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THEORY OF THERMOELASTICITY

Dorin Iesan

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16. Abstract This book traces the history of the development of the theory of thermoelasticity, which examines the interactions between the deformation of elastic media and the thermal field. The fundamental problems of the theory are presented. In addition, recent results of studies on the subject are presented. Reference is made primarily to media with generalized anisotropy, or to isotropy media. Thermo-mechanical problems and mathematical formulations and resolutions are emphasized.					
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Preface

The theory of thermoelasticity examines the interactions between the deformation of elastic media and the thermal field. The beginnings of this theory are found in the works of Duhamel and Neumann who considered the equations of the linear theory in the case of homogeneous and isotropic media. The reasoning from the thermodynamic viewpoint of these equations was carried out by Biot, after Voigt, Jeffreys, Lessen and Duke made various attempts in these directions.

The theory of thermoelasticity has advanced considerably through the recent years. The large number of investigators and their studies which present interest both from the viewpoint of their technique and the theoretical importance of the problems which are considered illustrate these facts. Among the monographs devoted to the subject of thermoelasticity we want to mention those written by Melan and Parkus, Boley and Weiner, Nowacki, Kovalenko, Grindei, Parkus, Carlson.

In this book the fundamental problems of the theory of thermoelasticity are presented. The book also contains recent results which are not included in other treatises. Of course it was not possible to present all aspects of this theory and it was not even attempted. The goal of the book is to present to the reader the basis of thermoelasticity and some of the remarkable results achieved in this field. The presented questions refer to media with a generalized anisotropy or to isotropy media. In order to maintain the unity of the exposition the generalized theories of thermoelasticity were not considered. Both the thermomechanical sense of the problem and the mathematical formulation and resolutions were emphasized in this work.

Without being exhaustive, the bibliography contains, in addition to the papers cited in the text, also studies which give a broader picture of the literature and the investigations of the theory of thermoelasticity.

The Author

TABLE OF CONTENTS

	Page
Chapter I. Deformation of the continuous medium	1
1. Description of the deformation	1
2. Displacement vector. Deformation tensors	3
3. Conditions of compatibility	8
4. The invariants of the deformation tensors	13
5. The transformation of the surface and volume elements	15
Chapter II. Principles of mechanics and thermodynamics	17
6. Velocity and acceleration	17
7. Several important material differentials	19
8. The deformation of velocity tensor. The spin tensor	21
9. Objective tensors	23
10. The principle of the conservation of mass. The equation of continuity	25
11. The principle of the impulse. The principle of the kinetic moment	27
12. The tension tensor	30
13. The equations of motion	33
14. The Piola-Kirchhoff tension tensors	36
15. The principle of energy	39
16. The heat flux vector. The energy equation	43
17. Statements about the principles introduced	46
18. The principle of entropy	49
Chapter III. The equations of thermoelasticity	51
19. Constituent equations	51
20. The consequences of the laws of thermodynamics	56
21. Nonlinear equations of thermoelasticity	62
22. Isotropic media. Incompressible media	66
23. The equations of the linear theory of thermoelasticity	69
24. Curvilinear coordinates	75
Chapter IV. Linear theory of dynamic thermoelasticity	94
25. Fundamental equations. Statement of the problem	94
26. Characterization of a mixed problem with the help of displacement and temperature	97
27. Another formulation of the problem	100
28. The theorem of uniqueness	103
29. The theorem of reciprocity	106
30. Applications of the reciprocity theorem	112
31. Variational theorems	116
32. Theorems of existence	124
33. Homogeneous and isotropic media	144
34. Formal representations of the solutions	146

	Page
35. Thermoelastic waves	151
36. The elastic space subjected to concentrated loads	164
37. The method of the potential in the study of vibrations	170
38. Short-time solutions. Lessen's method	184
39. Propagation of the discontinuity surfaces	199
Chapter V. The linear theory of thermoelasticity	231
40. Equations of thermoelastic equilibrium	231
41. Analogy with problems from the theory of elasticity	233
42. Formulas for the mean values of deformations and tensors	236
43. The theorem of reciprocity	238
44. The theorem of uniqueness	241
45. Variational theorems	243
46. The existence of the solution. Approximation method	249
47. Homogeneous and isotropic media	255
48. Special problems	261
49. Plane thermoelastic deformation	278
50. Generalized plane deformation	293
51. The problem of Saint Venant	299
Chapter VI. Nonlinear thermoelasticity	333
52. Successive approximations	333
53. The theory of thermoelasticity of the second order	337
54. A Stoppelli type theorem of existence and uniqueness	343
55. Plane deformation	350
56. Special problems	364
References	379

THE THEORY OF THERMOELASTICITY

Dorin Iesan

Chapter 1. DEFORMATION OF THE CONTINUOUS MEDIUM

1. Description of the Deformation

A continuous body or a continuous medium occupies a domain Ω ^{*} in space. This description does not correspond exactly to the physical concept of material bodies. It is a well known case that at the molecular scale matter has a complex structure and does not occupy a domain in the sense of the mathematical meaning of this word. In spite of this, the phenomena studied within the framework of the mechanics of continuous media describe quite well from the practical viewpoint the behavior of real media. A continuous medium is thus a mathematical model representing an idealization of the real medium.

Let us consider a continuous medium which at the moment $t = 0$ occupies the domain B of the boundary ∂B . The closing of B will be indicated by \bar{B} .

Let $OX_K (K=1,2,3,)$ be a system of rectangular cartesian coordinates, X_K the coordinates of a generic material point M , from the B domain, and \underline{R} the position vector of point M with respect to the system of coordinates under consideration.

Let us assume that the medium is deformed in such a way that at the moment t it occupies the domain \mathcal{B} of the $\partial \mathcal{B}$ boundary, while point M reaches into M^* . We will refer the medium which occupies the domain \mathcal{B} to another system of rectangular cartesian coordinates, fixed ox_i ($i = 1,2,3$). Let x_i be the coordinates of point M^* fixed with respect to the reference ox_i , while \underline{r} is the position vector of M^* with respect to this reference (Fig. 1).

*Numbers in the margin indicate pagination in the foreign text.

Biunivocal and bicontinuous correspondence is established between B and A in which the point M from B and the point M^* from A correspond to each other.

The deformation of the medium is defined by the relation

$$x_i = x_i(X_K, t), \quad (1.1)$$

which in view of what has been stated above may be resolved with respect to X_K

$$X_K = X_K(x_i, t). \quad (1.2)$$

Here and subsequently if not stated otherwise the indices will assume the values $(1, 2, 3)$. Occasionally the system of coordinates (x_1, x_2, x_3) will be indicated by x , and (X_1, X_2, X_3) by X .

If the coordinates X_K are fixed, the functions (1.1) determine the trajectory of the material point which at moment $t = 0$ would have the coordinates X_K ; if t is fixed, the functions (1.1) describe the transformation which takes the domain B into the domain A .

Let $(0, t_0)$ be a fixed time interval where $t_0 > 0$ may be infinity. In the following if not indicated otherwise we will assume that B is a region regulated in the sense of Kellog [232].

The functions (1.1) defined by $B \times (0, t_0)$ describe the /10
movement of the continuous medium under consideration. If not stated otherwise we will assume that these functions are of the C^2 class.

Obviously from the above it follows that the functional determinate

$$J = \left| \frac{\partial x_i}{\partial X_K} \right|, \quad (1.3)$$

is other than 0.

The X_K coordinates are called material coordinates and x_i coordinates are called spatial coordinates.

Let us indicate some of the designations and notations which the reader might encounter in the text. We state that a function f is continuous on portions on B if there is a subdivision of B_1, B_2, \dots, B_n of B in such a way that for each B_j ($j = 1, 2, \dots, n$) the restriction of f on B_j is limited on \bar{B}_j and continuous on $\bar{B}_j - D$, thereby D is a finite submagnitude (eventually empty) of B_j . Let S_1 and S_2 be parts of ∂B so that $S_1 \cup S_2 = \partial B, S_1 \cap S_2 = \emptyset$. We say that function g is regulated on the portions on S_α ($\alpha = 1, 2$) if g is continuous on the portions of S_α and every regulated point from S_α is a point of continuity for the function of g . A point $X \in \partial B$ is called regulated if the normal at ∂B is continuous in X .

Let g and \tilde{g} be regulated functions on the portions on ∂B . We will write $g = \tilde{g}$ on ∂B if $g(X) = \tilde{g}(X)$ in any regulated point $X \in \partial B$. Let f be a function of X and t defined on $B \times (0, t_0)$ and M and N be nonnegative numbers. We say that f is from the class $C^{M,N}$ if the functions

$$\partial^m f^{(n)} = \frac{\partial^m}{\partial X_1 \partial X_2 \dots \partial X_K} \left(\frac{\partial^n f}{\partial t^n} \right), \quad m \in \{0, 1, \dots, M\}, \quad n \in \{0, 1, \dots, N\},$$

$$m + n \leq \max \{M, N\},$$

exist and are continuous on $B \times (0, t_0)$.

We say that f is of the $C^{M,N}$ class on $\bar{B} \times [0, t_0)$ if f is of the class $C^{M,N}$ on $B \times [0, t_0)$ and for every $m \in \{0, 1, \dots, M\}$, $n \in \{0, 1, \dots, N\}$, $\partial^m f^{(n)}$ may be extended continuously to $\bar{B} \times [0, t_0)$. The class of the functions $C^{N,N}$ is indicated by C^N .

We will say that the function g is regulated on portions on $S_\alpha \times [0, t_0)$ if g is continuous on the portions on $S_\alpha \times [0, t_0)$ and for every $t \in [0, t_0)$ g is regulated on portions on S_α .

2. Displacement Vector. Deformation Tensors

We will designate by \underline{E}_K the versors of the axes OX_K and by \underline{e}_1

the versors of the axes ox_1 . Obviously we have

$$E_K E_L = \delta_{KL}, \quad e_i e_j = \delta_{ij}, \quad (2.1)$$

where δ_{KL} and δ_{ij} represent Kronecker's symbol.

If we take into account the notations introduced in paragraph 1, we can write

$$R = X_K E_K, \quad r = x_i e_i. \quad (2.2)$$

Let $\underline{b} = \vec{Oo}$ and $\underline{u} = \vec{MM'}$. The vector u is called the displacement /11 vector. We have

$$u = r - R + b. \quad (2.3)$$

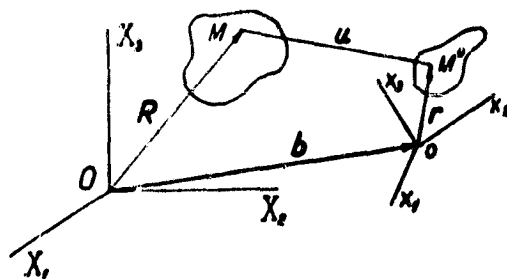


Figure 1

two coordinate systems coincide then δ_{iK} coincides with Kronecker's symbol.

We are introducing the constants

$$\delta_{iK} = e_i E_K. \quad (2.4)$$

These represent the directive cosines of the ox_1 axes with respect to the reference OX_K . If these

Let it be

$$u = u_i e_i = U_K E_K, \quad b = b_i e_i = B_K E_K. \quad (2.5)$$

From (2.3)-(2.5) we derive

$$u_i = x_i - \delta_{iK} X_K + b_i, \quad U_K = \delta_{iK} x_i - X_K + B_K. \quad (2.6)$$

If these two coordinate systems coincide then we can write $u_1 = x_1 - X_1$, etc.

Thus knowledge of the components of the displacement factor is equivalent to knowledge of the functions (1.1) or (1.2) which describe the deformation.

Thus the vector \underline{u} of the class C^2 on $B \times (0, t_0)$ class describes the movement of the continuous medium.

Let us consider the two positions of the medium: in the B domain and in the \mathcal{B} domain. If we take the differentials $d\underline{R}$ and $d\underline{r}$, we have

$$d\underline{R} = \underline{E}_K dX_K, \quad \underline{r} = x_{i,K} \underline{e}_i dX_K, \quad (2.7)$$

or

$$d\underline{R} = X_{K,i} \underline{E}_K dx_i, \quad d\underline{r} = \underline{e}_i dx_i, \quad (2.8)$$

where we note

$$X_{K,i} = \frac{\partial X_K}{\partial x_i}, \quad x_{i,K} = \frac{\partial x_i}{\partial X_K}. \quad (2.9)$$

By the agency of the relations (1.1) and (1.2) any function f of the variables x_i, t , is similarly a function of the variable X_K, t and inversely. In general we will use the following notations

$$f_{,M} = \frac{\partial f}{\partial X_M}, \quad f_{,i} = \frac{\partial f}{\partial x_i}. \quad (2.10)$$

The magnitudes $x_{i,K}$ and $X_{K,i}$ are called deformation gradients. /12
We have

$$x_{i,K} X_{K,j} = \delta_{ij}, \quad X_{K,i} x_{i,L} = \delta_{KL}. \quad (2.11)$$

If we introduce notations

$$\underline{C}_K = x_{i,K} \underline{e}_i, \quad \underline{e}_i = X_{K,i} \underline{E}_K, \quad (2.12)$$

from (2.7) and (2.8) it results that

$$d\underline{R} = \underline{e}_i dx_i, \quad d\underline{r} = \underline{C}_K dX_K. \quad (2.13)$$

Let us designate by dS and ds the magnitudes of the vectors $d\mathbf{R}$ and $d\mathbf{r}$ respectively. From (2.7), (2.8), (2.12), (2.13) we obtain

$$dS^2 = (d\mathbf{R})^2 = dX_K dX_K, \quad ds^2 = (d\mathbf{r})^2 = C_{KL} dX_K dX_L, \quad (2.14)$$

or

$$dS^2 = c_{ij} dx_i dx_j, \quad ds^2 = dx_i dx_i, \quad (2.15)$$

where

$$C_{KL} = C_K C_L = x_{i,K} x_{i,L}, \quad c_{ij} = c_i c_j = X_{K,i} X_{K,j}. \quad (2.16)$$

The quantities c_{ij} , C_{KL} are components of some second order symmetrical and positively defined tensors called Cauchy's deformation tensor and Green's deformation tensor, respectively.

In view of (2.14) and (2.15) we can write

$$ds^2 - dS^2 = 2E_{KL} dX_K dX_L = 2e_{ij} dx_i dx_j, \quad (2.17)$$

where

$$2E_{KL} = C_{KL} - \delta_{KL}, \quad 2e_{ij} = \delta_{ij} - c_{ij}. \quad (2.18)$$

The magnitudes E_{KL} , e_{ij} are the components of certain symmetrical tensors called lagrangian deformation tensors and eulerian deformation tensors respectively.

