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# On the Error Statistics of Viterbi Decoding and the Performance of Concatenated Codes

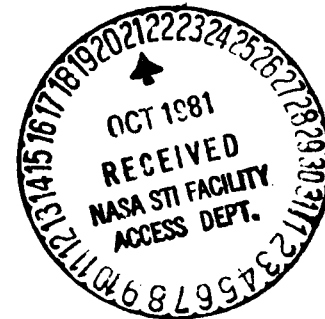
R. L. Miller  
L. J. Deutsch  
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## ABSTRACT

Computer simulation results are presented on the performance of convolutional codes of constraint lengths 7 and 10 concatenated with the (255, 223) Reed-Solomon code (a proposed NASA standard). These results indicate that as much as 0.8 dB can be gained by concatenating this Reed-Solomon code with a (10, 1/3) convolutional code, instead of the (7, 1/2) code currently used by the DSN. A mathematical model of Viterbi decoder burst-error statistics is developed and is validated through additional computer simulations.

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SECTION I  
INTRODUCTION

The purpose of this report is to present new results on the combined performance of short constraint length Viterbi-decoded convolutional codes and Reed-Solomon codes. When one coding scheme is superimposed upon another, the resulting combination is called a concatenated code. Those interested in learning about these coding schemes can find elementary presentations in Reference 1-1. Our interest is in their performance.

The DSN currently has both (7, 1/2) and (7, 1/3) Viterbi decoders. The performance of several convolutional codes of rates 1/2 and 1/3 with constraint lengths between 7 and 10 have been known for some time (References 1-2 and 1-3). At the time that the DSN Viterbi decoders were built, hardware speeds were not fast enough to build Viterbi decoders of constraint lengths beyond 7 that were sufficiently reliable and inexpensive. However, with current and expected technological advancements in mind, we have given another look at the possible performance of Viterbi decoders of constraint length 10 and rates 1/2 and 1/3.

This report not only extends previous Viterbi performance results, but also contains new performance results for convolutional codes concatenated with a (255, 223) Reed-Solomon code. (The performance of the DSN (7, 1/2) code concatenated with this Reed-Solomon code appears in Reference 1-4.) The Reed-Solomon bit-error probability depends not only on the (average) Viterbi bit-error rate, but also on the lengths of the Viterbi error bursts and the density of the errors within the bursts. Consequently, additional simulations are required to gather these statistics.

The Galileo Project and the International Solar Polar Mission are planning to employ a concatenated Reed-Solomon/Viterbi coding scheme for telemetering science and engineering data over the space communications channel. Even the Voyager mission has this capability on board. The reason for using a concatenated coding scheme over convolutional coding alone is that concatenated coding makes more efficient use of signal power to achieve bit-error probabilities in the  $10^{-5}$  range. Such low error rates are necessary to make data compression schemes workable. Data compression algorithms, while promising to remove substantial information redundancy, are very sensitive to transmission errors. In terms of cost vs benefits, a 0.8-dB (20%) improvement



from coding is comparable to the increased performance of a 64-m and 34-m array over a 64-m antenna alone. However, the cost of implementing such a coding scheme is not more than the cost of the electronics to array such a pair of DSN antennas.

A block diagram of a concatenated coding system is shown in Figure 1-1. Binary data generated on board the spacecraft are first encoded by the Reed-Solomon encoder. This encoder also interleaves the Reed-Solomon symbols so as to minimize the effect of error bursts on individual Reed-Solomon codewords. After this first level of coding, the data pass to the convolutional encoder. The modulators convert these binary data to a phase-modulated radio-frequency signal, which is amplified and sent out towards the Earth. Two modulation stages are actually performed in the transmitter. The binary data are first multiplied by a square-wave subcarrier, and then the resulting waveform is used to phase modulate a high-frequency sinusoidal carrier.

On the ground, the analog signal is detected and tracked by the receiver. A carrier reference is derived and is used to heterodyne the signal to subcarrier frequency. The subcarrier demodulator assembly (SDA) removes the square-wave subcarrier, and the symbol synchronizer assembly (SSA) attempts to recover the original coded bit stream. Due to channel noise (and other degradations caused within the receiver system), the SSA does not output the original binary sequence. Instead, it outputs a stream of quantized estimates of these bits. The Viterbi decoder takes these estimates as inputs and decodes the convolutional level of the coding. The Reed-Solomon decoder then deinterleaves the symbols and does the final decoding.

For the purpose of this report, only signal degradation caused by the Gaussian noise of the space channel is assumed. The comparisons made here should remain valid when additional system degradations are included. The simulations assumed that there are no losses from carrier and subcarrier tracking and demodulation, and that the Viterbi decoder retains node synchronization at all times. Studies of these degradations are being undertaken and the results will appear in future publications.

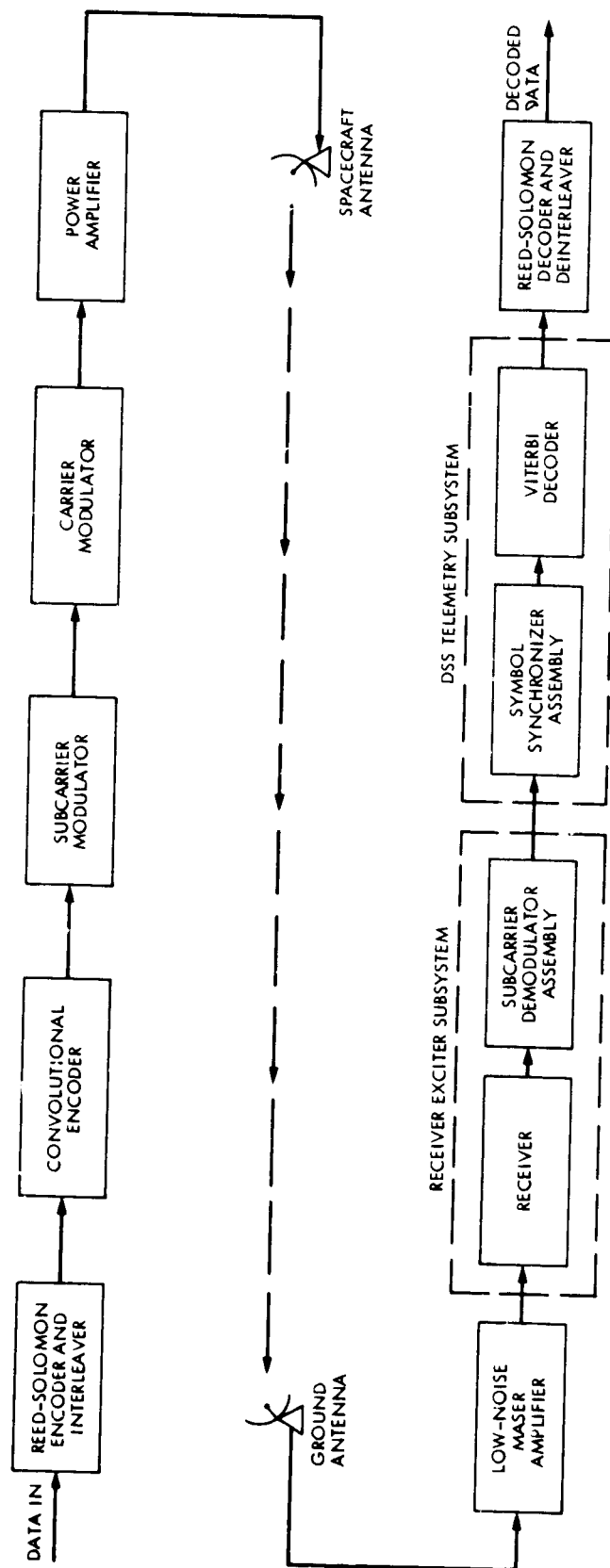


Figure 1-1. The Concatenated Coding System

## SECTION II

### SUMMARY

Figure 2-1 indicates the performances of several decoding schemes as a function of bit-energy-to-noise ratio. In particular, it shows the relative performances of several Viterbi-decoded convolutional codes including the (7, 1/2) code, which is the present standard for deep-space applications. Also shown in Figure 2-1 are Shannon's theoretical performance limits for rate 1/2 and rate 1/3 binary codes and the performance of uncoded transmission. The Shannon limits represent the best possible error performance for binary codes of these rates (Reference 2-1). It is easily seen that the (7, 1/2) code is 2.3 dB away from the theoretical limit at an error probability of  $5 \times 10^{-3}$ . Also, the (10, 1/3) code is less than 2 dB from Shannon's limit for rate 1/3 binary codes.

Also shown in Figure 2-1 are the results of concatenating these convolutional codes with an outer Reed-Solomon (255, 223) code. Ideal interleaving is assumed as well as no system losses other than Gaussian channel noise. The performance of the concatenated scheme is very sensitive to signal-to-noise ratio (SNR); a 1-dB change can result in a bit-error probability jump of several orders of magnitude. Consequently, the use of such a concatenated scheme should be accompanied by tight control of the signal-to-noise ratio of the communications link. Otherwise, the additional operating margin may negate the advantages derived from coding.

In addition to these error performance curves, a mathematical model of the burst-error statistics of Viterbi decoding is developed in this report. The model generates errors similar to those of Viterbi decoders by using a simple "Monte Carlo" technique. All that is needed to apply the model are three parameters that depend on the particular code and channel SNR. These parameters are tabulated in this report (see Appendix C) for several different convolutional codes and channel SNRs. The error sequences generated by this model are shown to behave similarly to actual error sequences when both are used to generate Reed-Solomon performance curves. The advantage of the model is that it may be used to generate Viterbi error sequences quickly and inexpensively for use in simulations when a very large amount of data is needed.

