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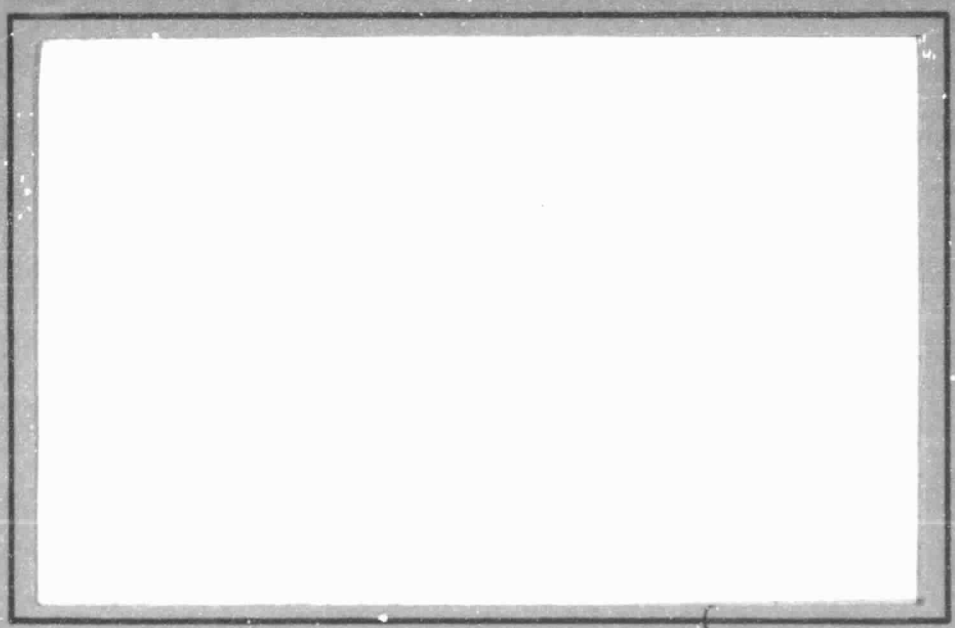

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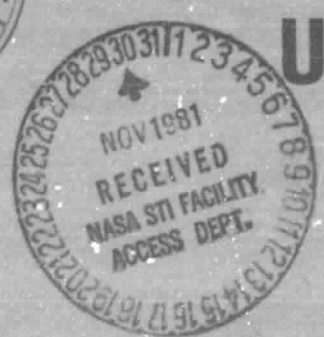

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Computational Methods of Robust Controller Design for Aerodynamic Flutter Suppression

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Sincere thanks is due to Professor Dennis Brewer for the elegant mathematical proof in the second paper of this report, to my very capable graduate research assistants Mr. Rasim Baykan and Mr. Alok Das for their major contribution to this work, and to Mrs. Mary Moore and Miss Sherry Layman for their fine typing of drafts and final papers.

TABLE OF CONTENTS

Introduction

The major focus of this research work has been the continuing development of "Riccati Iteration," a new numerical tool for the design and analysis of linear control systems. This report is arranged into four chapters consisting of two research papers, an extensive literature review, and a viewgraph summary of directions for future research work. The Masters degree thesis "Comparison of Algebraic Riccati Equation solvers" by Mr. Rasim Baykan was also supported by this research grant. An apology must be made for the fact that the numerical examples in chapters II and III are taken from other aerospace applications and computer-generated random cases rather than aerodynamic flutter models as originally proposed in this research effort.

POLE PLACEMENT AND ORDER REDUCTION IN TWO-TIME-SCALE CONTROL SYSTEMS THROUGH RICCATI ITERATION

Leonard R. Anderson*

Abstract

A transformation of variables taken from singular perturbations may be applied to two-time-scale linear systems in state space form to reduce the system to block-diagonal form with slow and fast modes decoupled. The transformation is easily computed by applying the new "Riccati Iteration." The iteration yields a solution to the nonsymmetric algebraic Riccati equation obtained by partitioning the original system matrix A. The numerical procedure is initiated with the trivial iterate Lo = 0, and is globally convergent to the desired unique time scale decoupling solution.

After transformation, the decoupled system may be used in controller design to achieve exact closed-loop pole placement in the slow subsystem without altering the poles of the fast subsystem. The decoupled form may also be used to reduce system order by sitting a small parameter to zero. Provided the fast subsystem is stable, the order reduction can be expected to yield a good approximation to the original system. These methods are demonstrated using the 16th order linear model of a turbofan engine.

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1. INTRODUCTION

Large scale linear control systems are encountered frequently in engineering problems such as the design of high performance aircraft[1,2], large space structures[3], or power systems[4]. A recurring theme in such problems is the need to reduce the originally high order of the system to facilitate simulation, eigenanalysis, or control system design studies. The question of order reduction has been widely researched and many order reduction schemes have been proposed[5,6]. The method presented here is related to the familiar eigenspace analysis of linear multivariable systems, but is particularly well suited to direct numerical implementation for large scale systems, and systems with "stiff" dynamics.

Consider the linear control system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ w &= Cx + Du \end{aligned} \tag{1}$$

where x, u and w are state, control and output variables of dimension n, m and l, respectively. As proposed by Chow and Kokotovic[7], system (1) will be classified as <u>two-time-scale</u> if the eigenvalue spectrum of the A matrix, represented as $\lambda(A)$, can be separated by absolute values into nonempty sets S and F with n_1 and $n_2 = n - n_1$ elements, respectively, such that

$$|s_i| \ll |f_j| \text{ for all } s_i \text{ in S, and } f_j \text{ in F.} \tag{2}$$

A naturally occurring system <u>small parameter</u> is the eigenvalue ratio

$$\mu = \frac{\max_i |s_i|}{\min_j |f_j|} \ll 1. \tag{3}$$

This parameter provides a measure of the system's time scale separation and identifies (1) as a singular perturbations problem. This particular small parameter was also proposed by Kelley[8].

More refined partitioning of the spectrum $\lambda(A)$ yielding three or more time scales (i.e., eigenvalue groups) may also be considered with two or more small parameters analogous to (3). The technique described below may be applied repeatedly for such cases. Only the two-time-scale case will be considered here.

2. THE LK TRANSFORMATION

Such two-time-scale systems may be conveniently transformed into decoupled subsystems by a two step transformation to new variables $y = T_2 T_1 x = Tx$ with

$$T_1 = \begin{bmatrix} I & 0 \\ L & I \end{bmatrix} \tag{4}$$

$$T_2 = \begin{bmatrix} I & K \\ 0 & I \end{bmatrix} \tag{5}$$

$$T = \begin{bmatrix} I + KL & K \\ L & I \end{bmatrix} \tag{6}$$

where the I are identity matrices of dimension n_1 and n_2. If the original system (1) is partitioned as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{7}$$

where A_{11} is $n_1 \times n_1$, etc., then the new system defined by transformation (4) will be block-triangular provided the $n_2 \times n_1$ matrix L satisfies the nonsymmetric algebraic Riccati equation (ARE)

$$LA_{11} - A_{22}L - LA_{12}L + A_{21} = 0. \tag{8}$$

If, in addition the $n_1 \times n_2$ matrix K satisfies the Lyapunov equation

$$K\tilde{A}_{22} - \tilde{A}_{11}K + A_{12} = 0 \tag{9}$$

where $\tilde{A}_{11} = A_{11} - A_{12}L$, $\tilde{A}_{22} = A_{22} + LA_{12}$, then (7) is transformed to the block-diagonal form

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} u \tag{10}$$

where $\tilde{B}_1 = (I+KL)\, B_1 + KB_2$, $\tilde{B}_2 = LB_1 + B_2$.

Such transformations have appeared in singular perturbations work[9,10,11] and were introduced into the controls literature by Kokotovic[12]. One attractive feature of transformation T is that it is always nonsingular and has the explicit inverse

$$T^{-1} = \begin{bmatrix} I & -K \\ -L & I+LK \end{bmatrix}.$$

Since $\tilde{A}$ is similar to matrix A, the eigenvalues of A become the eigenvalues of A_{11} and A_{22}, i.e.

$$\lambda(A) = \lambda(\tilde{A}_{11}) \cup \lambda(\tilde{A}_{22}).$$

It is well known that ARE has many solutions[13,14]. If we express the A matrix in spectral form as $A = MJQ$ where M is a modal matrix, J is diagonal (assuming A nondefective), and $Q=M^{-1}$, then solutions to ARE can be expressed

in terms of partitions of M. MJQ is partitioned conformably with (7) so that M_{11}, J_1 and Q_{11} have dimensions $n_1 \times n_1$, etc. It can be shown by matrix algebra that L is a solution to ARE if

1) M_{11} is nonsingular

2) $$L = -M_{21}M_{11}^{-1} = Q_{22}^{-1}Q_{21} \tag{11}$$

Also, if L satisfies (11), then $\tilde{A}_{11} = M_{11}J_1M_{11}^{-1}$. In general there will be one solution to ARE corresponding to each partition of the eigenvalues of A into A_{11} and A_{22}.

The solution L of particular interest in two-time-scale systems is the unique solution to ARE which yields

$$\lambda(\tilde{A}_{11}) = S, \quad \lambda(\tilde{A}_{22}) = F \tag{12}$$

and from here on it is assumed that L satisfies ARE and (12). The Lyapunov equation (9) will have a unique solution[15] provided $\tilde{A}_{11}$ and $\tilde{A}_{22}$ have no common eigenvalues, which is guaranteed by (12). The solution K of (9) can also be expressed in terms of the modal matrix M as

$$K = M_{11}Q_{12} = -M_{12}Q_{22}.$$

One method of computing L is to compute n_1 eigenvectors $\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}$ spanning the slow eigenspace and apply (11). However, as recently shown,[13,16] L can be computed as the limit of the Riccati Iteration[18] algorithm

$$L_{i+1} = (A_{22} + L_iA_{12})^{-1}(L_iA_{11} + A_{21})$$

initialized with $L_0 = 0$ which is globally convergent to the desired decoupling L. If we define the residual

$$R_i = L_iA_{11} - A_{22}L_i - L_iA_{12}L_i + A_{21}$$

then in the limit as $i \to \infty$, $\dfrac{||R_{i+1}||}{||R_i||} \to \mu$.

Thus L is readily computed for strongly two-time-scale systems (i.e. those with very small μ).

3. ORDER REDUCTION

The original system (1) can therefore be considered as the decoupled subsystems

$$\dot{y}_1 = \tilde{A}_{11}y_1 + \tilde{B}_1 u \tag{13a}$$

$$\dot{y}_2 = \tilde{A}_{22}y_2 + \tilde{B}_2 u \tag{13b}$$

which model the system slow and fast dynamic parts, respectively. The appropriate initial condition for (13a, b) is $y(0) = Tx(0)$. The decoupled form can yield a saving in simulation studies since only the fast subsystem (13b) need by integrated with a small time step. If the original system (1) is integrated directly, then one must treat the entire n^{th} order system as fast to obtain accurate numerical solutions.

As noted above, order reduction is often required in the study of large scale systems, and this may be conveniently done using decoupled subsystems (13a, b). Using the spectral norm,

$$\frac{||\tilde{A}_{11}||}{||\tilde{A}_{22}||} \leq \mu$$

so we can write (13a, b) in the standard singularly perturbed form

$$\begin{aligned} \dot{y}_1 &= \tilde{A}_{11} y_1 + \tilde{B}_1 u \\ \mu \dot{y}_2 &= \hat{A}_{22} y_2 + \hat{B}_2 u \end{aligned} \tag{14}$$

where $\hat{A}_{22} = \mu \tilde{A}_{22}$ and $\hat{B}_2 = \mu \tilde{B}_2$. The zeroth order approximation to (14) obtained by setting the small parameter to zero is then

$$\dot{y}_1 = \tilde{A}_{11} y_1 + \tilde{B}_1 u \tag{15a}$$

$$\hat{y}_2 \equiv -\tilde{A}_{22}^{-1} \tilde{B}_2 u \tag{15b}$$

with initial condition

$$y_1(0) = (I+KL)x_1(0) + K x_2(0) \tag{15c}$$

and the approximation $\hat{x}$ to the original state x is given by

$$\hat{x} \equiv \begin{bmatrix} I \\ -L \end{bmatrix} y_1 + \begin{bmatrix} -K \\ I+LK \end{bmatrix} \hat{y}_2. \tag{15d}$$

If the fast subsystem (13b) has eigenvalues with large negative real part, i.e.

$$\text{Re}\,(f_j) << -\max_i |s_i| \quad \text{for all } f_j \text{ in } F \tag{16}$$

then the n_1th order approximation x can be expected to yield a good approximation for x outside of an initial boundary layer where fast, stable dynamics may predominate.

Note that if one is primarily interested in output variables w rather than state variables x, the reduced order model (15) may be written in a compact matrix form similar to (1). Transform and partition the output matrix as

$$[\tilde{C}_1 \quad \tilde{C}_2] = \tilde{C} = CT^{-1}$$

to yield

$$\dot{y}_1 = \tilde{A}_{11}\, y_1 + \tilde{B}_1\, u$$
$$w = \tilde{C}_1\, y_1 + \tilde{D}\, u \qquad (17)$$

where

$$\tilde{D} = D - \tilde{C}_2 \tilde{A}_{22}^{-1}\, \tilde{B}_2.$$

This order reduction method has been shown to compare favorably with other standard methods[17].

4. POLE PLACEMENT

Dynamic systems are often modeled as linear control systems for the purpose of designing a feedback controller to achieve specific closed-loop eigenvalue locations. Through application of the LK transformation, an n-dimensional eigenvalue placement problem can be reduced to separate eigenvalue placement problems of dimension n_1 and n_2.

It is well known that if a constant linear control system is <u>controllable</u>, i.e.

$$\text{rank } (B \;\; AB \;\; \ldots \;\; A^{n-1}B) = n, \qquad (18)$$

then there exists at least one real m x n dimensional feedback matrix H such that the closed-loop eigenvalues given by

$$\lambda(A+BH)$$

can be placed arbitrarily (as long as complex eigenvalues appear in conjugate pairs). If the original system (1) is controllable, it can be shown by linear algebra that the slow and fast subsystems (13a, b) are also controllable. Transfer of observability from the original system to the decoupled subsystems follows by a dual argument.

