AN ANALYSIS OF SIMULTANEOUS SATELLITE VISIBILITY TIME SPANS FOR TWO EARTH OBSERVATION STATIONS

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ABSTRACT

Analysis was performed to estimate the statistical visibility time spans of earth orbiting satellites as seen simultaneously by a ground station and a ship. The analysis covers topics such as time average population, average population times and also the percentage visibility times for a given number of satellites. These results are useful for specific communications satellite applications. Numerical results are obtained for various configurations of ground station and ship.

SECTION 1 - INTRODUCTION

This report is concerned with the analysis of the number and also the time of satellites mutually observed by both a ground station and a ship. Unlike the relatively simple case of a single observation station for which the region of observation is the volume bounded by a cone, the present more complicated case has a region of observation determined by the intersection of two cones. This region has a volume determined only by the separation distance between the ground station and the ship; but it also has a directional property determined by the relative position of the ship with respect to the ground station. Because the analysis becomes extremely complex, it is necessary to make certain simplifying assumptions.

The first assumption is that the satellites presently orbiting the earth may be broadly classified into a few categories. This simplification is supported by the fact that[(1)] since 1977 approximately 635 satellites have been launched and these may be characterized as in Table 1.1.

Table 1.1

Class	Average Period	Average Inclination	Average Altitude	Number
I	100 min.	80°	800 km	440
II	12 hr.	60°	20,000 km	106
III	24 hr.	0°	36,000 km	57
IV	Others			32

Thus, instead of having to deal with the volume of the region of observation, the analysis deals with the areas at the various altitudes. In this analysis, only Class I and II satellites are considered. Class III satellites are considered separately because they are geosynchronous. Class IV satellites are irregular and will not be considered at all.

(1) NASA, Satellite Situation Report, Volume 21, Number 1, February 28, 1981.

The second assumption is that within each of the two categories considered, the satellites have circular orbits which are uniformly distributed in terms of equatorial crossing and, moreover, the satellites are also uniformly distributed along the orbital arcs.

Section 2 deals with the derivation of the number density of satellites in this statistical distribution. Section 3 deals with the determination of the common region of observation of both the ground station and the ship. Section 4 is concerned with the computation of the time average population of satellites within the mutual region of observation. Section 5 briefly discusses the computation of the average population times of these satellites in the same region. Section 6 summarizes the results of this study for Class I and II satellites.

Readers who are strictly interested in the numerical results may go directly to Section 6 and omit the intervening sections which deal with the mathematical analysis.

SECTION 2 - STATISTICAL DESCRIPTION OF ORBITING SATELLITES

2.1 Distribution Function

Consider a statistical description of a system of N satellites as previously described in which the circular orbits are uniformly distributed in terms of equatorial crossing and the satellites are uniformly distributed along the orbital arc. Consider Figure 2.1 which illustrates a given orbit with inclination i. Let θ be the latitude, ϕ be the right ascension measured from the equatorial crossing, and σ be the orbital arc measured also from the equatorial crossing.

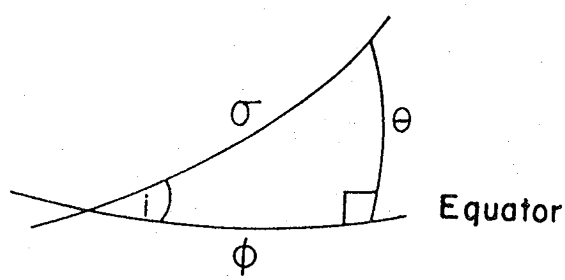

Figure 2.1

Consider Figure 2.2 which illustrates the area element dA_o at the equator

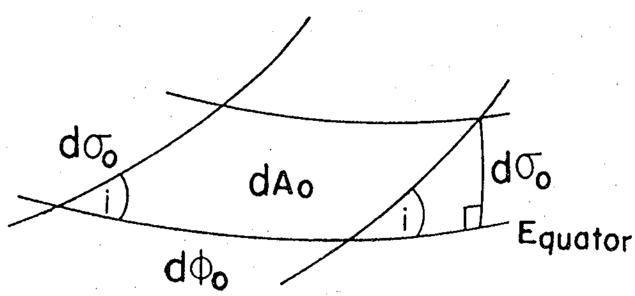

Figure 2.2

It is obvious that dA_o is given by

$$dA_o = r^2 \, d\phi_o \, d\theta_o \tag{2.1}$$

where r is the radius of the orbit. Let f_o denote the density of the satellites at the equator. Then, the number dN of satellites contained in dA_o is given by

$$dN = f_o \, dA_o \tag{2.2}$$

As these satellites move to latitude θ and right ascension ϕ, the corresponding area dA is then given by

$$dA = r^2 \cos\theta \, d\phi d\theta \tag{2.3}$$

and the density f is then obtained from

$$dN = f \, dA \tag{2.4}$$

Substitution of Equations (2.1) - (2.3) into (2.4) yields

$$f = \frac{f_o \, d\phi_o \, d\theta_o}{\cos\theta \, d\phi d\theta} \tag{2.5}$$

However, from Figure 2.1, we obtain the following spherical trigonometric formula

$$\sin\theta = \sin i \sin \sigma \tag{2.6}$$

so that at latitude θ we have

$$\cos\theta \, d\theta = \sin i \cos\sigma \, d\sigma \tag{2.7}$$

and at the equator we have

$$d\theta_o = \sin i \; d\sigma_o \tag{2.8}$$

Moreover, it is easily verified that we also have

$$d\phi = d\phi_o \tag{2.9}$$

$$d\sigma = d\sigma_o \tag{2.10}$$

Substitution of Equations (2.7) - (2.10) into (2.5) yields

$$f = \frac{f_o}{\cos \sigma} \tag{2.11}$$

which states that the density is inversely proportional to the cosine of the arc length.

Next, we obtain the equatorial density f_o as follows:

$$\begin{aligned} N &= \int_{\theta\,min}^{\theta\,max} \int_{o}^{2\pi} f \, r^2 \cos\theta \, d\phi \, d\theta \\ &= 2\pi r^2 \int_{\theta\,min}^{\theta max} f \cos\theta \, d\theta \\ &= 2\pi r^2 \int_{-\pi/2}^{\pi/2} f \sin i \cos\sigma \, d\sigma \\ &= 2\pi r^2 \int_{-\pi/2}^{\pi/2} f_o \sin i \, d\sigma \\ &= 2\pi^2 r^2 f_o \sin i \end{aligned} \tag{2.12}$$

in which Equations (2.7) and (2.11) have been used.