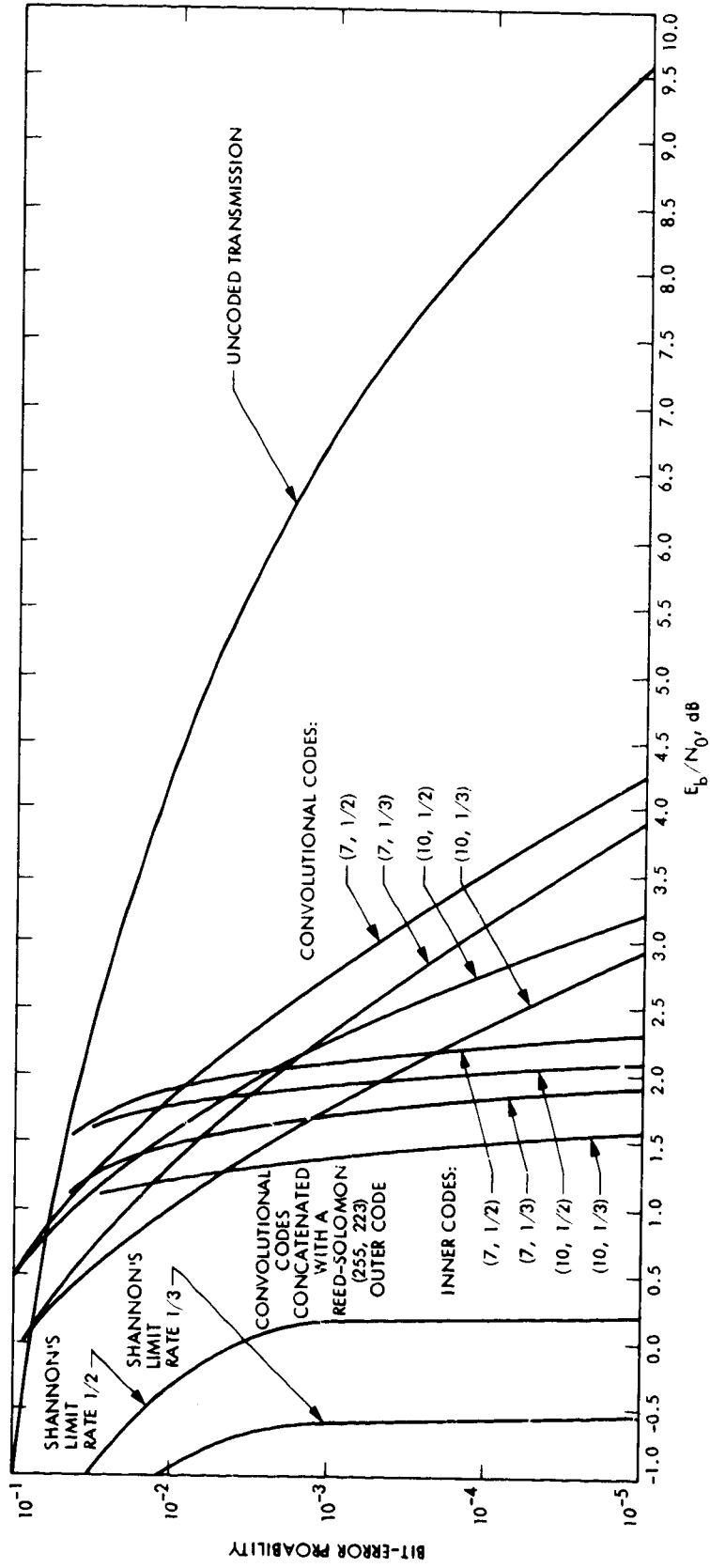


Figure 2-1. Comparison of Several Coding Schemes

### SECTION III

#### SIMULATED PERFORMANCE OF SEVERAL CODING SCHEMES

The key to computing the performance of the concatenated coding system is determining the Reed-Solomon symbol-error statistics. This information cannot be deduced from Viterbi bit-error performance curves. Consequently, extensive simulations were performed on the Xerox Data Systems Sigma 5 computer to calculate both the Viterbi bit-error and Reed-Solomon symbol-error statistics. Each data point was generated by processing 900,000 bits through a modification of the software Viterbi decoder developed by J.W. Layland. The simulations assumed that there were no system losses due to receiver noise or lack of synchronization. The only degradation present in the simulation was that of the random number generator simulating additive white Gaussian noise to reflect the channel SNR. Also, sufficient Reed-Solomon symbol interleaving was assumed so that the symbol-error events were independent. This is referred to as ideal interleaving. It is worth noting that interleaving to a depth of 5 is nearly ideal for the DSN (7, 1/2) inner convolutional code at SNRs above 2.0 dB.

Figure 3-1 shows the results of these simulations. In addition to the plots of Viterbi bit-error probability,  $p$ , as a function of channel SNR,  $(E_b/N_0)$ , each part of Figure 3 displays the Reed-Solomon symbol-error probability,  $\pi$ . The Reed-Solomon bit and word-error probabilities are calculated from  $\pi$  and other burst statistic information derived from these simulations. These calculations and the theory behind them are described in detail in Appendix A. The Reed-Solomon performance curves are plotted against a concatenated channel SNR, which is 0.58 dB greater than that of the Viterbi channel due to the overhead of the Reed-Solomon parity symbols.

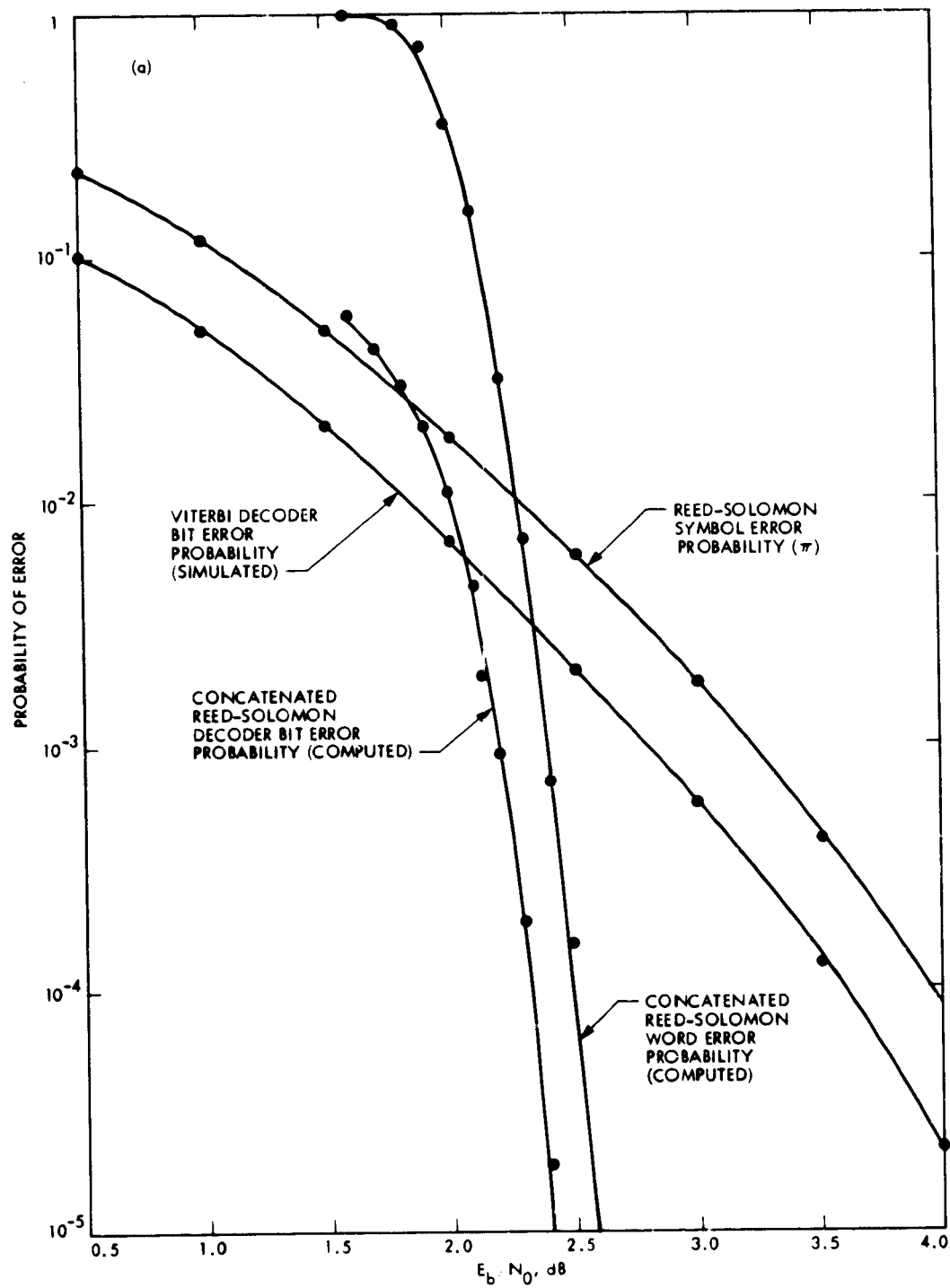


Figure 3-1. Performance Statistics for Ideally Interleaved Concatenated Coding Scheme (Assuming No System Losses): (a) 3233013 (7, 1/2) Convolutional Code and (255, 223) Reed-Solomon Code; (b) 7376147 (7, 1/3) Convolutional Code and (255, 223) Reed-Solomon Code; (c) 3103320323 (10, 1/2) Convolutional Code and (255, 223) Reed-Solomon Code; (d) 7461776427 (10, 1/3) Convolution Code and (255, 223) Reed-Solomon Code