The deformation tensors thus introduced may be expressed by means of the components of the displacement vector. Thus if we take into account the relation

$$\mathbf{C}_K = \mathbf{r}_{,K} = \mathbf{R}_{,K} + \mathbf{u}_{,K} = \mathbf{E}_K + U_{M,K} \mathbf{E}_M, \quad (2.19)$$

$$\mathbf{c}_i = \mathbf{R}_{,i} = \mathbf{r}_{,i} - \mathbf{u}_{,i} = \mathbf{e}_i - u_{s,i} \mathbf{e}_s,$$

we obtain

$$C_{KL} = \delta_{KL} + U_{K,L} + U_{L,K} + U_{M,K} U_{M,L}, \quad (2.20)$$

$$c_{ij} = \delta_{ij} - u_{i,j} - u_{j,i} + u_{s,i} u_{s,j}.$$

From (2.18) and (2.20) it follows that

$$\begin{aligned} 2E_{KL} &= U_{K,L} + U_{L,K} + U_{M,K}U_{M,L}, \\ 2e_{ij} &= u_{i,j} + u_{j,i} - u_{i,k}u_{k,j}. \end{aligned} \quad (2.21)$$

The relations (2.20) and (2.21) are called deformation displacement relations or geometric equations. In some cases their displacement vector may be assumed to have the form of $u = \epsilon u'$, where ϵ is a parameter the powers equal or higher than two of which may be neglected and u' is a vector which does not depend on ϵ . Thus the linear theory of deformation of the theory of small deformations is obtained. In this theory a single system of coordinates will be used and we will indicate the components of the displacement factor with respect to the system by u_i . Thus we have $x_i = X_i + u_i$. However we should observe the fact that within the limitations of the theory the partial derivatives of the components of the displacement factor with respect to the spatial coordinates coincide with the partial derivatives with respect to the corresponding material coordinates. Thus

$$\frac{\partial u_i}{\partial X_j} = \frac{\partial u_i}{\partial x_j} \frac{\partial x_j}{\partial X_i} = \frac{\partial u_i}{\partial x_j} \left(\delta_{ij} + \frac{\partial u_j}{\partial X_i} \right) = \frac{\partial u_i}{\partial x_j} + O(\epsilon^2) \text{ s.a.m.d.}$$

It follows from (2.21) that in the linear theory of the deformation the lagrangian and eulerian deformation tensors coincide. In this case we will note that components of the deformation tensor by ϵ_{ij} . We have

$$2\epsilon_{ij} = u_{i,j} + u_{j,i} \quad (2.22)$$

In the linear theory of deformation the coordinates of point M are usually indicated by x_i , the coordinates of point M* being obviously $x_i + u_i$.

The relations (2.22) represent the relations of the deformation-displacement called the geometric equations in the linear theory of deformation.

3. Conditions of Compatibility

The two positions of the continuous medium under consideration B and \mathcal{B} are placed in three dimensional euclidian space. Relations (1.1) will be considered as defining the transformation of coordinates from the cartesian rectangular coordinates x_i to the random curved coordinates X_K . The magnitudes C_{KL} are components of the metric tensor in the curvilinear coordinate system X_K . If we have been given the components C_{KL} will we be able to find the transformation $x = x(X)$? Let us note by $R_{KLMN}^{(0)}$ the Riemann-Christoffel tensor formed with the tensor C_{KL} .

$$R_{KLMN}^{(0)} = \frac{1}{2} (C_{KN,LM} + C_{LM,KN} - C_{KM,LN} - C_{LN,KM}) + \\ + C_{KS}^{-1} ([LM, S] [KN, R] - [LN, S] [KM, R]), \quad (3.1)$$

where

$$C_{KS} C_{SL}^{-1} = C_{KS}^{-1} C_{SL} = \delta_{KL}, \\ [KL, M] = \frac{1}{2} (C_{KM,L} + C_{LM,K} - C_{RL,M}). \quad (3.2)$$

This case is known (see for example Haimovici [168])

Theorem 3.1. In view of the fact that the functions C_{KL} of the class C^2 which verified the conditions $R_{KLMN}^{(0)} = 0$,

are given in B and are coefficients of a positively defined form /14
there exist in the neighborhood \underline{a} of a given point from \mathcal{B} ,
a system of curvilinear coordinates X in which these functions are
components of the metric tensor.

The same problem may be posed for the tensor c_{ij} . In other words if we assume that we have been given the symmetrical and positively defined tensors c_{ij} and C_{KL} , in order to be able to consider these tensors as metric tensors it is necessary that

$$R_{KLMN}^{(0)} = 0, \quad R_{ijkl}^{(0)} = 0. \quad (3.3)$$

It is a known fact that in the three dimensional space the Riemann-Christoffel tensor has only six components which do not annul themselves identically. Thus from (3.3) we will obtain six conditions of compatibility for C_{KL} and six conditions of compatibility for c_{ij} .

In view of the relations (2.18) these conditions of compatibility may be expressed by means of the tensors E_{KL} and e_{ij} .

In the case of the linear theory the conditions of compatibility become

$$e_{ij,k} + e_{ki,j} - e_{ik,j} - e_{ji,k} = 0. \quad (3.4)$$

We will demonstrate

Theorem 3.2. The deformation tensor ϵ_{ij} of the class C^2 on B satisfies the conditions of compatibility (3.4). If B is a simple connected domain and ϵ_{ij} is a symmetrical tensor of the class C^2 on B which satisfies the conditions of compatibility (3.4) then there exist functions u_i of the class C^3 on B which satisfy the equations (2.22).

The first part of the theorem is obtained easily in view of the deformation displacement relations (2.22). In order to demonstrate the second part of the theorem let us consider the system

$$u_{i,j} + u_{j,i} = 2\epsilon_{ij}, \quad (3.5)$$

where ϵ_{ij} are given functions which satisfy (3.4). If we note

$$u_{i,j} - u_{j,i} = 2\omega_{ij} \quad (3.6)$$

we have from (3.5) and (3.6)

$$u_{i,j} = \epsilon_{ij} + \omega_{ij}. \quad (3.7)$$

The conditions of integrability of the systems (3.7) are

$$\epsilon_{ij,k} + \omega_{ij,k} = \epsilon_{ik,j} + \omega_{ik,j}, \quad (3.8)$$

these relations may be put under the form of

$$\epsilon_{ij,k} - \epsilon_{ik,j} + \omega_{ij,k} - \omega_{ik,j} = 0. \quad (3.9)$$

By the permutation of the indices we obtain

$$\epsilon_{jk,i} - \epsilon_{ji,k} + \omega_{jk,i} - \omega_{ji,k} = 0, \quad (3.10)$$

$$\epsilon_{ki,j} - \epsilon_{kj,i} + \omega_{ki,j} - \omega_{kj,i} = 0. \quad (3.11)$$

If we add member for member relations (3.9) and (3.10) and /15 subtract from the result (3.11) we derive

$$\omega_{ij,k} = \epsilon_{ik,j} - \epsilon_{kj,i}. \quad (3.12)$$

We obtain a system of equations in unknowns ω_{ij} . The conditions of integrability for the system are

$$\epsilon_{ij,k} + \epsilon_{ki,j} - \epsilon_{ik,j} - \epsilon_{jk,i} = 0,$$

these conditions are identical with those in (3.4). In view of the fact that these conditions are assumed to be fulfilled it follows that the system (3.12) determines the functions ω_{ij} . Let x_0 be a fixed point from B. Then for every $x \in B$, we have

$$\omega_{ij} = \int_{x_0}^x (\epsilon_{ik,j} - \epsilon_{kj,i}) d\xi_k + \omega_{ij}^0, \quad (3.13)$$

the integral from (3.13) is independent of the paths in B from x_0 to x .

It may be noted that if we take $\omega_{ij}^0 = -\omega_{ji}^0$ then the antisymmetry of the magnitudes ω_{ij} is preserved. If the functions ω_{ij} are determined then the system (3.7) for which the conditions of

integrability are fulfilled allows us to determine the displacement vector. We obtain

$$u_i = \int_{x_0}^x (\epsilon_{ij} + \omega_{ij}) d\xi_j + u_i^0, \quad (3.14)$$

where u_i^0 are the values of the functions u_i in the point x_0 .

If we note

$$\omega_{ij}^0 = \int_{x_0}^x (\epsilon_{ik,j} - \epsilon_{jk,i}) d\xi_k, \quad (3.15)$$

then it follows from (3.13) and (3.14) that

$$u_i = \int_{x_0}^x (\epsilon_{ij} + \omega_{ij}^0) d\xi_j + \omega_{ij}^0(x_j - x_j^0) + u_i^0. \quad (3.16)$$

The above integrals are independent of the paths in B from x_0 to x . The thus defined functions u_i represent the components of a displacement vector corresponding to the deformation tensor ϵ_{ij} .

The relations (3.16) indicate that the displacement vector is determined as far as a rigid displacement, as it was anticipated.

The six distinct conditions of integrability from (3.4) are

$$\epsilon_{ii,j} + \epsilon_{jj,i} = 2\epsilon_{ij,i}, \quad (i \neq j, \text{ not added}), \quad (3.17)$$

$$\epsilon_{ii,j} + \epsilon_{jj,i} = \epsilon_{jj,i} + \epsilon_{ii,j}, \quad (i \neq j \neq k \neq i, \text{ not added}).$$

The vector ω the components of which are given by

$$\omega_i = \frac{1}{2} \epsilon_{ijk} \omega_{jk}, \quad (3.18)$$

is called the rotation vector.

We used e_{ijk} for designating the symbol of permutation which, /16 as is known, is defined in the following manner

$$e_{ijk} = \begin{cases} 1, & \text{if the values of } i, j, k, \text{ form an even permutation} \\ -1, & \text{if the values of } i, j, k, \text{ form an odd permutation} \\ 0, & \text{if at least 2 indices are equal} \end{cases} \quad (3.19)$$

Obviously we have

$$\omega = \frac{1}{2} \text{rot } u. \quad (3.20)$$

It may be easily seen that the relations (3.4) may be written in the form of

$$e_{ijp} e_{ljq} \varepsilon_{rs, pq} = 0. \quad (3.21)$$

Similarly on the basis of the fact that

$$e_{rij} e_{rsm} = \delta_{is} \delta_{jm} - \delta_{im} \delta_{js}, \quad (3.22)$$

the relation (3.12) obtains the form of

$$\omega_{ij, k} = e_{ijr} e_{rsm} \varepsilon_{ks, m}. \quad (3.23)$$

If we keep in mind (3.23) and relations

$$\int_{x_0}^x \omega_{ij, k} d\xi_j = (x_j - x_j^0) \omega_{ij}^0 + \int_{x_0}^x (x_j - \xi_j) \omega_{ij, k} d\xi_k,$$

from (3.14) we obtain another form for the components of the displacement vector

$$u_i = u_i^0 + (x_j - x_j^0) \omega_{ij}^0 + \int_{x_0}^x [\varepsilon_{ij} + e_{ijr} e_{rsm} (x_j - \xi_j) \varepsilon_{ks, m}] d\xi_k. \quad (3.24)$$

Similarly, from (3.23) we can derive

$$\omega_{ij} = \omega_{ij}^0 + \int_{x_0}^x e_{ijr} e_{rsm} \varepsilon_{ks, m} d\xi_k. \quad (3.25)$$

In case of a multiple-connected domain the problem may be studied by carrying out suitable sections with the help of which a simply connected domain is obtained.

At the margins of the sections, the components of the displacement vector will have in general different values, be it u_1^- , u_1^+ depending how the x tends to go toward a point on the section from one part or from the other part of the vector. If the displacements are continuous the following supplementary conditions must be fulfilled on the sections $u_1^- = u_1^+$. In the contrary case certain discontinuities will appear in the body; the functions u_1 will not return to the same values after /17 going around any closed contour.

Let us assume that the domain B is $(M + 1)$ connected. The functions u_i and ω_{ij} determined by (3.24) and (3.25) may have many forms. In order to ensure the uniformity of these functions it is necessary and sufficient that

$$\begin{aligned} \int_{L_n} [\varepsilon_{ij} + e_{ijr} e_{rrm}(x_j - \xi_j) \varepsilon_{kr,m}] d\xi_k &= 0, \\ \int_{L_n} e_{rrm} \varepsilon_{kr,m} d\xi_k &= 0, \quad (n = 1, 2, \dots, M), \end{aligned} \quad (3.26)$$

where L_n are simple closed curves from B , each of which surrounds one and only one of the cavities.

Conditions (3.26) may be written in the following equivalent form

$$\begin{aligned} \int_{L_n} [\varepsilon_{ij} - e_{ijr} e_{rrm} \xi_j \varepsilon_{kr,m}] d\xi_k &= 0, \\ \int_{L_n} e_{rrm} \varepsilon_{kr,m} d\xi_k &= 0, \quad (n = 1, 2, \dots, M). \end{aligned} \quad (3.27)$$

4. The Invariants of the Deformation Tensors

Let us consider in the three dimensional euclidian space a second

order symmetrical tensor a_{ij} . As is known the coefficients of the polynomial in λ

$$|a_{ij} - \lambda \delta_{ij}| = -\lambda^3 + I_1(a) \lambda^2 - I_2(a) \lambda + I_3(a), \quad (4.1)$$

are invariant at their reference transformation. Similarly any invariant of the matrix (a_{ij}) is a function of the invariants $I_j(a)$. From (4.1) we obtain

$$I_1(a) = a_{ii}, \quad I_2(a) = \frac{1}{2} (a_{ii} a_{jj} - a_{rs} a_{rs}), \quad I_3(a) = |a_{ij}|. \quad (4.2)$$

Often in place of the invariant $I_r(a)$ other invariant expressions are used. Magnitudes $I_r(a)$ are called principal invariants of the tensor a_{ij} . For the deformation tensors C_{KL} , E_{KL} , c_{ij} , and e_{ij} we will indicate the respective principal invariants by $I_1(C)$, $I_1(E)$, $I_1(c)$ and $I_1(e)$.

In view of the relations (2.18) it follows that

$$\begin{aligned} I_1(C) &= 3 + 2I_1(E), \quad I_1(c) = 3 - 2I_1(e), \\ I_2(C) &= 3 + 4I_1(E) + 4I_2(E), \quad I_2(c) = 3 - 4I_1(e) + 4I_2(e), \\ I_3(C) &= 1 + 2I_1(E) + 4I_2(E) + 8I_3(E), \quad I_3(c) = 1 - 2I_1(e) + 4I_2(e) - 8I_3(e). \end{aligned} \quad (4.3)$$

Let us note the fact that because

$$I_3(C) = |C_{KL}| = |x_{i,K}|^2, \quad (4.4)$$

it follows from (1.3) that

$$I_3(C) = J^2. \quad (4.5)$$

/18

Similarly we obtain

$$I_3(o) = j^3, \quad (4.6)$$

where

$$j = |X_{k,i}|. \quad (4.7)$$

Obviously

$$jJ = 1, \quad (4.8)$$

so that

$$I_3(C) I_3(o) = 1. \quad (4.9)$$

5. The Transformation of the Surface and Volume Elements

Let us consider the vectors $d\mathbf{R}_1 = \mathbf{E}_1 dX_1$, $d\mathbf{R}_2 = \mathbf{E}_2 dX_2$, $d\mathbf{R}_3 = \mathbf{E}_3 dX_3$ which after deformation become $d\mathbf{r}_1 = \mathbf{C}_1 dX_1$, $d\mathbf{r}_2 = \mathbf{C}_2 dX_2$, $d\mathbf{r}_3 = \mathbf{C}_3 dX_3$ respectively. Let dA_3 indicate the surface of the rectangle determined by the vectors $d\mathbf{R}_1$ and $d\mathbf{R}_2$ while $d\sigma_3$ is the surface of the parallelogram determined by the vectors $d\mathbf{r}_1$ and $d\mathbf{r}_2$. We thus have

$$d\sigma_3 = \mathbf{C}_1 \times \mathbf{C}_2 dA_3 = e_{ijk} x_{i,1} x_{j,2} e_k dA_3. \quad (5.1)$$

The relation (5.1) may be written in the form of

$$d\sigma_3 = J X_{3,i} e_i dA_3, \quad (5.2)$$

because

$$J = |x_{i,k}| = e_{ijk} x_{i,1} x_{j,2} x_{k,3}.$$

In the same way we obtain

$$d\sigma_1 = J X_{1,i} e_i dA_1, \quad d\sigma_2 = J X_{2,i} e_i dA_2, \quad (5.3)$$

where the meaning of the new notation is obvious.