Substitution of Equation (2.12) into (2.11) yields

$$f = \frac{N}{2\pi^2 r^2 \sin i \cos \sigma} \tag{2.13}$$

which expresses the density in terms of the total number, the radius, the inclination and the orbital arc. However, it is more convenient to obtain an expression in terms of latitude than orbital arc. This is accomplished as follows: Using the identity

$$\cos^2 \sigma = 1 - \sin^2 \sigma \tag{2.14}$$

and also Equation (2.6), we obtain

$$\sin i \cos \sigma = \sqrt{(\sin^2 i - \sin^2 \theta)} \tag{2.15}$$

so that Equation (2.13) becomes

$$f = \frac{N}{2\pi^2 r^2 \sqrt{(\sin^2 i - \sin^2 \theta)}} \tag{2.16}$$

2.2 Angular Separation

The system of N satellites under discussion is considered to be uniformly distributed in terms of equatorial crossing and also along the orbital arc. It is easily verified that the angular separations between the satellites are given by

$$\Delta\phi = \frac{2\pi}{\sqrt{N}} \tag{2.17}$$

$$\Delta\sigma = \frac{2\pi}{\sqrt{N}} \tag{2.18}$$

SECTION 3 - REGION OF OBSERVATION

3.1 Geocentric Conical Angle

Consider a ground station G on the earth's surface. Let β denote the conical observation angle at the earth's surface, α the conical angle subtended at the earth's center, r_e the mean radius of the earth, h the satellite's altitude, and a the conical distance as illustrated in Figure 3.1.

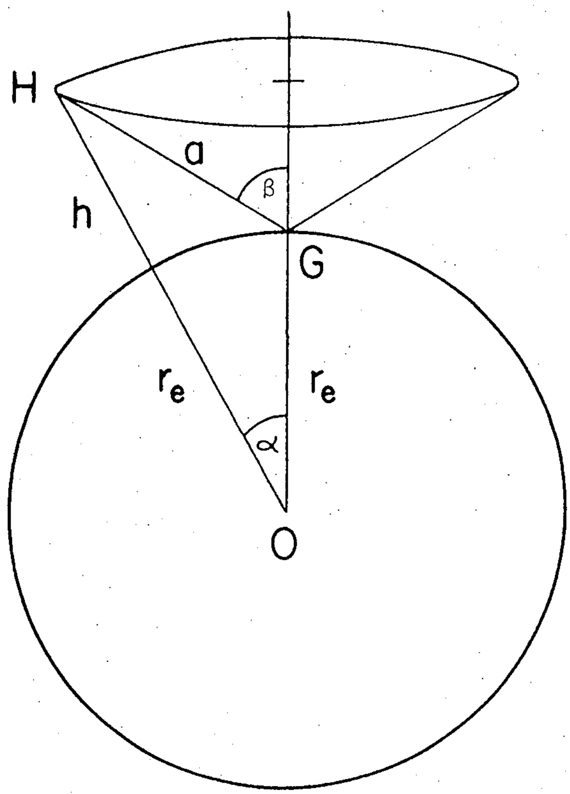

Figure 3.1

The geocentric conical angle α may be obtained as follows: For the triangle OGH, we have the sine formula

$$a = \frac{(r_e + h) \sin \alpha}{\sin (\pi - \beta)} \tag{3.1}$$

and the cosine formula

$$a^2 = r_e^2 + (r_e + h)^2 - 2r_e (r_e + h) \cos \alpha \tag{3.2}$$

Substitution of Equation (3.1) into (3.2) and use of the identity

$$\sin^2 \alpha = 1 - \cos^2 \alpha \tag{3.3}$$

yield the following quadratic equation for cos α

$$\cos^2\alpha - 2\left(\frac{r_e}{r_e + h}\right) \sin^2\beta \cos\alpha + \left[\left(\frac{r_e}{r_e + h}\right)^2 \sin^2\beta - \cos^2\beta\right] = 0 \tag{3.4}$$

whose solution is given by

$$\cos\alpha = \left(\frac{r_e}{r_e + h}\right) \sin^2\beta \pm \cos\beta \sqrt{\left[1 - \left(\frac{r_e}{r_e + h}\right)^2 \sin^2\beta\right]} \tag{3.5}$$

It may be verified that the physically acceptable solution is the one which yields the smaller angle α, i.e., the one with the positive sign in Equation (3.5). The other solution yields the larger angle α which results in a cone going into the earth, which is thus rejected.

3.2 Boundaries of Intersection Region

Consider Figure 3.2 which illustrates a ground station G and a ship S, and also the region of observation Ω common to both of them. Let C be the central point of the great circular arc GS, and γ the angle GOC.

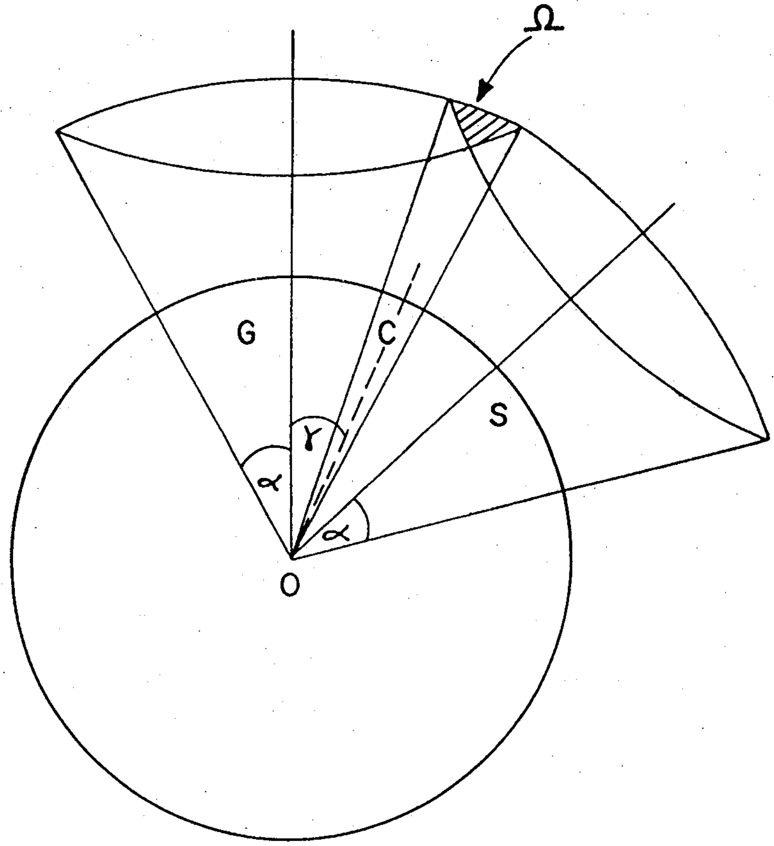

Figure 3.2

For simplicity, let G be on the equator and let S be at latitude θ_s and longitude ψ_s with respect to G, as illustrated in Figure 3.3.