The design of a feedback matrix H to achieve a specific closed-loop eigenvalue location then follows directly. Suppose it is desired to relocate the n_1 slow open-loop eigenvalues S to n_1 new eigenvalue locations S'. If one can find an m x n_1 feedback matrix $\tilde{H}_1$ which satisfies

$$(\tilde{A}_{11} + \tilde{B}_1 \tilde{H}_1) = S', \qquad (19)$$

then system (13a, b) with feedback $u = \tilde{H}_1 y_1$ becomes

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} \quad \begin{bmatrix} \tilde{A}_{11}+\tilde{B}_1\tilde{H}_1 & 0 \\ \tilde{B}_2\tilde{H}_1 & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} . \qquad (20)$$

Thus the fast eigenvalues F are unchanged and the corresponding feedback matrix for (1) is given by

$$H = [\tilde{H}_1 \quad 0]\, T. \qquad (21)$$

If in addition to relocating the slow mode eigenvalues S it is also necessary to relocate the fast open-loop eigenvalues F to closed-loop values F' then this can be done as follows. After solving for feedback matrix H_1 in (19) find the unique (assuming S' and F have no common eigenvalues) $n_2 \times n_1$ matrix P satisfying the Lyapunov equation

$$P(\tilde{A}_{11}+\tilde{B}_1\tilde{H}_1) - \tilde{A}_{22}P + \tilde{B}_2\tilde{H}_1 = 0.$$

The block-triangular system (20) can then be transformed to a new block-diagonal form

$$\begin{bmatrix} \dot{y}_1 \\ \dot{\tilde{y}}_2 \end{bmatrix} = \begin{bmatrix} \tilde{\tilde{A}}_{11} & 0 \\ 0 & \tilde{A}_2 \end{bmatrix} \begin{bmatrix} y_1 \\ \tilde{y}_2 \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{\tilde{B}}_2 \end{bmatrix} \tilde{u} \qquad (22)$$

where $\tilde{y}_2 = Py_1 + \tilde{y}_2$, $\tilde{\tilde{A}}_{11} = \tilde{A}_{11} + \tilde{B}_1\tilde{H}_1$, and $\tilde{\tilde{B}}_2 = P\tilde{B}_1 + \tilde{B}_2$ and $\tilde{u} = u - \tilde{H}_1 y_1$.

Now relocate fast eigenvalues F to new values F' by finding an $m \times n_2$ feedback matrix H_2 satisfying

$$\lambda(\tilde{A}_{22} + \tilde{\tilde{B}}_2\tilde{H}_2) = F'. \qquad (23)$$

The appropriate gain matrix for the original system (1) is then given by

$$u = Hx, \quad H = \begin{bmatrix} (\tilde{H}_1 + \tilde{H}_2P) & \tilde{H}_2 \end{bmatrix} T. \qquad (24)$$

Thus, the block-decoupling transformation T can be used to exactly relocate both slow and fast eigenvalues via state feedback.

Many variations of this approach are possible to facilitate the feedback matrix design task. For example, by repeated application of the block-decoupling transformation either of the decoupled subsystems (13a, b) could be further transformed into block-diagonal form and a complete eigenvalue relocation could be achieved in three consecutive design steps analogous to the above two design steps (19) and (23). In many practical applications, the fast modes are all stable and well damped so only the slow eigenvalues need be relocated.

5. A TURBOFAN EXAMPLE

The example considered here is the 16th order model of a turbofan engine which was the theme problem for a recent conference on control of linear multivariable systems[1]. This model is the linearization of a detailed nonlinear simulation (at the sea level maximum non-afterburner thrust point). The state variables consist of shaft speeds, temperatures and pressures; the five control inputs are fuel flow, nozzle area, two vane positions and compressor bleed; and the five output variables are net thrust, total airflow, a temperature and two stall margins. For $n_1 = 15$, 5 and 3, the resulting small parameters are 0.304, 0.371 and 0.383. Since these values are not particularly small relative to one, we might call this system weakly two-time-scale.

Selecting $n_1 = 5$, the L and K matrices were computed as described in Anderson[13]. The response of the full 16th order system and reduced 5th order system containing the slow dynamics are compared in Figures 1 and 2. The response of

total thrust and fan speed to step inputs in fuel flow rate and inlet guide vane position demonstrate that good agreement is achieved between full and reduced order response except during an initial boundary layer transient where fast dynamics are significant. The response to a guide vane step input provides a severe test of the reduced order model since this control variable is located at the front of the engine and some time is required for its effects to propagate to the net thrust.

Pole placement for this example was carried out with the goal of increasing the speed of the thrust response. Applying the methods of the previous section, the three slowest modes -0.65, -1.90 and -2.62 were relocated to approximately -6.6. Since there are five control inputs in this example and B_1 is nonsingular, it was possible to shift the above three poles and leave the eigenvectors of the slow subsystem unchanged. Let the slow subsystem have spectral form

$$A_{11} = M_{11} J_1 M_{11}^{-1}$$

where J_1 = diag (-0.65, -1.90, -2.62, -6.72 $\pm$ i 1.31).

Then feedback matrix $\tilde{H}_1$ was chosen as

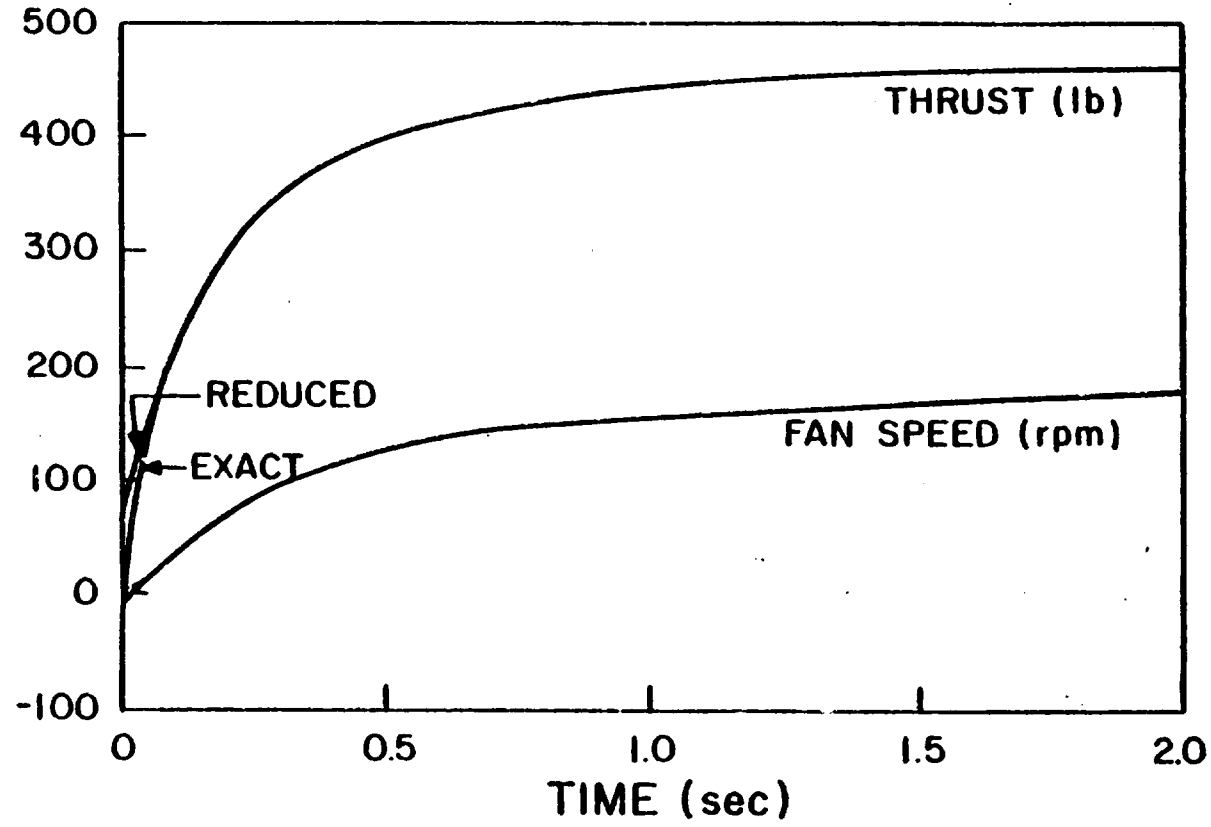
$$\tilde{H}_1 = \tilde{B}_1^{-1} M_{11} \Delta M_{11}^{-1}$$

where Δ = diag (-6, -5, -4, 0, 0) and the model response for the resulting H (c.f. equation (21)) is illustrated in Figure 3. As expected, the response is much quicker with this feedback.

6. CONCLUSIONS

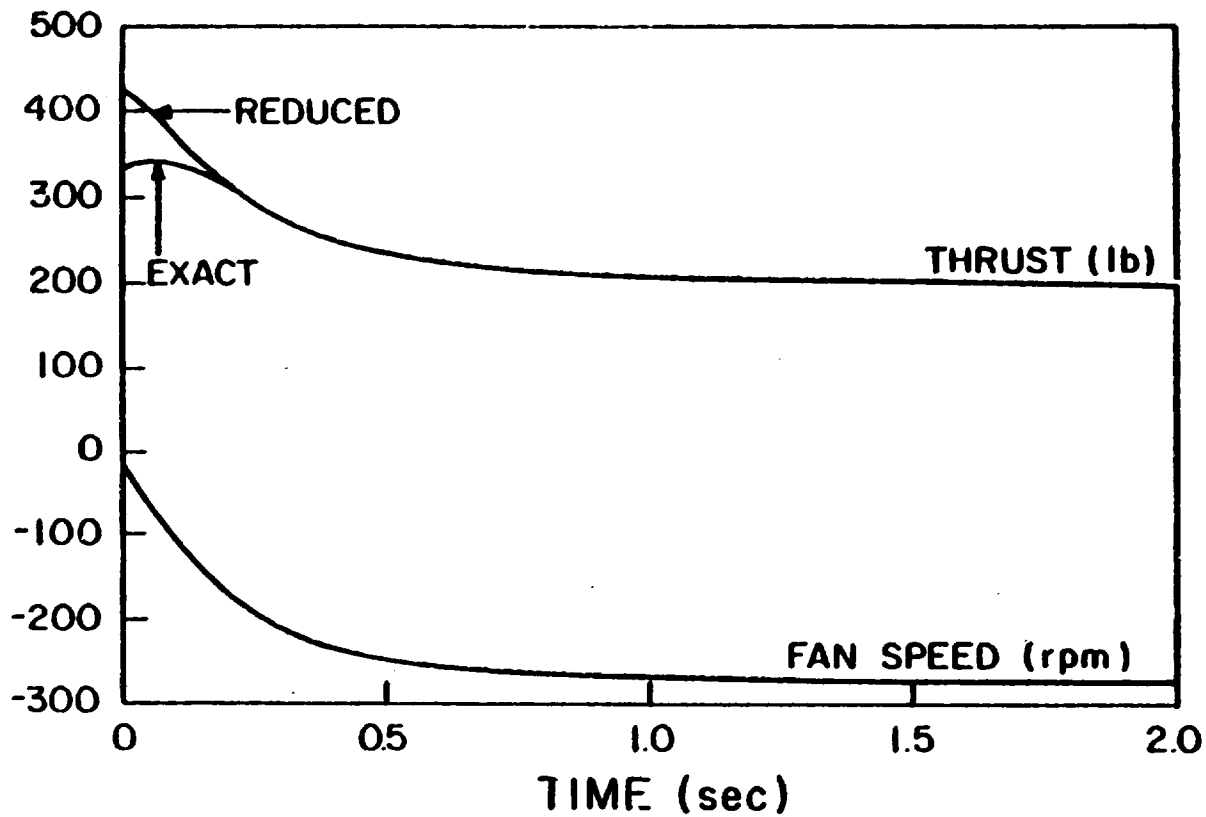
A method of time scale separation based on the block-decoupling LK transformation is proposed. Methods of applying this transformation to order reduction and pole placement design tasks are described and demonstrated for a turbofan engine example. As previously shown, the L and K matrices are easily computed for large scale systems, particularly those systems with large time scale separation and correspondingly small μ.

Figure 1



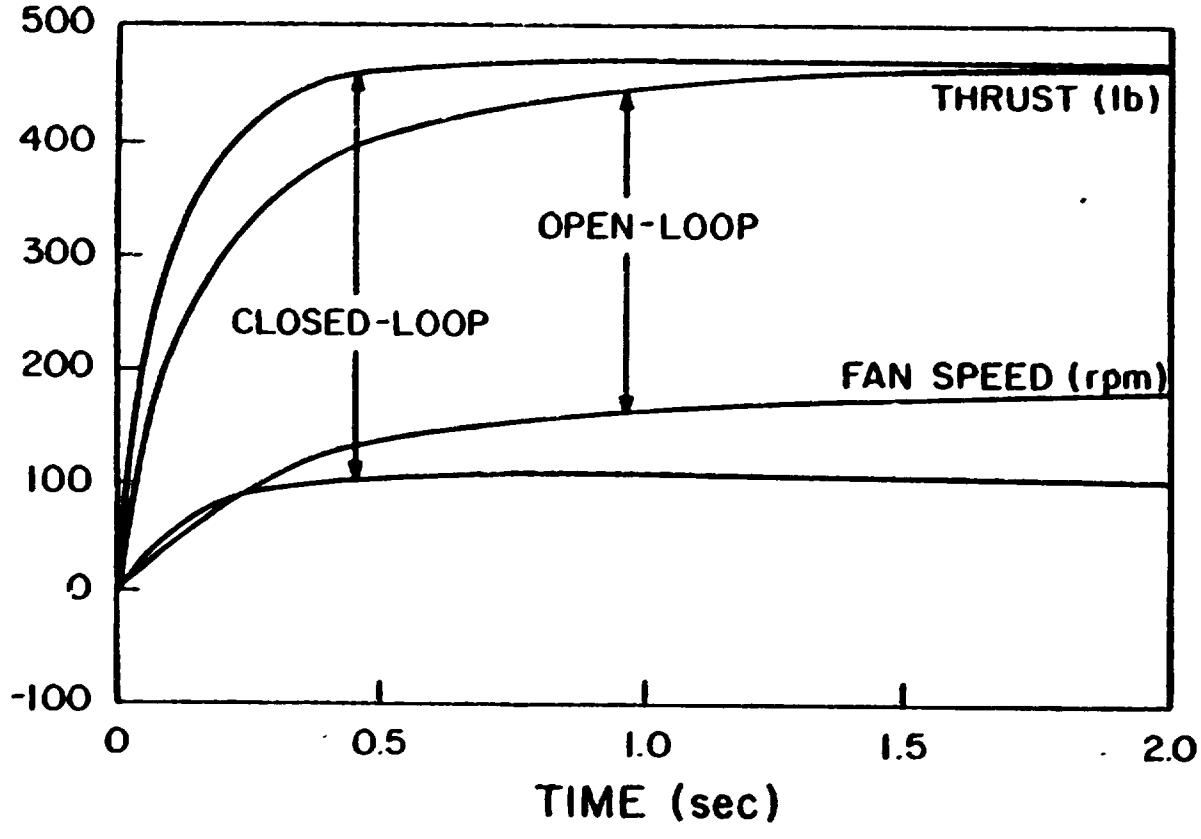
Response of thrust and fan speed to a 500 lb/hr step change in fuel flow rate

Figure 2



Response of thrust and fan speed to a 10 deg. step change in inlet guide vane position

Figure 3



Open- and closed-loop response of thrust and fan speed to step changes in fuel flow rate

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Numerical Solution of the Symmetric Riccati Equation through Riccati Iteration

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A. Rasim Baykan***

Abstract

This research paper presents a new numerical method for solving the symmetric algebraic Riccati equation from optimal control. This algorithm employs the "Riccati iteration" which has been successfully used to solve time-scale decoupling problems in structural vibrations. The algorithm is related to the subspace iteration method, and the rate of convergence to the solution is governed by the relative separation between the stable and unstable eigenvalues in the Hamiltonian system of equations. Provided there is adequate eigenvalue separation and ignoring roundoff error, the algorithm is globally convergent to the desired Riccati solution. The method is demonstrated for a set of 8th order random examples. Preliminary accuracy and timing comparisons with other standard methods of solving the symmetric Riccati equations are presented.