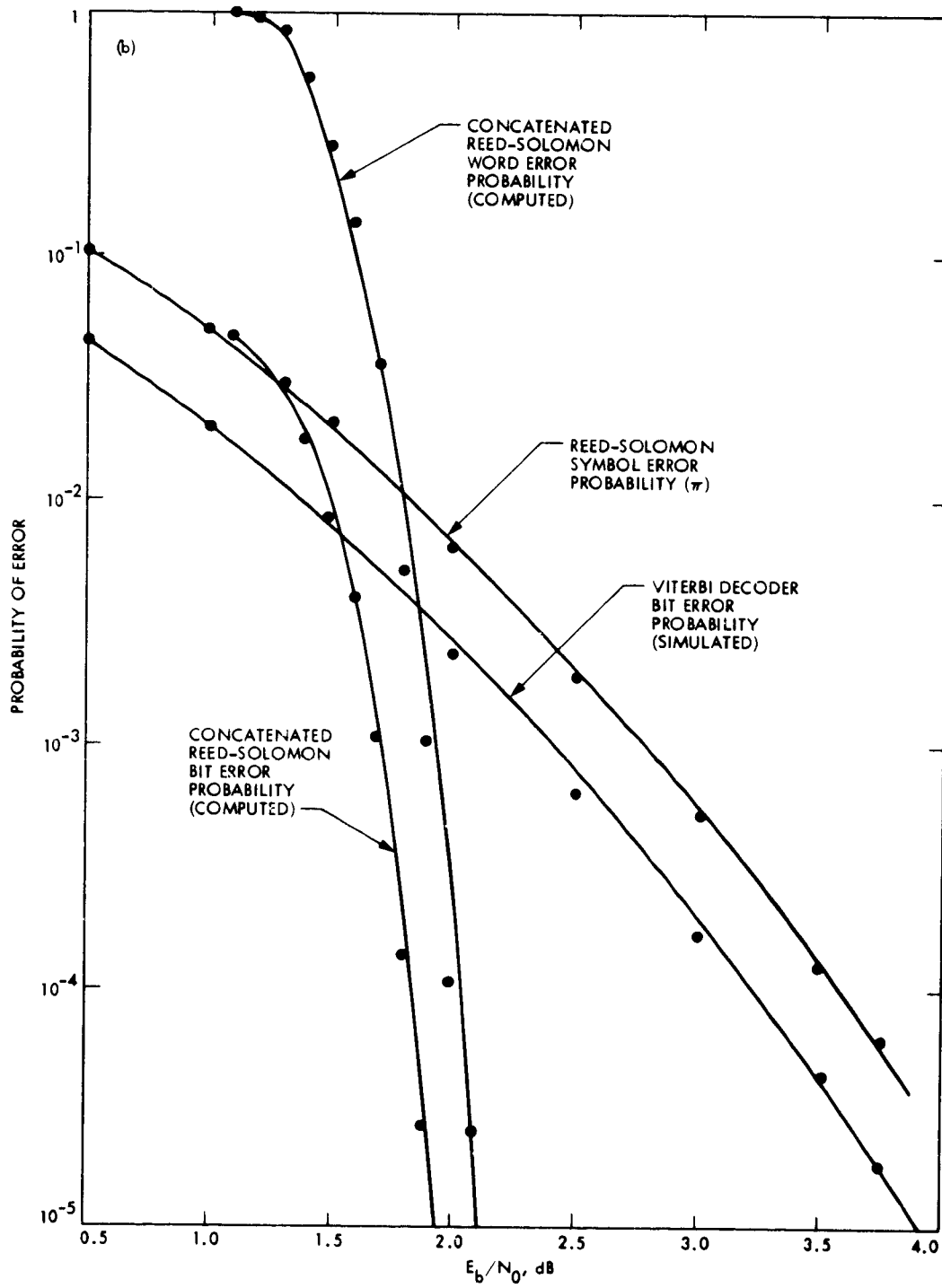


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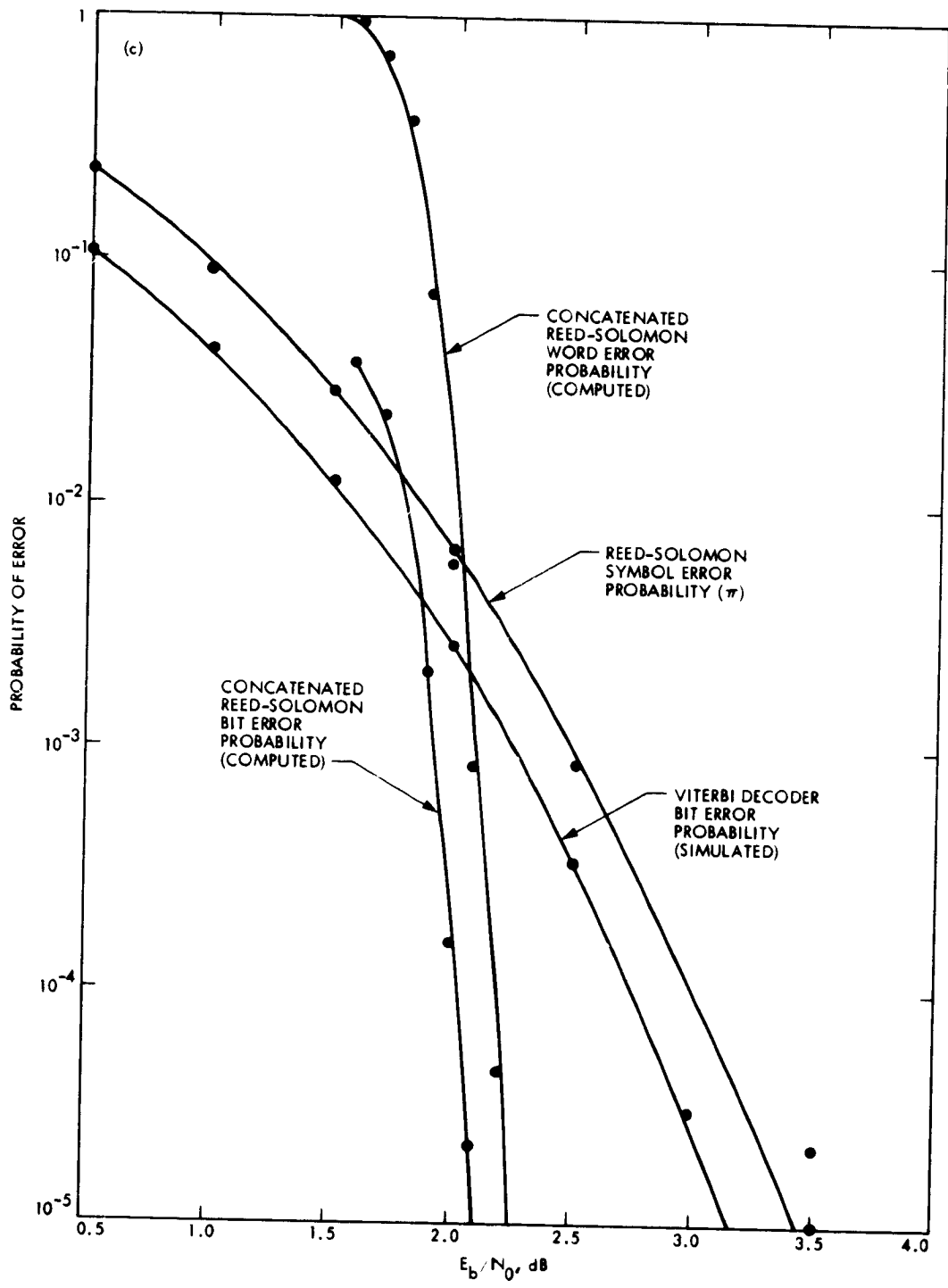


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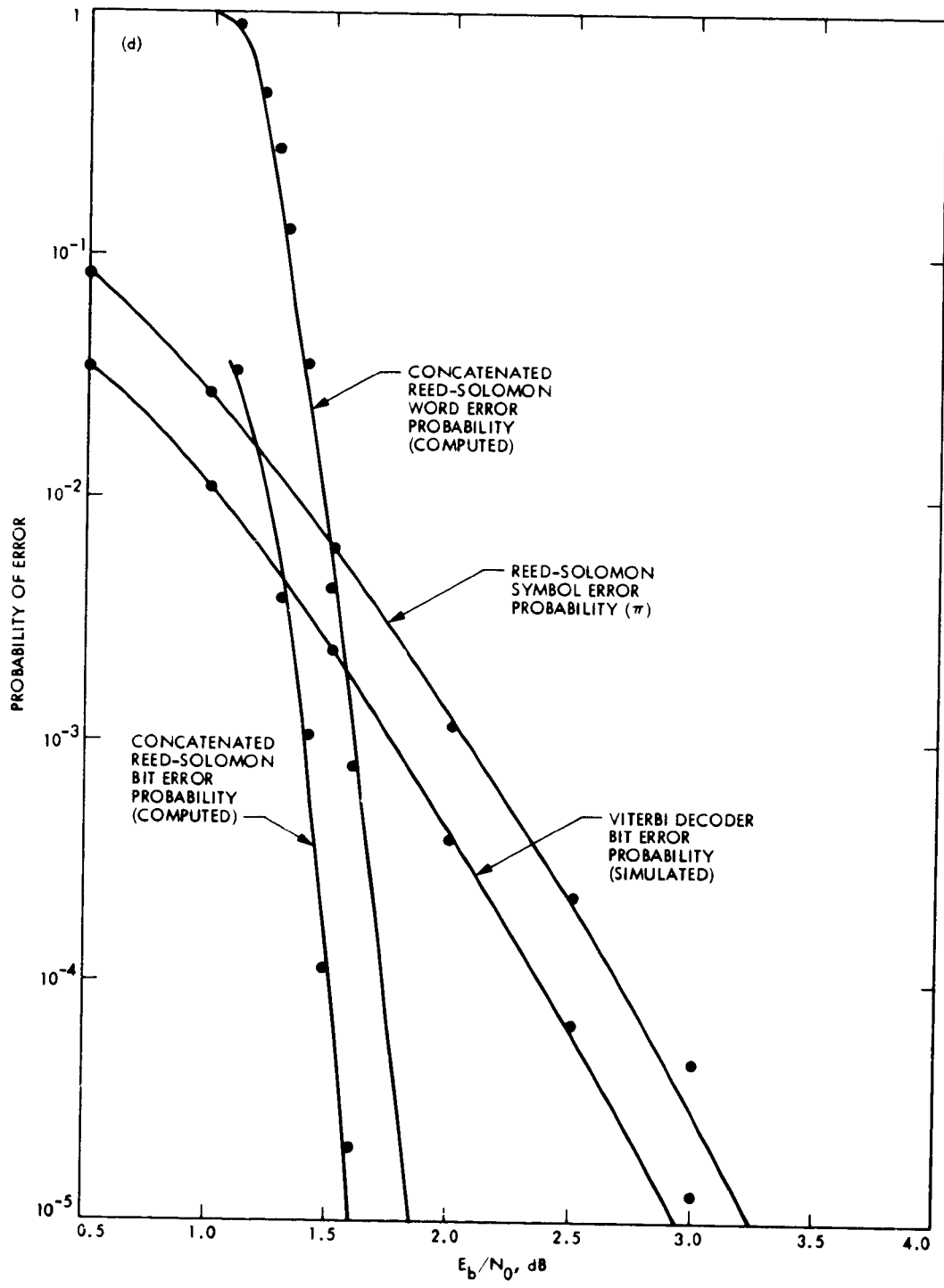


Figure 3-1 (contd)

SECTION IV  
A GEOMETRIC MODEL OF VITERBI BURST-ERROR STATISTICS

The software simulations described in Section III produced a large amount of Viterbi decoder burst-error statistics, which were then studied. It was found that Viterbi decoder error bursts, as well as waiting times between bursts, were very nearly geometrically distributed.

Three parameters are needed to define these distributions. They are the average burst length,  $\bar{B}$ , the average waiting time,  $\bar{W}$ , and the average density of errors in a burst,  $\theta$ . Given these parameters, Viterbi decoder burst lengths,  $B$ , were observed to be distributed according to

$$\text{pr}(B = m) = p(1 - p)^{m-1} \quad (m > 0)$$

where

$$p = 1/\bar{B}.$$

Errors within bursts occur randomly with probability  $\theta$ . Waiting times,  $\bar{W}$ , were observed to be distributed according to

$$\text{pr}(W = n) = q(1 - q)^{n-K+1} \quad (n \geq K - 1)$$

where  $K$  is the constraint length of the code and

$$q = 1/(\bar{W} - K + 2).$$

Derivations of these formulae may be found in Appendix B.

A Monte Carlo software routine was written to generate Viterbi error sequences directly from these formulae. The advantage of doing this is that the Viterbi software decoder requires about  $2^{K-7}$  hours per million bits of computer time (XDS Sigma-5 computer), while the geometric model requires an average of five minutes per billion bits.

To validate the geometric model, Reed-Solomon word- and bit-error rates were calculated using both the Viterbi software decoder and the geometric model routine. No interleaving was used so that the effects of the error bursts appeared to the maximum extent possible. These results were compared and are shown for various codes in Figure 4-1. (Some of the curves exhibited in this figure run off the edge of the page since the next data point was too low to be plotted on the same scale.) It is easily seen that the geometric model closely approximates the actual data in all cases.