If we note

$$da = d\sigma_1 + d\sigma_2 + d\sigma_3, \quad (5.4)$$

we have

$$da = JX_{K,i} dA_K e_i. \quad (5.5)$$

Setting $da = da_1 e_1$, it results

$$da_1 = JX_{K,i} dA_K, \quad (5.6)$$

these relations will be useful in the following.

The volume determined by the vectors $d\underline{r}_1, d\underline{r}_2, d\underline{r}_3$ is given by

$$dv = d\sigma_3 \underline{e}_3 dX_3 = JX_{3,i} e_i dA_3 x_{j,3} e_j dX_3 = JdV, \quad (5.7)$$

where dV is the volume of the parallelepiped determined by the vectors $d\underline{R}_1, d\underline{R}_2, d\underline{R}_3$.

/19

If we keep in mind (4.3), (4.5) we can write

$$dv = JdV = \sqrt{I_3(\bar{O})} dV = \sqrt{1 + 2I_1(E) + 4I_2(E) + 8I_3(E)} dV. \quad (5.8)$$

In the case of the linear theory we obtain

$$\frac{dv}{dV} = 1 + I_1(\varepsilon) = 1 + \varepsilon_{ii}, \quad (5.9)$$

or

$$\frac{dv - dV}{dV} = I_1(\varepsilon), \quad (5.10)$$

this relation expresses the fact that in the linear theory, $I_1(\varepsilon)$ represents the variation of the volume per nondeformed unit volume.

6. Velocity and Acceleration

Through the intermediary of relations (1.1) and (1.2) any quantity f which is a function of the variables x_1, t is also a function of the variables X_K, t and vice versa. We will indicate the dependence of f on a certain system of variables by writing $f(x, t)$ or $f(X, t)$. Let the function f be of the class C^1 on $B \times (0, t_0)$.

The differential of function f with respect to time, maintaining coordinate X_K constant is called the material differential of the function f and will be indicated by df/dt or by \dot{f} .

If $f = f(X, t)$, then we have

$$\dot{f} = \frac{\partial f}{\partial t}. \quad (6.1)$$

If $f = f(x, t)$, we obtain in view of (1.1)

$$\dot{f} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial t}. \quad (6.2)$$

Obviously the concept of material differential may be considered also in connection with vector or tensor quantities.

The velocity vector is defined by

$$\mathbf{v} = \dot{\mathbf{r}}. \quad (6.3)$$

If we observe

$$\mathbf{v} = v_i \mathbf{e}_i, \quad (6.4)$$

then

$$v_i = \frac{\partial x_i}{\partial t}. \quad (6.5)$$

The velocity vector may be expressed with the help of the displacement vector. From (6.3) and (2.3) we have

$$\mathbf{v} = \dot{\mathbf{u}}, \quad (6.6)$$

and if we take into account notations (2.5) we can write

$$v_i = \dot{u}_i. \quad (6.7)$$

Setting

$$\mathbf{u} = U_K(X, t) \mathbf{E}_K, \quad \mathbf{v} = V_K(X, t) \mathbf{E}_K, \quad (6.8)$$

we obtain

$$V_K = \frac{\partial U_K}{\partial t}. \quad (6.9)$$

In view of (6.5), the material derivative of the function $f(x, t)$ may also be written in the following manner

$$\dot{f} = \frac{\partial f}{\partial t} + f_{,i} v_i. \quad (6.10)$$

The acceleration vector is defined by

$$\mathbf{a} = \dot{\mathbf{v}}. \quad (6.11)$$

We can write

$$\mathbf{a} = \frac{\partial V_K(X, t)}{\partial t} \mathbf{E}_K = \frac{\partial^2 U_K(X, t)}{\partial t^2} \mathbf{E}_K. \quad (6.12)$$

Similarly if $\underline{a} = a_i \underline{e}_i$ and $\underline{v}_1 = v_1(x, t)$, we have

$$a_i = \frac{\partial v_i}{\partial t} + v_{i,j} v_j. \quad (6.13)$$

7. Several Important Material Differentials

(α) The material differential of the deformation gradient. We will show that the following relations take place.

$$\frac{d}{dt}(x_{i,k}) = v_{i,j} x_{j,k}, \quad \frac{d}{dt}(X_{k,i}) = -v_{j,i} X_{k,j}. \quad (7.1)$$

In view of the fact that in the operation of material differentiation the X_K coordinates are kept fixed, we can write

$$\frac{d}{dt}(x_{i,k}) = \frac{\partial}{\partial X_K} \left(\frac{dx_i}{dt} \right) = v_{i,j} x_{j,k},$$

which proves the first formula from (7.1).

Carrying out now the material differential of the relation (2.11)₁ we obtain

$$X_{M,j} \frac{d}{dt}(x_{i,M}) + x_{i,M} \frac{d}{dt}(X_{M,j}) = 0,$$

whence, on the basis of (7.1)₁, (7.2)₂ can be easily obtained.

If we consider $dx_i = v_{i,k} dX_k$, from (7.1) we can derive

/22

$$\frac{d}{dt}(dx_i) = v_{i,j} dx_j. \quad (7.2)$$

(β) The material differential of the square of an element of arc. We will show that

$$\frac{d}{dt}(ds^2) = 2d_{ij} dx_i dx_j, \quad (7.3)$$

where

$$2d_{ij} = v_{i,j} + v_{j,i}. \quad (7.4)$$

In view of (7.2) it follows

$$-\frac{d}{dt}(ds^2) = 2 \frac{d}{dt}(dx_i) dx_i = 2v_{i,j} dx_j dx_i = (v_{i,j} + v_{j,i}) dx_i dx_j,$$

which demonstrates the relation (7.3)

(γ) The material differential of the volume element. Let us show that

$$\frac{d}{dt}(dv) = v_{i,i} dv. \quad (7.5)$$

In view of the fact that $dv = J dV$, we have

$$\begin{aligned} \frac{d}{dt}(dv) &= \frac{dJ}{dt} dV = \frac{\partial J}{\partial x_{i,k}} \frac{d(x_{i,k})}{dt} dV = \\ &= J X_{k,i} v_{i,j} x_{j,k} dV = v_{i,i} J dV, \end{aligned}$$

which demonstrates relation (7.5). Here we use the fact that

$$\frac{\partial J}{\partial x_{i,k}} = J X_{k,i},$$

and also the relation (7.1)₁. Let us remember the fact that

$$\dot{J} = J v_{i,i}. \quad (7.6)$$

(δ) The material differential of a volume integral. Let $\mathcal{P}(t)$ be a random regular domain from the continuous medium considered at time t followed in its movement. Let us assume that this domain comes from the domain $P = \mathcal{P}(0)$.

The following integral is considered

$$I = \int_{\mathcal{P}} f(x, t) dv, \quad (7.7)$$

in which the domain \mathcal{P} and the function f assumed to be of the class C^1 are dependent on the time.

Let us demonstrate that

/23

$$\dot{F} = \int_{\mathcal{P}} (\dot{f} + f v_{i,i}) dv. \quad (7.8)$$

For this reason let us transform first of all the considered integral into an integral extended to P

$$F = \int_{\mathcal{P}} f(x, t) dv = \int_P f(x(X, t), t) J dV. \quad (7.9)$$

Obviously we have

$$\dot{F} = \int_P \frac{d}{dt} (fJ) dV. \quad (7.10)$$

Coming back to the spatial variables and taking into account (7.6) we obtain

$$\dot{F} = \int_{\mathcal{P}} \frac{1}{J} \frac{d}{dt} (fJ) dv = \int_{\mathcal{P}} (\dot{f} + f v_{i,i}) dv,$$

which was to be demonstrated. The relation (7.8) may also be written in the form of

$$\dot{F} = \int_{\mathcal{P}} \left[\frac{\partial f}{\partial t} + (f v_{i,i}) \right] dv = \int_{\mathcal{P}} \frac{\partial f}{\partial t} dv + \int_{\partial \mathcal{P}} f v_i n_i da, \quad (7.11)$$

where $\partial \mathcal{P}$ is the boundary of \mathcal{P} .

8. The Deformation Velocity Tensor. The Spin Tensor

The deformation velocity tensor d_{ij} and the spin tensor w_{ij} are defined by their relations

$$2d_{ij} = v_{i,j} + v_{j,i} \quad (8.1)$$

$$2w_{ij} = v_{i,j} - v_{j,i} \quad (8.2)$$

These tensors represent the symmetrical and the antisymmetrical part, respectively, of the tensor $v_{i,j}$. The vector \underline{w} with the component

$$w_i = \frac{1}{2} e_{ijk} w_{kj} = \frac{1}{2} e_{ijk} v_{k,j} \quad (8.3)$$

is called rotation velocity. We have

$$\underline{w} = \frac{1}{2} \text{rot}_s \underline{v}. \quad (8.4)$$

Let us establish certain relations between the material differentials of the deformation tensors and the deformation velocity. We will assume that $E_{KL} = E_{KL}(X, t)$, $e_{ij} = e_{ij}(x, t)$. We will show that the following relations are involved

[24]

$$\dot{E}_{KL} = d_{ij} x_{i,K} x_{j,L} \quad (8.5)$$

$$e_{ij} = d_{ij} - e_{ki} v_{k,j} - e_{kj} v_{k,i}. \quad (8.6)$$

In view of (2.16), (2.18), (7.1), we can write

$$\begin{aligned} 2\dot{E}_{KL} &= \dot{E}_{KL} = \frac{d}{dt} (x_{i,K} x_{i,L}) = v_{i,j} x_{j,K} x_{i,L} + v_{i,j} x_{j,L} x_{i,K} = \\ &= (v_{i,j} + v_{j,i}) x_{i,K} x_{j,L}, \end{aligned}$$

which demonstrates (8.5). In order to establish (8.6), we will use (7.1), (8.5) and the relation

$$e_{ij} = E_{KL} X_{K,i} X_{L,j}.$$

Thus we are able to write

$$\begin{aligned} e_{ij} &= E_{KL} X_{K,i} X_{L,j} + E_{KL} \frac{d}{dt} (X_{K,i}) X_{L,j} + E_{KL} X_{K,i} \frac{d}{dt} (X_{L,j}) = \\ &= d_{ij} - E_{KL} v_{m,i} X_{K,m} X_{L,j} - E_{KL} v_{m,j} X_{L,m} X_{K,i} = \\ &= d_{ij} - e_{mj} v_{m,i} - e_{mi} v_{m,j}. \end{aligned}$$

From (8.6) results the fact that in the linear theory we have

$$i_{ij} = d_{ij}. \quad (8.7)$$

9. Objective Tensors

The magnitudes which are independent of the movement of the person who observes them are called objective ones. Thus the distance between two points is objective. The velocity of a material point is not an objective magnitude.

Two movements of the continuous medium described by the functions $x_i(X, t)$ and $x'_i(X, t')$ respectively are called objectively equivalent if

$$x'_i(X, t') = Q_{ij}(t) x_j(X, t) + b_i(t), \quad t' = t - a, \quad (9.1)$$

where a is a constant, b_i represents a translation and Q_{ij} satisfies the relations

$$Q_{ij} Q_{ik} = Q_{jk} Q_{ki} = \delta_{jk}, \quad |Q_{ij}| = 1. \quad (9.2)$$

Thus two objectively equivalent movements differ only by the benchmark and the reference time. The two movements could be made to coincide by superimposing a rigid movement and by changing their 25 origin of time. If in case of such a transformation the components of a vector $A_i(X, t)$ are changed according to the law

$$A'_i(X, t') = Q_{ir}(t) A_r(X, t), \quad (9.3)$$

then this vector is called an objective one.

The tensor $A_{ij}(X, t')$ is called objective if

$$A'_{ij}(X, t') = Q_{ir}(t) Q_{js}(t) A_{rs}(X, t). \quad (9.4)$$

In view of (9.1) we are able to write

$$\frac{dx'_i}{dt'} = Q_{ij} \dot{x}_j + \dot{Q}_{ij} x_j + \dot{b}_i,$$

whence

$$v'_i = Q_{ij} v_j + \dot{Q}_{ij} x_j + \dot{b}_i. \quad (9.5)$$

This relation indicates if the velocity vector is not objective.

Let us examine the tensors d_{ij} and w_{ij} . From (9.5) we obtain

$$v'_{i,j} \equiv \frac{\partial v'_i}{\partial x'_j} = Q_{ir} v_{r,s} \frac{\partial x_r}{\partial x'_j} + \dot{Q}_{ir} \frac{\partial x_r}{\partial x'_j}.$$

Because

$$\frac{\partial x_r}{\partial x'_i} = Q_{ri},$$

we are able to write

$$v'_{i,j} = Q_{ik} Q_{jr} v_{k,s} + \dot{Q}_{ik} Q_{jr}. \quad (9.6)$$

If we derive (9.2) in relation to time we have

$$Q_{im} \dot{Q}_{jm} + Q_{im} Q_{jm} = 0. \quad (9.7)$$

From (9.6) and (9.7) we obtain

$$d'_{ij} = Q_{ir} Q_{js} d_{rs}, \quad (9.8)$$

a relation which expresses the fact that the deformation velocity tensor is objective.

Similarly from (9.6) and (9.7) we can derive

$$w'_{ij} = Q_{ij} Q_{kl} w_{kl} + Q_{ij} Q_{kl} \quad (9.9)$$

Consequently the spin tensor is not an objective tensor.

10. The Principle of the Conservation of Mass. The Equation of Continuity /26

The fundamental principles of the mechanics and thermodynamics of continuous media are: the principle of the conservation of mass, the principle of the pulse, the principle of the kinetic moment, the principle of energy and the principle of entropy.

Let us start with the study of the principle of the conservation of mass. Let us assume that we are given a strictly positive function ρ , of the C^1 class on $\bar{B} \times (0, t_0)$ and continuous on $\bar{B} \times [0, t_0]$ called mass density; the mass of any given portion \mathcal{P} from the continuous medium \mathcal{B} is given by

$$m(\mathcal{P}) = \int_{\mathcal{P}} \rho \, dv. \quad (10.1)$$

The principle of the conservation of mass states that the mass is conserved, in other words the mass of any portion from B is the same as the mass of the same portion after deformation. Let P be the domain from B which by deformation becomes the domain \mathcal{P} from \mathcal{B} .

