order partial differential equations by solving a problem consisting of a surrogate second-order system together with the original boundary conditions and supplementary relations obtained from the first-order system.

2. The surrogate-equation technique can be applied successfully to the solution of inviscid flow problems across the entire spectrum of subsonic, transonic, and supersonic conditions.

3. The surrogate-equation technique provides a means for formulating problems involving the first-order equations describing inviscid flow in such a way as to allow the use of fully conservative, type-dependent differencing and iterative solution procedures.

4. The surrogate-equation technique provides a means for solving inviscid flow problems without resort to assuming the existence of either a velocity potential function or a stream function. Hence the surrogate-equation approach is more general than either of these other approaches and does not share their limitations. In particular, the surrogate-equation technique can be employed to obtain solutions to flow problems where any combination of rotationality, transonic conditions, or three dimensionality is present.

5. An application of the surrogate-equation technique to the transonic small-disturbance equations results in an algorithm that, on the basis of the computational experimentation reported herein, appears to have a computational efficiency several times greater than that of the standard perturbation-potential algorithm.

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, November 19, 1980

Appendix A

Jacobian Matrices and Some of Their Properties

Recall that if the vector of unknowns w is defined such that $w=\{\rho,\rho u,\rho v, E\}^T$, the Jacobian matrices A and B are defined as $A=\partial f/\partial w$, where f and g are as given in equation (10) of the main text. The matrices A and B can be written explicitly as

$$A=-\begin{bmatrix} 0 & -1 & 0 & 0 \\ \frac{3-\gamma}{2}u^2+\frac{1-\gamma}{2}v^2 & (\gamma-3)u & (\gamma-1)u & 1-\gamma \\ uv & -v & -u & 0 \\ \frac{\gamma Eu}{\rho}+(1-\gamma)u(u^2+v^2) & -\frac{\gamma E}{\rho}+\frac{\gamma-1}{2}(3u^2+v^2) & (\gamma-1)uv & -\gamma u \end{bmatrix}$$

$$B=-\begin{bmatrix} 0 & 0 & -1 & 0 \\ uv & -v & -u & 0 \\ \frac{3-\gamma}{2}v^2+\frac{1-\gamma}{2}u^2 & (\gamma-1)u & (\gamma-3)v & 1-\gamma \\ \frac{\gamma Ev}{\rho}+(1-\gamma)v(u^2+v^2) & (\gamma-1)uv & \frac{-\gamma E}{\rho}+\frac{\gamma-1}{2}(3v^2+u^2) & -\gamma v \end{bmatrix}$$

To examine some of the properties of A and B, we first apply the similarity transformations S and S_1, as given by Turkel (ref. 50; see also refs. 51 through 53) to create the matrices A_1 and B_1, which are similar to A and B, respectively.

$$S = \begin{bmatrix} \frac{\gamma-1}{2\rho c}(u^2+v^2) & -\frac{(\gamma-1)u}{\rho c} & -\frac{(\gamma-1)v}{\rho c} & \frac{\gamma-1}{\rho c} \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 & 0 \\ -\frac{v}{\rho} & 0 & \frac{1}{\rho} & 0 \\ \frac{-c^2}{p}+\frac{\gamma-1}{2p}(u^2+v^2) & -\frac{(\gamma-1)u}{p} & -\frac{(\gamma-1)v}{p} & \frac{\gamma-1}{p} \end{bmatrix}$$

$$S_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1 = S_1 S A S^{-1} S_1^{-1}$$

$$B_1 = S_1 S B S^{-1} S_1^{-1}$$

These transformations yield

$$A_1 = \begin{bmatrix} u+c & 0 & 0 & 0 \\ 0 & u-c & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{bmatrix} \qquad B_1 = \begin{bmatrix} v & 0 & \frac{\sqrt{2}}{2}c & 0 \\ 0 & v & -\frac{\sqrt{2}}{2}c & 0 \\ \frac{\sqrt{2}}{2}c & -\frac{\sqrt{2}}{2}c & v & 0 \\ 0 & 0 & 0 & v \end{bmatrix}$$

Hence it is clear that A is singular for $u = \pm c$ and $u = 0$ and B is singular for $v = \pm c$ and $v = 0$. Also the eigenvalues of BA^{-1} are clearly

$$\lambda_{1,2} = \frac{v}{u}$$

$$\lambda_{3,4} = \frac{uv \pm c\sqrt{u^2 + v^2 - c^2}}{u^2 - c^2}$$

Appendix B

Simplified Expression of Euler Equations

Given the Euler equations in the form $f_x+(TF)_y=0$, where $T\equiv BA^{-1}$, we can carry out the indicated differentiation to arrive at

$$f_x+Tf_y+T_yf=0$$

Since $g=Tf$, we can write $(g-Tf)_y=0$, which can then be reexpressed as

$$g_y-Tf_y-T_yf=0$$

But it follows immediately that $g_y-Tf_y=0$ and hence that $T_yf=0$. Thus the Euler equations can be written in simplified form as $f_x+Tf_y=0$, which appears as equation (11) in the main text.

Similarly $T_xf=0$ and consequently

$$T_xf_y=T_yf_x$$

This latter observation allows reexpression of the surrogate second-order system for the Euler equations

$$\left[\frac{\partial^2}{\partial x^2}-T^2\frac{\partial^2}{\partial y^2}-(TT_y-T_x)\frac{\partial}{\partial y}\right]f=0$$

as

$$\left[\frac{\partial^2}{\partial x^2}-T^2\frac{\partial^2}{\partial y^2}-(TT_y+T_yT)\frac{\partial}{\partial y}\right]f=0$$

This equation can then be rewritten in the form

$$\left[\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial y}\left(T^2\frac{\partial}{\partial y}\right)\right]f=0$$

which appears as equation (14) in the main text.

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16. Abstract

A novel numerical procedure for the iterative solution of inviscid flow problems is described, and its utility for the calculation of steady subsonic and transonic flow fields is demonstrated. Application of the surrogate-equation technique defined herein allows the formulation of stable, fully conservative, type-dependent finite difference equations for use in obtaining numerical solutions to systems of first-order partial differential equations, such as the steady-state Euler equations. Steady, two-dimensional solutions to the Euler equations for both subsonic, rotational flow and supersonic flow and to the small-disturbance equations for transonic flow are presented.

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