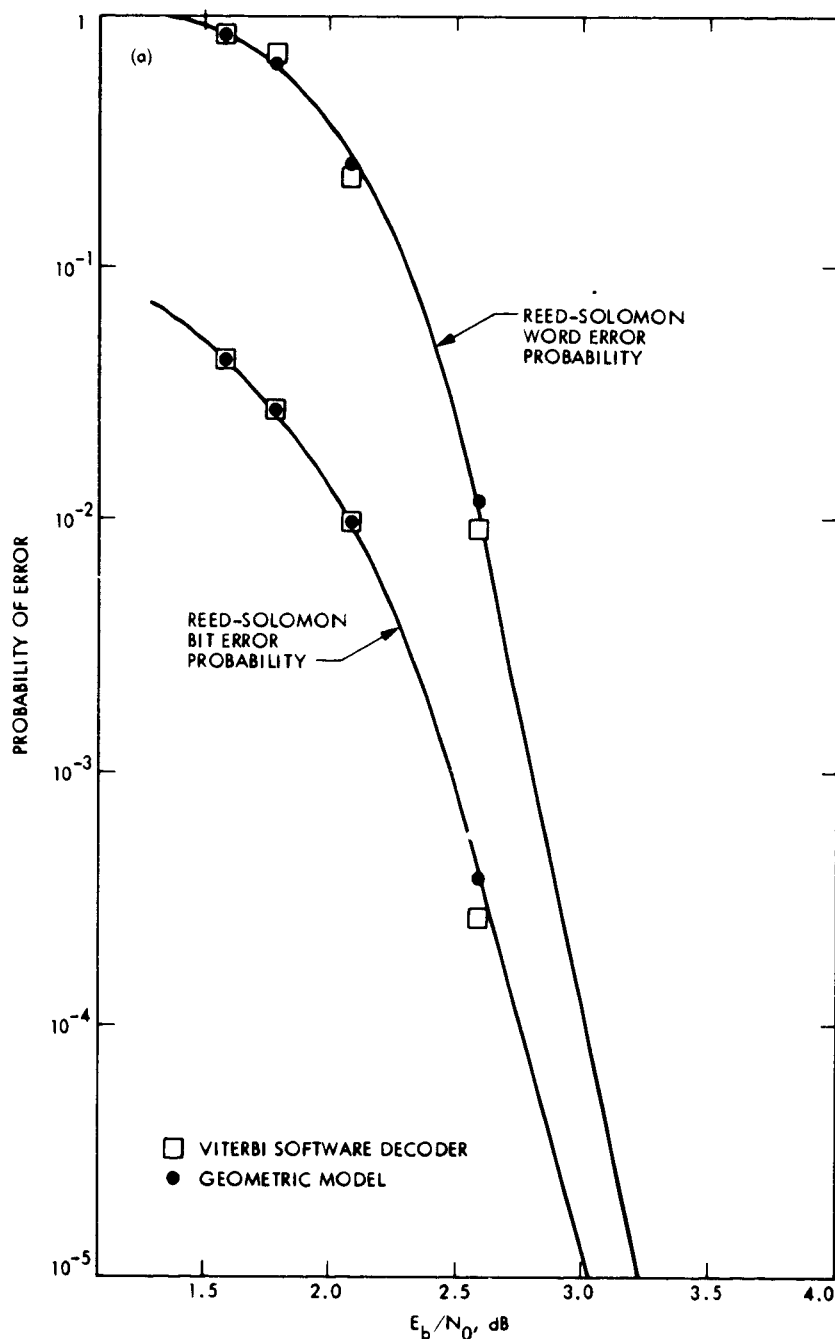
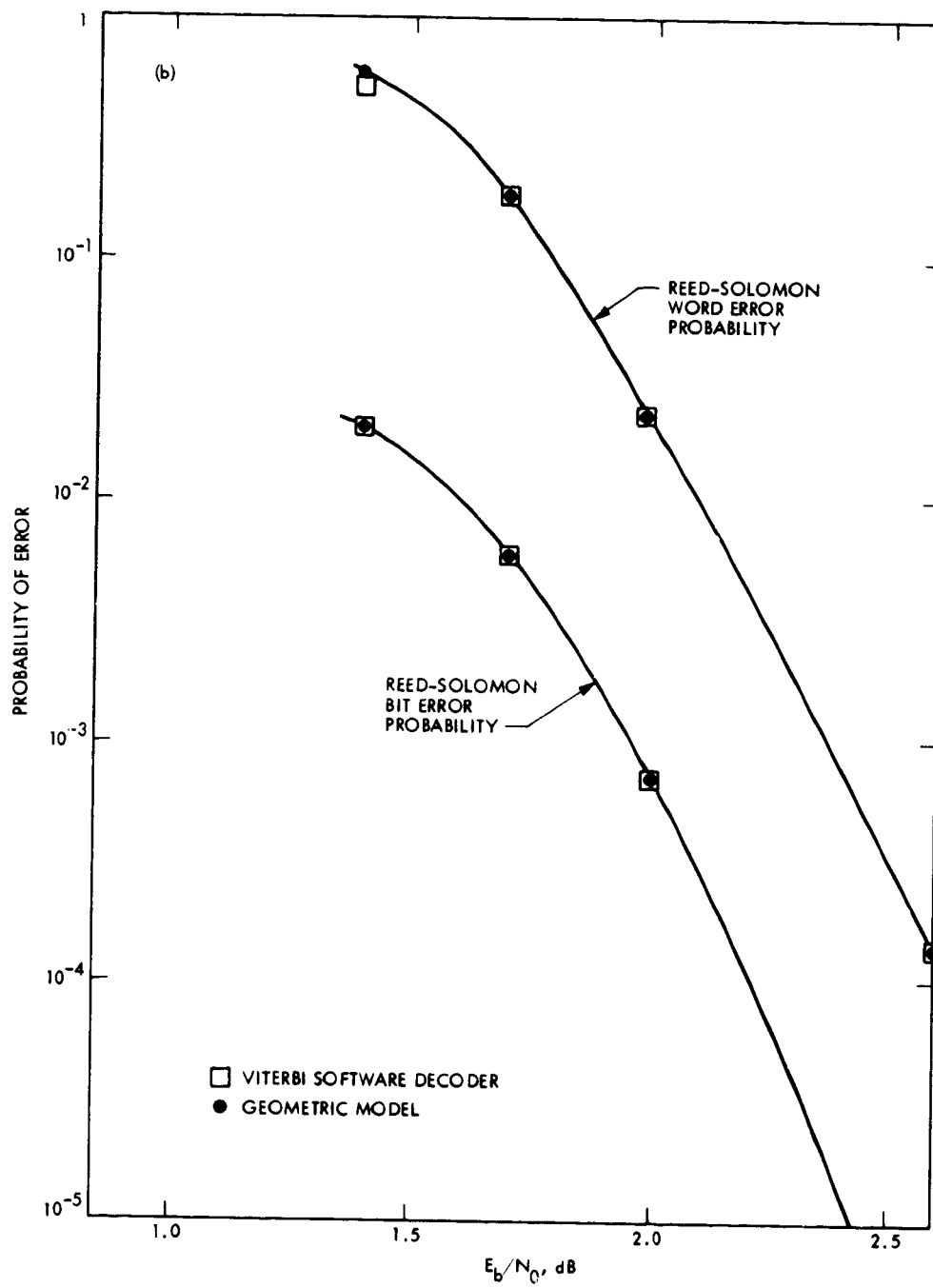


Figure 4-1. Noninterleaved Performance Statistics for Concatenated Coding Scheme (Assuming no System Losses): (a) 3233013 (7, 1/2) Convolutional Code and (255, 223) Reed-Solomon Code; (b) 7376147 (7, 1/3) Convolutional Code and (255, 223) Reed-Solomon Code; (c) 31311223 (8, 1/2) Convolutional Code and (255, 223) Reed-Solomon Code; (d) 313321013 (9, 1/2) Convolutional Code and (255, 223) Reed-Solomon Code; (e) 3103320323 (10, 1/2) Convolutional Code and (255, 223) Reed-Solomon Code; (f) 7461776427 (10, 1/3) Convolutional Code and (255, 223) Reed-Solomon Code