If we designate the density of the continuous medium at the initial moment by ρ_0 , then the principle of the conservation of mass may be expressed in the form of

$$\int_{\mathcal{P}} \rho \, dv = \int_P \rho_0 \, dV. \quad (10.2)$$

From (10.2) follows the relation

$$\rho_0 = J \rho, \quad (10.3)$$

which is called the continuity equation. If the motion is known, the relation (10.3) determines the density ρ . The function ρ_0 is prescribed. In view of the fact that ρ_0 does not depend on time we obtain from (10.3)

$$\dot{J}\rho + J\dot{\rho} = 0.$$

If we consider (7.6), it follows

$$\dot{\rho} + \rho v_{i,i} = 0. \quad (10.4)$$

This is another form of the continuity equation. It may be expressed also in the following manner.

$$\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i} = 0. \quad (10.5)$$

From (10.3), on the basis of the relations (4.3) and (4.5) it follows that we have in the linear theory

$$\rho = \rho_0(1 - \varepsilon_{ii}). \quad (10.6)$$

In ending this paragraph we will consider the material derivative of an integral of the form of (7.7), in which $f = \rho\varphi$. In consideration of (7.8) we derive

$$\frac{d}{dt} \int_V \rho\varphi \, dv = \int_V \left[\frac{d}{dt} (\rho\varphi) + \rho\varphi v_{i,i} \right] dv = \int_V [\rho\dot{\varphi} + \varphi(\dot{\rho} + \rho v_{i,i})] dv.$$

If we consider (10.4), we obtain the important formula

/27

$$\frac{d}{dt} \int_V \rho\varphi \, dv = \int_V \rho \frac{d\varphi}{dt} \, dv. \quad (10.7)$$

11. The Principle of the Impulse. The Principle of the Kinetic Moment

In Paragraph 2 we have seen that a motion of the considered medium is defined by a displacement vector \underline{u} of the class C^2 on $B \times (0, t_0)$.

A motion is called allowable if $u, \dot{u}, \ddot{u}, u, \kappa, \dot{u}, \kappa,$ are continuous on $B \times [0, t_0)$.

Given an allowable motion and a portion \mathcal{P} of the body under consideration at the moment t , then the vector

$$I(\mathcal{P}) = \int_{\mathcal{P}} \rho \dot{u} dv, \quad (11.1)$$

is by definition the impulse of the portion \mathcal{P} at the moment t .

The kinetic moment (with respect to point o) of the portion \mathcal{P} at the moment t is, by definition, the vector

$$H_o(\mathcal{P}) = \int_{\mathcal{P}} \rho r \times \dot{u} dv. \quad (11.2)$$

A system of forces associated with the body in motion is defined as follows:

(a) For any given time t , there is given a vector $f(x, t)$, $x \in \mathcal{B}$. This vector is called the specific mass force; it represents the force per unit mass exerted on the point x at the moment t by bodies external with respect to \mathcal{B} . The vector $F_m(\mathcal{P})$ is defined by

$$F_m(\mathcal{P}) = \int_{\mathcal{P}} \rho f dv,$$

and is called the resultant of the mass forces exerted on the portion \mathcal{P} at the moment t .

(β) For any moment t and for any unitary vector \underline{n} there is given a vector $\underline{t}_{(n)}(x, t)$, $x \in \mathcal{B}$. If \mathcal{P} is a regular and oriented surface from \mathcal{B} , with a normal unitary vector \underline{n} , then $\underline{t}_{(n)}(x, t)$ is the force per unit surface in x and at the moment t , exerted by the portion of the body located on that part of \mathcal{P} toward which \underline{n} is directed on the portion from \mathcal{B} located at the other part. The vector $\underline{t}_{(n)}(x, t)$ is called the tension vector.

The resultant of the surface force exerted on the portion of \mathcal{P} from \mathcal{B} is defined by

$$F_s(\mathcal{P}) = \int_{\partial \mathcal{P}} \underline{t}_{(n)}(x, t) da,$$

where $\partial \mathcal{P}$ is the boundary of the domain \mathcal{P} , and \underline{n} is the versor of the normal exterior at $\partial \mathcal{P}$.

If $x \in \partial \mathcal{B}$, and \underline{n} is the versor of the normal external, /28
to $\partial \mathcal{B}$ in x , then $\underline{t}_{(n)}(x, t)$ is the specific force of the surface in (x, t) .

(γ) The resultant of the forces exerted on the portion of \mathcal{P} from \mathcal{B} is defined by

$$F(\mathcal{P}) = \int_{\mathcal{P}} \rho f dv + \int_{\partial \mathcal{P}} \underline{t}_{(n)} da. \quad (11.3)$$

(δ) The vector $\underline{t}_{(n)}$ is continuous on $\bar{B} \times [0, t_0)$ and of the class $C^{1,0}$ on $B \times (0, t_0)$, and \underline{f} is a continuous vector on $\bar{B} \times [0, t_0)$.

By definition the resultant moment (with respect to point o) of the forces which act on a portion \mathcal{P} from \mathcal{B} is

$$M_o(\mathcal{P}) = \int_{\mathcal{P}} \rho \mathbf{r} \times \mathbf{f} dv + \int_{\partial \mathcal{P}} \mathbf{r} \times \underline{t}_{(n)} da. \quad (11.4)$$

By analogy with the mass force and the tension vector the concepts of mass moment and surface moment may be introduced. A continuous medium in which mass moments and surface moments are present is called a polar continuous medium. We will assume in what follows that the medium under consideration is nonpolar.

The principle of the impulse states that for any portion \mathcal{P} from \mathcal{B} and any t the following equation occurs

$$\dot{\mathbf{I}}(\mathcal{P}) = \mathbf{F}(\mathcal{P}). \quad (11.5)$$

The principle of the kinetic moment states that for any portion \mathcal{P} from \mathcal{B} and any t the following relation takes place

$$\dot{\mathbf{H}}_0(\mathcal{P}) = \mathbf{M}_0(\mathcal{P}). \quad (11.6)$$

The ordered aggregate $[\mathbf{u}, \mathbf{t}_{(n)}, \mathbf{f}]$, where \mathbf{u} is an allowable motion and $\mathbf{f}, \mathbf{t}_{(n)}$ is a system of forces is called a dynamic process if, for any portion \mathcal{P} from \mathcal{B} and for any t , the relations (11.5) and (11.6) are satisfied.

In view of (10.7), (11.1)-(11.4), relations (11.5) and (11.6) are written in the following form

$$\int_{\mathcal{P}} \rho \dot{\mathbf{u}} \, dv = \int_{\mathcal{P}} \rho \mathbf{f} \, dv + \int_{\partial \mathcal{P}} \mathbf{t}_{(n)} \, da, \quad (11.7)$$

$$\int_{\mathcal{P}} \rho \mathbf{r} \times \dot{\mathbf{u}} \, dv = \int_{\mathcal{P}} \rho \mathbf{r} \times \mathbf{f} \, dv + \int_{\partial \mathcal{P}} \mathbf{r} \times \mathbf{t}_{(n)} \, da. \quad (11.8)$$

In Paragraphs 12 and 13 we will present the local consequences of the principle of the impulse and the principle of the kinetic moment.

12. The Tension Tensor

Theorem 12.1. If $[u, t_{(n)}, f]$ is a dynamic process then for any given unitary vector \underline{n} we have

$$t_{(n)} = -t_{(-n)}. \quad (12.1)$$

Demonstration (Gurtin [163]). In view of the fact that the domain occupied by a body is limited, in view of the properties of ρ , \underline{u} and \underline{f} , it follows that the function /29

$$k(t) = \sup_{x \in B} |\rho(f - \underline{u})|,$$

is finite on $[0, t_0]$.

From (11.7) we derive that

$$\left| \int_{\partial \mathcal{P}} t_{(n)} da \right| \leq k V(\mathcal{P}), \quad (12.2)$$

on $[0, t_0]$, where $V(\mathcal{P})$ is the volume of \mathcal{P} .

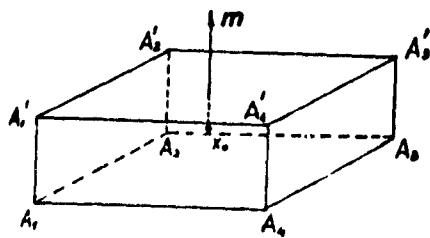


Figure 2

Let us consider a point $x_0 \in \mathcal{P}$ and unit vector \underline{m} . We will apply (12.2) for the case when \mathcal{P} is a parallelepiped, \mathcal{P}_ϵ , with a center x_0 (figure 2),

having faces $A_1 A_2 A_3 A_4$ and $A'_1 A'_2 A'_3 A'_4$ normal to \underline{m} . We will assume that these faces are squares with a side ϵ and will designate them by

σ_ε^+ and σ_ε^- , respectively. We will assume similarly that the edge $A_1 A_1'$ is equal to ε^2 . Let us designate with ω_ε the reunion of the parallel faces with \underline{m} .

We have

$$\partial \mathcal{D}_\varepsilon = \sigma_\varepsilon^+ \cup \sigma_\varepsilon^- \cup \omega_\varepsilon, \quad (12.3)$$

and

$$V(\mathcal{D}_\varepsilon) = \varepsilon^4, \quad \mathcal{A}(\sigma_\varepsilon^\pm) = \varepsilon^2, \quad \mathcal{A}(\omega_\varepsilon) = 4\varepsilon^3, \quad (12.4)$$

where for every surface σ , $\mathcal{A}(\sigma)$ designates the surface of σ . From (12.2), (12.4), we derive

$$\frac{1}{\varepsilon^2} \int_{\partial \mathcal{D}_\varepsilon} t_{(n)} da \rightarrow 0 \quad \text{when } \varepsilon \rightarrow 0. \quad (12.5)$$

In view of the fact that $\underline{t}_{(p)}$ is continuous on \mathcal{D} for any fixed unit vector p , it follows from (12.4)

$$\frac{1}{\varepsilon^2} \int_{\sigma_\varepsilon^\pm} t_{(\pm m)} da \rightarrow t_{(\pm m)}(x_0, t), \quad \frac{1}{\varepsilon^2} \int_{\omega_\varepsilon} t_{(n)} da \rightarrow 0 \quad \text{when } \varepsilon \rightarrow 0. \quad (12.6)$$

Keeping (12.3), (12.5), (12.6) in mind we obtain

$$t_{(m)}(x_0, t) + t_{(-m)}(x_0, t) = 0,$$

this relation proves the theorem inasmuch as \underline{m} and x_0 are arbitrary.

The vector $\underline{t}_{(n)}(x, t)$ is designated by $\underline{t}(x, t; \underline{n})$. We will designate the components of vector $\underline{t}_{(n)}$ by $t_1(x, t; \underline{n})$ or \underline{t}_1 . /30

Theorem 12.2 If $[u, t_{(n)}, f]$ is a dynamic process, then there is a tensor t_{ij} of class $C^{1,0}$ on $B \times (0, t_0)$ which is continuous on $\bar{B} \times [0, t_0)$ so that for every unit vector \underline{n} we have

$$t_i = t_{ij} n_j, \quad (12.7)$$

where n_1 are components of the vector \underline{n} .

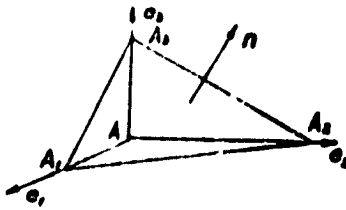


Figure 3

Proof. Let us assume that the portion \mathcal{P} , including its limit is internal with respect to the domain and has the shape of a tetrahedron $AA_1A_2A_3$ (fig. 3) in which the vectors $\overrightarrow{AA_1}$ have vectors \underline{e}_1 respectively as the right versors. Let x_1 be the coordinates of point A , and \underline{n} the versor of the external normal with respect to the plane $A_1A_2A_3$. The tension vector which acts on the surface element with an external normal \underline{e}_1 is $\underline{t}(\underline{e}_1)$. To simplify the description we will designate this vector by \underline{t}_1 . Obviously vectors \underline{t}_1 depend only on the point and time.

Let σ be the face of the tetrahedron with the external unit normal \underline{n} and σ_1 the face of the tetrahedron normal to \underline{e}_1 . Obviously the external unit normal at the face σ_1 is $-\underline{e}_1$. The tension vector which acts on this face will be $-\underline{t}_1$.

The principle of the impulse in the form of (11.7) applied to the tetrahedron under consideration becomes

$$\int_{\mathcal{P}} \rho(\underline{n} - \underline{f}) dv = \int_{\sigma} \underline{t}_{(n)} da - \int_{\sigma_1} \underline{t}_1 da - \int_{\sigma_2} \underline{t}_2 da - \int_{\sigma_3} \underline{t}_3 da. \quad (12.8)$$

If we designate

$$\underline{t}_{(n)} = t_i \underline{e}_i, \quad \underline{t}_i = t_{ij} \underline{e}_j, \quad (12.9)$$

the relation (12.8) will be written as

$$\int_{\mathcal{P}} \rho(a_i - f_i) dv = \int_{\sigma} t_i da - \int_{\sigma_1} t_{1i} da - \int_{\sigma_2} t_{2i} da - \int_{\sigma_3} t_{3i} da. \quad (12.10)$$

In view of the fact that $\underline{t}_{(p)}$ is continuous for any fixed unit vector \underline{p} , we derive from (12.10)

$$[\rho(a_i - f_i)]^* I^*(\mathcal{P}) = t_i^* \mathcal{A}(\sigma) - t_{1i}^* \mathcal{A}(\sigma_1) - t_{2i}^* \mathcal{A}(\sigma_2) - t_{3i}^* \mathcal{A}(\sigma_3), \quad (12.11)$$

where $[p(a_i - f_i)]^*$ represents the value of the function $p(a_i - f_i)$ in a given point M_1 inside the tetrahedron, t_1^* represents the value of the function t_1 in a point N_1 of the face σ and t_{j1}^* represents the value of the function t_{j1} in a given point F_{j1} of the face σ_j .

Let h be the height of the tetrahedron derived from A . We have /31

$$V(\mathcal{P}) = \frac{1}{3} h \mathcal{A}(\sigma), \quad \mathcal{A}(\sigma_i) = n_i \mathcal{A}(\sigma), \quad (12.12)$$

where $\underline{n} = n_1 \underline{e}_1$.

From (12.11) we obtain

$$[p(a_i - f_i)]^* \frac{h}{3} = t_1^* - t_{j1}^* n_j. \quad (12.13)$$

If in (12.13) we make h tend toward zero, we derive

$$t_i(x, t; \underline{n}) = t_{ji}(x, t) n_j(x), \quad (12.14)$$

which demonstrates (12.7).

It should be noted that the formula (12.14) is true also when $\underline{n} = \pm \underline{e}_1$.

The fact that the functions t_{ij} are of the class $C^{1,0}$ on $B \times (0, t_0)$ and continuous on $\bar{B} \times [0, t_0)$ results from the fact that $t_1(x, t; \underline{n})$ has these properties regardless what \underline{n} would be.

The relation (12.14) may also be written in the form of

$$t_{(n)} = t_i n_i. \quad (12.15)$$

This relation expresses the dependence of the tension vector $\underline{t}_{(n)}$ on the vector \underline{n} ; it is known under the name of Cauchy's formula. The tensor t_{ij} is called Cauchy's tension tensor.