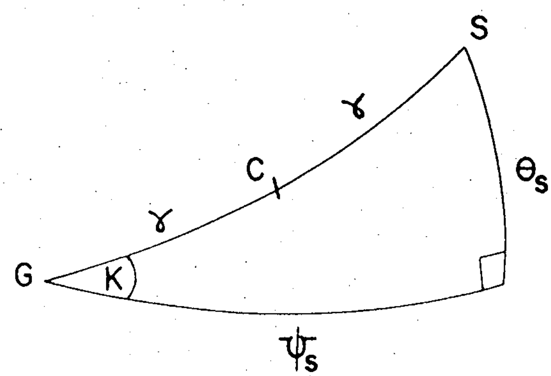

Figure 3.3

Then, from spherical trignometry, the arc length 2γ between G and S is given by

$$\cos 2\gamma = \cos \theta_s \cos \psi_s \tag{3.6}$$

and the inclination κ of S with respect to G is given by

$$\sin \theta_s = \sin \kappa \sin 2\gamma \tag{3.7}$$

Next, consider Figure 3.4 which illustrates the boundaries R and L of the intersection region Ω. It is to be noted that these boundaries are not arcs of great circles, but are arcs of small circles.

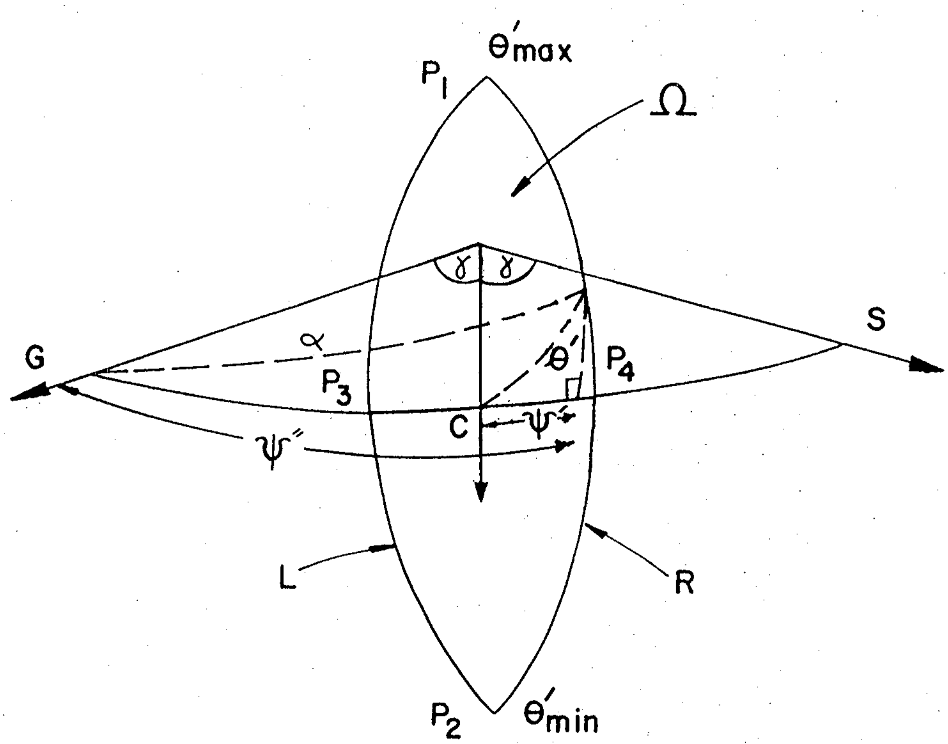

Figure 3.4

In order to obtain expressions for the boundaries R and L, it is convenient to consider the arc GS as the equator in an oblique coordinate system. First, consider the curve R.

Let θ' be the latitude and ψ'' be the longitude of a point with respect to G. Then, from spherical trigonometry, the equation of the curve R is given by

$$\cos\alpha = \cos\theta' \cos\psi'' \tag{3.8}$$

However, if ψ' denotes the longitude measured from C, then we have

$$\psi'' = \psi' + \gamma \tag{3.9}$$

and Equation (3.8) becomes

$$\cos\alpha = \cos\theta' \cos(\psi' + \gamma) \tag{3.10}$$

which is the equation for the boundary R in the oblique geographical system having C as the origin of latitude and longitude. Similarly, the equation for the boundary L is given by

$$\cos\alpha = \cos\theta' \cos(\psi' - \gamma) \tag{3.11}$$

The points of intersection of the curves R and L are given by P_1 ($\psi' = 0$, $\theta' = \theta'_{max}$) and P_2 ($\psi' = 0$, $\theta' = \theta'_{min}$) where

$$\theta'_{max} = \cos^{-1}\left(\frac{\cos\alpha}{\cos\gamma}\right) \tag{3.12}$$

$$\theta'_{min} = -\theta'_{max} \tag{3.13}$$

3.3 Regular to Oblique Geographic Transformation

Let $\vec{r} = (x,y,z)$ denote the coordinates of a point in the regular geographic system, and $\vec{r}' = (x', y', z')$ denote the corresponding coordinates of the same point in the oblique geographic system. Figure 3.5 illustrates the angular rotations to accomplish the necessary coordinate transformation.

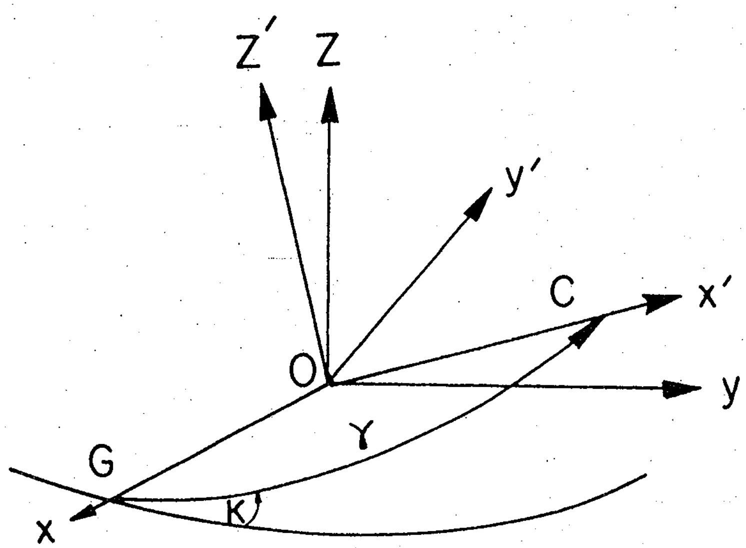

Figure 3.5

Let A denote the transformation matrix from $\vec{r}$ to $\vec{r}'$, i.e.,

$$\vec{r}' = A\vec{r} \tag{3.14}$$

Then, it is well-known that A is given by

$$A = \begin{bmatrix} c_\gamma & c_\kappa s_\gamma & s_\gamma s_\kappa \\ -s_\gamma & c_\kappa c_\gamma & c_\gamma c_\kappa \\ 0 & -s_\kappa & c_\kappa \end{bmatrix} \tag{3.15}$$