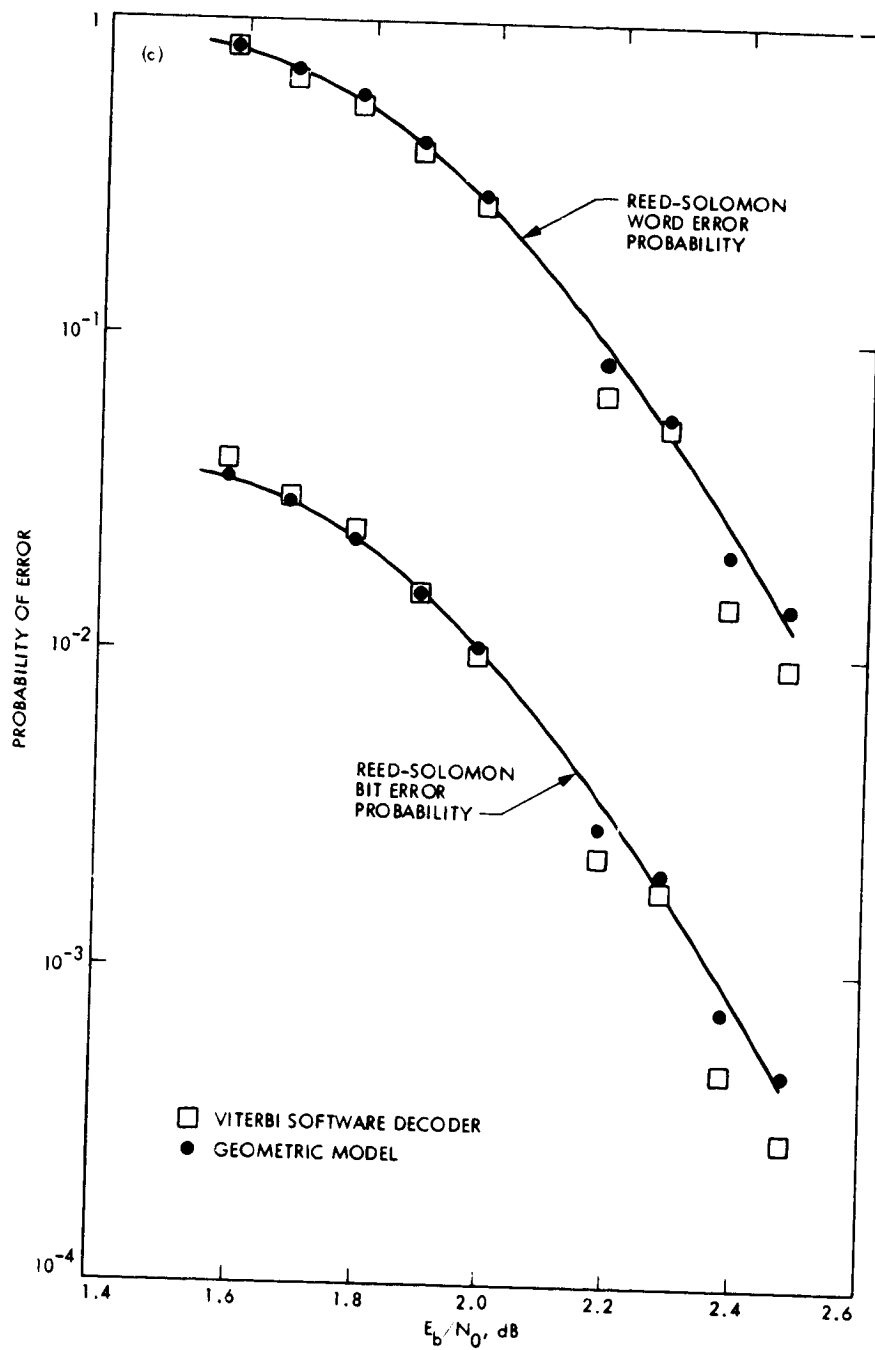


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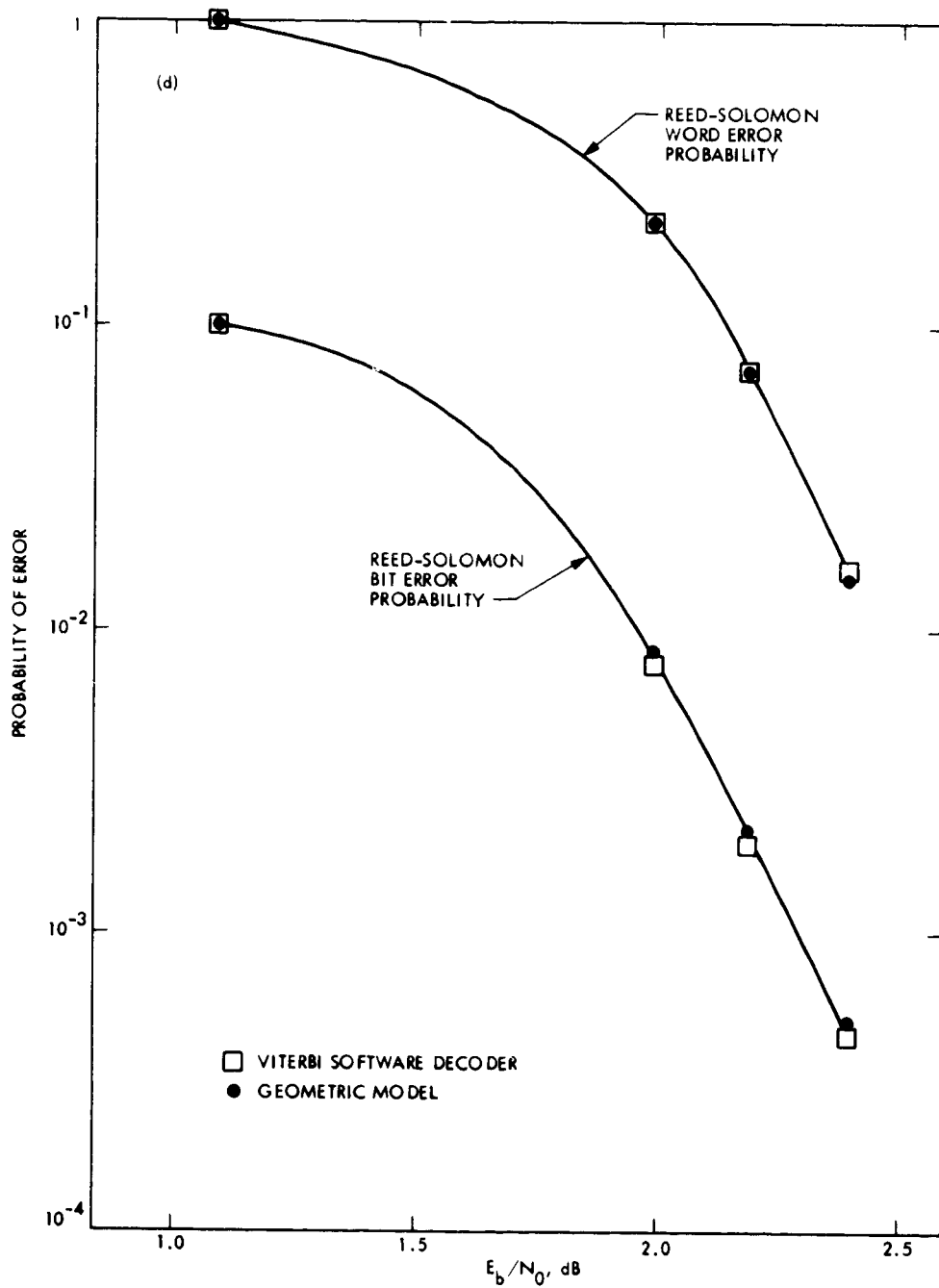


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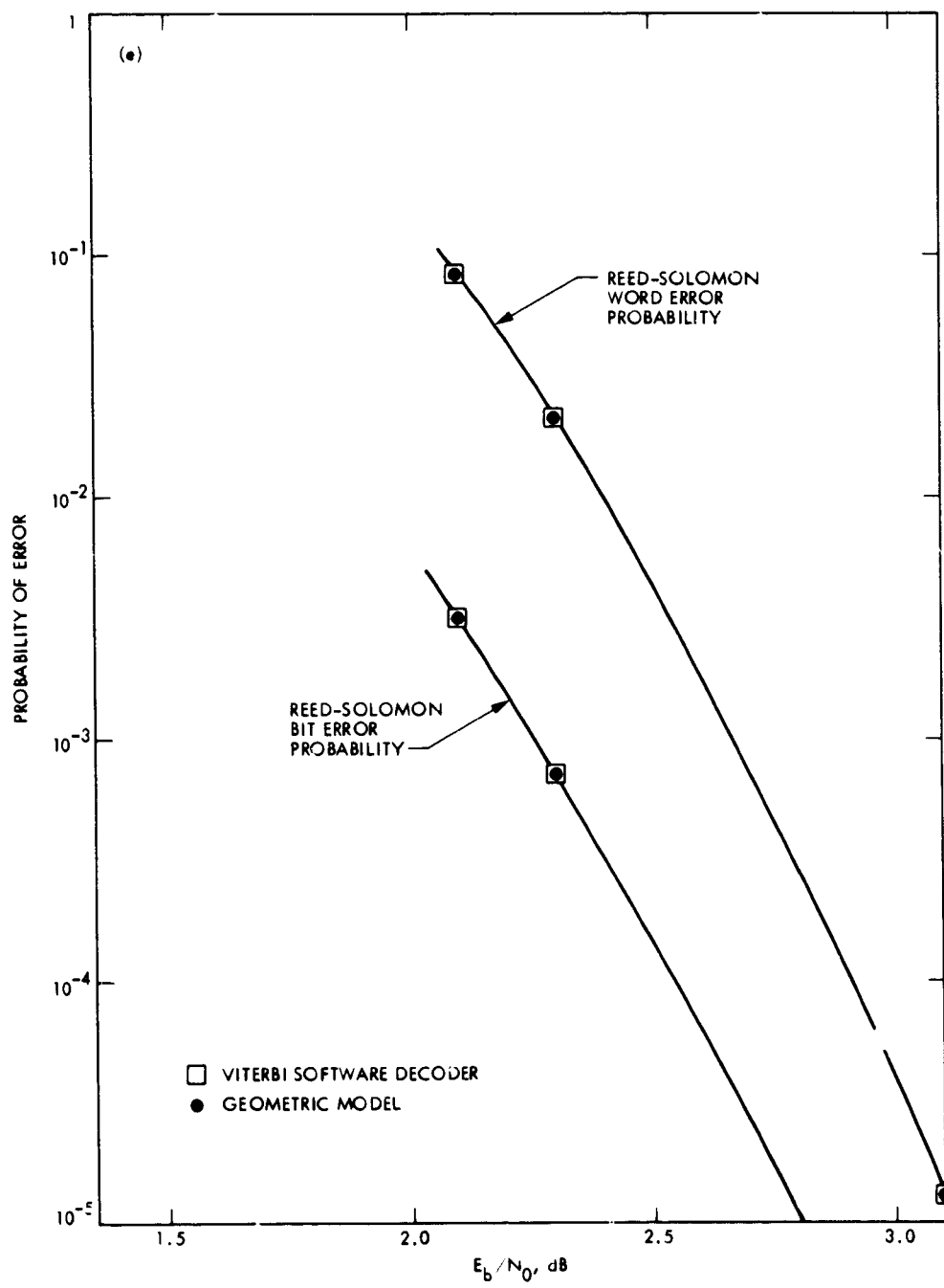


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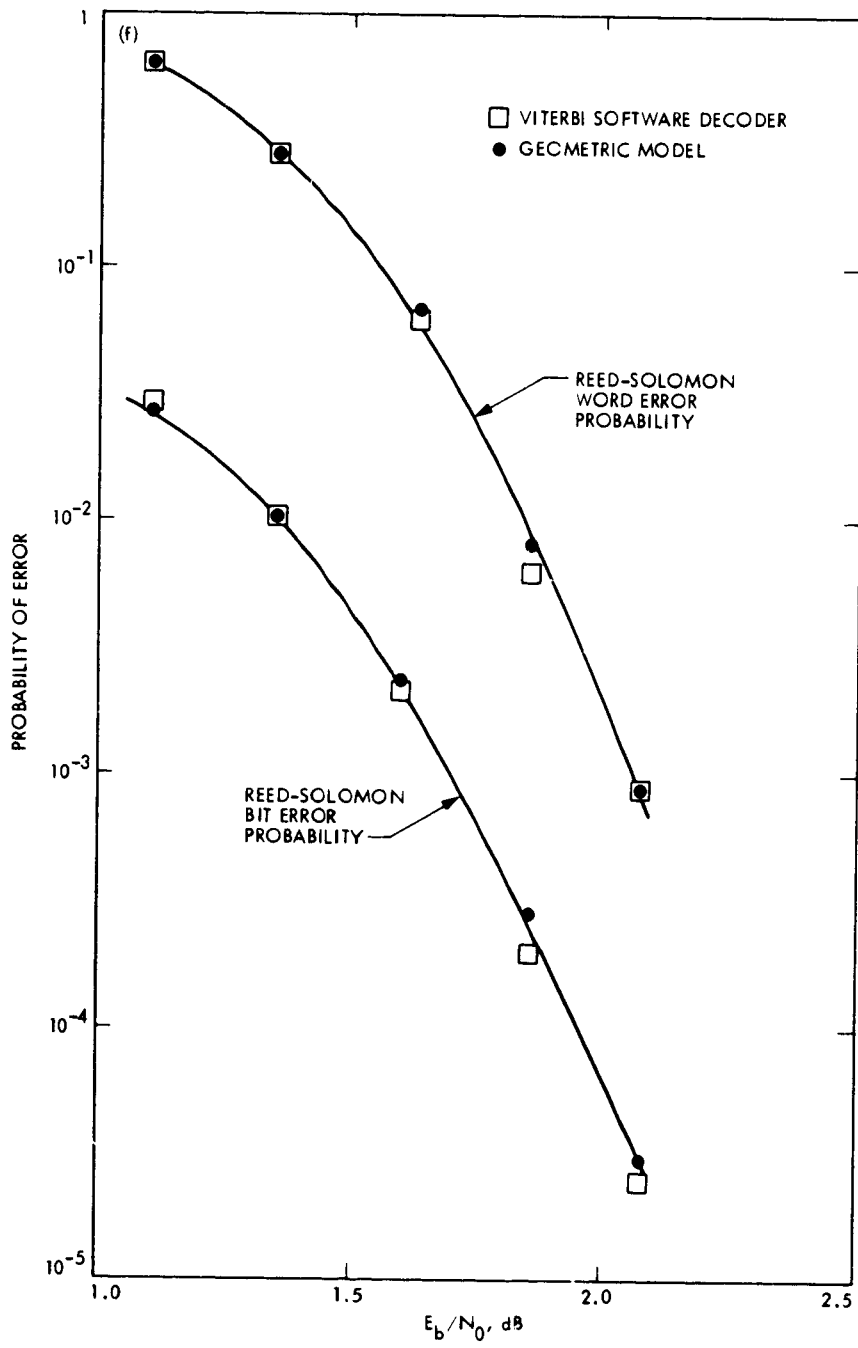


Figure 4-1 (contd)

## SECTION V CONCLUSION AND DISCUSSION

The software model of the Viterbi decoder used in this study can be enhanced by the inclusion of some of the system losses that arise in actual hardware devices. By adding these degradations to the model, their effects can be studied and possible remedies can be incorporated into future system designs.