13. The Equations of Motion

Theorem 13.1 (Cauchy-Poisson Theorem). Let \underline{u} be an admissible

motion and $\underline{f}, \underline{t}_{(n)}$ a system of forces. Then $[u, t_{(n)}, f]$ is a dynamic process if and only if the following conditions are fulfilled.

(i) There is a tension tensor t_{ij} such that $t_i(x, t; \underline{n})$ is represented in the form (12.7) and this tensor is symmetrical,

$$t_{ij} = t_{ji}, \quad (13.1)$$

(ii) u_i , t_{ij} and f_i satisfy the equation

$$t_{ji,i} + \rho f_i = \rho \ddot{u}_i. \quad (13.2)$$

Proof. If $[u, t_{(n)}, f]$ is a dynamic process, then according to theorem 12.2 there is a tension tensor t_{ij} such that (12.7) takes place. Let us show that the relations (13.1) and (13.2) occur. We will write the principle of the impulse and the principle of the kinetic moment in the form of

$$\int_V \rho \ddot{u}_i dv = \int_V \rho f_i dv + \int_{\partial V} t_i da, \quad (13.3)$$

$$\int_V \rho e_{ijk} x_j \ddot{u}_k dv = \int_V \rho e_{ijk} x_j f_k dv + \int_{\partial V} e_{ijk} x_j t_k da, \quad (13.4)$$

where e_{ijk} is the permutation symbol defined by (3.19).

In view of the fact that the relations (12.7) take place, /32
applying the divergence theorem, we have

$$\int_{\partial V} t_i da = \int_V t_{ji,i} dv, \quad \int_{\partial V} e_{ijk} x_j t_k da = \int_V (e_{ijk} x_j t_{k,i} + e_{ijk} t_{jk}) dv. \quad (13.5)$$

Relations (13.3) and (13.4) may thus be written in the following manner

$$\int_V (\rho \ddot{u}_i - \rho f_i - t_{ji,i}) dv = 0, \quad (13.6)$$

$$\int_{\mathcal{V}} [e_{ijk} x_j (\rho \ddot{u}_k - \rho f_k - t_{rk,r}) - e_{ijk} t_{jk}] dv = 0. \quad (13.7)$$

In view of the fact that the functions under the integral sign are continuous and the domain \mathcal{V} is arbitrary, equations (13.2) result from (13.6). These equations are called equations of motion of the continuous medium under consideration.

On the basis of equations (13.2), the relation (13.7) implies

$$e_{ijk} t_{jk} = 0,$$

that is, relations (13.1).

In the opposite case, if we presuppose that (1), (11) take place and that the aggregate $[\underline{u}, \underline{t}_{(n)}, \underline{f}]$ is given, where \underline{u} is the admissible motion, $\underline{t}_{(n)}, \underline{f}$ is a system of forces comprising a dynamic process.

From the fact that relations (13.2) take place, we obtain (13.6) as the result. Because tensor t_{1j} exists so that (12.7) takes place, (13.5) results. From (13.6) and (13.5) we obtain

$$\int_{\mathcal{V}} \rho (\ddot{u}_i - f_i) dv = \int_{\partial \mathcal{V}} t_i da,$$

that is, the principle of the impulse.

In view of the fact that relations (13.1) and (13.2) take place, relation (13.7) is true. In view of (13.5)₂ we derive from (13.7) the form (13.4) of the principle of the kinetic moment.

From equations (13.2) and the continuity of ρ, \ddot{u}_i, f_i on $\bar{B} \times [0, t_0)$ it results that $t_{j1,j}$ are continuous functions on $\bar{B} \times [0, t_0)$.

From the above it results that the tension tensor t_{1j} has the following

properties.

- 1) t_{ij} is symmetrical and of the class of $C^{1,0}$ on $B \times (0, t_0)$,
- 2) t_{ij} and $t_{ij,j}$ are continuous on $B \times [0, t_0]$.

A tensor with these properties is an admissible tension tensor.

In view of theorem 13.1 it results that the specification of a dynamic process is equivalent with the specification of the ordered magnitude $[u_i, t_{ij}, f_i]$ where

- (α) u_i is an admissible movement,
- (β) t_{ij} is an admissible tension tensor,
- (γ) f_i are continuous functions on $B \times [0, t_0]$,
- (δ) u_i, t_{ij}, f_i satisfy the equations of motion

$$t_{ij,j} + \rho f_i = \rho \ddot{u}_i.$$

Thus, a dynamic process is an ordered array of functions u_i, t_{ij}, f_i with properties (α) -- (δ).

/33

14. The Piola-Kirchhoff Tension Tensors

As has been stated before the tension vector is associated with a material surface \mathcal{S} from \mathcal{B} and is measured on the unit area of this surface. However, the area of the deformed surface is not known and therefore it is necessary to introduce a tension vector to act on \mathcal{S} which is to be measured on the unit area of the surface \mathcal{S}_0 , where \mathcal{S}_0 is the surface from B which by deformation becomes the surface \mathcal{S} from \mathcal{B} .

Let P be the domain from B which is transformed by deformation into the domain \mathcal{P} from \mathcal{B} . The relation (11.7) may also be written in the following manner

$$\int_P \rho_0 \ddot{u} dV = \int_P \rho_0 f dV + \int_{\partial P} T_{(N)} dA, \quad (14.1)$$

where $T_{(N)}$ is the tension that acts on $\partial \mathcal{P}$ and is measured on the unit

area of the ∂P surface. Let $\underline{N} = N_K \underline{E}_K$ be the versor of the normal exterior to ∂P . Proceeding as in Paragraph 12, from (14.1) we obtain a relation analogous with (12.15)

$$\mathbf{T}_{(N)} = \mathbf{T}_K N_K, \quad (14.2)$$

where \underline{T}_K represent the tension vector associated with a surface from \mathcal{B} , corresponding to the plane of the external normal \underline{E}_K from \mathcal{B} .

Let dA be the oriented element of area from \mathcal{B} with an external unit normal \underline{N} which, by deformation, becomes the element da with an external unit normal \underline{n} .

We have

$$\begin{aligned} t_{(n)} da &= \mathbf{T}_{(N)} dA, \quad t_i n_i da = \mathbf{T}_K N_K dA, \\ da_i &= n_i da, \quad dA_K = N_K dA, \quad t_i da_i = \mathbf{T}_K dA_K, \end{aligned} \quad (14.3)$$

where we used (12.15) and (12.2).

In view of the relations (5.6), from (14.3) it follows

$$\mathbf{T}_K = J X_{K,i} t_i. \quad (14.4)$$

From (14.4) we obtain

$$t_i = \frac{1}{J} x_{i,K} \mathbf{T}_K. \quad (14.5)$$

We will note

$$\mathbf{T}_K = T_{Ki} e_i. \quad (14.6)$$

T_{K1} is called the Piola-Kirchhoff tensor of the first kind.

From (12.9), (14.4), (14.6) it follows

$$T_{Ki} = J X_{K,i} t_{ij} \quad (14.7)$$

and thus

$$t_{ij} = \frac{1}{J} x_{i,K} T_{Kj} \quad (14.8)$$

The Piola-Kirchhoff tensor of the second kind, T_{KL} is defined ^{/34}
by

$$T_{Ki} = x_{i,L} T_{KL} \quad (14.9)$$

From (14.8) and (14.9) we derive

$$t_{ij} = \frac{1}{J} x_{i,K} x_{j,L} T_{KL} \quad (14.10)$$

and thus

$$T_{KL} = J X_{K,i} X_{L,j} t_{ij} \quad (14.11)$$

In view of the symmetry of the tensor t_{ij} it follows

$$x_{i,K} T_{Kj} = x_{j,K} T_{Ki}, \quad T_{KL} = T_{LK} \quad (14.12)$$

Let us express the equation of motion (13.2), with the help of the tensions T_{Ki} or T_{KL} . We will use the fact that

$$\left(\frac{1}{J} x_{i,K} \right)_{,i} = 0, \quad (14.13)$$

these relations may be easily verified.

From (13.2), (14.8) and (14.13) we can derive the following form of the equations of motion

$$T_{Ki,K} + \rho_0 f_i = \rho_0 \ddot{u}_i, \quad (14.14)$$

which can be easily derived also from (14.1). Obviously, the equation (14.14) may also be written in the following way

$$(T_{KL} x_{i,L})_{,K} + \rho_0 f_i = \rho_0 \ddot{u}_i \quad (14.15)$$

In the linear theory of deformation the tensions are of the same order of magnitude and displacement. In this case we followed the convention of using a single system of coordinates and have shown that from (2.6), (4.3), (4.5) it follows

$$J = 1 + \varepsilon u, \quad X_{j,i} = \delta_{ji} + u_{j,i}. \quad (14.16)$$

In view of (14.7), (14.10), (14.16) it follows that in the linear theory the tensors t_{ij} , T_{Ki} , T_{KL} coincide.

$$T_{ij} = t_{ij}. \quad (14.17)$$

15. The Principle of Energy

Let us associate with each dynamic process an internal energy per unit mass $e(x,t)$; the internal energy of the portion \mathcal{P} from the body at the moment t being defined by

$$E(\mathcal{P}) = \int_{\mathcal{P}} e \, dv. \quad (15.1)$$

By definition the kinetic energy of a continuous medium which occupies the domain \mathcal{D} at the moment t is /35

$$K(\mathcal{D}) = \frac{1}{2} \int_{\mathcal{D}} \rho \dot{u}^2 dv. \quad (15.2)$$

The power of the applied forces of the portion \mathcal{D} from \mathcal{D} is defined by

$$W(\mathcal{D}) = \int_{\mathcal{D}} \rho f \dot{u} dv + \int_{\partial \mathcal{D}} t_{(n)} \dot{u} da. \quad (15.3)$$

In continuation we will study the principle of energy in the case in which two forms of energy are present, mechanical energy and thermal energy. In the following the thermomechanical behavior of a continuous medium will be our subject.

A system of heat sources associated with a body in motion is defined by the following:

(α) For any t there is a function $r(x, t)$, $x \in \mathcal{B}$ given. This function is called the output of the heat sources per unit mass. If \mathcal{D} is a portion of the body \mathcal{B} , then the integral

$$Q_1(\mathcal{D}) = \int_{\mathcal{D}} \rho r dv,$$

represents the amount of heat transmitted in \mathcal{D} by radiation per unit time.

(β) For every t and for any unitary vector \underline{n} there is given a function $h_{(n)}(x, t)$, $x \in \mathcal{B}$. If \mathcal{S} is an oriented and regulated surface from \mathcal{B} with a unitary normal \underline{n} , then $h_{(n(x))}(x, t)$ represents the amount of heat per unit area of the surface \mathcal{S} and within the unit of time in x and at the moment t , which passes from the portion of the body located in the part of \mathcal{S} toward which \underline{n} is directed towards the portion from \mathcal{B} placed at the other part. This function is called the heat flux. The integral

$$Q_2(\mathcal{D}) = \int_{\partial \mathcal{D}} h_{(n)} da,$$

represents the amount of heat that enters by the border of the portion \mathcal{P} , by conduction in a unit of time.

(γ) The amount of heat that enters in a unit time into the portion \mathcal{P} is

$$Q(\mathcal{P}) = \int_{\mathcal{P}} \rho r \, dv + \int_{\partial \mathcal{P}} h_{(n)} \, da. \quad (15.4)$$

(δ) The function $h_{(n)}$ is continuous on $\bar{B} \times [0, t_0]$ and is of the class $C^{1,0}$ on $B \times (0, t_0)$ and the function r is continuous on $\bar{B} \times [0, t_0]$.

Let us associate with each dynamic process and entropy per unit mass $\eta(x, t)$; the entropy Γ of the portion \mathcal{P} of the body at the moment t being defined by

$$\Gamma(\mathcal{P}) = \int_{\mathcal{P}} \rho \eta \, dv. \quad (15.5)$$

Finally let us introduce a function $T(x, t) > 0$, called absolute 36 temperature.

The principle of energy (or the first law of thermodynamics) states that for every portion \mathcal{P} from \mathcal{B} and for every t the following relation applies

$$\dot{K}(\mathcal{P}) + \dot{E}(\mathcal{P}) = W(\mathcal{P}) + Q(\mathcal{P}). \quad (15.6)$$

In view of (10.7), (15.1)-(15.4), the relation (15.6) takes the shape of

$$\int_{\mathcal{P}} \rho u_i \ddot{u}_i \, dv + \int_{\mathcal{P}} \rho \dot{e} \, dv = \int_{\mathcal{P}} \rho f_i \dot{u}_i \, dv + \int_{\partial \mathcal{P}} t_i \dot{u}_i \, da + \int_{\mathcal{P}} \rho r \, dv + \int_{\partial \mathcal{P}} h_{(n)} \, da. \quad (15.7)$$

The ordered set $[u, T, e, t_{(n)}, \gamma, h_{(n)}, f, r]$ in which

- (i) u is an admissible motion,
 - (ii) $t_{(n)}, f$ is a system of forces,
 - (iii) $h_{(n)}, r$ is a system of heat sources,
 - (iv) T is a positive function of the class $C^{2,1}$ on $B \times (0, t_0)$ being continuous together with \dot{T} and $T_{,K}$ on $\bar{B} \times [0, t_0]$,
 - (v) e is a function of class C^1 on $B \times (0, t_0)$ and continuous together with e on $\bar{B} \times [0, t_0]$,
 - (vi) η is a function of class C^1 on $B \times (0, t_0)$ and continuous together with $\dot{\eta}$ on $\bar{B} \times [0, t_0]$,
- form a thermodynamic process if the principle of the impulse, the principle of the kinetic moment and the principle of energy are satisfied.

The conditions of regularity imposed on the functions which characterize the thermodynamic process are somewhat more restrictive than those necessary in this chapter.

In view of the definition of the dynamic process it follows that $\mathcal{S} = [u, T, e, t_{(n)}, \gamma, h_{(n)}, f, r]$ is a thermodynamic process if

- 1) $[u, t_{(n)}, f]$ is a dynamic process,
- 2) $h_{(n)}, r$ is a system of heat sources,
- 3) T, e, η have the properties indicated in (iv)-(vi),
- 4) \mathcal{S} satisfies the principle of energy.

Theorem 15.1. If $[u, t_{(n)}, f]$ is a dynamic process then the first law of thermodynamics may be written in the following form

$$\int_{\mathcal{S}} \rho \dot{e} dv = \int_{\mathcal{S}} t_{ij} d_{ij} dv + \int_{\mathcal{S}} \rho r dv + \int_{\partial \mathcal{S}} h_{(n)} da, \quad (15.8)$$

where d_{ij} is the tensor of the velocity of deformation.