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I. Introduction

It is well known that in order to solve the infinite-time regulator problem [1,2] with linear system

$$\dot{x} = Ax + Bu \tag{1}$$

and quadratic performance index (the LQR problem)

$$J = 1/2 \int_0^\infty (x^T Q x + u^T R u) dt \tag{2}$$

one must compute the unique positive definite solution S to the symmetric algebraic Riccati equation

$$0 = SA + A^T S - SBR^{-1}B^T S + Q \, . \tag{3}$$

Here x is an n-dimensional state vector, u is an m-dimensional control vector, A and B are constant matrices Q and R are positive semi-definite and positive definite, respectively. Also it is assumed that the matrix pair (A,B) is controllable, and the pair $(A,Q^{1/2})$ is observable and that B has full rank. Then assuming that any constraints on the size of u(t) are not violated, the linear feedback control law which minimizes the performance index J is given by

$$u = Kx \text{ where } K = -R^{-1}B^T S \, . \tag{4}$$

Algebraic Riccati equations similar to (3) also play a fundamental roll in discrete-time optimal control problems, in optional estimation problems, and is many other areas of applied mathematics [3-7]. See Jones [8] for basic necessary and sufficient conditions for solutions of equations of this type.

Among the most widely used solution methods for solving algebraic Riccati equations are the Schur vector method described by Laub [9], methods employing eigenvectors of the Hamiltonian system [1,2,10,11] and Newton-Raphson

methods [1,2,12]. The Riccati iteration method differs from these methods in that neither eigenvalue computation nor solution of equations of the Lyapunov-type are required. The algorithm is suprisingly simple, requiring only repetitive solution of linear equations of order n. However, caution must be exercised when applying the method to LQR problems with nearly zero eigenvalues, or eigenvalues with large imaginary part.

II. The eigenvector solution to the algebraic Riccati equation

Given the linear system (1) and performance index (2), we can form the control Hamiltonian and apply the necessary conditions for optimal control to obtain the Hamiltonian system of equations

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \equiv H \begin{bmatrix} x \\ y \end{bmatrix}, \tag{5}$$

where y is an n-dimensional vector of costate or adjoint variables. The 2n x 2n matrix H has symmetric eigenvalues. That is, if λ is an eigenvalue of H then $-\lambda$ is also an eigenvalue of H.

If matrix H is represented in spectral form

$$H = M \Lambda M^{-1} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} -\Lambda_1 & 0 \\ 0 & \Lambda_1 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

where the matrices are partitioned conformally with (5), and Λ_1 contains all the eigenvalues with positive real part ordered as

$$\Lambda_1 = \text{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n) \text{ with } |\lambda_i| \leq |\lambda_{i+1}|, \; i = 1, 2, \ldots n-1, \tag{6}$$

then the desired solution, S, to the Riccati equation is given by

$$SM_{11} = M_{21} \text{ or } Q_{22} S = -Q_{21} . \tag{7}$$

To demonstrate this fact, apply the similarity transformation

$$T = \begin{bmatrix} I & 0 \\ -S & I \end{bmatrix} \quad , \quad T^{-1} = \begin{bmatrix} I & 0 \\ S & I \end{bmatrix} \tag{8}$$

to the Hamiltonian system to obtain

$$\hat{H} \equiv THT^{-1} = \begin{bmatrix} A - BR^{-1}B^TS & -BR^{-1}B^T \\ -SA-A^TS+SBR^{-1}B^TS-Q & -A^T+SBR^{-1}B^T \end{bmatrix} , \tag{9}$$

or in spectral form

$$\hat{H} = \begin{bmatrix} I & 0 \\ -M_{21}M_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} -\Lambda_1 & 0 \\ 0 & \Lambda_1 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -Q_{22}^{-1}Q_{21} & I \end{bmatrix}$$

$$= \begin{bmatrix} -M_{11}\Lambda_1 M_{11}^{-1} & M_{11}J_1Q_{12}+M_{12}J_2Q_{22} \\ 0 & Q_{22}^{-1}\Lambda_1 Q_{22} \end{bmatrix} \tag{10}$$

Since the closed-loop system for feedback law (4) is

$$\dot{x} = Ax + Bu = (A-BR^{-1}B^TS)x \tag{11}$$
$$\equiv \tilde{A}x$$

the eigenvalues $-\Lambda_1$ become the eigenvalues $\lambda(\tilde{A})$ of the closed-loop system. Given efficient eigenanalysis software such as EISPACK [13], one can directly compute the eigenvalues and eigenvectors of the Hamiltonian system (5) and solve the appropriate system of linear equations (7) for the Riccati solution S.

If $-\Lambda_1$ includes complex conjugate eigenvalues λ_j, $\bar{\lambda}_j$ with corresponding complex conjugate eigenvectors $u_j = \alpha + i\beta$, $\bar{u}_j = \alpha - i\beta$, then complex arithmetic can be avoided in the set of linear equations (7) by replacing u_j with α and $\bar{u}_j$ with β.

As an alternative to computing eigenvectors $\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}$, one can compute the n Schur vectors [9] spanning the stable eigenspace and solve (7) with M_{11}, M_{21} replaced by corresponding Schur vectors. Schur vectors are defined as follows. Given any nxn matrix C with eigenvalues $\lambda_1, \ldots, \lambda_n$ there exists a unitary matrix V such that $V^H C V$ is upper triangular with diagonal elements $\lambda_1, \ldots, \lambda_n$. The columns of V are the Schur vectors of C, possibly complex, corresponding to this particular ordering of eigenvalues.

If some of the eigenvalues λ_i are complex, then one can avoid complex Schur vectors by employing a (real) orthogonal matrix $\tilde{V}$ such that $\tilde{V}^T A \tilde{V}$ is real and nearly upper triangular. The only nonzero subdiagonal elements in $\tilde{V}^T A \tilde{V}$ will be due to real 2x2 diagonal blocks corresponding to complex eigenvalues λ_1. This is termed the real Schur form of C, and the columns of $\tilde{V}$ are real Schur vectors.

To return to the problem of computing the solution (7) to the symmetric Riccati equation (3), one can compute the 2n real Schur vectors

$$\tilde{V} = \begin{bmatrix} \tilde{V}_{11} & \tilde{V}_{12} \\ \tilde{V}_{21} & \tilde{V}_{22} \end{bmatrix} \tag{12}$$

of the 2nx2n matrix H so that $\begin{bmatrix} \tilde{V}_{11} \\ \tilde{V}_{21} \end{bmatrix}$ provides a basis for the eigenspace corresponding to stable eigenvalues $-\Lambda_1$.

The computation is best performed using the subroutine HQR3 [14] with small modifications. In its standard form, HQR3 orders the eigenvalues of H

by modulus with $|\lambda_1| \leq |\lambda_2| \leq \ldots$ on the diagonal of $\tilde{V}^T H \tilde{V}$. The ordering statements in HQR3 may be modified so that either

1) all stable (negative real part) eigenvalues are placed in the upper left diagonal block of $\tilde{V}^T H \tilde{V}$ and unstable eigenvalues are placed in the lower right block, with no regard to ordering within each block, or
2) the eigenvalues of H are ordered along the diagonal of $\tilde{V}^T H \tilde{V}$ by real part from most negative to most positive.

Only modification (2) is implemented in the numerical examples which follow.

III. Riccati iteration solution to the algebraic Riccati equation

Before proceeding to the solution of the symmetric Riccati equation (3), we will first describe in detail the solution of the closely related time-scale decoupling problem using Riccati iteration [15,16], and then extend these results to (3).

The time-scale decoupling problem can be stated as follows. Given an nxn diagonalizable matrix C with eigenvalues

$$s_1, s_2, \ldots, s_{n_1}, f_1, f_2, \ldots, f_{n_2}$$

not necessarily distinct and satisfying

$$|s_1| \leq |s_2| \leq \ldots \leq |s_{n_1}| < |f_1| \leq |f_2| \leq \ldots \leq |f_{n_2}|,$$

$$\text{with} \quad \mu = \frac{|s_{n_1}|}{|f_1|}, \tag{13}$$

if parameter μ is small relative to one, we say that the matrix is two-time-scale [17,18]. Let the diagonalization of C be given by $C = EJF$ where $F = E^{-1}$ and J is diagonal with diagonal entries

$$s_1, s_2, \ldots, s_{n_1}, f_1, f_2, \ldots, f_{n_2}.$$

Partition the matrices C, E, F and J as

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \quad E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$$

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \quad J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$$

where $C_{11}, E_{11}, F_{11}, J_1$ are $n_1 x n_1$, etc. For $k = 0, 1, 2, \ldots$ partition the matrix C^k similarly as

$$C^k = \begin{bmatrix} S_k & T_k \\ U_k & V_k \end{bmatrix} .$$

The Riccati iteration and its convergence properties for the time-scale decoupling problem [19] are introduced through

Theorem 1. Given $\mu < 1$ and V_k nonsingular for $k = 1, 2, 3, \ldots$, the iteration

$$\left.\begin{aligned} L_0 &= 0 \\ L_{k+1} &= (C_{22} + L_k C_{12})^{-1}(L_k C_{11} + C_{21}) \end{aligned}\right\} \qquad (14)$$

$$k = 0, 1, 2, \ldots$$

is well defined and

$$L_k \longrightarrow F_{22}^{-1} F_{21} \text{ as } k \longrightarrow \infty$$

with convergence in the sense of matrix norm. For any ixj matrix B, ||B||

will be given by

$$||B|| = \sup_{||x||_1 \leq 1} ||Bx||_j$$

where $||\cdot||_1$, $||\cdot||_j$ denote convenient norms on R^1 and R^j.

The proof of Theorem 1 is presented in the Appendix.

Furthermore, the rate of convergence of L_k to $F_{22}^{-1}F_{21}$ is controlled by the small parameter μ as shown in the Appendix. Convergence would be expected to be unreasonably slow for μ close to one.

To apply the Riccati iteration to the symmetric Riccati equation (3) we need to shift the Hamiltonian matrix (5) as

$$\tilde{H} = H + \lambda_s I \tag{15}$$

so that the stable, left half-plane eigenvalues $-\lambda_1, -\lambda_2, \ldots, -\lambda_n$ of H become the slow or small eigenvalues of $\tilde{H}$ analogous to $s_1, s_2, \ldots, s_{n_1}$.

We can now state the formal conditions for convergence of the Riccati iteration for the symmetric Riccati equation (3) as

Theorem 2. Given the Hamiltonian system (5) and given shift distance λ_s satisfying[1]

[1]Note, this redefines μ for the sequel, cf. (13).

$$\mu = \frac{\max_{i=1,\ldots,n} |-\lambda_i + \lambda_s|}{\min_{i=1,\ldots,n} |\lambda_i + \lambda_s|} < 1 \tag{16}$$

the Riccati iteration

$$\left.\begin{aligned} S_0 &= 0 \\ S_{i+1} &= (-A^T + \lambda_s I - S_i B R^{-1} B^T)^{-1}(S_i(A + \lambda_s I) - Q) \end{aligned}\right\} \tag{17}$$

converges to solution (7) with asymptotic rate

$$||S_i + Q_{22}^{-1} Q_{21}|| \leq c\mu^k$$

where the matrix norm is as in Theorem 1, and constant c is independent of k. The proof of Theorem 2 follows directly from the proof of Theorem 1 in the Hamiltonian setting.

The key practical difficulty in implementing iteration (17) is the estimation of the shift distance λ_s. Two choices for the shift distance are proposed, namely

$$\lambda_s \cong \max_{i=1,\ldots,n} |\lambda_i| \tag{18}$$

or

$$\lambda_s \cong \frac{\max_{i=1,\ldots,n} |\lambda_i| + \min_{i=1,\ldots,n} |\lambda_i|}{2} \tag{19}$$

The shifting strategy (19) will in general yield higher rates of convergence since the stable eigenvalues of H become nearly centered about the origin in $\tilde{H}$. Shift strategy (18) is recommended for symmetric Riccati equations with eigenvalues having large imaginary part, since this minimizes the chances of no convergence, or convergence to a solution other than the desired solution (7). Both shifting strategies can be implemented numerically in approximate form.

The computational algorithm for the symmetric Riccati equation is therefore

Algorithm 1.