The Viterbi decoders currently used by the DSN suffer loss of node synchronization at low SNRs. This means that if the signal is too weak, the decoder cannot decide which of the two code symbols associated with each data bit should be first. The concatenated coding system described in this report allows transmission of data at SNRs lower than those required for a convolutional-only scheme. This means that node synchronization losses will be higher in the concatenated scheme.

Synchronization problems are also associated with the Reed-Solomon code. A method for determining Reed-Solomon symbol and word boundaries is needed. If a packet telemetry system such as the one proposed by the End-to-End Information System (EEIS) (Reference 5-1) is to be implemented, then a frame synchronization device is also required.

For this report, only the error correcting capability of the Reed-Solomon code was considered. However, this code is also capable of correcting a number of erasures, i.e., Reed-Solomon symbols that are suspected to be in error. The (255, 223) code can correct  $E$  errors and  $e$  erasures in each code-word as long as  $2E + e < 33$ . If erasures can be detected, then the performance of the concatenated codes may improve by as much as 0.3 dB.

It should be noted that the loss of node synchronization and subsequent recovery by the Viterbi decoder may cause a deletion or insertion of a bit into the data stream entering the Reed-Solomon decoder. When this occurs, Reed-Solomon symbol and word synch will be lost. In the proposed EEIS packet telemetry scheme, a node synch failure could result in a loss of over 8000 information bits. Consequently, the sensitivity of the concatenated coding to node synchronization losses is potentially greater than that of convolutional coding alone.

The effects of carrier, subcarrier, symbol, and bit tracking in the system are also important to the overall performance of the coded channel since poor carrier tracking increases the number of Viterbi decoder bit errors.

The strict error-rate requirements of data compression are a major reason for investigating concatenated coding schemes. These requirements stem from the removal of redundant information, hence compression. As an example, one data compression scheme (Reference 5-2), reduces the number of bits per picture by over one half without loss of information. The reconstructed compressed data, however, are more sensitive to transmission errors than the original data. Hence, error correcting codes must be used. Notice that the concatenated schemes described in this report more than double the number of bits that are transmitted per information bit. This seems to neutralize the useful effects of data compression. Actually, this is not the case since an SNR of 9.5 dB would be required if no coding were employed to achieve an error rate of  $10^{-5}$  (see Figure 2-1), whereas the concatenated scheme with the (7, 1/2) inner code requires only 2.3 dB, and only 1.6 dB is required when the (10, 1/3) inner code is used. It might be beneficial to consolidate data compression and channel coding into a one-step process.

APPENDIX A  
EXPLANATION OF SIMULATIONS AND CALCULATIONS

It is well known that the bit errors produced by Viterbi decoding in the presence of noise are not independent. Instead, they tend to group together in error clumps known as "bursts." This happens because error events in a Viterbi decoder are caused by excursions from the correct path in the code trellis structure (Reference 1-1). The calculation required to generate the performance curves of Section III required a careful tabulation of these error bursts for the various convolutional codes under consideration. The theory behind these calculations is outlined in this section.

A. THE DEFINITION OF "BURST"

Denote the constraint length of the convolutional code under consideration by  $K$ . Consider a sequence of bits output by the Viterbi decoder of the form

$$\underbrace{ccc \dots c}_{K-1} \quad e \quad \underbrace{xxx \dots x}_B \quad e \quad \underbrace{ccc \dots c}_{K-1}$$

where the letter  $c$  represents a correctly decoded bit, an  $e$  represents a bit error, and an  $x$  may be either correct or in error. Suppose also that there is no string of  $K - 1$  consecutive  $c$ 's in the sequence  $xxx \dots x$ . Then the string  $exxx \dots xe$  is called a "burst" of length  $B$ . The motivation behind this definition of a burst is that a string of  $K - 1$  consecutive correct bits will return the Viterbi decoder to the correct decoding path. A string of  $c$ 's between two bursts will be referred to as a "waiting time."

B. THE CALCULATION OF  $p$  AND  $\pi$

The calculation of the Viterbi bit-error rate,  $p$ , by the Viterbi software decoder amounts to simply counting the number of bit errors made and dividing by the total number of bits examined. The calculation of the Reed-Solomon input symbol error probability,  $\pi$ , is more involved and is described below.

The Reed-Solomon code considered in this report has code words consisting of 255 eight-bit Reed-Solomon symbols. The quantity  $\pi$  is therefore the probability that a set of eight consecutive Viterbi-decoded bits contains at least one error. If Viterbi bit errors were independent, one would expect  $\pi = 1 - (1 - p)^8$ . However, Viterbi bit errors are certainly not independent and, in fact, this estimate for  $\pi$  is more than double the correct value for small  $p$ .

One way to obtain a good estimate for  $\pi$  is to partition the output bit stream of the Viterbi decoder into disjoint sets of eight consecutive bits and observe how many of these sets contain bit errors. However, since no a priori knowledge of symbol synchronization is assumed in these simulations, it is better to average over all sets of eight consecutive bits. This amounts to sliding a window of size eight over the output of the Viterbi decoder. The actual algorithm used in the simulations is as follows.

Suppose  $n$  bits are decoded by the Viterbi decoder. Then the number of symbol errors should be  $n\pi/8$ . On the other hand, a burst of length  $B$  will corrupt, on the average,  $(B + 7)/8$  symbols. Since waiting times are of length at least  $K - 1$ , and  $K \geq 7$  for this report, it is extremely rare for two bursts to corrupt the same symbol. If there were  $N$  bursts of length  $B_i$  ( $1 \leq i \leq N$ ), then

$$\begin{aligned} \text{number of symbol errors} &= \sum_{i=1}^N \frac{B_i + 7}{8} \\ &= \frac{N}{8} (\bar{B} + 7) \end{aligned}$$

where  $\bar{B}$  is the average burst length. It follows that

$$\pi = \frac{N(\bar{B} + 7)}{n} \tag{A-1}$$

If  $\theta$  is defined as the average density of errors in a burst, then Equation (A-1) may be rewritten in the form

$$\pi = \frac{7N}{n} + \frac{P}{\theta}.$$

The quantities  $n$ ,  $N$ ,  $p$ , and  $\theta$  are easily tabulated by the software Viterbi decoder.

### C. ESTIMATION OF UNCERTAINTY IN THE CALCULATIONS

Since only a finite number of bits may be examined in any simulation, it is advantageous to have a measure of how well the estimates of  $p$  and  $\pi$  described above reflect their actual values. Again let  $N$  be the number of bursts observed. Let  $X_i$  be the number of bit errors in the  $i^{\text{th}}$  burst ( $1 \leq i \leq N$ ). Assume that the  $X_i$ 's are independent and identically distributed (i.i.d.) with first and second moments  $E(X)$  and  $E(X^2)$  respectively. Assume also that  $N$  is Poisson distributed with mean  $\lambda$  and that  $N$  is independent of each  $X_i$ . The the number of bit errors observed is

$$S_N = \sum_{i=1}^N X_i.$$

In Reference A-1 it is shown that

$$E(S_N) = E(N)E(X)$$

and

$$\text{Var}(S_N) = E(N)\text{Var}(X) + (E(X))^2\text{Var}(N).$$

Since  $N$  is Poisson,  $E(N) = \text{Var}(N) = \lambda$ . Let  $\beta = \sigma(S_N)/E(S_N)$  be the fractional uncertainty in the measurement of  $S_N$ . Then the quantities  $E(X)$ ,  $E(X^2)$ , and  $\lambda$  are estimated by the software Viterbi decoder to give an estimate of  $\beta$ .

To estimate the uncertainty in the calculation of  $\pi$ , let  $B_i$  be the length of the  $i^{\text{th}}$  burst and let  $T_N$  be the number of symbol errors occurring in  $N$  bursts. From Equation (A-1) it follows that

$$T_N = \frac{1}{8} \sum_{i=1}^N (B_i + 7)$$

so that

$$E(T_N) = \lambda \left( \frac{7 + E(B)}{8} \right)$$

and

$$\text{Var}(T_N) = \frac{\lambda}{64} (49 + 14E(B) + E(B^2)).$$

If  $\gamma = \sigma(T_N)/E(T_N)$  is the fractional uncertainty in the calculation of  $\pi$ , then

$$\gamma = \frac{1}{7 + E(B)} \sqrt{\frac{49 + 14E(B) + E(B^2)}{\lambda}}.$$

As before,  $E(B)$  and  $E(B^2)$  are easily tabulated by the software Viterbi decoder.

Figure A-1 shows plots of  $p$  and  $\pi$  (the same as those in Figure 3-1) with error bars representing the uncertainty in their calculation included.

#### D. REED-SOLOMON WORD ERROR PROBABILITY, $p_{\text{word}}$

To calculate the Reed-Solomon word-error probability, sufficient symbol interleaving is assumed so that the symbol errors may be considered to be independent. Since a word error occurs exactly when there are 17 or more symbol errors in a 255-symbol word,