Proof. If $[u, t_{(n)}, f]$ is a dynamic process then according to the theorem 12.2, the relations (12.7) take place. In this case the second term of the right member of the relation (15.7) may be written

in the following form

$$\int_{\sigma} t_i \dot{u}_i d\sigma = \int_{\sigma} t_{ji} v_i n_j d\sigma = \int_{\sigma} (t_{ji} v_i)_{,j} dv = \int_{\sigma} (t_{ji,j} v_i + t_{ji} v_{i,j}) dv.$$

In view of this fact the relation (15.7) becomes

/37

$$\begin{aligned} \int_{\sigma} v_i (\rho \ddot{u}_i - \rho f_i - t_{ji,j}) dv + \int_{\sigma} \rho \dot{e} dv = \int_{\sigma} t_{ji} v_{i,j} dv + \\ + \int_{\sigma} \rho r dv + \int_{\sigma} h_{(n)} d\sigma. \end{aligned} \quad (15.9)$$

On the basis of the theorem 13.1, u_i , t_{ij} , f_i satisfy the equations (13.2) and therefore (15.9) is reduced to

$$\int_{\sigma} \rho \dot{e} dv = \int_{\sigma} t_{ji} v_{i,j} dv + \int_{\sigma} \rho r dv + \int_{\sigma} h_{(n)} d\sigma. \quad (15.10)$$

If we take into account (13.1), relation (15.10) will take the shape of (15.8),

16. The Heat Flux Vector. The Energy Equation

Theorem 16.1. If $[\underline{u}, T, e, \underline{t}_{(n)}, n, h_{(n)}, \underline{f}, r]$ is a thermodynamic process, then for every given unit vector \underline{n} we have

$$h_{(n)} = -h_{(-n)}. \quad (16.1)$$

The demonstration of this theorem is similar to the demonstration of theorem 12.1. However this time relation (15.8) is applied for the parallelepiped $P_{\underline{e}}$, used to prove theorem 12.1.

Theorem 16.2. If $[\underline{u}, T, e, \underline{t}_{(n)}, n, h_{(n)}, \underline{f}, r]$ is a thermodynamic process then there is a vector \underline{q} of the class $C^{1,0}$ on $B \times (0, t_0)$ and continuous on $\bar{B} \times [0, t_0)$ such that for every unit vector \underline{n} the following relation takes place.

$$h_{(n)} = q_i n_i, \quad (16.2)$$

where q_i are components of vector \underline{q} . The vector \underline{q} is called the heat flux vector.

The theorem is proven the same way as theorem 12.2. In this case the relation (15.8) of the tetrahedron used for proving theorem 12.2 is applied. With respect to the meaning of the components of vector \underline{q} let us mention that the heat flux that enters a tetrahedron through the face σ_1 is $-q_1$.

Theorem 16.3. Let $[\underline{u}, T, e, \underline{t}_{(n)}, \eta, h_{(n)}, \underline{f}, r]$ exist with properties given in (i) -- (vi) respectively from Paragraph 15. Then $[\underline{u}, T, e, \underline{t}_{(n)}, \eta, h_{(n)}, \underline{f}, r]$ is a thermodynamic process if, and only if

- (α) $[\underline{u}, \underline{t}_{(n)}, \underline{f}]$ is a dynamic process,
- (β) there is a heat flux vector \underline{q} so that $h_{(n)}$ is represented in the form of (16.2),
- (γ) the relation (the energy equation) takes place

$$\rho \dot{e} = t_{ij} d_{ij} + q_{i,i} + \rho r. \quad (16.3)$$

Proof. If the considered set is a thermodynamic process then $[\underline{u}, \underline{t}_{(n)}, \underline{f}]$ is a dynamic process. Similarly, according to theorem 16.2, there is a heat flux vector \underline{q} such that (16.2) takes place. /38 Let us show that relation (16.3) is satisfied. In view of (16.2), the relation (15.8) may be written in the form of

$$\int_{\mathcal{V}} (\rho r - t_{ij} d_{ij} - q_{i,i}) dV = 0. \quad (16.4)$$

In view of the fact that the function below the integral sign is continuous and the domain of \mathcal{V} is arbitrary, (16.3) results from (16.4)

In a reciprocal manner let us assume that (α) -- (γ) take place and let us show that the set $[\underline{u}, T, e, \underline{t}_{(n)}, \eta, h_{(n)}, \underline{f}, r]$ with the properties (i) -- (vi) of Paragraph 15 is a thermodynamic process.

From the fact that (β) and (γ) occur, (15.8) follows. Taking into account (α) it follows that the relations (13.1) and (13.2) take place. Therefore, we can write (15.8) in the form of (15.10) and from this we can derive (15.9). Afterwards, (15.7) is derived.

From equation (16.3) it follows that $q_{1,1}$ is a continuous function on $\bar{B} \times [0, t_0)$. From the above it follows that the heat flux vector has the following properties

- (i) q_1 is of the class $C^{1,0}$ on $\bar{B} \times (0, t_0)$,
- (ii) q_1 and $q_{1,1}$ are continuous on $\bar{B} \times [0, t_0)$.

A vector having the properties (i), (ii) will be called an admissible heat flux vector.

From the statement made in Paragraph 13 concerning the thermodynamic processes and theorem 16.3, it follows that the specification of a thermodynamic process is equivalent with the specification of the ordered magnitude $[u_1, T, e, t_{1j}, n, q_1, f_1, r]$ whereby

- 1) $[u_1, t_{1j}, f_1]$ is a dynamic process,
- 2) q_1 are components of an admissible heat flux vector,
- 3) T, e, n are position and time functions with properties mentioned in Paragraph 15,
- 4) r is a continuous function on $\bar{B} \times [0, t_0)$,
- 5) the local equation of energy (16.3) is satisfied.

Thus the thermodynamic process is an ordered set of functions $[u_1, T, e, t_{1j}, n, q_1, f_1, r]$ with the properties given in 1)--5).

As in Paragraph 14 we will introduce the concept of heat flux referred to the area of a nondeformed body. We must consider the function $H_{(N)}, Q_K$ so that

$$h_{(n)} da = q_i n_i da = q_i da_i = H_{(N)} dA = Q_K N_K dA = Q_K dA_K. \quad (16.5)$$

The relations (16.5) are similar to the relations (14.3). We used here some of the designations from Paragraph 14.

The vector Q of the component Q_K is called the heat flux vector referred to the area of a nondeformed body.

In view of relations (5.6), it follows from (16.5)

$$Q_K = J X_K, i q_i. \quad (16.6)$$

From this we derive

$$q_i = \int x_{i,K} Q_K. \quad (16.7)$$

If we keep in mind (14.13) we can write

$$q_{i,i} = \frac{1}{J} Q_{K,K}, \quad (16.8)$$

and thus equation (16.3) will take the form of

$$J \rho \dot{e} = J t_{ij} d_{ij} + Q_{K,K} + J p r. \quad (16.9)$$

Using the relations (8.5), (10.3) and (14.10), the energy equation may be written in the following way

$$\rho_0 \dot{e} = T_{KL} \dot{E}_{KL} + Q_{K,K} + \rho_0 r. \quad (16.10)$$

In the linear theory the displacement vector \underline{u} and the heat flux vector \underline{q} are of the form of $\underline{u} = \epsilon \underline{u}'$, $\underline{q} = \epsilon \underline{q}'$ where ϵ is a parameter whose powers equal to or higher than 2 are negligible and \underline{u}' and \underline{q}' are vectors which do not depend on ϵ . In view of (14.16), it follows from (16.6) that, in the linear theory, we have

$$q_i = Q_{i.} \quad (16.11)$$

17. Statements about the Principles Introduced

Green and Rivlin [143] have shown that the principle of the conservation of mass, the principle of the impulse and the principle of kinetic momentum may be derived from the law of the conservation of energy using the conditions of invariance and the superposition of a rigid motion. Let us present this fact. If we consider (15.1), (15.2), applying the formula (7.8), the principle of energy (15.6) may be written in the form of

$$\int_V \left[(v_i \dot{v}_i + \dot{e}) \rho + \left(-\frac{1}{2} v_i v_i + e \right) (\dot{\rho} + \rho v_{i,i}) \right] dv =$$

$$= \int_V \rho (f_i v_i + r) dv + \int_{\partial V} (t_i v_i + h_{(n)}) da. \quad (17.1)$$

We no longer use the formula (10.7) because one assumes the principle of conservation of mass.

Let us assume that a considered medium reached a given state at the moment t by means of a certain motion. We will consider another motion which is obtained from the given motion by the superposition of a translation motion with constant speed. The functions $\rho, \dot{e}, t_i, f_i, h_{(n)}$ remain unchanged by the superposition of this rigid motion. The components of the velocity vector are, however, changed into $v_i + \alpha_i$, where α_i are arbitrary constants. The equation (17.1) is true regardless what the velocity vector is, and when v_i is replaced by $v_i + \alpha_i$.

Thus we have

$$\int_V \left\{ [(v_i + \alpha_i) \dot{v}_i + \dot{e}] \rho + \left[-\frac{1}{2} (v_i + \alpha_i)^2 + e \right] (\dot{\rho} + \rho v_{i,i}) \right\} dv = \quad (17.2)$$

$$= \int_V [\rho (f_i (v_i + \alpha_i) + r)] dv + \int_{\partial V} [t_i (v_i + \alpha_i) + h_{(n)}] da.$$

From (17.1) and (17.2) we obtain

40

$$\left\{ \int_V [\rho (f_i - \dot{v}_i) - v_i (\dot{\rho} + \rho v_{i,i})] dv + \int_{\partial V} t_i da \right\} \alpha_i -$$

$$- \frac{1}{2} \alpha_i \alpha_i \int_V (\dot{\rho} + \rho v_{i,i}) dv = 0, \quad (17.3)$$

this relation is true regardless of the constant α_i . The integrals from (17.3) are independent of α_i . If we replace α_i by $\beta \alpha_i$, where β is an arbitrary scalar quantity, it can be easily derived that

(17.3) implies the relations

$$\int_V (\dot{\rho} + \rho v_{i,i}) dv = 0, \quad (17.4)$$

$$\int_V [\rho(f_i - \ddot{r}_i) - v_i (\dot{\rho} + \rho v_{i,i})] dv + \int_{\partial V} t_i da = 0. \quad (17.5)$$

Obviously (17.4) represents the principle of the conservation of mass. Now in view of (10.4), the relation (17.5) becomes

$$\int_V \rho \ddot{u}_i dv = \int_V \rho f_i dv + \int_{\partial V} t_i da,$$

which represents the principle of impulse.

In view of (10.4), (12.7), (13.2), relation (17.1) becomes

$$\int_V \rho \dot{e} dv = \int_V (\rho r + t_{ji} v_{i,j}) dv + \int_{\partial V} h_{(n)} da. \quad (17.6)$$

Let us consider a movement of the medium which differs from the movement given by the superposition of a rigid motion with a constant angular velocity, as the body has the same spatial orientation at the moment t . The functions ρ , \dot{e} , r , t_{ji} , $h_{(n)}$ remain unchanged by the superposition of the above-mentioned motion. The relation (17.6) is true also when $v_{i,j}$ is replaced by $v_{i,j} + \Omega_{ij}$, where Ω_{ij} is an antisymmetric constant and arbitrary tensor. Thus we have

$$\int_V \rho \dot{e} dv = \int_V [\rho r + t_{ji} (v_{i,j} + \Omega_{ij})] dv + \int_{\partial V} h_{(n)} da. \quad (17.7)$$

From (17.6) and (17.7) it follows that

$$\int_V \Omega_{ii} t_{ii} dv = 0,$$

from which we obtain (13.1). On the basis of theorem 13.1, the principle of kinetic momentum can be derived.

18. The Principle of Entropy

/41

The thermodynamic processes are subject to the action of the second law of thermodynamics, known also under the name of the principle of entropy. This principle states that for any thermodynamic process the inequality

$$\frac{d}{dt} \int_V \rho \eta dv - \int_V \frac{\rho r}{T} dv - \int_{\partial V} \frac{h^{(n)}}{T} da \geq 0, \quad (18.1)$$

must be satisfied regardless what the portion from would be at any moment t .

The inequality (18.1) is known under the name of the Clausius-Duhem inequality.

If we consider the relations (10.7) and (16.2), the inequality (18.1) may be written in the form of

$$\int_V \left[\rho \dot{\eta} - \frac{\rho r}{T} - \left(\frac{q_i}{T} \right)_{,i} \right] dv \geq 0.$$

This inequality is equivalent to the inequality

$$\rho T \dot{\eta} - \rho r - q_{i,i} + \frac{1}{T} q_i T_{,i} \geq 0, \quad (18.2)$$

which represents the local form of the Clausius-Duhem inequality.

If we consider (10.3), (16.7) and (16.8), the inequality (18.2) assumes the form of

$$\rho_0 T \dot{\eta} - \rho_0 r - Q_{K,K} + \frac{1}{T} Q_K T_{,K} \geq 0. \quad (18.3)$$

Often, instead of function e the function of free energy ψ is introduced, defined by,

$$\psi = e - T \eta. \quad (18.4)$$

is introduced.

The equation of energy takes the form of

$$\rho_0 (\dot{\psi} + T \dot{\eta} - T \dot{\eta}) = T_{KL} E_{KL} + Q_{K,K} + \rho_0 r. \quad (18.5)$$

If we replace the expression $\rho_0 r + Q_{K,K}$ from (18.5) in the inequality (18.3), we obtain

$$-\rho_0 (\eta T + \psi) + T_{KL} E_{KL} + \frac{1}{T} Q_K T_{,K} \geq 0. \quad (18.6)$$

In the following instead of the function e , we will use the function ψ introduced by (18.4). In view of the definition of the thermodynamic process, it can easily be seen that the free energy has the properties from (v) Paragraph 15. Obviously a thermodynamic process is described by the ordered set $[u_1, T, \psi, t_{1j}, n, q_1, f_1, r]$.

19. Constituent Equations

Two bodies of the same form and mass may behave differently when they are subjected to the same strains. In that case, we say that the two bodies are made of different materials.

The principles studied in Chapter II are applicable for any continuous medium regardless of its internal constitution. Therefore certain relations are needed to define the various classes of continuous media corresponding their different behavior. These relations are called constitutive relations or constitutive equations. In any case, it can easily be seen that the equations derived until now are not sufficient to characterize the unknowns.

In order to illustrate a certain behavior which characterizes one class of continuous media, a mathematical model was developed. Thus, various types of ideal continuous media are introduced: elastic, fluid, viscoelastic media, etc.

A continuous medium is called elastic if

$$\begin{aligned}\psi(X, t) &= \hat{\psi}[x_{i,M}(X, t), T(X, t), T_{,N}(X, t), X], \\ T_{KL}(X, t) &= \hat{T}_{KL}[x_{i,M}(X, t), T(X, t), T_{,N}(X, t), X], \\ Q_K(X, t) &= \hat{Q}_K[x_{i,M}(X, t), T(X, t), T_{,N}(X, t), X], \\ \gamma_i(X, t) &= \hat{\gamma}_i[x_{i,M}(X, t), T(X, t), T_{,N}(X, t), X].\end{aligned}\tag{19.1}$$

The relations (19.1) are the constitutive equations of the elastic medium. Let us assume that the functions $\hat{\psi}, \hat{T}_{KL}, \hat{Q}_K, \hat{\gamma}_i$

are of the class C^1 on the domain \mathcal{D} of the elements of the form of $(x_{i,K}, T, T_{,L}, X)$, where $|x_{i,K}| \neq 0$, $T > 0$ and $X \in B$.