where the symbols s and c respectively denote the sine and cosine functions of the argument which appears as the subscript. Next, it is also noted that r and r' may be respectively expressed in terms of their latitude and longitude as follows:

$$\left.\begin{aligned} x &= r\ c_\theta c_\psi \\ y &= r\ c_\theta s_\psi \\ z &= r\ s_\theta \end{aligned}\right\} \qquad (3.16)$$

$$\left.\begin{aligned} x' &= rc_{\theta'} c_{\psi'} \\ y' &= rc_{\theta'} s_{\psi'} \\ z' &= rs_{\theta'} \end{aligned}\right\} \qquad (3.17)$$

Thus, Equations (3.14) - (3.17) may now be used to express the oblique latitude and longitude in terms of the regular ones. The final results are given by

$$s_\theta' = - s_\kappa c_\theta s_\psi + c_\kappa s_\theta \qquad (3.18)$$

$$\tan\ (\psi' + \gamma) = \frac{c_\kappa c_\theta s_\psi + s_\kappa s_\theta}{c_\theta c_\psi} \qquad (3.19)$$

3.4 Oblique to Regular Geographic Transformation

Next, to obtain the regular latitude and longitude in terms of the oblique ones, we proceed as follows: We note that

$$\vec{r} = A^T \vec{r}\,' \qquad (3.20)$$

where A^T denotes the transpose matrix of A.

Then, proceeding as before but now using Equations (3.20) and (3.15) - (3.17), we obtain

$$s_{\theta} = s_{\gamma}\ s_{\kappa}\ c_{\theta'}\ c_{\psi'} + c_{\gamma}\ s_{\kappa}\ c_{\theta'}\ s_{\psi'} + c_{\kappa}\ s_{\theta'} \tag{3.21}$$

$$\tan\psi = \frac{c_{\kappa}\ c_{\theta'}\ s_{(\psi' + \gamma)} - s_{\kappa}\ s_{\theta'}}{c_{\theta'}\ c_{(\psi' + \gamma)}} \tag{3.22}$$

SECTION 4 - TIME AVERAGE POPULATION

4.1 Exact Folmulation

Let N_Ω denote the number of satellites (time average population) within the common domain of observation Ω. It is obviously given by

$$N_\Omega = \iint_\Omega f \, d\Omega \tag{4.1}$$

where the density f is given by Equation (2.16) and the element of area $d\Omega$ is given by

$$d\Omega = r^2 \cos\theta \, d\psi \, d\theta \tag{4.2}$$

It appears that the above integral may be trivially expressed in terms of the regular latitude θ and longitude ψ as follows:

$$N_\Omega = \int_{\theta\ min}^{\theta\ max} \int_{\psi_L(\theta)}^{\psi_R(\theta)} \frac{N \cos\theta \, d\psi \, d\theta}{2\pi^2 \sqrt{(\sin^2 i - \sin^2\theta)}} \tag{4.3}$$

where $\psi_R(\theta)$ denotes the expression obtained by solving for ψ in terms of θ using the equation for the R curve given by Equations (3.10), (3.18), and (3.19), and similarly for $\psi_L(\theta)$ in terms of the L curve. Not only is this process difficult, but it is noted that the integral on the RHS of Equation (4.3) may not even be valid or, worse yet, amenable to numerical evaluation even in principle. This point becomes evident by combining Figures 3.3 and 3.4. It is possible that the location of the ship S with respect to the ground station G can give rise to the case where, in performing the integration with respect to ψ, the process does not take place from the L curve to the R curve and, furthermore, in performing the integration with respect to θ, the process also does not take place from $\theta\,min$ to $\theta\,max$. This difficulty can be circumvented by writing the element of area $d\Omega$ as follows

$$d\Omega = r^2 \cos\theta' \, d\psi' \, d\theta' \tag{4.4}$$

so that the integral becomes

$$N_\Omega = \int_{\theta' min}^{\theta' max} \int_{\psi'_L(\theta')}^{\psi'_R(\theta')} \frac{N \cos\theta' \, d\psi' \, d\theta'}{2\pi^2 \sqrt{\;} (\sin^2 i - \sin^2\theta)}$$

$$= \frac{N}{2\pi^2} \int_{\theta' min}^{\theta' max} \int_{\psi'_L(\theta')}^{\psi'_R(\theta')} \frac{\cos\theta' \, d\psi' \, d\theta'}{\sqrt{\left[s_i^{\,2} - (s_\gamma s_\kappa c_{\theta'} c_{\psi'} + c_\gamma s_\kappa c_{\theta'} \; s_{\psi'} + c_\kappa s_{\theta'})^2\right]}} \tag{4.5}$$

in which Equation (3.21) has been used. It is to be noted the ψ' integration will always proceed from the L curve to the R curve, and the θ' integration will always proceed from $\theta'\,min$ to $\theta'\,max$.

4.2 Approximate Formulation

An approximate formulation may be obtained by going back to the original Equation (4.1) which may be used to yield the following

$$N_\Omega = f_{ave}\Omega \tag{4.6}$$

where

$$\Omega = \iint_{\Omega} r^2 \cos\theta' \, d\psi' \, d\theta'$$

$$= 4r^2 \int_0^{\theta'_{max}} \int_0^{\psi_R'(\theta')} \cos\theta' \, d\psi' \, d\theta'$$

$$= 4r^2 \int_0^{\theta'_{max}} \cos\theta' \left[\cos^{-1} \left(\frac{\cos\alpha}{\cos\theta'} \right) - \gamma \right] d\theta' \qquad (4.7)$$

in which Equations (3.10) and (3.12) have been used. This integral may be evaluated numerically once the relative position of the ship S is specified. The average value of f to be used may be obtained by averaging the 4 values at the mid-points on the axes of symmetry of Ω. These, in turn, may be obtained by averaging the values at the center C and those at the extremities P_i illustrated in Figure 3.4. Thus, we may write

$$f_{ave} = \frac{1}{8} \left[f(P_1) + f(P_2) + f(P_3) + f(P_4) + 4f(C) \right] \qquad (4.8)$$

SECTION 5 - AVERAGE POPULATION TIMES

5.1 Exact Formulation

First consider Figure 2.1, for which we may write the following spherical trigonometric formulas

$$\sin \theta = \sin i \sin \sigma \tag{5.1}$$

$$\cos \sigma = \cos \theta \cos \phi \tag{5.2}$$

where the relevant quantities have already been previously defined in Section 2. Next, consider Figure 5.1 which illustrates the ground station at G, when the satellite crosses the equator at N_1. Subsequently, when the satellite has moved to latitude θ, the rotation of the earth has taken the ground station to the point G.