1) Form the Hamiltonian matrix

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}$$

2) Beginning with the 2n vector

$$x_0 = (2n)^{-1/2} [1\ 1 \ldots\ 1]^T$$

perform a few power steps (five or more)

$$\tilde{x}_{i+1}^T = x_i^T H$$

$$x_{i+1} = \tilde{x}_{i+1} / ||\tilde{x}_{i+1}||, \quad i = 0, 1, \ldots 5$$

where $||\cdot||$ is the Euclidean or Frobenius norm.

3) Obtain an initial estimate of the shift distance

$$\alpha_1 = ||\tilde{x}_1||$$

$$\alpha_{i+1} = \frac{\alpha_i + ||\tilde{x}_{i+1}||}{2}, \quad i = 1,2, \ldots, 5$$

$$\lambda_{s,0} = \alpha_6.$$

4) Estimate a limit for λ_{max}

$$\lambda_{limit} = 1.5\ \lambda_{s,0}$$

and partition the vector x_6 as

$$x_6 = [u_0\ u_0']^T$$

with u_0 of dimension n.

5) Initialize $S_0 = 0$, $j = 0$, and form the inverse power vector

$$v_0 = (n)^{-1/2}\ [1\ 1 \ldots 1]^T$$

of dimension n.

6) Form the matrices

$$H_{22,j} = -A^T + \lambda_{s,j} I - S_j B R^{-1} B^T$$

$$G_j = S_j (A + \lambda_{s,j} I) - Q.$$

7) Perform a power step and estimate λ_{max}.

$$\tilde{u}_{j+1}^T = u_j^T H_{22,j}$$

$$\lambda_{max,j} = \max\ (0\ ,\ \min\ (\lambda_{limit},\ ||\tilde{u}_{j+1}|| - \lambda_{s,j}))$$

$$u_{j+1} = \tilde{u}_{j+1} / ||\tilde{u}_{j+1}||$$

8) Solve the linear system for S_{j+1} and $\tilde{v}_{j+1}$

$$H_{22,j}\ [S_{j+1}\ \tilde{v}_{j+1}] = [G_j\ v_j] \qquad (20)$$

9) If $j \geq 1$, find the relative change in S_j.

$$\delta_j = ||S_{j+1} - S_j|| / ||S_j||$$

and terminate the iteration if δ_j is less than or equal to some specified convergence tolerance ε, or has reached a minimum value.

10) Normalize the inverse power vector $\tilde{v}_{j+1}$ and estimate λ_{min}.

$$\lambda_{min,j} = \max\left(0\ ,\ \min\left(\lambda_{max,j},\ \frac{1}{||\tilde{v}_{j+1}||} - \lambda_{s,j}\right)\right)$$

$$v_{j+1} = \tilde{v}_{j+1}/||\tilde{v}_{j+1}||.$$

11) Update the shift distance by either

$$\lambda_{s,j+1} = \frac{3\lambda_{s,j} + \lambda_{max,j}}{4} \tag{21}$$

or

$$\lambda_{s,j+1} = \frac{6\lambda_{s,j} + \lambda_{max,j} + \lambda_{min,j}}{8} \tag{22}$$

12) Increase j by one and go to step 6.

Obviously, if one computes the updated shift distance by the more conservative (21), then one need not include the inverse power vector v_j in the iteration. The power and inverse power estimates of $\lambda_{min} = |\lambda_1|$ and $\lambda_{max} = |\lambda_n|$ are based on the fact that the eigenvalue spectrum of $H_{22,j}$ converges to the eigenvalues Λ_1 increased by $\lambda_{s,j}$.

This iterative technique for solving the symmetric Riccati equation has similarities to one proposed by Farrar and DiPietro [20]. The methods differ in that their method is initialized using eigenvectors, and employs iterative improvement requiring solution of a Lyapunov-type equation at each step.

IV. Numerical examples

The eigenvector and Schur vector solution methods and the Riccati iteration method were evaluated in a limited comparative test. The test consisted of

sets of linear-quadratic-regulator problems generated as follows. The elements of the A and B matrices of system (1) were chosen as random integers from a uniform distribution over the internal [-10, +10], with 25% of the elements set arbitrarily to zero. The order n of the system was varied between 5 and 20, and the control dimension m was held at one. The positive semidefinite weighting matrix Q was constructed from a Choleski factorization

$$Q = PP^T$$

where P is upper-triangular. The upper-triangular elements of P are also integers chosen uniformly from the internal [-10, +10], with 25% of these elements set to zero. In all cases the R matrix, a scalar, had a value of 2.

All methods were either obtained or coded in the FORTRAN language in double precision, and executed in batch mode on the VPI&SU IBM 370/158 computer.

Table 1 presents the results for a set of ten random cases of order eight generated as described above and solved by Algorithm 1. The table includes the theoretical rate of convergence μ for each of the shift strategies (18) and (19), the value of the Riccati residual $||R_i||$ where

$$R_i = S_iA + A^TS_i - S_iBR^{-1}B^TS_i + Q \tag{23}$$

for the converged solution S_i, the maximum relative error between the eigenvalues of the closed-loop system (11) and the stable eigenvalues of the Hamiltonian system (5), the number of iterations in Algorithm 1 to reach a minimum value of δ_i, and the corresponding central processor execution time in seconds. As noted earlier, the shift strategy (18), (21) is significantly slower in convergence than (19), (22). However, the faster method (19), (22) fails to converge for case 8 as expected since $\mu > 1$.

To further describe the convergence characteristics of the Riccati iteration, the values of $\log_{10}||R_i||$ and $\log_{10}||\delta_i||$ versus iteration

number are plotted in Figures 1 and 2. These figures correspond to cases 2 and 9, and represent the most rapid and the slowest convergence of the ten cases, respectively. As shown, the theoretical limit μ, cf. (15), accurately predicts the true rate of convergence of the algorithm for each of the two shifting strategies. The specific A, B, Q and R matrices for cases 2 and 9 are listed in Tables 2 and 3.

For comparison purposes, the symmetric Riccati equation for these ten cases is also solved by the eigenvector and Schur vector methods. This data is listed in Table 4. As in Table 1, we specify the value of the norm of the Riccati residual $||R||$ corresponding to the computed solution S, the maximum relative error between the eigenvalues of the closed-loop system and the Hamiltonian system, and the execution time in seconds. The execution times include both the time to compute eigenvectors or Schur vectors and the time to solve the linear system (7). The numerical method used to solve both the linear systems (7) and (20) in this study was LU decomposition with partial pivoting without iterative improvement of accuracy. Specifically, the Fortran subroutines DECOMP and SOLVE were used, but not the subroutine IMPRUV from the widely used Forsythe and Moler text [21].

For the eigenvector solution method, all eigenvalues and eigenvectors of the Hamiltonian matrix were computed using EISPACK subroutines ELMHES, ELTRAN and HQR2. Then eigenvectors corresponding to stable eigenvalues were selected to form the real $\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}$ matrix. For the Schur vector solution method, the real Schur vectors $\begin{bmatrix} \tilde{V}_{11} \\ \tilde{V}_{21} \end{bmatrix}$ spanning the stable eigenspace were computed using EISPACK subroutines ORTHES and ORTRAN, and Stewart's subroutines HQR3 and EXCHNG. The HQR3 subroutine was modified [22] so that the eigenvalues $\tilde{V}^T H \tilde{V}$ are ordered along the diagonal from most negative to most positive real part.

As shown in comparison of Tables 1 and 4, the Riccati iteration is able to

produce final Riccati residuals two orders of magnitude smaller on the average than the eigenvector and Schur vector methods. However, the Riccati iteration execution times are slower than the eigenvector and Schur vector methods by a factor of ten, on the average.

The choice of method clearly depends on both the parameter μ for the class of problems under consideration, and the relative importance of minimizing computer execution time versus software simplicity. In general, as the order n of the LQR problem increases, the parameter μ increases and approaches or exceeds one. Therefore, the eigenvector/Schur vector algorithms would still be the method of choice for problems of large order.

V. Conclusions

A new numerical algorithm for solving the symmetric Riccati equation from the linear-quadratic-regulator problem has been presented and compared with standard methods. A formal proof of convergence for the Riccati iteration is presented, and numerical examples confirm the theoretical rate of convergence. The strengths of this new algorithm are its simplicity, accuracy and theoretically transparent basis. Riccati iteration may be particularly useful for low order adaptive control algorithms or control system design studies where one must update the Riccati solution as system matrices or performance index weighting matrices slowly change. The primary weakness of this method is that it is slower than standard methods as shown here, and the rate of convergence is dependent on sufficient stable/unstable eigenvalue separation in the Hamiltonian system.

Future research tasks include:

- finding an appropriate conformal transformation for the Hamiltonian system to increase stable/unstable eigenvalue separation and/or decrease the imaginary component of eigenvalues.
- Applying a doubling algorithm [23] to this method to increase rate of convergence.

Appendix - Proof of Theorem 1

To prove theorem 1 the following lemma will be useful.

Lemma: For $k = 0, 1, 2, \ldots$

$$L_k = V_k^{-1} U_k .$$

Proof: We proceed by induction on k:

For $k = 0$

$$L_0 = V_0^{-1} U_0 = I \cdot 0 = 0$$

as required. Note that partitioning the product

$$C^k C = C^{k+1}$$

yields

$$U_k C_{11} + V_k C_{21} = U_{k+1}$$
$$U_k C_{12} + V_k C_{22} = V_{k+1} .$$

Assuming the induction hypothesis,

$$(C_{22} + L_k C_{12}) = C_{22} + V_k^{-1} U_k C_{12} = V_k^{-1} V_{k+1}$$

which is nonsingular by hypothesis.

$$\begin{aligned} \text{Therefore } L_{k+1} &= (C_{22} + L_k C_{12})^{-1} (L_k C_{11} + C_{21}) \\ &= (V_{k+1}^{-1} V_k)(V_k^{-1} U_k C_{11} + C_{21}) \\ &= V_{k+1}^{-1} V_k V_k^{-1} U_{k+1} = V_{k+1}^{-1} U_{k+1} . \end{aligned}$$

This completes the proof of the lemma.

Proof of Theorem 1

Since $C = EJF$, where $F=E^{-1}$, we have $C^k = (EJF)^k = EJ^kF$.

Partitioning the product EJ^kF yields

$$U_k = E_{21}J_1{}^kF_{11} + E_{22}J_2{}^kF_{21}$$
$$V_k = E_{21}J_1{}^kF_{12} + E_{22}J_2{}^kF_{22}$$

where J_1 and J_2 are diagonal matrices with diagonal entries $s_1, s_2, \ldots, s_{n_1}$ and $f_1, f_2, \ldots, f_{n_2}$ respectively. Therefore, by the lemma

$$L_k = (E_{21}J_1{}^kF_{12} + E_{22}J_2{}^kF_{22})^{-1}(E_{21}J_1{}^kF_{11} + E_{22}J_2{}^kF_{21})$$

for $k = 0, 1, 2, \ldots$.

For brevity let

$$D_k = (E_{21}J_1{}^kF_{12} + E_{22}J_2{}^kF_{22}).$$

An easy computation shows that

$$D_k^{-1} = (I-D_k^{-1}(E_{21}J_1{}^kF_{12}))(E_{22}J_2{}^kF_{22})^{-1},$$

therefore

$$L_k = (I - D_k^{-1}E_{21}J_1{}^kF_{12})(E_{22}J_2{}^kF_{22})(E_{21}J_1{}^kF_{11} + E_{22}J_2{}^kF_{21})$$
$$= (I - D_k^{-1}E_{21}J_1{}^kF_{12})(F_{22}^{-1}J_2{}^{-k}E_{22}^{-1}E_{21}J_1{}^kF_{11} + F_{22}^{-1}F_{21}).$$

Hence,

$$L_k - F_{22}^{-1}F_{21} = F_{22}^{-1}J_2{}^{-k}E_{22}^{-1}E_{21}J_1{}^kF_{11}$$
$$-(D_k^{-1}E_{21}J_1{}^kF_{12})(F_{22}^{-1}J_2{}^{-k}E_{22}^{-1}E_{21}J_1{}^kF_{11})$$
$$-(D_k^{-1}E_{21}J_1{}^kF_{12})(F_{22}^{-1}F_{21}).$$

Note that $||J_1|| \le |s_{n_1}|$ and $||J_2{}^{-1}|| \le \frac{1}{|f_1|}$,

so $||J_2{}^{-k}||\ ||J_1{}^k|| \le \frac{|s_{n_1}|^k}{|f_1|^k} = \mu^k$.

Furthermore,

$$D_k^{-1}E_{21}J_1^kF_{12} = (E_{21}J_1^kF_{12} + E_{22}J_2^kF_{22})^{-1}E_{21}J_1^kF_{12}$$

$$= F_{22}^{-1}J_2^{-1}E_{22}^{-1}(I + E_{21}J_1^kF_{12}F_{22}^{-1}J_2^{-k}E_{22}^{-1})^{-1}E_{21}J_1^kF_{12}\,.$$

Now $||(I + T)^{-1}|| \leq (1-||T||)^{-1}$ if $||T|| < 1$,

so $||D_k^{-1}E_{21}J_1^kF_{12}|| \leq \frac{a\mu^k}{1 - a\mu^k}$

for sufficiently large k,

where $a = ||F_{22}^{-1}||\ ||E_{22}^{-1}||\ ||E_{21}||\ ||F_{12}||$.

Combining these estimates yields

$$||L_k - F_{22}^{-1}F_{21}|| \leq \frac{b\mu^k}{1-a\mu^k} + \frac{a\mu^k}{1-a\mu^k}\ ||F_{22}^{-1}F_{21}||$$

where $b = a||F_{11}||/||F_{12}||$.

Since $\mu < 1$, $\mu^k \longrightarrow 0$ as $k \longrightarrow \infty$. This estimate completes the proof.