The constitutive equations represent a restriction imposed on a thermodynamic process. A thermodynamic process in which relations (19.1) take place is called an admissible thermodynamic process.

The constitutive relations which define a material must follow certain principles (Noll [311], Truesdell and Noll [421], Eringen [107], Jaunzemis [220], Misicu [290], Truesdell [422]). For the equations (19.1), these principles are reduced to: the principle of objectivity and the principle of material invariants.

The principle of objectivity states that the constitutive equations must be independent of the observer. In view of what has been said in Paragraph 9, it follows that this principle may also be formulated in the following manner: the constitutive equations must be the same in any two objectively equivalent motions. /43

Let x' and x be two objectively equivalent motions and $F = [x_{i,K}]$, $Q = [Q_{ij}]$, where the functions Q_{ij} satisfy (9.2).

The restriction imposed by the principle of objectivity on the constitutive equations consist of the fact that these equations must satisfy the relations

$$\begin{aligned}\hat{\psi}(F, T, T_{,K}, X) &= \hat{\psi}(QF, T, T_{,L}, X), \\ \hat{T}_{KL}(F, T, T_{,N}, X) &= \hat{T}_{KL}(QF, T, T_{,N}, X), \\ \hat{Q}_K(F, T, T_{,N}, X) &= \hat{Q}_K(QF, T, T_{,N}, X), \\ \hat{\eta}(F, T, T_{,N}, X) &= \hat{\eta}(QF, T, T_{,N}, X),\end{aligned}\tag{19.2}$$

for any specific orthogonal matrix Q .

Use of the whole orthogonal group to describe the objectivity could eliminate, in some cases, essential properties of the materials (Rivlin [351], Eringen [107], Green and Adkins [141]), and therefore we considered specific orthogonal transformations.

In view of the fact that the functions $\hat{\psi}, \hat{T}_{KL}, \hat{Q}_K, \hat{n}$ are subject to the same type of conditions, we will study only the function $\hat{\psi}$. The problem is to find out how $\hat{\psi}$ should depend on F so that the relation (19.2)₁ is satisfied regardless of the specific orthogonal transformation Q . We will use the following theorem.

Theorem of polar decomposition. Any nonsingular matrix A may be written in the form

$$A = RM = NR,$$

where R is an orthogonal matrix and M and N are symmetric and positively defined matrices. The matrices M , N and R are uniquely determined.

We recommend the work of Ericksen [106] to prove the theorem.

In view of the fact that matrix F is nonsingular, according to the above theory, we may write

$$F = RU, \tag{19.3}$$

where U is a symmetrical and positively defined matrix and R is an orthogonal matrix.

The relations (19.2) may be satisfied regardless of the orthogonal matrix Q . We will designate by A^* the transposed matrix A . Let us choose $Q = R^*$. In this case the first relation from (19.2) is written as

$$\hat{\psi}(F, T, T_{,K}, X) = \hat{\psi}(R^*, N, T, T_{,L}, X). \tag{19.4}$$

In view of the fact that R is a specific orthogonal matrix (determined as $F > 0$), we have $R^*F = U$, and thus (19.4) becomes

$$\hat{\psi}(F, T, T_{,N}, X) = \hat{\psi}(U, T, T_{,L}, X). \quad (19.5)$$

From (19.3) we obtain

$$F^*F = UR^*RU = U^2. \quad (19.6)$$

Keeping in mind the definition of matrix F and the relations (2.16), (2.18), we have

$$U^2 = [x_i \cdot x_{i,L}] = [C_{KL}] = [2E_{KL} + \delta_{KL}].$$

It follows that $\hat{\psi}$ depends on F by means of the components of the deformation tensor E_{MN} . The same result applies also for $\hat{T}_{KL}, \hat{Q}_M, \hat{\eta}$. We will write

$$\begin{aligned} \psi &= \tilde{\psi}(E_{MN}, T, T_{,K}, X), \\ T_{KL} &= \tilde{T}_{KL}(E_{MN}, T, T_{,S}, X), \\ Q_K &= \tilde{Q}_K(E_{MN}, T, T_{,L}, X), \\ \eta &= \tilde{\eta}(E_{MN}, T, T_{,K}, X). \end{aligned} \quad (19.7)$$

It can easily be seen that the functions (19.7) satisfy the conditions (19.2) regardless of the orthogonal matrix Q . There thus follows

Theorem 19.1. The constitutive equations (19.1) satisfy the principle of objectivity if and only if they can be written in the form (19.7)

There are continuous media which present certain symmetries as far as their physical properties are concerned. Thus, for certain

materials their physical properties are the same when (X_1, X_2, X_3) are changed into $(X_1, X_2, -X_3)$. Obviously it is possible to imagine more general transformations of the material coordinates which do not modify the physical properties of the material. We will designate with $\{S\}$ the subgroup of orthogonal transformations of the material coordinates which characterize the symmetry properties of a continuous medium.

The principle of the material invariants is applied in the case when such symmetry properties exist. This principle may be stated in the following manner: the constitutive equations of a material must be invariant with respect to the group of transformations $\{S\}$ which characterize the symmetry properties of the material.

According to this principle the constitutive equations must be invariant at any transformation of the form of

(19.8)

$$X'_K = S_{KL} X_L,$$

$$S_{ML} S_{MK} = S_{KM} S_{LM} = \delta_{KL},$$

from $\{S\}$.

If the group $\{S\}$ coincides with the group of specific orthogonal transformations, the material is called hemitropic.

If $\{S\}$ coincides with a whole group of orthogonal transformations, then the material under consideration is called isotropic. Media which are not isotropic are called anisotropic.

If the functionals (19.1) do not depend explicitly on X , the corresponding medium is called homogeneous; in the contrary case it is called inhomogeneous.

/45

Sometimes the internal constitution of the material or the method of treatment restricts the class of deformation to which the material in question may be subjected. Thus, incompressible media represent an example of materials which cannot be arbitrarily deformed. As is

well known, a body is incompressible if, during any kind of deformation, its density remains unchanged; $\rho = \rho_0(X)$. The possible deformations of these media are those which respect the conditions of incompressibility.

In view of (4.5) and (10.3) it follows that incompressible materials may be characterized by the condition

$$I_3(C) = |C_{KL}| = |2E_{KL} + \delta_{KL}| = 1. \quad (19.9)$$

Thus, incompressibility implies a connection between the components of the deformation tensor.

It is possible to imagine continuous media which have been subjected to certain restrictions by the fabrication process, limiting the class of the possible deformations. Thus, a body which is reinforced by a system of fine and inextensible wires, parallel with a certain direction, will be considered inextensible in that direction. These types of restriction imposed on the deformation tensor are called connections.

In the following, except when stated otherwise (when we will consider the condition (19.9)), we will assume that we do not have connections. They have been studied in various works: Green and Adkins [141], Green, Naghdi and Trapp [145], Trapp [419], Gurtin and Guidugli [164], etc.

20. The Consequences of the Laws of Thermodynamics

Theorem 20.1. An admissible thermodynamic process satisfies the inequality (18.6) if and only if

(i) the functions $\tilde{\psi}$, \tilde{T}_{KL} , $\tilde{\eta}$ are independent of the material gradient of temperature

$$\psi = \tilde{\psi}(E_{MN}, T, X), \quad T_{KL} = \tilde{T}_{KL}(E_{MN}, T, X), \quad \eta = \tilde{\eta}(E_{MN}, T, X), \quad (20.1)$$

(ii) the functions \tilde{T}_{KL} and $\tilde{\eta}$ are determined by the function $\tilde{\Psi}$ by

$$\tilde{T}_{KL} = \frac{1}{2} \rho_0 \left(\frac{\partial \tilde{\Psi}}{\partial E_{KL}} + \frac{\partial \tilde{\Psi}}{\partial E_{LK}} \right), \quad (20.2)$$

$$\tilde{\eta} = - \frac{\partial \tilde{\Psi}}{\partial T}, \quad (20.3)$$

(iii) the functions \tilde{Q}_K satisfy the inequality

$$\tilde{Q}_K T_{,K} > 0. \quad (20.4)$$

Proof. In view of (19.7) it follows that an admissible thermodynamic process is compatible with the inequality (18.6) if and only if

/46

$$\begin{aligned} & - \left[\frac{1}{2} \rho_0 \left(\frac{\partial \tilde{\Psi}}{\partial E_{KL}} + \frac{\partial \tilde{\Psi}}{\partial E_{LK}} \right) - T_{KL} \right] \dot{E}_{KL} - \rho_0 \left(\frac{\partial \tilde{\Psi}}{\partial T} + \eta \right) \dot{T} - \\ & - \rho_0 \frac{\partial \tilde{\Psi}}{\partial T_{,N}} \dot{T}_{,N} + \frac{1}{T} \tilde{Q}_K T_{,K} > 0, \end{aligned} \quad (20.5)$$

on \mathcal{D} . We assumed that both E_{LK} and E_{KL} enter into $\tilde{\Psi}$.

(20.5) indicates the sufficiency of the conditions (i) -- (iii).

In this paragraph we will sometimes use the designations

$$\underline{G} = T_{,K} E_K = G_K E_K.$$

In order to prove that these conditions are necessary we will first establish

LEMMA 20.1. If $(F^*, T^*, G^*, X^*) \in \mathcal{D}$, $t^* \in (0, t_0)$, are matrix A , a scalar quantity τ and a vector \underline{a} exist in a thermodynamic process so that

$$F(X^*, t^*) = F^*, \quad T(X^*, t^*) = T^*, \quad G(X^*, t^*) = G^*$$

$$\dot{F}(X^*, t^*) = A, \quad \dot{T}(X^*, t^*) = \tau, \quad \dot{G}(X^*, t^*) = a.$$

Proof. There is an open sphere $\Sigma \subset B$ with the center in X^* and an open interval $\mathcal{T} \subset (0, t_0)$ which contains t^* so that

$$\det [F^* + (t - t^*)A] \neq 0,$$

$$T^* + \tau(t - t^*) + [G^* + (t - t^*)a](R - R^*) > 0,$$

for any $(X, t) \in \Sigma \times \mathcal{T}$, where $R - R^* = (X_K - X_K^*)E_K$. Let \underline{r} and T be defined by $\Sigma \times \mathcal{T}$ by

$$x_i = [F_{iL}^* + (t - t^*)A_{iL}](X_L - X_L^*),$$

$$T = T^* + \tau(t - t^*) + [G^* + (t - t^*)a](R - R^*).$$

It can be easily seen that the corresponding admissible thermodynamic process has all the required properties.

Let us prove now the necessity of conditions (i) -- (iii). If we apply the inequality (20.5) to the process constructed by lemma 20.1, in $X = X^*$, $t = t^*$, we have

$$\begin{aligned} & - \left[\frac{1}{2} \rho_0 \left(\frac{\partial \tilde{\psi}}{\partial E_{KL}} + \frac{\partial \tilde{\psi}}{\partial E_{LK}} \right) - T_{KL} \right]^* D_{KL} - \rho_0 \left(\frac{\partial \tilde{\psi}}{\partial T} + \eta \right)^* \tau - \\ & - \rho_0 \left(\frac{\partial \tilde{\psi}}{\partial T_{,K}} \right)^* a_N + \frac{1}{T^*} \tilde{Q}_K^* G_K^* > 0, \end{aligned} \quad (20.6)$$

where by $[\]^*$ we noted the value of the function from the parentheses /47 in F^*, T^*, G^*, X^* . Similarly we use the notation

$$2D = A^T A - I, \quad D = [D_{KL}], \quad I = [\delta_{KL}], \quad a = a_N E_N, \quad G^* = G_K^* E_K.$$

In view of the fact that in (20.6) $F^*, T^*, G_K^*, X^*, t^*, D_{KL}, \tau$ and a_N are arbitrary, (20.1) - (20.4) may be derived from these.

This result is due to the investigations carried out by Green and Adkins [141], Truesdell and Toupin [420], Coleman and Noll [63], Coleman and Mizel [65].

Theorem 20.2. In an admissible thermodynamic process the equation of the energy becomes

$$\rho_0 T \dot{\eta} = Q_{K,K} + \rho_0 r. \quad (20.7)$$

Proof. It was shown that the equation of the energy may be written in the form of (18.5). In view of theorem 20.1 we obtain

$$\rho_0 \dot{\psi} = T_{KL} \dot{E}_{KL} - \rho_0 \eta \dot{T}. \quad (20.8)$$

(20.7) follows from (20.8) and (18.5).

Theorem 20.3. If the temperature gradient is annulled, then the vector of the heat flux is zero, i.e.,

$$\tilde{Q}_K(E_{MN}, T, 0, X) = 0. \quad (20.9)$$

Proof. In view of (20.4) it follows that the function

$$f(\xi_s) = \xi_K \tilde{Q}_K(E_{MN}, T, \xi_s, X),$$

where E_{MN} , T and X are set and $f(0) = 0$ is also non-negative. Therefore, f has a minimum equal to zero in $\xi = 0$. Therefore

$$\frac{\partial f}{\partial \xi_K} = \tilde{Q}_K + \xi_L \frac{\partial \tilde{Q}_L}{\partial \xi_K},$$

(20.9) follows from this.

This result was determined by Pipkin and Rivlin [342] (see also Coleman and Noll [63], Coleman and Mizel [65]). It should be noted that from the relations

$$\frac{\partial \hat{\psi}}{\partial x_{i,K}} = \frac{\partial \bar{\psi}}{\partial E_{MN}} \frac{\partial E_{MN}}{\partial x_{i,K}}, \quad E_{MN} = \frac{1}{2} (x_{i,M} x_{i,N} - \delta_{MN}),$$

$$\frac{\partial E_{MN}}{\partial x_{i,K}} = \frac{1}{2} (x_{i,M} \delta_{NK} + x_{i,N} \delta_{MK}),$$

we can deduce

/48

$$\frac{\partial \hat{\psi}}{\partial x_{i,K}} = \frac{1}{2} x_{i,M} \left(\frac{\partial \bar{\psi}}{\partial E_{MK}} + \frac{\partial \bar{\psi}}{\partial E_{KM}} \right). \quad (20.10)$$

In view of (14.9) and (20.2) it follows that the Piola-Kirchhoff tensor of the first kind has the form of

$$T_{Ki} = \rho_0 \frac{\partial \hat{\psi}}{\partial x_{i,K}}. \quad (20.11)$$

From (2.6) we have

$$x_{i,K} = u_{i,K} + \delta_{iK}. \quad (20.12)$$

Therefore

$$T_{Ki} = \rho_0 \frac{\partial \hat{\psi}}{\partial u_{i,K}}. \quad (20.13)$$

If we introduce the notation

$$\rho_0 \hat{\psi} = \sigma, \quad (20.14)$$

then it follows from (14.9) and (20.2) that

$$T_{KL} = \frac{1}{2} x_{i,L} \left(\frac{\partial \tilde{\sigma}}{\partial E_{KL}} + \frac{\partial \tilde{\sigma}}{\partial E_{LK}} \right) = \frac{1}{2} (u_{i,L} + \delta_{iL}) \left(\frac{\partial \tilde{\sigma}}{\partial E_{KL}} + \frac{\partial \tilde{\sigma}}{\partial E_{LK}} \right). \quad (20.15)$$

The function σ represents the free energy per unit of initial volume.