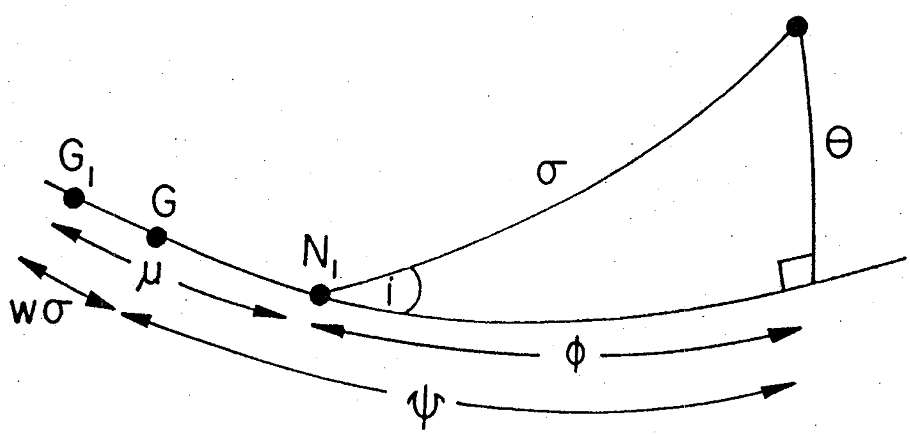

Figure 5.1

Then, it is obvious that the following relation holds for both direct ($i < \pi/2$) and retrogade ($i > \pi/2$) orbits

$$\mu + \phi = \psi + \omega\sigma \tag{5.3}$$

where

$$\omega \equiv \frac{P}{P_e} \tag{5.4}$$

μ = longitude of satellite crossing measured from ground station

ϕ = right ascension of satellite measured from equatorial crossing

ψ = longitude of satellite measured from ground station

σ = orbital arc of satellite measured from equatorial crossing

ω = ratio of satellite orbital period P to earth rotational period P_e

Substitution of Equation (5.3) into (5.2) yields

$$\cos\sigma = \cos\theta \cos (\psi + \omega\sigma - \mu) \tag{5.5}$$

Equations (5.1) and (5.5) express the latitude and longitude in terms of the orbital arc. Symbolically, we may write

$$\theta = \theta \ (\sigma; i) \tag{5.6}$$

$$\psi = \psi \ (\sigma; i, \mu) \tag{5.7}$$

In turn, these equations may be substituted into Equations (3.18) and (3.19) to yield expressions for the oblique latitude θ' and longitude ψ' in terms of orbital arc σ. Thus, we have

$$\theta' = \theta' \ (\theta, \psi; \kappa, \gamma) = \theta' \ (\sigma; i, \mu, \kappa, \gamma) \tag{5.8}$$

$$\psi' = \psi' \ (\theta, \psi; \kappa, \gamma) = \psi' \ (\sigma; i, \mu, \kappa, \gamma) \tag{5.9}$$

which constitute 2 equations in the 3 unknowns θ', ψ' and σ. If we wish to determine the point of intersection with the R curve bounding one side of the common region of observation Ω, we also have Equation (3.10) which is

$$\cos \chi = \cos \theta' \cos (\psi' + \gamma) \tag{5.10}$$

Substitution of Equations (5.8) and (5.9) into (5.10) yields a complicated equation for σ which may then be solved numerically to obtain the value $\sigma = \sigma_R$ corresponding to the intersection point. Next, to obtain the point of intersection with the L curve, we have Equation (3.11) which is really Equation (5.10) with γ replaced by $-\gamma$. Thus, the same process may be used to obtain the value $\sigma = \sigma_L$ corresponding to the intersection point. Thus, the population time τ of the satellite within the region Ω is exactly given by

$$\tau = \frac{P}{2\pi} (\sigma_R - \sigma_L) \tag{5.11}$$

Let μ_c denote the value of μ which corresponds to the orbit passing through the central point C. The above process is first performed with a value $\mu = \mu_c + \Delta\phi_o$ where $\Delta\phi_o$ is a random number in the range $0 \leq \Delta\phi_o < \Delta\phi$ where $\Delta\phi$ is given by Equation (2.17) which is

(5.12)

The process is then repeated with values $(\mu + n\Delta\phi)$ where $n = \pm 1, \pm 2, \ldots$ until no more orbits intersect the region Ω. After this, the entire above process is then repeated with other random values of $\Delta\phi_o$. The average population times are then obtained by averaging the results of all these processes.

5.2 Approximate Formulation

Consider Figure 5.2 which illustrates the spherical triangle formed by the equator, the meridian and the arc length of the central point C measured from the ground station G. This spherical triangle is fixed on the rotating earth.

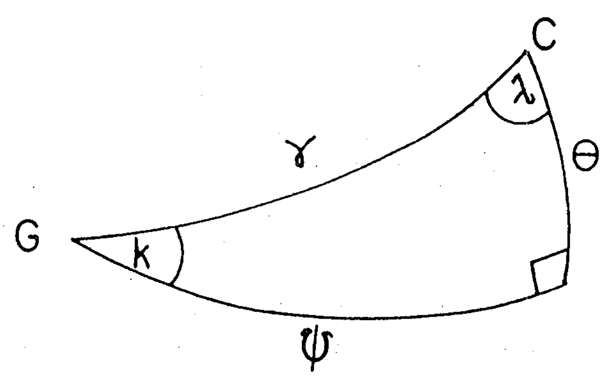

Figure 5.2

Then, we have the following spherical trigonometric formulas

$$\sin\theta = \sin\kappa \ \sin\gamma \tag{5.13}$$

$$\cos\gamma = \cos\theta \ \cos\psi \tag{5.14}$$

$$\sin\gamma \ \sin\lambda = \sin\psi \tag{5.15}$$

which may be used to compute the latitude and longitude of C and also the angle λ the arc GC makes with the meridian through C.

Next, consider Figure 5.3 which illustrates the spherical triangle formed by the equator, the meridian and the orbital arc of a satellite just passing through the point C. This spherical triangle is fixed on the celestial sphere, which is inertial.