Table 1 Summary of Riccati Iteration Performance for ten 8th order random examples

	$\lambda_s = \frac{\lvert\lambda_{max}\rvert + \lvert\lambda_{min}\rvert}{2}$, cf. (19), (21)					$\lambda_s = \lvert\lambda_{max}\rvert$, cf. (18), (20)				
case	μ	$\lvert\lvert R\rvert\rvert_{min}$	max. rel. e-value error	iterations	time (sec)	μ^*	$\lvert\lvert R\rvert\rvert_{min}$	max. rel. e-value error	iterations	time (sec)
1	0.656	4.0×10^{-5}	2.4×10^{-7}	85	3.2	0.792	4.3×10^{-5}	4.5×10^{-7}	144	5.5
2	0.613	2.5×10^{-6}	1.8×10^{-9}	86	3.2	0.730	4.1×10^{-7}	2.9×10^{-9}	137	5.2
3	0.704	2.0×10^{-4}	3.8×10^{-8}	92	3.4	0.826	1.6×10^{-4}	5.5×10^{-8}	146	5.4
4	0.730	7.8×10^{-10}	1.3×10^{-12}	106	4.1	0.838	8.7×10^{-10}	1.2×10^{-12}	181	6.8
5	0.692	6.8×10^{-10}	1.7×10^{-12}	91	3.5	0.818	9.0×10^{-10}	3.0×10^{-12}	159	6.0
6	0.913	9.0×10^{-9}	2.2×10^{-12}	160	6.1	0.760	9.7×10^{-10}	1.9×10^{-13}	127	4.8
7	0.768	4.6×10^{-10}	9.8×10^{-12}	136	5.3	0.868	6.6×10^{-5}	6.4×10^{-11}	146	5.6
8	1.106	no convergence ———				0.930	7.2×10^{-5}	2.1×10^{-9}	343	13.3
9	0.927	8.7×10^{-11}	2.7×10^{-13}	452	17.0	0.962	8.2×10^{-11}	3.5×10^{-13}	893	33.8
10	0.716	5.5×10^{-13}	4.1×10^{-14}	106	4.2	0.825	7.3×10^{-13}	1.8×10^{-4}	180	6.9

Table 2 System and performance index matrices for case #2

$$A = \begin{bmatrix} -8 & -8 & 6 & -4 & -7 & 4 & 0 & 0 \\ -2 & -10 & -6 & -4 & 0 & -3 & 1 & 1 \\ -5 & 9 & 2 & 5 & -6 & 8 & 0 & -7 \\ -7 & -4 & -4 & 10 & 5 & 2 & 0 & -3 \\ 3 & 5 & -1 & 2 & 1 & -10 & 0 & 6 \\ -8 & 2 & -7 & -9 & -8 & 4 & -8 & 0 \\ -7 & -1 & -6 & 9 & -10 & -8 & -6 & 7 \\ -4 & 7 & 0 & 4 & 0 & 6 & -1 & 2 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 0 & -7 & -7 & 5 & 6 & -7 & -2 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 48 & 18 & 7 & -3 & -5 & -10 & 6 & -9 \\ 18 & 27 & -10 & 11 & -1 & 3 & 0 & 0 \\ 7 & -10 & 30 & -6 & 1 & -5 & 0 & 0 \\ -3 & 11 & -6 & 33 & 0 & 2 & 0 & 0 \\ -5 & -1 & 1 & 0 & 6 & -3 & -4 & 6 \\ -10 & 3 & -5 & 2 & -3 & 10 & 0 & 0 \\ 6 & 0 & 0 & 0 & -4 & 0 & 4 & -6 \\ -9 & 0 & 0 & 0 & 6 & 0 & -6 & 9 \end{bmatrix}$$

$$R = 2$$

Table 3 System and Performance index matrices for case #9

$$A = \begin{bmatrix} -8 & -8 & -7 & 0 & -4 & 10 & -9 & 0 \\ 4 & -9 & -6 & -4 & 7 & 0 & -4 & -7 \\ 1 & 7 & -10 & -1 & -3 & 0 & 10 & -2 \\ 0 & 6 & 10 & 0 & 0 & 9 & 2 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 7 & -2 & -3 & 2 & -4 & -6 & 2 & 0 \\ 8 & -8 & 8 & 0 & 0 & 0 & 0 & -8 \\ 5 & 0 & 0 & 0 & -2 & -1 & -5 & -6 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 0 & 5 & 8 & 6 & 0 & 0 & 0 & -6 \end{bmatrix}$$

$$Q = \begin{bmatrix} 44 & 7 & -6 & 15 & 17 & -7 & -12 & 6 \\ 7 & 20 & 8 & 7 & 7 & -11 & -4 & -2 \\ -6 & 8 & 21 & 6 & -4 & -4 & 4 & -4 \\ 15 & 7 & 6 & 29 & 11 & -7 & -6 & 4 \\ 17 & 7 & -4 & 11 & 14 & -7 & -10 & 4 \\ -7 & -11 & -4 & -7 & -7 & 17 & 2 & 4 \\ -12 & -4 & 4 & -6 & -10 & 2 & 8 & -4 \\ 6 & -2 & -4 & 4 & 4 & 4 & -4 & 4 \end{bmatrix}$$

$$R = 2$$

Table 4 Summary of eigenvector and Schur vector solution performance for ten 8th order random examples

	eigenvector solution (EISPACK)			Schur vector solution (HQR3)		
case	$\|\|R\|\|_{min}$	max. rel. e-value error	time (sec)	$\|\|R\|\|_{min}$	max. rel. e-value error	time (sec)
1	1.3×10^{-2}	5.9×10^{-8}	0.40	4.0×10^{-3}	5.9×10^{-8}	0.62
2	2.8×10^{-4}	4.8×10^{-11}	0.45	7.6×10^{-5}	5.1×10^{-11}	0.63
3	2.3×10^{-2}	3.8×10^{-9}	0.46	6.4×10^{-5}	1.4×10^{-9}	0.55
4	1.6×10^{-8}	2.6×10^{-12}	0.47	2.0×10^{-8}	3.1×10^{-12}	0.63
5	1.8×10^{-8}	2.1×10^{-12}	0.46	1.9×10^{-8}	6.6×10^{-13}	0.57
6	2.2×10^{-8}	2.0×10^{-11}	0.40	1.3×10^{-8}	2.8×10^{-12}	0.57
7	2.5×10^{-8}	2.0×10^{-12}	0.42	1.1×10^{-8}	1.2×10^{-12}	0.61
8	2.0×10^{-4}	1.2×10^{-10}	0.47	3.9×10^{-5}	3.8×10^{-11}	0.56
9	5.8×10^{-4}	1.6×10^{-12}	0.42	3.1×10^{-9}	3.8×10^{-13}	0.56
10	6.5×10^{-11}	1.1×10^{-13}	0.41	2.4×10^{-11}	6.2×10^{-14}	0.59

Fig. 1 The Convergence of Case #2

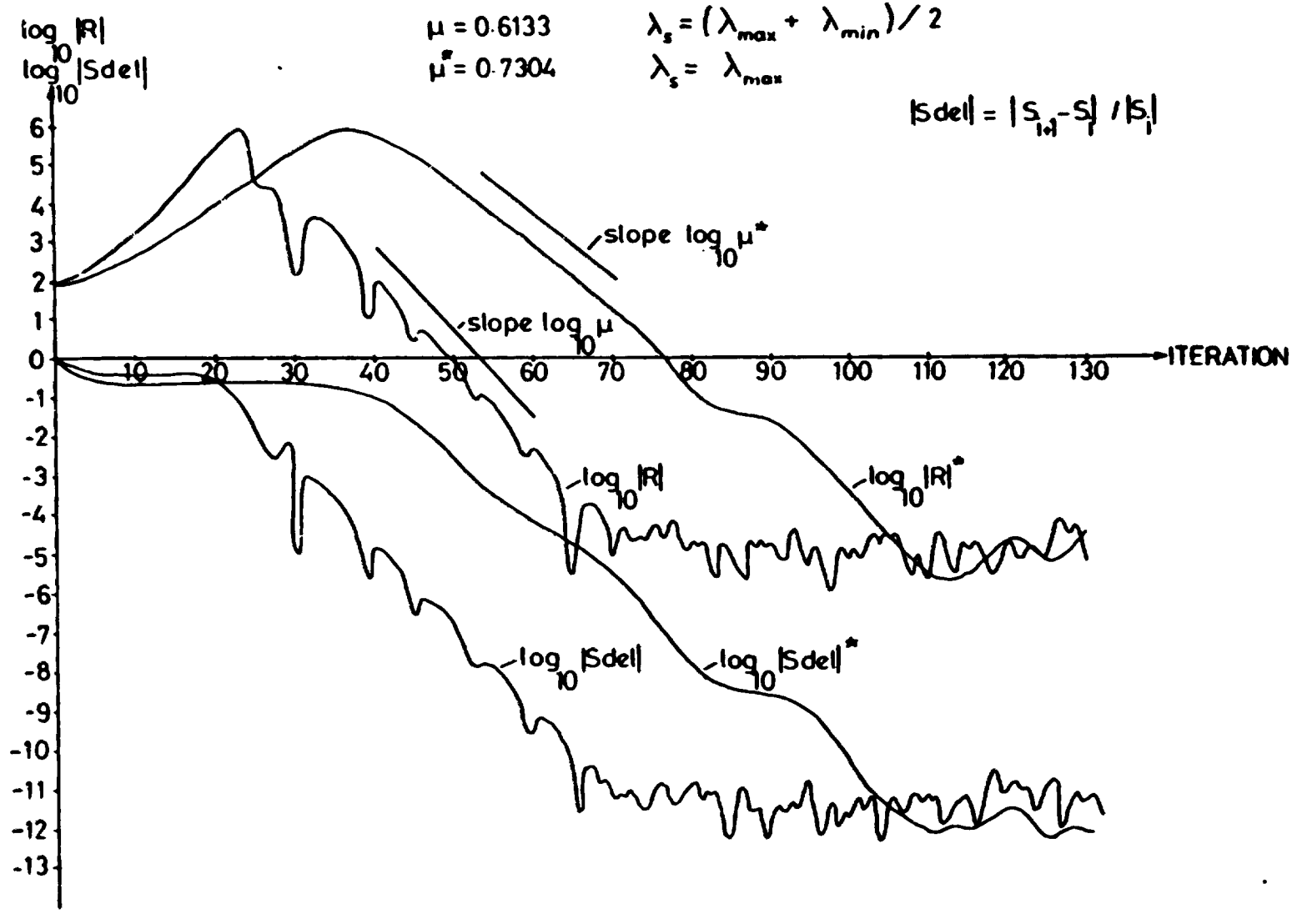
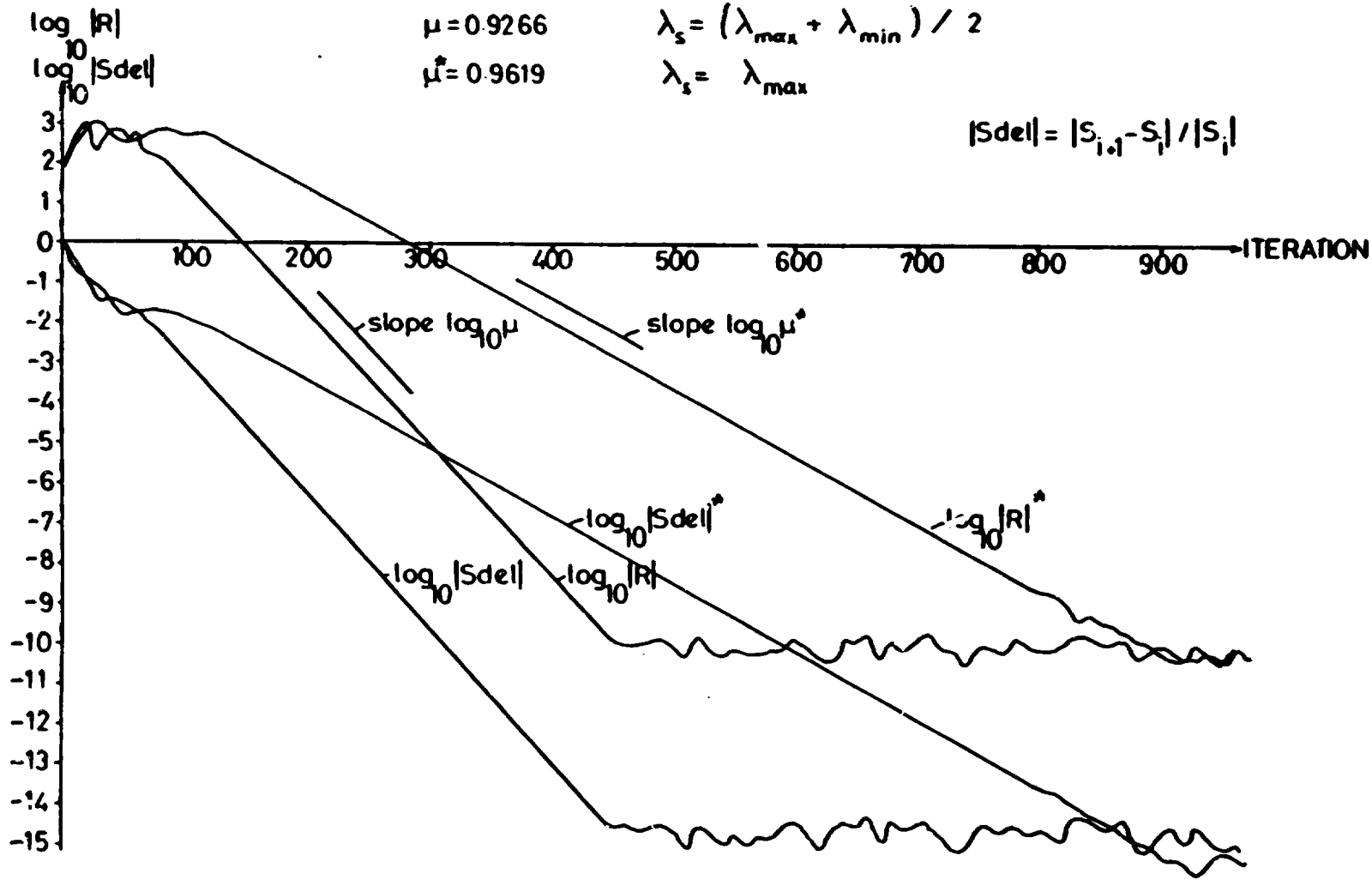


Fig. 2 The Convergence of Case #9



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A Literature Review of Robust Controller Design Methods

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Abstract

This paper presents a literature survey on the methods available for designing robust controllers. A number of methods for reducing the trajectory/performance index sensitivity in linear regulators are described. It is shown with the help of an example that decrease in system sensitivity to variation in parameters is obtained at the cost of a higher value for the performance index. A method for reducing the eigenvalue sensitivity is also discussed.

