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NOTATION

P	projection operator from X onto $\tilde{X}$
X	Banach space of periodic, continuous functions on $[0,1] \times [0,1]$ for the first case and similar definitions for the other cases
$\tilde{X}$	piecewise linear subspace of X
$\bar{X}$	Banach space isomorphic to $\tilde{X}$
Δ_{if}	$\equiv \Delta_i \Delta_f$; $\lvert\Delta_{if}\rvert = \lvert\Delta_i\rvert \lvert\Delta_f\rvert$; if $\lvert\Delta\rvert = \lvert\Delta_{if}\rvert \forall_{ij}$, then $\lvert\Delta_{if}\rvert = \lvert\Delta^2\rvert$
Δ_{ij}	$\equiv$ ijth rectangle of the domain
ϕ	linear extension of ϕ_0 to X; $\phi \equiv \phi_0 P$
ϕ_0	a mapping creating an isomorphism between $\tilde{X}$ and $\bar{X}$
$\omega_s(\delta)$	modulus of continuity of the kernel function $h(s,t)$ relative to s; $\omega_s(\delta) = \sup\lvert h(s+\sigma,t) - h(s,t)\rvert (s \geq 0,\ t \leq 1,\ \lvert\sigma\rvert \leq \delta)$
$\forall$	for all
$\exists$	there exists
$\lVert\ \rVert$	norm in the appropriate space
$\ni$	such that
$\Rightarrow$	implies
$C^{\eta}(S)$	η times continuously differentiable on S
p	pressure
u	component velocity in x-direction
v	component velocity in y-direction
$\Delta[\ ]$	$\partial^2[\ ]/\partial x^2 + \partial^2[\ ]/\partial^2[\ ]/\partial y^2$
$\Delta\Delta[\ ]$	$\partial^4[\ ]/\partial x^2 + 2\dfrac{\partial^2\partial^2[\ ]}{\partial x^2 \partial y^2} + \partial^4[\ ]/\partial y^4$
∂S	boundary of domain S
$[\ ]_\alpha$	$\dfrac{\partial}{\partial\alpha}[\ ]$

A COMPUTER SOLUTION TO THE STATIONARY NAVIER-STOKES EQUATIONS IN TWO DIMENSIONS WITH PROVEN CONVERGENCE

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SUMMARY

A general computer solution for the stationary Navier-Stokes equations is developed. It is explicitly proved that this solution converges to the true solution as the grid size shrinks to zero.

INTRODUCTION

Existing algorithms for solving the Navier-Stokes equations have one critical deficiency: if ψ^* were the true solution and $\tilde{\psi}$ the computer solution, the relationship between $\tilde{\psi}$ and ψ^* would be strictly heuristic [ ]. In this report, however, the relationship between ψ^* and $\tilde{\psi}$ will be explicitly demonstrated, and it will be proved that $\tilde{\psi}$ converges to ψ^* as the grid size shrinks to zero.

MAIN DEVELOPMENT

The two-dimensional stationary Navier-Stokes equations are typically expressed by the set of equations

$$uu_x + vu_y + q_x - \nu\Delta u + f_1 = 0 \tag{1}$$

$$uv_x + vv_y + q_y - \nu\Delta v + f_2 = 0 \tag{2}$$

$$u_x + v_y = 0 \tag{3}$$

with

$$u(\partial s) = -b_2 \ , \qquad v(\partial s) = b_1 \tag{4}$$

Equations (1) and (2) express the conservation of momentum and equation (3) expresses conservation of mass, where u and v denote the velocity components in the x and y directions, respectively, and ∂s denotes the boundary of the domain of consideration s.

Solving the set of equations (1)-(4) is equivalent to solving the equation

$$\nu\Delta\Delta\psi + \psi_y\Delta\psi_x - \psi_x\Delta\psi_y + f_{1_y} - f_{2_x} = 0 \tag{5}$$

with

$$\psi_x(\partial s) = b_1 , \quad \psi_y(\partial s) = b_2 \tag{6}$$

for the stream function ψ (see ref. 1), where $-\psi_y = u$ and $+\psi_x = v$.

In reference 1, it is shown that solving equations (5) and (6) is equivalent to solving a sequence of linear partial differential equations:

$$\nu\Delta\Delta\tilde{\psi}_m + \psi_{m_y}\Delta\tilde{\psi}_{m_x} + \Delta\psi_{m_x}\tilde{\psi}_{m_y} - \Delta\psi_{m_y}\tilde{\psi}_{m_x} - \psi_{m_x}\Delta\tilde{\psi}_{m_y} + P(\psi_m) = 0 \tag{7}$$

with

$$\tilde{\psi}_m(\partial s) = 0 , \quad \frac{\partial\psi_m}{\partial n}(\partial s) = 0 , \quad m = 0,1,2, \ldots \tag{8}$$

where

$$P(\psi) \equiv \nu\Delta\Delta\psi + \psi_y\Delta\psi_x - \psi_x\Delta\psi_y + f_{1_y} - f_{2_x}$$

$$\tilde{\psi}_m \equiv \psi_{m+1} - \psi_m$$

By inspection, we see that each element of this sequence is of the form

$$\nu\Delta\Delta\tilde{\psi} + \psi_y\Delta\tilde{\psi}_x + \Delta\psi_x\tilde{\psi}_y - \Delta\psi_y\tilde{\psi}_x - \psi_x\Delta\tilde{\psi}_y + P(\psi) = 0 \tag{9}$$

with

$$\tilde{\psi}(\partial s) = 0 \,, \qquad \tilde{\psi}_n(\partial s) = 0 \tag{10}$$