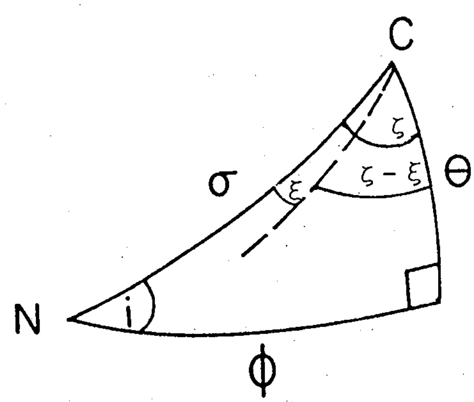

Figure 5.3

Then, we have the following spherical trigonometric formulas

$$\sin\theta = \sin i \ \sin\sigma \tag{5.16}$$

$$\cos\sigma = \cos\theta \ \cos\phi \tag{5.17}$$

$$\sin\sigma \ \sin\zeta = \sin\phi \tag{5.18}$$

which may be used to compute the orbital arc, the right ascension and also the angle ζ the arc NC makes with the meridian through C.

However, because of the earth's rotation, the satellite's ground track does not really make an angle ζ with the meridian through C. Rather, it is deflected through an angle ξ which is, in general, given by

$$\tan \xi = \frac{\omega \cos\theta \cos\zeta}{1 - \omega \cos i} \tag{5.19}$$

where ω is defined by Equation (5.4). (It may be verified that this deflection causes direct orbits to be more inclined and retrograde orbits to be less inclined as viewed by their ground tracks.) Thus, the angle between the satellite ground track and the meridian at point C is given by $(\zeta - \xi)$, as shown in Figure 5.3. Next, consider Figure 5.4 which illustrates the inclination η of the orbital arc with the oblique equator

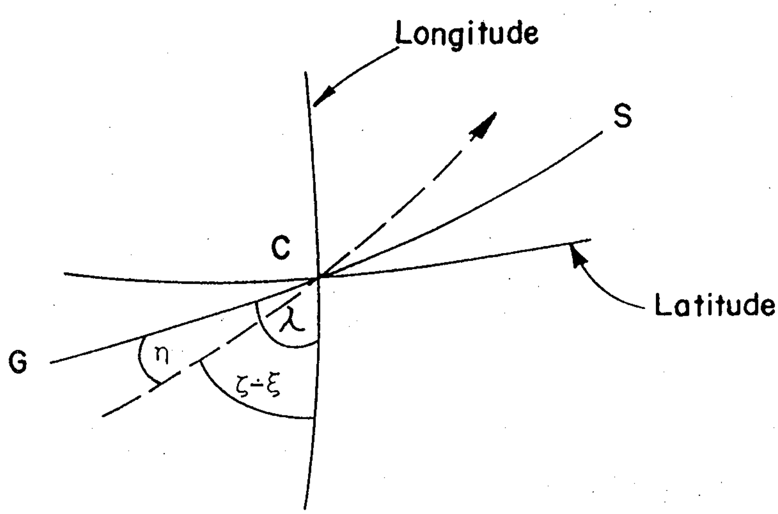

Figure 5.4

Hence, it is seen that we have

$$\left.\begin{aligned} \eta &= \lambda - \zeta + \xi && \text{for ascending orbits} \\ \eta &= \lambda + \zeta - \xi - \pi && \text{for descending orbits} \end{aligned}\right\} \tag{5.20}$$

It may also be verified that these equations are algebraically valid for both direct and retrograde orbits.

Next, consider Figure 5.5 which illustrates the case of a satellite just passing through the point D which is displaced by $\Delta\gamma$ from the central point C. This corresponds to an orbit whose equator crossing is displaced by $\Delta\phi$ from the point N.

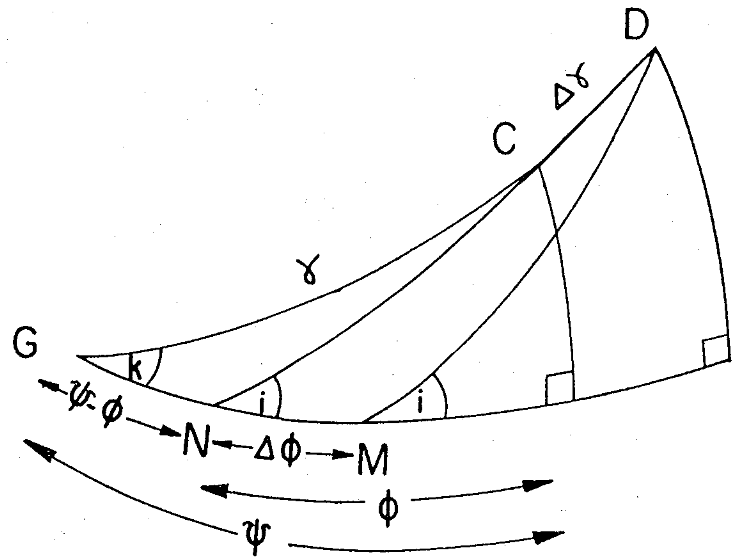

Figure 5.5

Then, using spherical trigonometric formulas, it may be shown that $\Delta\gamma$ is related to $\Delta\phi$ by the following equation

$$\tan(\gamma + \Delta\gamma) = \frac{\tan i \sin(\psi - \phi + \Delta\phi)}{\tan i \cos\kappa \cos(\psi - \phi + \Delta\phi) - \sin\kappa} \tag{5.21}$$

Thus, by replacing γ with $(\gamma+\Delta\gamma)$, Equations (5.13) - (5.20) may be used to compute the inclination η of the new orbital arc with the oblique equator. It may be verified that Equation (5.21) is also algebraically valid for both the cases of $i > \kappa$ and $i < \kappa$. Moreover, it is also valid for both direct and retrograde orbits. Furthermore, it is valid for arbitrary finite differences $\Delta\phi$ and $\Delta\gamma$, but considerable care must be exercised when taking the inverse tangent to obtain $(\gamma + \Delta\gamma)$ in the correct quadrant corresponding to the increment $\Delta\phi$.

Up to this point, no approximations have been made. It is now assumed that the satellite's ground track is an arc of a great circle lying within the region Ω and making an angle η with the oblique equator GS. Figure 5.6 illustrates the cases of orbital arcs passing through the points C and D.