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Introduction

The design of a controller for a system requires a mathematical model of the system. In most cases some of the parameters of the system will change with time. The changes in the parameters could occur due to a variety of reasons for example, aging of components, environmental changes, etc. It has been found that for some systems even small changes in certain key parameters could appreciably degrade the performance of the control system. Hence, it is very important to estimate the effect of changes in the system parameters on its overall performance and to use controllers which minimize this degradation in performance.

The problem then is, given a specified structure of the controller, to find the particular controller which yields a system with minimum sensitivity to variation in system parameters. Such controllers are usually called robust or insensitive controllers. The robust controllers with constant feedback gain matrix will be considered. Adaptive controllers provide an attractive alternative to the problem of varying system parameters. We will not discuss adaptive controllers here.

A large amount of literature is available on robust controller design using both classical and modern control theories. A brief literature survey is presented here and some of the more important methods are described. In the analysis, the system parameters are assumed to take an unknown but constant value around their known nominal value.

Appendix A contains a bibliography on robust controllers containing 173 references. The bibliography covers the following journals and conference proceedings from 1970 to 1980.

Allerton Conference on Circuit and System Theory
IEEE Transactions on Automatic Control
International Journal of Control
Joint Automatic Control Conference
IEEE Decision and Control Conference
IFAC Symposium on System Stability and Adaptivity (1968, 1973)
Automatica

Some other important references are also included.

Parameter Sensitivity Reduction in Linear Regulators:

A considerable amount of work has been done on the parameter sensitivity reduction in linear regulators. Most of the literature on this topic falls in one of the following catagories:

a) Trajectory sensitivity reduction

b) Performance index sensitivity reduction

In both of these approaches to sensitivity reduction, the main objective is to achieve a trade off between optimality in the nominal performance and sensitivity to small parameter variations. In a recent paper Yedavalli and Skelton [1] treat the problem of trajectory sensitivity and performance index sensitivity in a unified way. We will now discuss the trajectory sensitivity reduction and combined trajectory and performance sensitivity reduction.

Trajectory sensitivity reduction. In this approach a quadratic trajectory sensitivity term is included in the integrand of the performance index. One of the initial papers on this is by Kahne [2]. This paper served as a starting point for a number of research efforts. Kahne implements the control in an open loop manner.

Let the linear time invariant system be given by:

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(o) = x_o \qquad (1)$$

where $x(t)$ is the state vector of dimension n

$u(t)$ is the control vector of dimension m

A and B are the state and control matrices of appropriate dimensions.

α is a time invariant parameter of the system x, A may depend on α. B and u are taken to be independent of α.

In the linear regulator problem we determine the optimal control vector $u^*(t)$ which minimizes the performance index (subject to system eq. (1)).

$$J = \frac{1}{2} x^T(T)\, Fx(T) + \frac{1}{2}\int_0^T [x^T(t)\, Q(t)\, x(t) + u^T(t)\, R(t)\, u(t)]\, dt \qquad (2)$$

(2)

where T is the final time.

F and Q(t) are positive semidefinite nxn matrices

and R(t) is positive definite mxm matrix.

To incorporate the trajectory sensitivity into the design, consider the trajectory sensitivity vector

$$\sigma = \left.\frac{\partial x}{\partial \alpha}\right|_{\alpha = \alpha \text{ nominal}}$$

Differentiating (1) with respect to α gives

$$\dot{\sigma} = A\sigma + A_\alpha x,\ \sigma(0) = 0 \qquad (3)$$

where $A_\alpha = \frac{\partial A}{\partial \alpha}$

A trajectory sensitivity term is incorporated in the performance index and the problem becomes:

find u(t) that minimizes

$$J = \frac{1}{2} x^T(T) Fx(T) + \frac{1}{2}\int_0^T \{x^T Q x + u^T R u + \sigma^T s \sigma\}\ dt \qquad (4)$$

subject to

$$\begin{aligned} \dot{x} &= Ax + Bu,\ x(0) = x_0 \\ \dot{\sigma} &= A\sigma + A_\alpha x,\ \sigma(0) = 0 \end{aligned} \qquad (5)$$

Kahne solved this problem in the standard way.

This method has two major drawbacks:

1) For each parameter considered the order of the system of equations to be solved increases by n. Kahne showed that this order could be somewhat reduced by using the fact that the feedback matrix obtained in symmetric.

2) The more important problem is that in a closed loop implementation, as is usually the case, $\sigma(t)$ no longer represents the trajectory sensitivity. This was pointed out by Kreindler [3].

We will now formulate the problem for a closed loop implementation. In this case u and B will depend on the parameter α. Then eq. (1) gives

$$\dot{\sigma} = A_\alpha x + A\sigma + B_\alpha u + B \frac{\partial u}{\partial \alpha} \quad \sigma(0) = 0 \tag{6}$$

Using the linear feedback law

$$u = K_1(t)x + K_2(t)\sigma \tag{7}$$

we get

$$\frac{\partial u}{\partial \alpha} = K_1\sigma + K_2 \frac{\partial \sigma}{\partial \alpha} \tag{8}$$

Substituting in eq. (6),

$$\dot{\sigma} = (A_\alpha + B_\alpha K_1)x + (A + B_\alpha K_2 + BK_1)\sigma + BK_2 \frac{\partial \sigma}{\partial \alpha}, \ \sigma(0) = 0 \tag{9}$$

The second order derivative $\frac{\partial \sigma}{\partial \alpha}$ is neglected because the only way it can be obtained is by differentiating eq. (9) with respect to α, this introduces $\frac{\partial^2 \sigma}{\partial \alpha^2}$ term and so on. The solution of eq. (9) with $\frac{\partial \sigma}{\partial \alpha}$ neglected will provide an approximate trajectory sensitivity vector p(t).

If p(t) is used in the feedback law, then $u = K_1 x + K_2 p$ (10)

and substituting in eq. (1) we get

$$\dot{x} = (A + BK_1)x + BK_2 p, \ x(0) = x_0 \tag{11}$$

Differentiating (11) with respect to α (neglecting $\frac{\partial p}{\partial \alpha}$ and letting $\frac{\partial x}{\partial \alpha} = p$) gives

$$\dot{p} = (A_\alpha + B_\alpha K_1)x + (A + BK_1 + B_\alpha K_2)p, \ p(0) = 0 \tag{12}$$

Exact differentiation of (11) yields

$$\dot{\sigma} = (A_\alpha + B_\alpha K_1)x + B_\alpha K_2 p + (A + BK_1)\sigma + BK_2 \frac{\partial p}{\partial \alpha}, \ p(0) = 0 \tag{13}$$

where $\frac{\partial p}{\partial \alpha}$ can be obtained from

$$\frac{\partial \dot{p}}{\partial \alpha} = (A_\alpha + B_\alpha K_1)\sigma + (A + BK_1 + B_\alpha K_2)\frac{\partial p}{\partial \alpha}, \frac{\partial p}{\partial \alpha}(o) = o \tag{14}$$

The difference between p and σ can be reduced by taking $u = \hat{K}_1 x + \hat{K}_2 p$ where $\hat{K}_1$ and $\hat{K}_2$ are chosen so as to reduce the difference between p and σ. Then,

$$\dot{x} = (A + B\hat{K}_1)\ x + B\hat{K}_2\ p \tag{15}$$

$$\dot{p} = (A_\alpha + B_\alpha \hat{K}_1)x + (A + B\hat{K}_1 + B_\alpha \hat{K}_2)p,\ p(o) = o \tag{16}$$

$$\text{and } \frac{\partial \dot{p}}{\partial \alpha} = (A_\alpha + B_\alpha \hat{K}_1)\sigma + (A + B\hat{K}_1 + B_\alpha \hat{K}_2)\frac{\partial p}{\partial \alpha}, \frac{\partial p(o)}{\partial \alpha} = o \tag{17}$$

A number of authors have used this formulation to solve the problem of closed loop trajectory sensitivity. Some of the important works will now be discribed.

Kreindler [3] used the approximate trajectory sensitivity vector p(t) in his formulation. Using equations (1) and (16) he constructed an augmented system of order 2n.

Take $Z = \begin{bmatrix} x \\ p \end{bmatrix}$

then

$$\dot{Z} = \bar{A}Z + \bar{B}u,\ Z(o) = Z_o \tag{18}$$

where $\bar{A} = \begin{bmatrix} A & 0 \\ (A_\alpha + B_\alpha \hat{K}_1) & (A + B\hat{K}_1 + B_\alpha \hat{K}_2) \end{bmatrix}$, $\bar{B} = \begin{bmatrix} B \\ o \end{bmatrix}$

and $Z_o = \begin{bmatrix} x_o \\ o \end{bmatrix}$

also $u = [\hat{K}_1\ \hat{K}_2]Z$

The augmented problem solved by Kreindler was to find $\hat{K}_1$ and $\hat{K}_2$ so as to minimize

$$J = \int_o^T (Z^T \bar{Q} Z + u^T R u)dt \tag{19}$$

subject to (18)

Where

$$\bar{Q} = \begin{bmatrix} Q & 0 \\ 0 & S \end{bmatrix}$$

Q and S are positive semidefinite nxn matrices and R is a positive definite mxm matrix.

This is now a standard linear regulator problem and the solution is given by:

$$K = [K_1 \; K_2] = -R^{-1} \bar{B}^T \bar{P}_1 \tag{20}$$

where $\bar{P}_1$ is the solution of the matrix Riccati eq.

$$-\dot{\bar{P}}_1 = \bar{A}^T \bar{P}_1 + \bar{P}_1 \bar{A} - \bar{P}_1 \bar{B} R^{-1} \bar{B}^T \bar{P}_1 + \bar{Q}, \; \bar{P}_1(T) = 0 \tag{21}$$

As an initial estimate for $\hat{K}_1$ and $\hat{K}_2$, Kreindler takes them to be K_1 and K_2, but then eq. (21) is no longer one of the Riccati type because $\bar{A}$ depends on K_1. Nevertheless he chooses K_1 and K_2 to satisfy eqs. (20) and (21).

This approach works well as long as p does not differ much from the true trajectory sensitivity vector σ.

Rao and Soudack [4] modified the above procedure so that one can get a better approximation for σ. They also use p in the cost functional but use a generalized version of Eq.(16).

$$\dot{p} = Cx + Dp, \; p(0) = 0 \tag{22}$$

Matrices C and D contain as many parameters as one may consider necessary to get a close approximation for σ by the procedure described below. From eq. (22) we get

$$\frac{\partial \dot{p}}{\partial \alpha} = C\sigma + D \frac{\partial p}{\partial \alpha}, \; \frac{\partial p}{\partial \alpha}(0) = 0 \tag{23}$$

In this case the augmented system of equations are

$$\dot{Z} = \bar{A}Z + \bar{B}u, \; Z(0) = Z_0 \tag{24}$$

where

$$\bar{A} = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \text{ and } Z_o = \begin{bmatrix} x_o \\ o \end{bmatrix}$$

The procedure is to choose a set of values for the parameters in C and D and with the new expression for $\bar{A}$ to find a K by solving eqns. (20) and (21). This will minimize J given by (19). With this K, the parameters in C and D are varied so as to minimize $J_2 = \int_0^T ||p - \sigma||^2 \, dt$

For this, eqns (11), (13), (22) and (23), have to be integrated simultaneously for $\alpha = \alpha_{nominal}$. The new values of C and D are used to again solve the Riccati equation to get a new K. This procedure is repeated until J_2 reduces to an acceptable level.

This procedure may give better results than Kreindler's procedure, but its computational complexity is far greater than that of the latter.

Fleming and Newmann [5] use the exact trajectory sensitivity vector σ in the performance index but use the approximate trajectory sensitivity vector p in the feedback law. They define the augmented state vector Z as

$$Z = \begin{bmatrix} x \\ p \\ \sigma \\ \frac{\partial p}{\partial \alpha} \end{bmatrix}$$

Using eqns. (11), (12), (13) and (14) the augmented eqns can be written as (of order 4n)

$$\dot{Z} = \bar{A}Z, \; Z(o) = Z_o \tag{25}$$

where

$$A = \begin{bmatrix} A + BK_1 & BK_2 & 0 & 0 \\ A_\alpha + B_\alpha K_1 & A + BK_1 + B_\alpha K_2 & 0 & 0 \\ A_\alpha + B_\alpha K_1 & B_\alpha K_2 & A + BK_1 & BK_2 \\ 0 & 0 & A_\alpha + B_\alpha K_1 & A + BK_1 + B_\alpha K_2 \end{bmatrix}$$

$$\text{and} \quad Z_0 = \begin{bmatrix} x_0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The performance index is

$$J = \int_0^T (x^T Q x + \sigma^T s \sigma + u^T R u) dt$$

This can be written as

$$J = \int_0^T Z^T (\bar{Q} + E^T K^T R K E) Z \, dt \tag{26}$$

where

$$\bar{Q} = \begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & S & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} I_{2n} & 0_{2n} \end{bmatrix} \text{ and } K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

The problem then becomes to choose K so as to minimize J subject to (25). Clearly this is not a linear regulator problem in its standard form and the authors describe a few computational methods of solving it.