Lemma 1:

A solution of equations (9) and (10) is a solution of equation (11), and vice versa:

$$\tilde{\psi}_m(x',y') + \int_S \tilde{\psi}_m(x,y)K_m(x',y',x,y)\,dx\,dy = f(\psi_m) \tag{11}$$

where

$$f(\psi_m) = \frac{1}{\nu}\int_S P(\psi_m)G\,dS$$

and

$$K_m(x',y',x,y) = -\Delta G_y(x',y',x,y)\psi_{m_x}(x,y) + \Delta G_x\psi_{m_y} + 2G_{xy}(\psi_{m_{yy}} - \psi_{m_{xx}})$$

$$+ \psi_{m_{xy}}(G_{xx} - G_{yy})$$

and

G denotes the biharmonic Green's function of the first type on S (see ref. 2 for the proof).

Consequently, the initial problem — equations (1)-(4) — has been reduced to one of solving a sequence of Fredholm integral equations of the second kind, as expressed in equation (11). In particular, if ψ^* denotes the true solution to equations (1)-(4), then

$$\lim_{m\to\infty} \psi_m = \psi^* - \psi_o$$

Of course, one could stop with equations (9) and (10) and strictly solve them numerically, which has been done with great success for the nonstationary case in reference 3. However, if one were to computerize equations (9) and (10), one would end up in the following situation:

1. From reference 1, we would have a sequence of solutions ψ_m converging to the true solution ψ^*. Also,

$$\|\psi^* - \psi_m\| \leq 2^{2^m - m} \left[\frac{H_4}{\nu\left(1 - \frac{1}{\nu} M_{\psi_o} H_3\right)} \right]^{2^{m+1}-1} \|P(\psi_o)\|^{2m} .$$

2. If ψ_{mn} were the actual computer solution for ψ_m, it would be practically impossible to determine the accuracy of ψ_{mn} relative to ψ_m, since equation (9) is an unbounded operator equation.

However, because of the results of Lemma 1, the original problem has been reduced to one of solving a sequence of bounded linear operator equations; moreover, the equations are in a form in which it is possible to explicitly determine the accuracy between the computer solution and the true solution. The one real drawback with the integral equation representation is that the kernel is a function of the biharmonic Green's function (see eq. (11)), an area primarily familiar to those in structural mechanics.

Therefore, it remains to develop a computer solution with error estimates for equation (11). To accomplish this, we must first determine the character of the kernel of equation (11).

The kernel of equation (11) is of the form

$$K_m = -\Delta G_y \psi_{m_x} + \ldots$$

In reference 4, it is demonstrated that G can be expressed in the form

$$G = \frac{1}{8\pi} r^2 \log r + v$$

with v a regular biharmonic function in S with continuous fourth derivatives throughout S. Let $G = \bar{G} + v$, where

$$\bar{G} = \frac{1}{8\pi} r^2 \log r$$

By differentiating $\bar{G}$, it follows directly that

$$\Delta\bar{G}_y = \frac{1}{8\pi}\left(\frac{4\sin\theta}{r}\right)$$

$$\Delta\bar{G}_x = \frac{1}{8\pi}\left(\frac{4\cos\theta}{r}\right)$$

$$\bar{G}_{yy} - \bar{G}_{xx} = \frac{2}{8\pi}\left(\sin^2\theta - \cos^2\theta\right)$$

and

$$\bar{G}_{xy} = \frac{1}{8\pi}\left(2\cos\theta\sin\theta\right)$$

Therefore, the kernel $K_m(P,Q)$ of equation (11) is continuous except as P approaches Q and there it is of the form $1/r_{PQ}$. In reference 5, a numerical algorithm is developed for two-dimensional Fredholm integral equations of the second kind with kernels of the form $\log(1/r_{PQ})$. For this case it is proved (ref. 5) that the computer solution converges, as a function of grid size Δ, to the true solution (as $\Delta \to 0$). It just so happens that the results of reference 5 also hold true if the kernel is of the form $1/r_{PQ}$ as $P \to Q$ and continuous otherwise. The argument goes through in exactly the same manner for $1/r$ as it did for $\log(1/r)$. In essence, by substituting $1/r$ for $\log(1/r)$ in the development, and by slightly changing the proofs for the new singularity, the results immediately follow.

Let $\tilde{\psi}_m^{\Delta}$ be the computer solution of ψ_m — the solution of equation (11) — for partition Δ relative to reference 6. It therefore follows that

$$\tilde{\psi}_m^{\Delta} \to \psi_m \qquad \text{as} \qquad \Delta \to 0$$

Also, relative to reference 1, $\psi_m \to \psi^*$ as $m \to \infty$, where ψ^* is the true solution of the problem (1-4). Hence,

$$\tilde{\psi}_m^{\Delta} \to \psi^* \qquad \text{as} \qquad \Delta \to 0\ , \qquad m \to \infty$$

since

$$\|\tilde{\psi}_m^{\Delta} - \psi^*\| \le \|\tilde{\psi}_m^{\Delta} - \psi_m\| + \|\psi_m - \psi^*\| \to 0 \qquad \text{as} \qquad m \to \infty\ , \qquad \Delta \to 0$$

Hence, by applying the results proved in references 1, 2, 5, and 6, it has been possible to develop a computer solution $\tilde{\psi}_\Delta$ as a function of grid size Δ such that $\tilde{\psi}_\Delta$ converges to ψ^* as Δ approaches zero, providing of course that the conditions are satisfied as specified herein.

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