$$p_{\text{word}} = \sum_{j=17}^{255} \binom{255}{j} \pi^j (1 - \pi)^{255-j}.$$

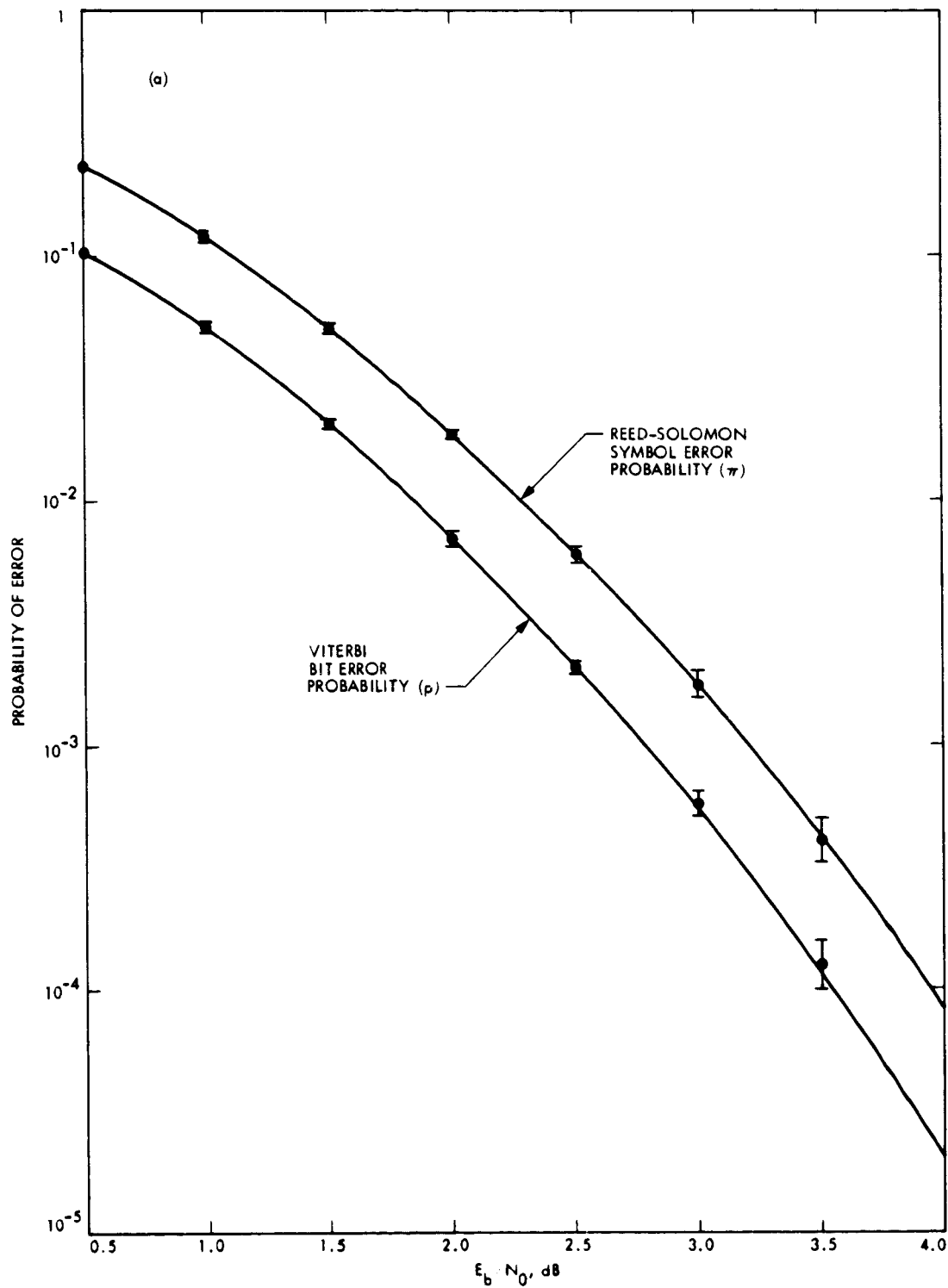


Figure A-1. Performance Statistics of Viterbi Decoder (Assuming No System Losses): (a) 3233013 (7, 1/2) Convolutional Code; (b) 7376147 (7, 1/3) Convolutional Code; (c) 3103320323 (10, 1/2) Convolutional Code; (d) 7461776427 (10, 1/3) Convolutional Code



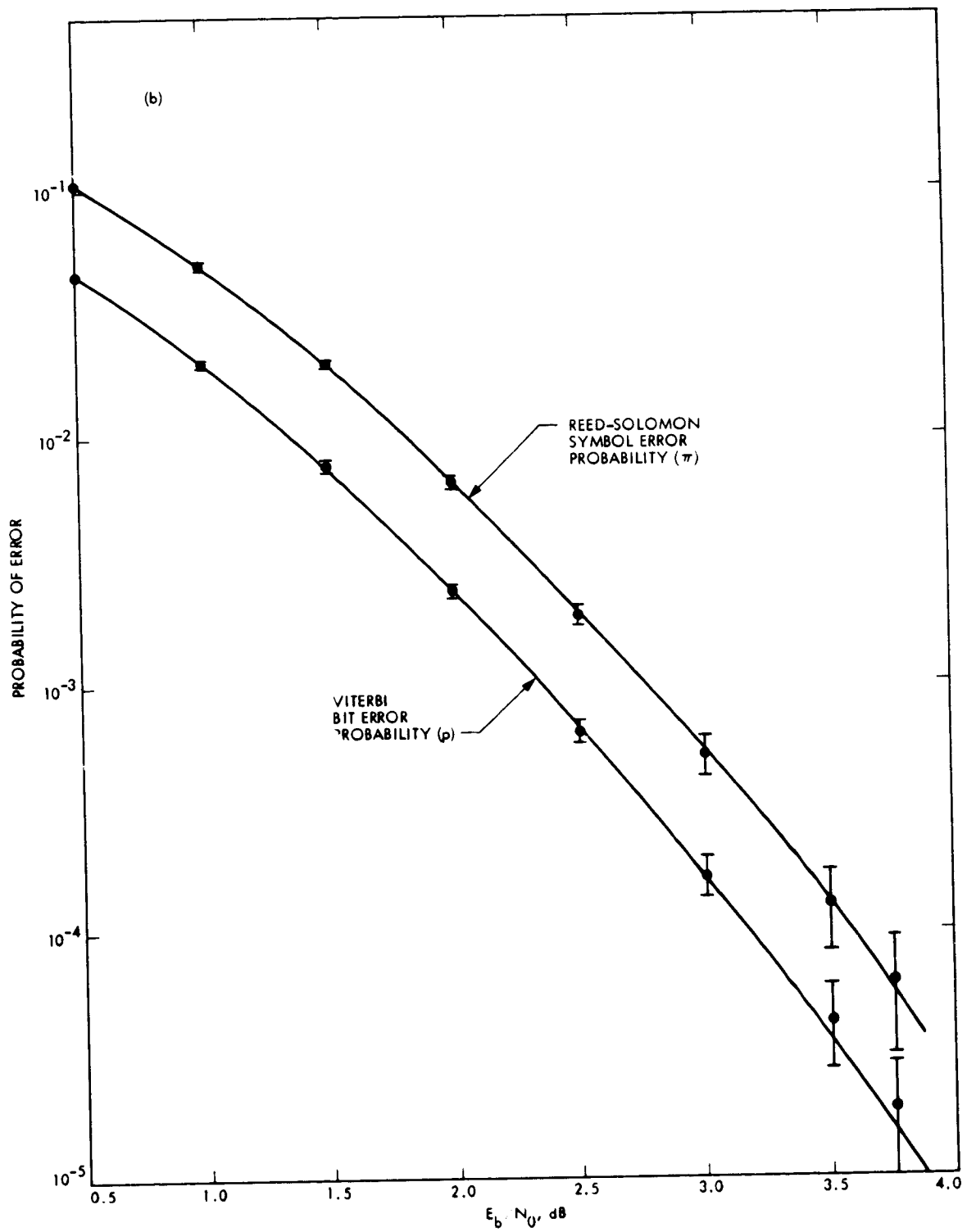


Figure A-1 (contd)

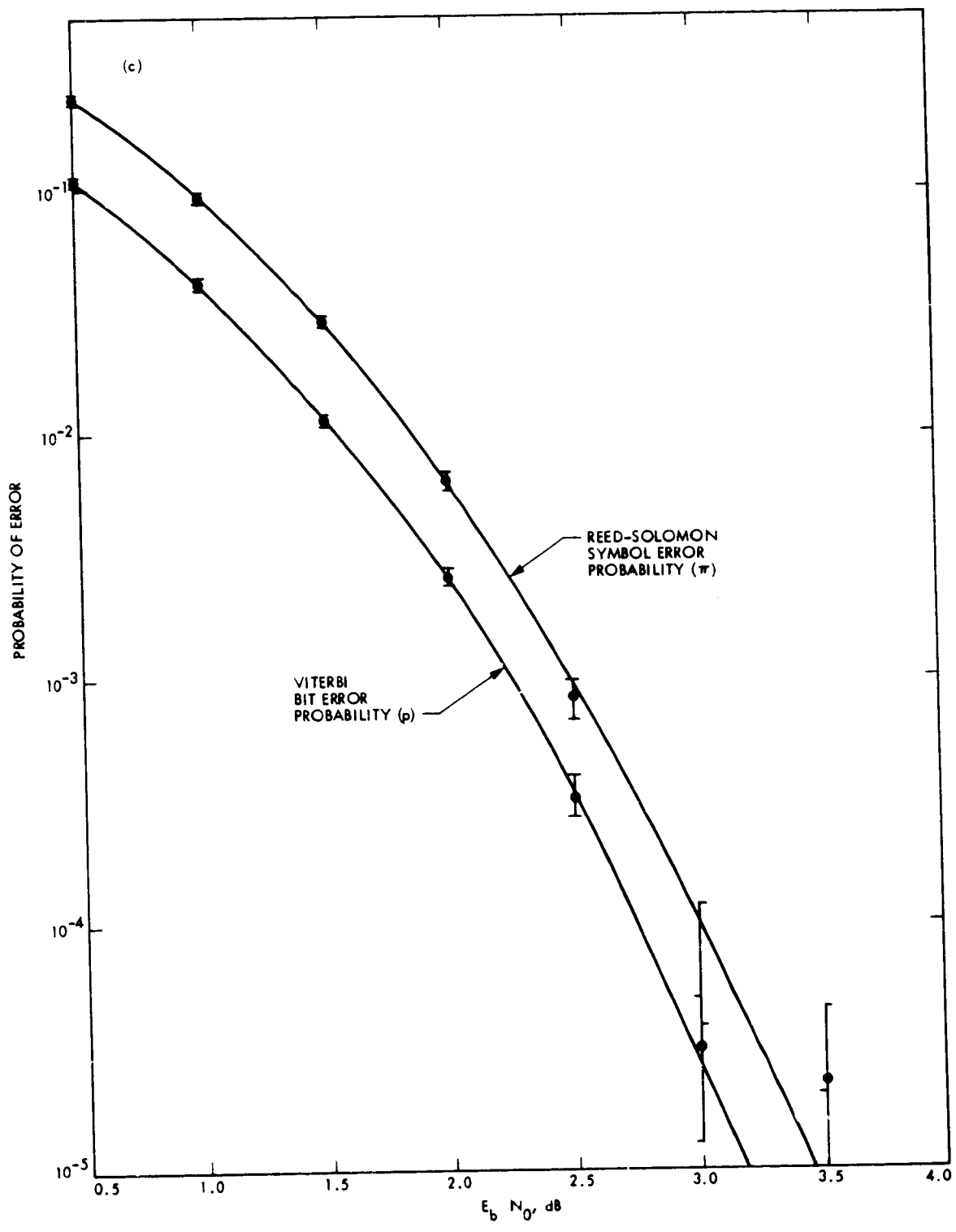


Figure A-1 (contd)