In the three approaches given till now, one either neglects the second order derivative term $\frac{\partial \sigma}{\partial \alpha}$ or uses successive minimization techniques. In an interesting correspondence Byrne and Burke [6] use a slightly different approach. They use the optimal control for the regulator problem without sensitivity constraints as an initial approximation to the optimal control with sensitivity constraints. Let the optimal control for the problem without trajectory constraints be

$$u = Kx \tag{27}$$

where K is given by

$$K = -R^{-1} B^T P_1 \tag{28}$$

and P_1 is the solution of the Riccati eq.

$$\dot{P}_1 + P_1A + A^TP_1 + Q - P_1BR^{-1}B^TP_1 = 0,\ P_1(T) = 0 \tag{29}$$

Differentiating u with respect to parameter α gives

$$\frac{\partial u}{\partial \alpha} = K\sigma + K_\alpha x \tag{30}$$

where $K_\alpha = \partial K/\partial\alpha$

and differentiating (28) with respect to α gives

$$K_\alpha = -R^{-1}B^T\Sigma - R^{-1}B_\alpha^T P_1 \tag{31}$$

where $\Sigma = \dfrac{\partial P_1}{\partial \alpha}$

Differentiating (29) with respect to α

$$\dot{\Sigma} + \Sigma(A-BR^{-1}B^TP_1) + (A-BR^{-1}B^TP_1)^T\Sigma + (P_1A_\alpha + A_\alpha^T P_1)$$

$$-P_1(B_\alpha R^{-1}B^T + BR^{-1}B_\alpha^T)\ P_1 = 0,\ \Sigma(o) = o \tag{32}$$

This equation is solved to give Σ. Substituting eq. (30) in eq. (6), we get

$$\dot{\sigma} = (A + BK)\ \sigma + (A_\sigma + BK_\sigma)\ x + B_\alpha u,\ \sigma(o) = o \tag{33}$$

Then using the augmented vector $Z = \begin{bmatrix} x \\ \sigma \end{bmatrix}$, the augmented system becomes

$$\dot{Z} = \bar{A}Z + \bar{B}u,\ Z(o) = Z_o \tag{34}$$

where

$$\bar{A} = \begin{bmatrix} A & 0 \\ A_\alpha + BK_\alpha & A + BK \end{bmatrix},\quad \bar{B} = \begin{bmatrix} B \\ B_\alpha \end{bmatrix},\quad Z_o = \begin{bmatrix} x_o \\ o \end{bmatrix}$$

The problem is now to minimize the performance index given by (19) subject to (34). This is a standard regulator problem and the solution is

$$u = -R^{-1}\bar{B}^T\bar{P}_1Z \tag{35}$$

where $\bar{P}_1$ is the solution of the equation

$$\dot{\bar{P}}_1 + \bar{P}_1 \bar{A} + \bar{A}^T \bar{P}_1 + \bar{Q} - \bar{P}_1 \bar{B} \bar{R}^{-1} \bar{B}^T \bar{P}_1 = 0, \; \bar{P}_1(T) = 0 \tag{36}$$

A number of comparisons have been made between these methods and the results of these depend very much on the problem at hand. But if one was to use a comparison index of some sort composed of reduction in sensitivity per unit increase in cost and the numerical effort required, then the routine of Kreindler and Byrne and Burke would probably come out ahead.

Combined trajectory and performance index sensitivity reduction: The parameter sensitivity of the performance index is important because the optimal control is obtained by minimizing the given performance index. Yahagi [7] worked on this problem and gave necessary conditions for an optimal output feedback control with reduced performance index sensitivity.

Yedavalli and Skelton [1] established relationship between the trajectory and performance index sensitivities. They exploit this relationship to present a unified way of reducing trajectory (output or state) sensitivity and performance index sensitivity. Their method also considers control sensitivity, something which is ignored by most authors. Although Yedavalli and Skelton developed their method for the general case of r parameters, we will describe it for the case of one parameter only. This somewhat reduces the notational complexity.

Using the notation developed earlier, the linear time invariant system is

$$\dot{x} = Ax + Bu, \; x(0) = x_0 \tag{37}$$

$$y = Cx \tag{38}$$

where

y is a k dimensional output vector and C is the output matrix.

The performance index of the unperturbed problem is

$$J = \int_0^\infty (y^T Q y + u^T R u) dt \tag{39}$$

where

Q and R are symmetric positive definite matrices. The standard solution of this is

$$u = Kx \tag{40}$$

$$\text{where } K = - R^{-1} B^T P_1 \tag{41}$$

and P_1 is the unique positive definite solution of the matrix Riccati equation

$$P_1 A + A^T P_1 - P_1 B R^{-1} B^T P_1 + C^T Q C = 0 \tag{42}$$

Let α be the uncertain parameter and A, B, C, x and y are continuous functions of α. The sensitivity vectors are

State trajectory sensitivity vector $\sigma = \partial x / \partial \alpha$

Output trajectory sensitivity vector $y_\alpha = \frac{\partial y}{\partial \alpha}$

Control trajectory sensitivity vector $u_\alpha = \frac{\partial u}{\partial \alpha}$

and performance index sensitivity $J_\alpha = \frac{\partial J}{\partial \alpha}$

From equations (37) and (38) we get

$$\dot{\sigma} = A_\alpha x + A\sigma + B_\alpha u + B u_\alpha, \ \sigma(o) = o \tag{43}$$

$$y_\alpha = C_\alpha x + C\sigma \tag{44}$$

Where the subscript α denotes partial derivative with respect to α

Using $Z = \begin{bmatrix} x \\ \sigma \end{bmatrix}$ and $W = \begin{bmatrix} y \\ y_\alpha \end{bmatrix}$ the augmented system is

$$\dot{Z} = \bar{A} z + \bar{B} u + \hat{B} u_\alpha, \ Z(o) = Z_o \tag{45}$$

$$W = \bar{C} Z \tag{46}$$

where

$$\bar{A} = \begin{bmatrix} A & 0 \\ A_\alpha & A \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ B_\alpha \end{bmatrix}, \quad \bar{\gamma} = \begin{bmatrix} 0 \\ B \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} C & 0 \\ C_\alpha & C \end{bmatrix} \quad \text{and} \quad Z_o = \begin{bmatrix} x_o \\ o \end{bmatrix}$$

The authors define a new performance index which in addition to the output sensitivity term also contains control sensitivity term.

$$J_s = \int_o^\infty \left\{ y^T Q y + y_\alpha^T Q_1 y_\alpha + u_\alpha^T R_1 u_\alpha + u^T R u \right\} dt \tag{47}$$

where Q_1 and R_1 are positive definite matrices.

The norms $y_\alpha^T Q_1 y_\alpha$, $u_\alpha^T R_1 u_\alpha$ and $|kJ_\alpha|$ are functions of time. They showed that J_s is an upper bound to these three norms.

where

k_1 is a nonzero scalar such that the matrix

$$S_c = \begin{bmatrix} S & \pm\alpha S \\ \pm\alpha S^T & \bar{S} \end{bmatrix} \quad \text{is positive definite}$$

with S = block diagonal [Q, R]

$\bar{S}$ = block diagonal $[Q_1, R_1]$

They also showed that if $\frac{\partial^2 y}{\partial \alpha^2}$ and $\frac{\partial^2 u}{\partial \alpha^2}$ are zero, then,

$$\int_o^\infty (y_\alpha^T Q_1 y_\alpha + u_\alpha^T R_1 u_\alpha)\, dt = \frac{1}{2} \beta \frac{\partial^2 J}{\partial \alpha^2} \tag{48}$$

This relates the trajectory sensitivity terms $y_\alpha^T Q_1 y_\alpha$ and $u_\alpha^T R_1 u_\alpha$ in (47) with the performance index sensitivity J_α.

All this implies that minimization of J_s achieves a trade-off between

unperturbed optimal cost J and the sensitivities (output, control and performance index). Although J_α is not included in the performance index J_s, performance index sensitivity reduction is obtained as a "byproduct".

Using equation (38) and (44)

$$y^T Qy + y_\alpha^T Q_1 y_\alpha = x^T C^T QCx + (x^T C_\alpha^T + \sigma^T C^T)Q\,(C_\alpha x + C\sigma)$$

then J_s becomes

$$J_s = \int_o^\infty \left\{ [x^T \ \sigma^T] \begin{bmatrix} (C^TQC + C_\alpha{}^TQ_1C_\alpha) & C_\alpha{}^TQ_1C \\ C^TQ_1C_\alpha & C^TQ_1C \end{bmatrix} \begin{bmatrix} x \\ \sigma \end{bmatrix} + [u^T u_\alpha^T] \begin{bmatrix} R & 0 \\ 0 & R_1 \end{bmatrix} \begin{bmatrix} u \\ u_\alpha \end{bmatrix} \right\} dt \tag{49}$$

The above expression requires u_α. Since the desired control law

$$u = k_1 x + k_2 \sigma \tag{50}$$

is not available, the authors use the control for the unperturbed problem (given by eqs. (40), (41) and (42)) to obtain an approximate u_α.

Then the approximate control sensitivity $u_{\alpha a}$ is given by

$$u_{\alpha a} = kp \tag{51}$$

where p is the approximate trajectory sensitivity.

Substituting equation (51) in eq. (43),

$$\dot{p} = A_\alpha x + (A + BK)p + B_\alpha u \tag{52}$$

Defining $Z_a = \begin{bmatrix} x \\ p \end{bmatrix}$, we get

$$\dot{Z}_a = A_1 Z_a + B_1 u, \ Z_a(o) = Z_{ao} \tag{53}$$

$$u_{\alpha_a} = \hat{K} Z_a \tag{54}$$

where

$$A_1 = \begin{bmatrix} A & 0 \\ A_\alpha & (A + BK) \end{bmatrix}, \quad B_1 = \begin{bmatrix} B \\ B_\alpha \end{bmatrix}, \quad Z_{a_o} = \begin{bmatrix} x_o \\ o \end{bmatrix} \text{ and } \hat{K} = [0 \; K]$$

The performance index J_s becomes (using $u_{\alpha a}$ for u_α)

$$J_s = \int_o^\infty (Z_a^T U_a Z_a + u^T R u) \, dt \tag{55}$$

$$\text{where } U_a = \begin{bmatrix} (C^T Q C + C_\alpha^T Q_1 C_\alpha) & C_\alpha^T Q_1 C \\ C^T Q_1 C_\alpha & (C^T Q_1 C + K^T R_1 K) \end{bmatrix}$$

The problem is to minimize J_s subject to equation (53). This is a standard regulator problem and the solution is

$$u = \bar{K} Z_a = \bar{K}_1 \; x + \bar{K}_2 p \tag{56}$$

$$\text{where } \bar{K} = -R^{-1} B_1^T \bar{P}_1 \tag{57}$$

$\bar{P}_1$ is the positive definite solution of the Riccati equation

$$\bar{P}_1 A_1 + A_1^T \bar{P}_1 - \bar{P}_1 B_1 R^{-1} B_1^T \bar{P}_1 + U_a = o \tag{58}$$

So the steps in the Yedavalli and Skelton procedure are

(a) Compute K given by eqns. (41) and (42)

(b) Form the matrices A_1, B_1 and U_a

(c) The desired control is given by equations (56), (57) and (58).

As pointed earlier, results of comparisons between the various methods depends on the problem used. Keeping this in mind we will now use a first order example for the comparison of the methods described above. This example has been used by a number of authors for this purpose.

The first order plant is

$$\dot{x} = \alpha^2 x + u, \; x(o) = 1$$

where α is the parameter with a nominal value of 1. The objective is to minimize the trajectory sensitivity.

The performance index is given by

$$J = \int_o^\infty (x^2 + u^2)\, dt$$

To aid in the comparison of the various methods, the sensitivity integral, S, is also evaluated along with J for each of the methods. S is given by

$$S = \int_o^\infty \sigma^2 dt$$

The results of this are given in table 1. Clearly for all the algorithms a comparison with the simple regulator reveals that the decrease in sensitivity measured by S is obtained at the expense of a higher cost given by J. The best method would be the one which for the least increase in J will give the maximum decrease in S. The method of Byrne and Burke is the best for this example. The methods of Rao and Saudack and Fleming and Newman require much more computation than the other methods.

Table 1

	Simple Regulator (No sensitivity)	Kriendler	Rao and Soudack	Fleming and Newmann	Byrne and Burke	Yedavalli and Skelton
J	2.414	2.450	2.443	2.478	2.432	2.466
S	0.354	0.176	0.189	0.140	0.137	0.154
Percentage Increase in J		1.49	1.20	2.65	0.75	2.15
Percentage Decrease in S		50.28	46.61	60.45	61.30	56.50

Eigenvalue Sensitivity

Besides the trajectory and performance index (cost) sensitivities there is another important means of estimating the sensitivity of the system to variations in its parameters. This is through the use of eigenvalue/eigenvector sensitivity. It should be mentioned here that eigenvalue/eigenvector sensitivity provides a less direct measure of the system sensitivity than trajectory or cost sensitivity. Designers working with classical design techniques tend to use eigenvalue sensitivity.

Crossby and Porter [8] developed expressions for eigenvalue and eigenvector sensitivities for linear time invariant systems. Reddy [9] also worked on the problem of determining the effects of variation in system parameters on eigenvalues. Although they provided explicit expressions for the sensitivity, these authors do not suggest a method of reducing the eigenvalue/eigenvector sensitivity. In an interesting paper Gourishankar and Ramar [10] combine the problems of reducing eigenvalue sensitivity and closed loop eigenvalues (pole) placements for a linear time invariant multivariable system. We will now discuss their approach in some detail.

It has been known for a long time that using complete state feedback the closed loop poles can be assigned to any desired location. It is also known that for multiinput systems a number of feedback matrices give the desired closed loop pole locations. These controllers will in general give different time responses. Designers usually use this freedom to obtain other desired characteristics in the system response. Gourishankar and Ramar use this freedom to minimize the sensitivity of the closed loop eigenvalues to the variation in the system parameters without effecting their desired location. In the following development, the closed-loop poles are taken to be distinct.

Using the notation of the previous chapter, the linear time invariant system is given by

$$\dot{x} = Ax + Bu \tag{1}$$

and the feedback law is

$$u = kx \tag{2}$$

where K is mxn time invariant feedback matrix.

The closed loop system is then given by

$$\dot{x} = (A + BK)x \tag{3}$$

Keeping in mind that n elements of the mxn feedback matrix are sufficient to place the n closed loop poles to their desired location, we subdivide the feedback matrix into two parts as follows

$$\bar{K} = [k_1^T \; k_2^T \; \text{---} k_{i-1}^T \; k_{i+1}^T \; \text{---} k_m^T]^T$$

and $\hat{k} = k_i$

where k_j is the jth row of the K matrix, $j = 1, 2, \text{-- } m$

The vector $\hat{k}$ is used to obtain the desired pole locations while the elements of the matrix $\bar{K}$ are chosen to reduce the eigenvalue sensitivity. Define

$$\bar{u} = [u_1 \; u_2 \; \text{----} \; u_{i-1} \; u_{i+1} \; \text{----} u_m]^T$$

and $\hat{u} = u_i$

Eq. (2) becomes

$$\bar{u} = \bar{K}x \tag{4}$$

and

$$\hat{u} = \hat{k}x \tag{5}$$

We can now write eq. (1) as

$$\dot{x} = (A + \bar{B}\,\bar{K})x + \hat{b}\,\hat{u} \tag{6}$$

where

$$\bar{B} = [b_1\ b_2\ \text{----}b_{i-1}\ b_{i+1}\ \text{----}b_m] \text{ and } \hat{b} = b_i$$

b_j's being the columns of the matrix B.