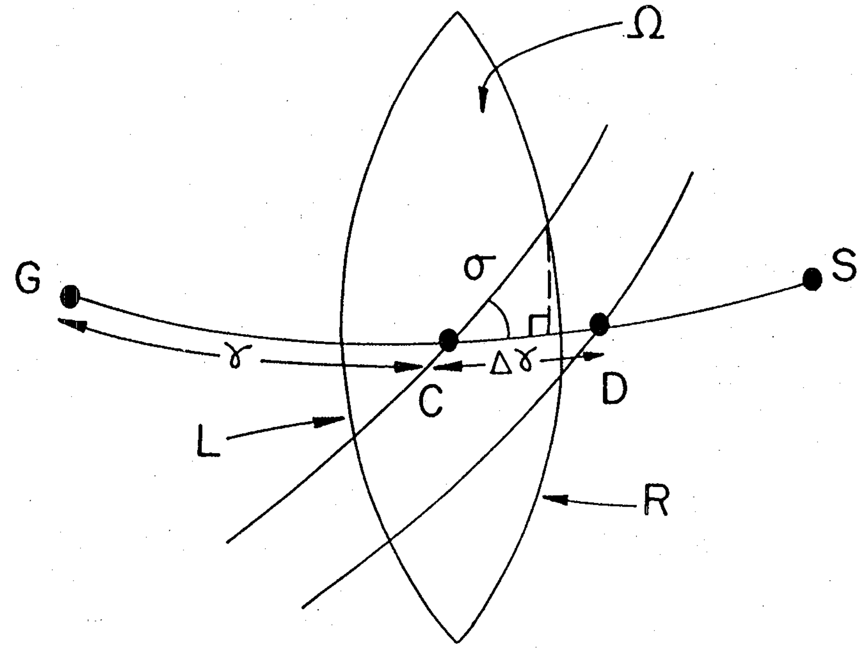

Figure 5.6

Now, it is possible to write the following two approximate relations for the orbit passing through the central point C

$$\sin \theta' = \sin\eta \sin \sigma' \tag{5.22}$$

$$\cos \theta' = \cos \theta' \cos \psi' \tag{5.23}$$

where σ' is the arc length measured from the oblique equatorial crossing. These two equations are the crude analogs of Equations (5.8) and (5.9) of the exact case. If we wish to determine the point of intersection with the R curve, we also have Equation (3.10) which is

$$\cos\alpha = \cos \theta' \cos (\psi' + \gamma). \tag{5.24}$$

However, instead of substituting Equations (5.22) and (5.23) into (5.24) to yield a complicated equation for σ', it turns out to be the case that an algebraic equation can be obtained involving $\sin\theta'$. This is accomplished as follows: From Equations (5.22) and (5.23), the following auxiliary equation is obtained

$$\sin\psi' = \frac{\tan\theta'}{\tan\eta} \tag{5.25}$$

Equation (5.24) is then written as

$$\begin{aligned} \cos\alpha &= \cos\theta' (\cos\psi' \cos\gamma - \sin\psi' \sin\gamma) \\ &= \cos\sigma' \cos\gamma - \cos\theta' \frac{\tan\theta'}{\tan\eta} \sin\gamma \\ &= \cos\gamma \sqrt{\left(1 - \frac{\sin^2\theta'}{\sin^2\eta}\right)} - \frac{\sin\theta'}{\tan\eta} \sin\gamma \end{aligned}$$

or equivalently

$$\cos\gamma\sqrt{(\sin^2\eta - \sin^2\theta')} = \sin\eta \cos\alpha + \cos\eta \sin\gamma \sin\theta'. \tag{5.26}$$

By squaring both sides of this equation, it is obvious that a quadratic equation is obtained involving $\sin\theta'$. After much simplification, it may be shown that we have

$$\frac{\sin\theta'}{\sin\eta} = \frac{-\cos\alpha \sin\gamma \cos\eta \pm \cos\gamma \sqrt{(\sin^2\alpha - \sin^2\gamma \sin^2\eta)}}{(1 - \sin^2\gamma \sin^2\eta)} \tag{5.27}$$

A little consideration will reveal that for the intersection point with the R curve, it is necessary to retain only the positive sign in the above equation. Thus, this expression corresponds to the value at $\theta' = \theta'_R$.

However, from Equation (5.22), it is seen that the value σ_R' is given by

$$\sin\theta_R' = \sin\eta \quad \sin\sigma'_R \tag{5.28}$$

Consequently, we have

$$\sin\sigma_R' = \frac{-\cos\alpha \sin\gamma \cos\eta + \cos\gamma \quad \sqrt{(\sin^2\alpha - \sin^2\gamma \sin^2\eta)}}{(1 - \sin^2\gamma \sin^2\eta)} \tag{5.29}$$

Next, to obtain the intersection point with the L curve, we have Equation (3.11) which is really Equation (5.24) with γ replaced by $-\gamma$. Thus, the same process may be used to obtain σ_L' which can be shown to be given by retaining the negative sign in Equation (5.27). Consequently, we have the following result

$$\sin\sigma_L' = \frac{\cos\alpha \sin\gamma \cos\eta - \cos\gamma\sqrt{(\sin^2\alpha - \sin^2\gamma \sin^2\eta)}}{(1 - \sin^2\gamma \sin^2\eta)} \tag{5.30}$$

which states that $\sigma_L' = -\sigma_R'$ as expected (only for the case of the orbit passing through the central point C). Thus, the population time τ of the satellite within the region Ω is approximately given by

$$\tau = \frac{P}{2\pi} \quad (\sigma_R' - \sigma_L') \tag{5.31}$$

which is the crude analog of Equation (5.11).

Next, to obtain the intersection point between the R curve and the orbit passing through the point D, a little consideration will reveal that it suffices to replace γ by $(\gamma + \Delta\gamma)$ and also use the corresponding value of η and then repeat the process above for computing σ_R' as given by Equation (5.29). However, to obtain the intersection point between the L curve and the orbit passing through D, a little more caution is now necessary. It now suffices to replace γ by $(-\gamma + \Delta\gamma)$ and also use the corresponding value of η and then repeat the process above for computing σ_R' but now retain the negative sign. This result yielding the value of σ_L' is no longer trivially the negative of σ_R' as for the special case of C.

The above process is first performed with a random value $\Delta\phi_o$ in the range $0 \leq \Delta\phi_o < \Delta\phi$ where $\Delta\phi$ is given by Equation (2.17) which is

$$\Delta\phi = \frac{2\pi}{\sqrt{N}} \tag{5.32}$$

The process is then repeated with values $(\Delta\phi_o + n\Delta\phi)$ where $n = \pm 1, \pm 2, \ldots$ until no more orbits intersect the region Ω. After this, the entire above process is then repeated with other random values of $\Delta\phi_o$. The average population times are then obtained by averaging the results of all these processes.