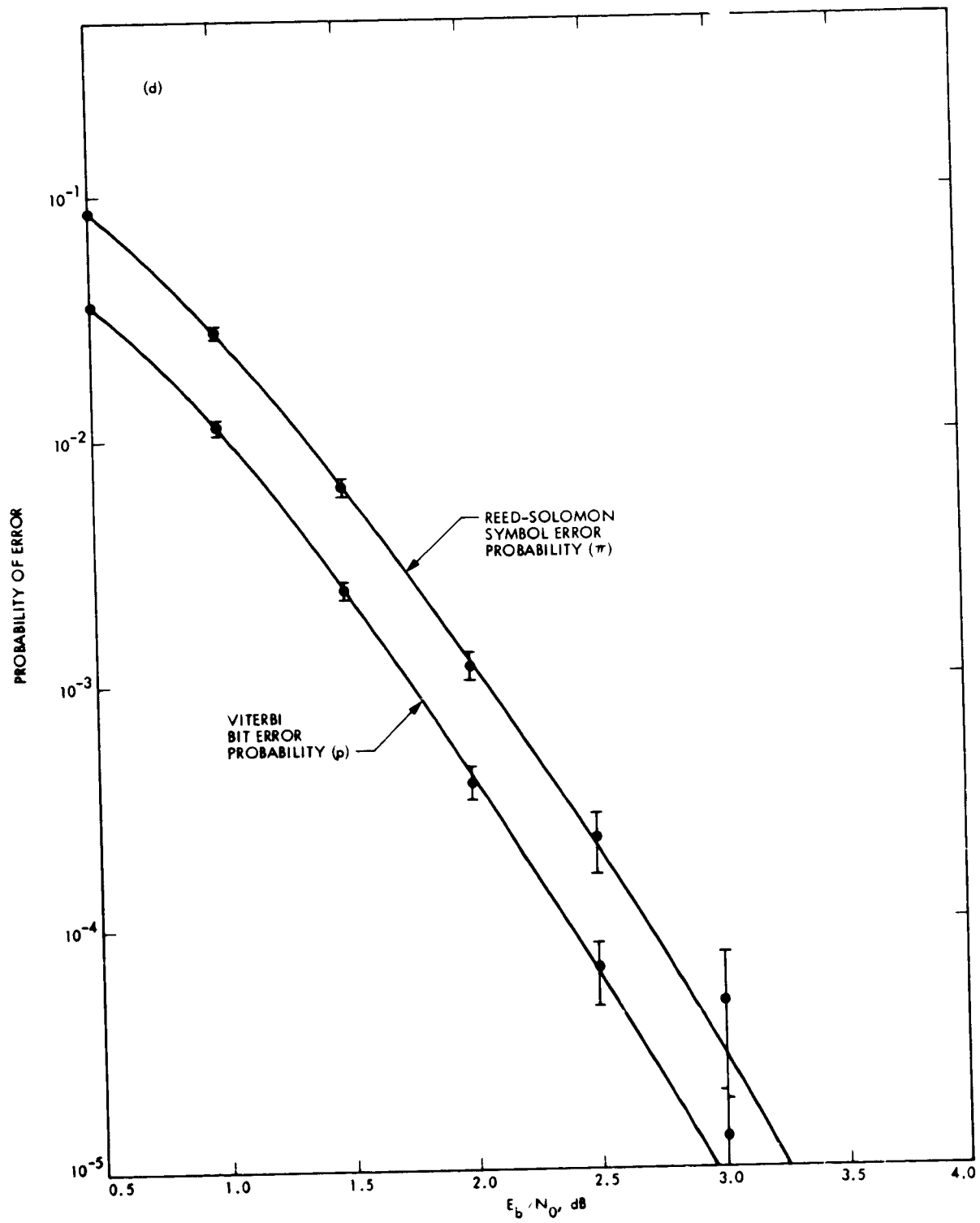


Figure A-1 (contd)

E. REED-SOLOMON BIT-ERROR PROBABILITY,  $P_{\text{bit}}$

Let  $\bar{B}$  and  $\theta$  be the average burst length and the average density of errors in a burst, respectively, for the convolutional inner code in question. If a Reed-Solomon word error occurs, then between 17 and 255 symbols are in error. Recall that the average number of symbols corrupted by a burst is  $(\bar{B} + 7)/8$ . Hence the bit-error probability in a corrupted symbol is  $\theta\bar{B}/(\bar{B} + 7)$ . It follows that

$$P_{\text{bit}} = \frac{\theta \bar{B}}{\bar{B} + 7} \sum_{j=17}^{255} \binom{j}{255} \binom{255}{j} \pi^j (1 - \pi)^{255-j}.$$

APPENDIX B  
DERIVATION OF THE GEOMETRIC MODEL OF BURST STATISTICS

A random variable  $X$  is said to be geometrically distributed with parameter  $p \in [0, 1]$  if

$$\text{pr}(X = s) = p(1 - p)^s \quad (s = 0, 1, 2, \dots).$$

For the purposes of this section, a random variable  $Y$  satisfies a "modified geometric distribution" of parameter  $p \in [0, 1]$  if there exists a positive integer  $d$  such that

$$\text{pr}(Y = s) = p(1 - p)^{s-d} \quad (s = d, d + 1, d + 2, \dots).$$

In this case,  $Y$  will be called  $d$ -geometrically distributed.

In Reference B-1, J. Omura showed by a random coding argument that burst lengths for an "average convolutional code" have a distribution that may be overbounded by a 1-geometric distribution. In this report, it will be shown, in fact, that for convolutional codes of constraint lengths seven through ten, burst lengths are very nearly 1-geometrically distributed. Moreover, the waiting times are  $K - 1$  geometrically distributed.

The tests that were used to exhibit these facts were essentially the same for burst lengths and waiting times. For this reason, only the test for burst lengths will be described below.

Suppose that a Viterbi decoder simulation is performed and  $N$  bursts are observed. Let  $B_i$  be the length of the  $i^{\text{th}}$  burst ( $i = 1, 2, 3, \dots, N$ ). Let  $B$  be the random variable representing burst length (so  $B_i$  is the  $i^{\text{th}}$  sample of the random variable  $B$ ). It must be shown that

$$\text{pr}(B_i = s) = p(1 - p)^{s-1} \quad (s = 1, 2, 3, \dots) \quad (\text{B-1})$$

for some  $p \in [0, 1]$ . The fact that these probabilities must sum to one forces  $p = 1/\bar{B}$ .

For each  $m = 1, 2, 3, \dots$ , let  $N_m$  be the number of bursts of length greater than or equal to  $m$ . If the burst lengths were indeed 1-geometrically distributed with parameter  $1/\bar{B}$ , then the expected value of  $N_m/N_n$  would be

$$\begin{aligned}
E(N_m/N_n) &= \left( \sum_{s=m}^{\infty} p(1-p)^{s-1} \right) / \left( \sum_{s=n}^{\infty} p(1-p)^{s-1} \right) \\
&= (1-p)^{m-n} = \left(1 - \frac{1}{\bar{B}}\right)^{m-n}.
\end{aligned}$$

In other words, for  $N$  sufficiently large,

$$1 - 1/\bar{B} = (N_m/N_n)^{1/(m-n)} \quad (\text{B-2})$$

which is a constant.

Since  $N$  is only moderately large in the simulations described in Section III (on the order of 200 to 500), the performance of this test can be improved by grouping bursts of several consecutive lengths into bins. Enough bursts were placed into each bin so that  $1 - 1/\bar{B}$  could be approximated to within 0.05 with 90% accuracy for each bin. These approximations remained reasonably constant between bins, indicating a successful test.

As remarked in Section IV, waiting times were found, by a similar test, to be  $K - 1$  geometrically distributed with parameter  $q = 1/(\bar{W} - K + 1)$ , where  $\bar{W}$  is the average waiting time and  $K$  is the constraint length of the code.

The geometric model of Viterbi burst-error statistics states that these bursts occur randomly according to these two modified geometric distributions. Errors within a burst occur essentially randomly (except for the fact that each burst starts and ends with an error) with probability  $\theta$ . To generate error sequences similar to those produced by a Viterbi decoder, only the quantities  $\bar{B}$ ,  $\bar{W}$ , and  $\theta$  must be known. These are tabulated for several codes and channel SNRs in Appendix C.

## APPENDIX C

### TABLES OF VITERBI DECODER BURST STATISTICS

The following tables give values of some important parameters describing the burst statistics of Viterbi decoding. These are the average burst length,  $\bar{B}$ , the average waiting time,  $\bar{W}$ , and the average density of errors in a burst,  $\theta$ . The tables show the values of these parameters for the DSN (7, 1/2) convolutional code as well as for (7, 1/3), (10, 1/2), and (10, 1/3) codes. A range of channel SNRs that is of interest in deep-space applications is considered.

Table C-1. Viterbi Decoder Burst  
 Statistics: 3233013  
 (7, 1/2) Convolutional  
 Code

$E_b/N_0$	$\bar{B}$	$\bar{W}$	$\theta$
0.5	25.84	131.2	0.564
0.75	23.46	158.5	0.566
1.0	21.07	220.5	0.571
1.1	19.78	275.8	0.574
1.2	19.27	293.2	0.574
1.3	18.02	371.2	0.573
1.4	17.46	430.6	0.573
1.5	17.01	474.1	0.578
1.6	15.76	600.8	0.578
1.7	15.21	702.2	0.579
1.8	14.32	847.0	0.586
1.9	13.50	931.7	0.584
2.0	12.89	1122	0.590
2.1	12.86	1530	0.589
2.5	10.17	3258	0.599
2.0	8.67	9596	0.584
3.5	6.70	3.7E04	0.630
4.0	4.40	2.0E5	0.591

Table C-2. Viterbi Decoder Burst  
 Statistics: 7376147  
 (7, 1/3) Convolutional  
 Code

$E_b/N_0$	$\bar{B}$	$\bar{W}$	$\theta$
0.5	16.80	228.3	0.596
0.6	15.79	258.6	0.598
0.7	15.31	290.1	0.601
0.8	14.70	308.2	0.602
0.9	13.94	355.5	0.605
1.0	13.24	440.1	0.612
1.1	13.13	473.5	0.611
1.2	12.13	567.1	0.613
1.3	12.01	663.4	0.615
1.4	11.40	787.2	0.620
1.5	11.30	980.8	0.624
1.6	10.79	1146	0.622
2.0	9.46	2556	0.635
2.5	7.53	8613	0.653
3.0	6.35	2.9E04	0.685
3.5	7.25	1.2E05	0.672



Table C-3. Viterbi Decoder Burst  
 Statistics: 2103320323  
 (10, 1/2) Convolutional  
 Code

$E_b/N_0$	$\bar{B}$	$\bar{W}$	$\theta$
0.5	37.98	162.4	0.511
0.6	35.99	184.8	0.512
0.7	32.72	221.7	0.517
0.8	30.11	248.5	0.515
0.9	28.07	292.9	0.518
1.0	26.98	353.0	0.523
1.1	25.76	440.9	0.522
1.2	25.16	526.7	0.530
1.3	22.86	601.0	0.530
1.4	21.15	857.6	0.537
1.5	21.13	983.6	0.531
1.6	20.86	1217	0.545
1.7	18.80	1566	0.541
2.0	16.95	4048	0.551
2.5	14.14	2.5E4	0.585
3.0	11.25	2.5E5	0.622

Table C-4. Viterbi Decoder Burst  
 Statistics: 7461776427  
 (10, 1/3) Convolutional  
 Code

$E_b/N_0$	$\bar{B}$	$\bar{W}$	$\theta$
0.5	25.29	398.1	0.533
0.6	24.84	455.3	0.532
0.7	22.06	549.4	0.539
0.8	21.37	642.4	0.541
0.9	20.76	813.0	0.540
1.0	19.34	990.1	0.540
1.1	19.09	1317	0.547
1.2	17.68	1606	0.546
1.3	16.33	2094	0.555
1.5	14.08	3245	0.566
2.0	11.21	1.6E5	0.566
2.5	8.20	6.8E5	0.646

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