Eq. (6) represents a single input system ($\hat{u} = u_i$) and the feedback vector $\hat{k}$ in eq. (5) can be designed to place the closed loop poles in their specified location. This can always be done as long as the pair $(A + \bar{B}\,\bar{K}, \hat{b})$ is completely controllable. Since $A + \bar{B}\,\bar{K}$ depends on $\bar{K}$, care has to be taken while selecting $\bar{K}$ so as to maintain the above mentioned pair completely controllable. The authors have mentioned that this is not a serious limitation as almost any $\bar{K}$ satisfies this requirement.

Substituting eq. (5) in (6) yields the closed loop system

$$\dot{x} = (A + \bar{B}\,\bar{K} + \hat{b}\,\hat{k})x \tag{7}$$

It is to be noted that the feedback vector $\hat{k}$ depends on $A + \bar{B}\,\bar{K}$ and hence on $\bar{K}$.

Morgan [11] gave the expression for the sensitivity of the closed loop eigenvalues to the variations in the elements of the system matrix A.

$$S_{j\ell}^{i} = \frac{\partial Si}{\partial a_{j\ell}} = \frac{1}{(\partial g(s)/\partial s)/_{s=si}} \text{ trace } [R(S_i)\,\frac{\partial \hat{A}}{\partial a_{j\ell}}] \tag{8}$$

where

$S_{j\ell}^{i}$ is the sensitivity of the eigenvalue S_i to a small variation in the element $a_{j\ell}$ of the system matrix A,

g(s) is the characteristic polynamial of the closed loop system,

$R(S_i)$= adjoint $(S_i\ I - \hat{A})$

and $\hat{A} = A + \bar{B}\,\bar{K} + \hat{b}\,\hat{k}$

The authors choose the elements of the matrix $\bar{K}$ so as to minimize the performance index

$$J = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{\ell=1}^{n} (S_{j\ell}^{i})^2 \tag{9}$$

If only the $a_{j\ell}$ element is changing, the performance index becomes

$$J = \sum_{i=1}^{n} (S_{j\ell}^{i})^2 \tag{10}$$

The authors mention that transforming to phase-variable form facilitates in the minimization of J.

The procedure can be summarized as follows

(a) Choose $\bar{K}$ so as to minimize J

(b) With this value of $\bar{K}$ find $\hat{k}$ which assigns the closed loop poles to the desired location.

Any row of matrix $\bar{K}$ can be used as $\hat{k}$. One could take the rows of $\bar{K}$ as $\hat{k}$ one at a time and calculate $J_{minimum}$ each time. Then choose that row of $\bar{K}$ as $\hat{k}$ for which the lowest $J_{minimum}$ was obtained. This is usually not done as the amount of computation increases enormously.

The design procedure is illustrated by the following linear time invariant second-order system with two inputs and two outputs,

$$\dot{x} = Ax + Bu \tag{11}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The nominal values of a_{11}, a_{12}, a_{21} and a_{22} are 0, 1, 0 and 0 respectively.

Only a_{21} varies from its nominal value. The desired location for the closed loop poles is $S_1 = -2$ and $S_2 = -3$. The feedback matrix is

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \begin{bmatrix} \hat{k} \\ \bar{k} \end{bmatrix}$$

then

$$B = [\hat{b}\ \bar{B}]$$

and $$\hat{A} = \begin{bmatrix} 0 & 1 \\ a_{21} & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} [k_{21}\ k_{22}] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [k_{11}\ k_{12}]$$

$$= \begin{bmatrix} (k_{11} + k_{21}) & (1 + k_{12} + k_{22}) \\ (a_{21} + k_{21}) & k_{22} \end{bmatrix}$$

This gives

$$\frac{\partial \hat{A}}{\partial a_{21}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\text{and } R(s) = \text{Adjoint} \begin{bmatrix} (s - k_{11} - k_{21}) & -(1 + k_{12} + k_{22}) \\ -(a_{21} + k_{21}) & (s - k_{22}) \end{bmatrix}$$

$$= \begin{bmatrix} (s - k_{22}) & (1 + k_{12} + k_{22}) \\ (a_{21} + k_{21}) & (s - k_{11} - k_{21}) \end{bmatrix}$$

Now trace $R(s) \frac{\partial A}{\partial a_{21}}$ = trace $\begin{bmatrix} (1 + k_{12} + k_{22}) & 0 \\ (s - k_{11} - k_{21}) & 0 \end{bmatrix}$

$$= (1 + k_{12} + k_{22})$$

$$g(s) = (s - k_{11} - k_{21})(s - k_{22}) - (1 + k_{12} + k_{22})(a_{21} + k_{21})$$

$$\therefore \frac{\partial g(s)}{\partial s} = 2s - (k_{11} + k_{21} + k_{22})$$

The eq. (8) now gives

$$S_{21}^1 = \frac{\partial S}{\partial a_{21}}\bigg|_{S=-2} = \frac{-(1 + k_{12} + k_{22})}{(k_{11} + k_{21} + k_{22} + 4)}$$

$$S_{21}^2 = \frac{\partial S}{\partial a_{21}}\bigg|_{S=-3} = \frac{-(1 + k_{12} + k_{22})}{(k_{11} + k_{21} + k_{22} + 6)}$$

The performance index becomes

$$J = (S_{21}^1)^2 + (S_{21}^2)^2$$

$$= (1 + k_{12} + k_{22})^2 \left\{ \frac{1}{(k_{11} + k_{21} + k_{22} + 4)^2} + \frac{1}{(k_{11} + k_{21} + k_{22} + 6)^2} \right\}$$

We have to choose $\bar{K}$ so as to minimize J. J takes its minimum value 0 if

$$k_{22} = -(1 + k_{12})$$
$$k_{11} + k_{21} + k_{22} + 4 \neq 0$$
and $$k_{11} + k_{21} + k_{22} + 6 \neq 0$$

Take $k_{21} = 0$

For $a_{21} = 0$, we get

$$g(s) = s^2 - s(k_{11} + k_{21} + k_{22}) + k_{11} k_{22} - k_{21} - k_{12} k_{21}$$

Substituting the values of k_{21} and k_{22} in the above expression,

$$g(s) = s^2 - s(k_{11} - k_{12} - 1) - k_{11} (1 + k_{12})$$

k_{11}, k_{12} have to be chosen so as to get the desired characteristic polynomial

$$(s + 2) (s + 3) = s^2 + 5s + 6$$

Equating the two polynomials, we get

$$k_{11} - k_{12} - 1 = -5$$

and $$k_{11} (1 + k_{12}) = -6$$

Solving for k_{12},

$$(k_{12} - 4) (1 + k_{12}) = -6$$

or $$k_{12}^2 - 3 k_{12} + 2 = 0$$

or $$(k_{12} - 2) (k_{12} - 1) = 0$$

$\therefore$ $k_{12} = 2, 1$

$k_{11} = -2, -3$

$k_{22} = -3, -2$

$k_{21} = 0$

This gives the desired feedback matrix as

$$K = \begin{bmatrix} -2 & 2 \\ 0 & -3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -3 & 1 \\ 0 & -2 \end{bmatrix}$$

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ROBUST CONTROLLER DESIGN FOR LARGE PARAMETER VARIATIONS

Leonard R. Anderson*

ASSUMPTIONS

1) The physical process depends on a few key parameters $\alpha_1, \ldots, \alpha_N$. (For wing flutter suppression these would be dynamic pressure, fuel load in wings, etc). N would not typically be larger than 2 or 3.

2) These N parameters vary over limits that are either specified, or to be determined in the design process.

$$\alpha_{i_{min}} \leq \alpha_i \leq \alpha_{i_{max}} \quad i = 1, \ldots, N$$

The range of each parameter is divided into a grid of a small number of equally spaced points, e.g., 3-7 points.

$$[\alpha_{i_{min}} ; \alpha_{i_{max}}] = [\alpha_{i_1} ; \alpha_{i_2} ; \ldots\ldots ; \alpha_{i_{Mi}}]$$

with $\alpha_{i_{min}} = \alpha_{i_1}$; $\alpha_{i_{max}} = \alpha_{i_{Mi}}$

The set of Σ_α of possible parameter values, therefore has $N_\alpha = M_1 \cdot M_2 \ldots M_N$ elements.

3) For each element $\vec{\alpha} \in \Sigma_\alpha$ the physical process can be modeled by a system of linear constant coefficient O.D.E.'s. For example, consider a wing flutter model.

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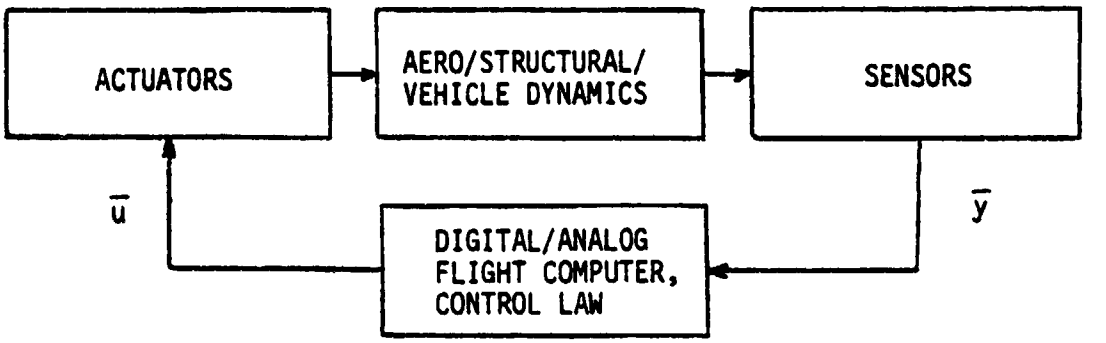
$$\dot{\vec{x}} = A(\vec{\alpha})\vec{x} + B(\vec{\alpha})\vec{u} + G(\vec{\alpha})\ \vec{w},\ \vec{x}\epsilon R^n,\ \vec{u}\ R^m$$
$$\vec{y} = H\vec{x} + \vec{v} \qquad \vec{y}\epsilon R^p$$



$$\dot{\vec{Z}} = Ac\vec{Z} + B_c\vec{y}$$
$$\vec{u} = K_1\vec{Z} + K_2\vec{y}$$

Typical Parameters that would vary might be

Actuators: Gain, Phase Error
Vehicle : Dynamic Pressure, Geometry
Sensors : Gain, Offset Error

4) The structure of the control law is specified, but values of constants in the control law are to be determined in the design process.

For example, suppose the sensor signals consist of accelerations, y_i, measured at various points on the vehicle/wing, and the control law is:

$$\begin{bmatrix} \dot{Z}_{1i} \\ \dot{Z}_{2i} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ C_{1i} & C_{2i} \end{bmatrix} \begin{bmatrix} Z_{1i} \\ Z_{2i} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y_i,\ i = 1, \ldots p$$

$$\vec{u} = K_1\vec{Z} + K_2\vec{y}$$

Let the elements of matrices K_1, K_2 and C_{11}, C_{21}, C_{12}, C_{22}, .. be represented by the <u>design vector</u> $\vec{k}$.

5) If we choose a parameter vector $\vec{\alpha}$ and a design vector $\vec{k}$, we have a well defined closed-loop linear system

$$\begin{bmatrix} \dot{\vec{x}} \\ \dot{\vec{Z}} \end{bmatrix} = \tilde{A}(\vec{k},\vec{\alpha}) \begin{bmatrix} \vec{x} \\ \vec{Z} \end{bmatrix} + \begin{bmatrix} G \\ 0 \end{bmatrix} \vec{w} + \begin{bmatrix} 0 \\ I \end{bmatrix} \vec{v}$$

6) For one design point $\vec{k}$ there are N_α closed-loop system matrices

$$\tilde{A}(\vec{k},\vec{\alpha}_1), \ldots, \tilde{A}(\vec{k},\vec{\alpha}_{N_\alpha})$$

corresponding to the possible discrete values of the parameters $\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\alpha}_{N_\alpha}$

We can compute the eigenvalues of each of these N_α closed-loop system matrices as

$$\lambda\ (\tilde{A}(\vec{k},\vec{\alpha}_i))$$

and collect them all into one set that depends only on the design point $\vec{k}$ as

$$\Lambda\ (\vec{k}) = \{\lambda(\tilde{A}(\vec{k},\vec{\alpha}_1)), \ldots, \lambda(\tilde{A}(\vec{k},\vec{\alpha}_{N_\alpha}))\}$$

7) Also, for specified initial state and noise covariance matrices, we can compute the RMS values of actuator signals, structural degrees of freedom, or other linearly related parameters for any given closed loop system $\tilde{A}(\vec{k},\vec{\alpha})$.

If the number N_α were not unreasonably large, we could find the <u>maximum</u> RMS values of these variables for the N_α closed-loop systems

$$\tilde{A}(\vec{k},\vec{\alpha}_1), \ldots, A(\vec{k},\vec{\alpha}_{N_\alpha})$$

8) Then, applying either random pattern search strategy (Bekey '81) or nonlinear programming without gradient evaluation, we can search over the space $\Sigma_{\vec{k}}$ of possible design points to achieve any of the following design objectives:

a) minimize RMS values of actuator signals while requiring the set $\Lambda(\vec{k})$ to remain in some favorable region of the complex plane.

b) maximize the range $\alpha_{i\ min}, \alpha_{i\ max}$ while constraining both RMS values of actuator signals, and location of $\Lambda(\vec{k})$ in the complex plane.

c) for specified limits on actuator signals and parameter ranges, move $\Lambda(\vec{k})$ as far to the left in the complex plane as possible.

IMPLEMENTATION PROBLEMS:

1) One must be able to effectively generate the linear models $A(\vec{\alpha})$, $B(\vec{\alpha})$, $G(\vec{\alpha})$ corresponding to the specified parameter ranges. This is probably not possible on-line, but will require these arrays of arrays be stored on disk before the controller design study is initiated.
2) It may be difficult to avoid local minima in the search for an optimal $\vec{k}$, and alternate starting points should be considered.
3) One must religiously avoid the curse of dimensionality and reduce the number of parameter points N_α to an absolute minimum (i.e., 9 or 15).
4) Care must be taken to construct a well-posed problem in which none of the design variables $\vec{k}$ may go to infinity.

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