Finally, it must be mentioned that in order to insure that the correct segment of the R circle (see Figure 5.6) is identified to yield the desired intersection point as given by the general analog of Equation (5.29), a little consideration will reveal that we must have η in the range $-\pi/2 < \eta \leq \pi/2$. Thus, if η is outside this range, we must accordingly add to or subtract π from η. Similarly, the same procedure applies to insure the identification of the correct segment of the L circle to yield the desired intersection point as given by the general analog of Equation (5.30). Furthermore, considerable thought will reveal that this assignment of the η range not only correctly gives the desired intersection points for orbits crossing the oblique equator within the observation region Ω, but also for the case of equatorial crossings outside it for a range of $\Delta\gamma$ exceeding $\pi/2$ measured from the central point C. The reasons for this are not apparent and, at first sight, it would seem that this assignment of η values outside the region Ω leads to incorrect answers. But this is not so because of the manner in which the inverse trigonometric functions are assigned their principal values. Thus, Equations (5.29) and (5.30) contain many subtle features in logic which automatically combine to yield, in mutual accord, the correct intersection points regardless of the equatorial crossing. In particular, additional consideration will reveal that it is only necessary to consider equatorial crossings such that the orbits intersect the oblique meridian through the central point C at an oblique latitude θ' not greater than θ^* given by

$$\theta^* = \min\left\{ \left|\eta\right| , \cos^{-1}\left(\frac{\cos\alpha}{\cos\gamma}\right)\right\} \tag{5.33}$$

This corresponds to a range $\Delta\gamma^*$ given by

$$\Delta\gamma^* = \sin^{-1}\left(\frac{\tan\ \theta^*}{\tan\left|\eta\right|}\right) \tag{5.34}$$

so that

$$\Delta\gamma^* = \min\left\{\pi/2,\ \sin^{-1}\left(\frac{\tan\left[\cos^{-1}\left(\frac{\cos\alpha}{\cos\gamma}\right)\right]}{\tan\left|\eta\right|}\right)\right\} \tag{5.35}$$

or equivalently

$$\Delta\gamma^* = \min\left\{\pi/2,\ \sin^{-1}\sqrt{\left(\frac{\cos^2\gamma - \cos^2\alpha}{\cos^2\gamma - \cos^2\alpha + \cos^2\alpha\ \tan^2\eta}\right)}\right\} \tag{5.36}$$

It is not difficult to see that if an orbit intersects the oblique equator outside the range $\Delta\gamma^*$ and also eventually intersects the observation region Ω, then this orbit would already have been counted as lying within the acceptable range on the other side of the central point.

SECTION 6 - RESULTS FOR ORBITING SATELLITES

6-1 Average Population Time Computations

Computations were performed, except for minor modifications, according to the method discussed in Section 5.2 to obtain the average population times for Class I and II satellites. The representative values of parameters used are shown in Table 6.1.

Table 6.1

Quantity	Class I	Class II
Period P (minutes)	100.9	717.9
Inclination i (degrees)	74.0	63.9
Altitude h (km)	800	20,178.5
Number N	400	100

The value of β, the conical observation angle at the earth's surface, is taken to be 80° for both the ground station and the ship. The ground station is taken to be at the origin of longitude and latitude while the ship is taken to be at various values of longitude ψ_s and latitude θ_s only in the first quadrant. It may be verified that for locations of the ship in the other quadrants, the corresponding results may be obtained by taking mirror reflections about the primary axes.

After the average population times τ have been obtained, the results were divided by the characteristic time T defined by

$$T = \frac{P}{\sqrt{N}} \tag{6.1}$$

to obtain the number of satellites visible to both the ground station and the ship. (T is the time for a satellite to travel the intra-satellite distance $\Delta\sigma$ where $\Delta\sigma$ is given by Equation (2.18).) The relevant results for Class I and II satellites are respectively summarized in Figures 6.1 and 6.2, each of which was obtained by averaging the results using ξ given by Equation (5.19) and those using $\xi = 0$.

Note: (1) Numbers in the boxes denote the number of satellites visible for the percent of time indicated.

(2) Numbers below the boxes denote the relative longitude and the absolute latitude of the ship.

2 30%
4 17%

(0,30)

2 84%
4 58%
6 2%

(0,20)

1 43%
2 26%
3 21%

(10,20)

4 100%
6 39%
10 12%

(0,10)

1 97%
2 75%
3 65%
4 46%

(10,10)

2 68%
3 43%
4 32%

(20,10)

4 100%
6 90%
8 56%
10 21%

(0,0)

4 100%
6 64%
8 17%

(10,0)

2 100%
4 63%
6 1%

(20,0)

2 32%
4 8%

(30,0)

Figure 6.1 - Results for 100 Minute Orbiting Satellites

Note: (1) Numbers in the boxes denote the number of satellites visible for the percent of time indicated.

(2) Numbers below the boxes denote the relative longitude and the absolute latitude of the ship.

6	100%
8	37%
12	58%
14	50%

(0,80)

5	100%
6	94%
7	88%
9	57%
10	32%
11	25%
12	7%

(40,80)

12	100%
14	75%
16	23%
18	15%
22	3%

(0,60)

2	100%
3	85%
4	78%
5	74%
6	60%

(40,60)

14	100%
18	60%
20	51%
22	34%
24	12%
26	7%

(0,40)

3	100%
4	93%
5	76%
6	49%
7	23%
8	19%
9	5%

(40,40)

2	100%
3	90%
4	83%
5	68%
6	56%
7	24%
8	19%

(80,40)

16	100%
18	97%
20	75%
24	54%
26	16%
28	5%

(0,20)

8	100%
10	93%
12	71%
13	24%
14	3%

(40,20)

5	100%
6	81%
7	20%
10	15%

(80,20)

20	100%
22	88%
24	63%
26	56%
28	39 %
30	29%
32	14%

(0,0)

14	100%
16	66%
18	56%
20	21%
24	8%
26	2%

(40,0)

6	100%
8	58%
10	19%
12	11%
14	6%

(80,0)

2	41%
4	10%

(120,0)

Figure 6.2 - Results for 12 Hour Orbiting Satellites

6.2 Time Average Population Computations

For the special case of the ship at the origin of longitude and latitude, the time average population N_Ω may be computed by Equation (4.3). Numerical integration yields a value of about 28.48% for N_Ω/N for Class II satellites. That is, on the average, 28.48 satellites are mutually visible to the ground station and the ship when they are together.

As a comparison, it may be shown that the ratio of the area of common visibility Ω to the area of the zonal belt A covered by the satellite orbits is given by

$$\frac{\Omega}{A} = \frac{\left[1 - \sin(\pi/2 - \alpha)\right]}{2 \sin i} \qquad (6.2)$$

when the ground station and ship are together. Hence, for Class II satellites, we obtain a value of about 22.4% for Ω/A. As expected, this value is smaller than that for N_Ω/N because the density f increases with latitude and hence contributes toward giving a higher value of N_Ω in the numerical integration.

The other comparison is made with the results displayed in the (0,0) box of Figure 6.2 which are seen to yield a smaller value than that for N_Ω/N. This is also to be expected because the approximation made in Section 5.2 assumed that the satellite orbits are arcs of great circles within the region Ω and hence yields a smaller value of the average population time τ than that obtained by considering the actual satellite ground track.