From (20.13) we obtain

$$T_{KL} = \frac{\partial \tilde{\sigma}}{\partial u_{i,K}}. \quad (20.16)$$

We shall write similarly

$$Q_K = \tilde{Q}_K(E_{MN}, T, T_L, X) = \hat{Q}_K(u_{i,M}, T, T_N, X). \quad (20.17)$$

Obviously, the internal energy has the form

$$e = \tilde{e}(E_{MN}, T). \quad (20.18)$$

The function

$$c^*(E_{KL}, T) = \frac{\partial \tilde{e}}{\partial T}, \quad (20.19)$$

is called specific mass heat. In view of (18.4) and (20.3), we obtain

$$c^*(E_{KL}, T) = T \frac{\partial \tilde{\eta}}{\partial T}. \quad (20.20)$$

In the theory of thermoelasticity a lagrangian description of the deformation is used. The problem of the thermoelastic deformation of the medium under consideration consists in the determination of the functions

$$x_i = x_i(X_K, t), \quad T = T(X_K, t), \quad (21.1)$$

which amounts to the determination of the components of the displacement vector \underline{u} and of the temperature T as functions of X_K and t .

From the facts established in the previous paragraphs it follows that the fundamental equations of the theory of thermoelasticity are

- equations of motion

$$T_{Kt,K} + \rho_0 f_i = \rho_0 \ddot{u}_i, \quad (21.2)$$

- equation of energy

$$\rho_0 T \dot{\eta} - Q_{L,L} = \rho_0 r, \quad (21.3)$$

- the constitutive equations

$$\sigma = \tilde{\sigma}(E_{KL}, T, X), \quad (21.4)$$

$$T_{Kt} = \frac{1}{2} (u_{i,L} + \delta_{iL}) \left(-\frac{\partial \tilde{\sigma}}{\partial E_{KL}} + \frac{\partial \tilde{\sigma}}{\partial E_{LK}} \right),$$

$$\tau_i = -\frac{1}{\rho_0} \frac{\partial \tilde{\sigma}}{\partial T},$$

$$Q_K = \tilde{Q}_K(E_{MN}, T, T_{L,L}, X).$$

- geometric equations

$$2E_{KL} = U_{K,L} + U_{L,K} + U_{M,K} U_{M,L}; \quad u_i = \delta_{Ki} U_K. \quad (21.5)$$

In the above relations $\hat{\sigma}(E_{MN}, T, X)$ and $\hat{Q}_K(E_{MN}, T, T_{,L}, X)$ are prescribed functions which are compatible with the principles formulated in the previous paragraphs, which characterize the thermo-mechanical properties of the material under consideration. The functions $\rho_0(X), f_1(X, t), r(X, t)$ are given functions which satisfy the conditions of regularity from Paragraphs 9, 11, and 15.

If we substitute (21.5) in (21.4) and substitute the expressions obtained in (21.2) and (21.3), we obtain four equations for the components of the vector \underline{u} and the function T .

Later on we will present other forms of the fundamental equations. If we consider the results given in Paragraph 20, the equations (21.2) -- (21.5) are substituted by

- equations of motion

$$T_{Kl,K} + \rho_0 f_l = \rho_0 \ddot{u}_l, \quad (21.6)$$

- equations of energy

/50

$$\rho_0 T \dot{\eta} - Q_{L,L} = \rho_0 r, \quad (21.7)$$

- constitutive equations

$$\sigma = \hat{\sigma}(u_{i,L}, T, X), \quad T_{Kl} = \frac{\partial \hat{\sigma}}{\partial u_{i,K}}, \quad \eta = -\frac{1}{\rho_0} \frac{\partial \hat{\sigma}}{\partial T}, \quad (21.8)$$

$$Q_K = \hat{Q}_K(u_{i,L}, T, T_{,N}, X).$$

By replacing the relations (21.8) in (21.6) and (21.7), we obtain four equations for the unknowns u_1 and T .

Initial conditions and conditions at the boundary of the body under consideration are also added to the fundamental equations (21.2) -- (21.5) (or (21.6) -- (21.8)). The initial conditions have the following shape.

$$u(X, 0) = \underline{a}(X), \dot{u}(X, 0) = \underline{b}(X), \eta(X, 0) = \eta_0(X), X \in \bar{B}, \quad (21.9)$$

where \underline{a} , \underline{b} , η_0 are prescribed continuous functions for \bar{B} .

The boundary conditions consist of mechanical and thermal conditions. The mechanical conditions are those known from the theory of elasticity where, at the boundary, the displacement vector or the tension vector is prescribed.

The thermal boundary conditions which appear most often are: (i) the temperature at the boundary is prescribed, (ii) the heat flux is prescribed at the boundary, (iii) the convection condition.

Mixed type mechanical and thermal conditions may also be considered. Thus, we will consider frequently the mixed problem characterized by the conditions

$$\begin{aligned} \underline{u} &= \underline{u}^* \text{ pc } \Sigma_1 \times [0, t_0], T_K N_K = \underline{P}^* \text{ pc } \Sigma_2 \times [0, t_0], \\ T &= \Theta \text{ pc } \Sigma_3 \times [0, t_0], Q_K N_K = H^* \text{ pc } \Sigma_4 \times [0, t_0], \end{aligned} \quad (21.10)$$

where Σ_s ($s = 1, 2, 3, 4$) are part of the boundary ∂B so that $\Sigma_1 \cup \Sigma_2 = \Sigma_3 \cup \Sigma_4 = \partial B$, $\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \emptyset$,

while \underline{u}^* , \underline{P}^* , Θ , H^* are prescribed. The components of vector \underline{u}^* are continuous on $\Sigma_1 \times [0, t_0]$, the components of vector \underline{P}^* are regular functions on portions on $\Sigma_2 \times [0, t_0]$ and are continuous with respect to t , Θ is a continuous function on $\Sigma_3 \times [0, t_0]$ and H^* is a regulated function on portions on $\Sigma_4 \times [0, t_0]$ and continuous with respect to t .

The convection condition assumes the form of

$$Q_K N_K = h(T - T_c) \text{ pc } \partial B \times [0, t_0], \quad (21.11)$$

where T_e is the temperature of the surrounding medium and $h(> 0)$ is the heat transfer coefficient. The functions T_e and h are prescribed.

If the solution of the thermoelasticity problem is determined, the density ρ is obtained from (10.3)

In the study of thermoelasticity problems it is generally assumed that the reference ox_1 coincides with the reference OX_K . The components of the displacement vector, the lagrangian deformation tensor, the Piola-Kirchhoff tensor of the first kind, the heat flux vector measured per unit area from the nondeformed body, with respect to the reference under consideration, will be designated by $u_1(X_j, t)$, $E_{1j}(X_s, t)$, $T_{1j}(X_s, t)$, $Q_1(X_s, t)$, respectively. In view of the fact that only the variables X_s and t are used, only partial derivatives with respect to these variables will appear, and we will indicate now by f_1 the derivative of the function f with respect to X_1 . The fundamental equations assume the form of

-equations of motion

$$T_{\mu,1} + \rho_0 f_1 = \rho_0 \ddot{u}_1, \quad (21.12)$$

- equation of energy

$$\rho_0 T \dot{\eta} - Q_{i,1} = \rho_0 r, \quad (21.13)$$

- the constitutive equations

$$\sigma = \bar{\sigma}(E_{ij}, T, X_s), \quad (21.14)$$

$$T_{\mu} = \frac{1}{2} (u_{i,1} + \delta_{i\mu}) \left(\frac{\partial \bar{\sigma}}{\partial E_{\mu}} + \frac{\partial \bar{\sigma}}{\partial E_{i\mu}} \right)$$

$$\eta = - \frac{1}{\rho_0} \frac{\partial \bar{\sigma}}{\partial T},$$

$$Q_i = \bar{Q}_i(E_{mn}, T, T_{ij}, X_s),$$

- the geometric equations

$$2E_{ij} = u_{i,j} + u_{j,i} + u_{i,k}u_{k,j}, \quad (21.15)$$

The constitutive equations written in the form of (21.8) become

$$T_{ij} = \frac{\partial \hat{\sigma}}{\partial u_{i,j}}, \quad \rho = \frac{1}{\rho_0} \frac{\partial \hat{\sigma}}{\partial T}, \quad Q_i = \hat{Q}_i(u_{j,k}, T, T_{j,k}, X_p). \quad (21.16)$$

22. Isotropic Media. Incompressible Media

The continuous isotropic media were defined in Paragraph 19. A thermoelastic medium is isotropic if the functions $\hat{\sigma}$ and \tilde{Q}_K depend similarly on E_{MN} and $T_{,L}$, regardless of their orthogonal transformations (19.8). In view of (20.1), (20.14) it results that the $\hat{\sigma}$ is an invariant of the deformation tensor E_{MN} and therefore

$$\hat{\sigma} = \hat{\sigma}[I_1(E), I_2(E), I_3(E), T, X]. \quad (22.1)$$

It is shown that for isotropic media the functions \tilde{Q}_K are expressed in the form of

$$\tilde{Q}_K = (\mathcal{C}_1 \delta_{KL} + \mathcal{C}_2 E_{KL} + \mathcal{C}_3 E_{KM} E_{ML}) T_{,L}, \quad (22.2)$$

where \mathcal{C}_i are functions of the variables

$$I_1(E); T_{,K} T_{,K}; T_{,K} T_{,L} E_{KL}; T_{,K} T_{,L} E_{LM} E_{MK}; T; X_N. \quad (22.3)$$

This result was established by Pipkin and Rivlin [342] for the case when \tilde{Q}_K depends polynomially on its variables. Green and Adkins [141], Koh and Eringen [237] extended the results for hemitropic media, and Truesdell and Noll [421] to the case when \tilde{Q}_K are general functions of $E_{MN}, T_{,L}, T, X$ with the regularity properties given in Paragraph 19.

Often instead of the invariant $I_1(E)$, the invariants J_1 defined as follows

$$\begin{aligned} J_1 &= I_1(E) = E_{LL}, \quad J_2 = I_1^2(E) - 2I_2(E) = E_{KL}E_{KL}, \\ J_3 &= 1 + 2I_1(E) + 4I_2(E) + 8I_3(E) = I_3(C) = |\delta_{KL} + 2E_{KL}|. \end{aligned} \quad (22.4)$$

are considered.

In order to obtain the form of the tension tensor in the case of isotropic media we will consider the relations

$$\frac{\partial J_1}{\partial E_{KL}} = \delta_{KL}, \quad \frac{\partial J_2}{\partial E_{KL}} = 2E_{KL}, \quad \frac{\partial J_3}{\partial E_{KL}} = 2J_3 C_{KL}^{(-1)}, \quad (22.5)$$

where

$$C_{KL}^{(-1)} = X_{K,L} \cdot X_{L,K} \quad (22.6)$$

In deriving the last relation from (22.5), we kept in mind that for the algebraic complement A_{KN} of the element C_{KN} from the determinant $I_3(C) \equiv J_3$ we have the relations

$$C_{KN}A_{NL} = J_3\delta_{KL}, \quad A_{KN} = \frac{\partial J_3}{\partial C_{KN}}.$$

Therefore

$$C_{KN}C_{NL}^{(-1)} = \delta_{KL},$$

it follows that

$$C_{NL}^{(-1)} = \frac{1}{J_3} \frac{\partial J_3}{\partial C_{NL}}, \quad \frac{\partial J_3}{\partial E_{KL}} = 2 \frac{\partial J_3}{\partial C_{KL}} = 2J_3 C_{KL}^{(-1)}.$$

From (21.4), (22.1), (22.5), we obtain

$$T_{KI} = (u_{i,L} + \delta_{iL}) \left(\frac{\partial \tilde{\sigma}}{\partial J_1} \delta_{KL} + 2 \frac{\partial \tilde{\sigma}}{\partial J_2} E_{KL} + 2J_3 \frac{\partial \tilde{\sigma}}{\partial J_3} C_{KL}^{(-1)} \right). \quad (22.7)$$

Let us consider the case of the incompressible media. Now the relation (20.5) takes place only for those deformations which satisfy the condition (19.9). If we consider (22.5), (22.6), it follows from (19.9)

$$C_{KL}^{(-1)} \dot{E}_{KL} = 0. \quad (22.8)$$

In this case, (20.5) and (22.8) imply

53

$$T_{KL} = p C_{KL}^{(-1)} + \frac{1}{2} \left(\frac{\partial \tilde{\sigma}}{\partial E_{KL}} + \frac{\partial \tilde{\sigma}}{\partial E_{LK}} \right), \quad (22.9)$$

where p is a lagrangian parameter. The relations (22.9) substitute the relations (20.2) which were established for an anisotropic medium in the absence of connections.

If the elastic medium under consideration is isotropic and incompressible, then we have

$$\tilde{\sigma} = \tilde{\sigma}(J_1, J_2, T, X), \quad (22.10)$$

and the relations (22.9) become

$$T_{KL} = p C_{KL}^{(-1)} + \frac{\partial \tilde{\sigma}}{\partial J_1} \delta_{KL} + 2 \frac{\partial \tilde{\sigma}}{\partial J_2} E_{KL}. \quad (22.11)$$

Here $p(X, t)$ is an unknown function which must be determined. This time equation (19.9) is to be added to the equations considered until now.

In case of isotropic and incompressible media, the relations (22.7) are substituted by

$$T_{KL} = (u_{i,L} + \delta_{iL}) \left(\frac{\partial \tilde{\sigma}}{\partial J_1} \delta_{KL} + 2 \frac{\partial \tilde{\sigma}}{\partial J_2} E_{KL} + p C_{KL}^{(-1)} \right). \quad (22.12)$$

From (14.10), (19.9), (22.6) and (22.11) it also results

$$t_{ij} = p\delta_{ij} + x_{i,k}x_{j,l}\left(\frac{\partial\bar{\sigma}}{\partial J_1}\delta_{kl} + 2\frac{\partial\bar{\sigma}}{\partial J_2}E_{kl}\right). \quad (22.13)$$

Obviously, in this case I_3 does not appear among the variables on which functions $\bar{\sigma}_i$ depend.

23. The Equations of the Linear Theory of Thermoelasticity

In the following we will designate by θ the temperature variation of the medium with respect to the absolute temperature T_0 of the reference state.

We have thus

$$T = T_0 + \theta. \quad (23.1)$$

The concept of the linear theory of deformation was introduced in Paragraph 2. The linear theory of thermoelasticity is characterized by the fact that each of the functions $u_i, \theta, t_{ij}, q_i, \eta$ are of the form of $\epsilon\phi$ whereby ϵ is a parameter whose powers greater than or equal to two are negligible, and ϕ does not depend on ϵ . As it was agreed, in this case we will use a single system of coordinates and will designate with x_r the material coordinates. Let us recall that in the linear theory the relations (2.22), (14.17), (16.11) take place. We are designating the tension tensor by t_{ij} and the components of the flux vector by q_i .

In the linear theory the free energy developed according to ϵ_{ij} and θ is also considered (see also Paragraph 53) up to the second order terms, and it is assumed that

$$\sigma = \bar{\sigma}(\epsilon_{ij}, \theta) = c_0 - c_1\theta + c_2\epsilon_{ij} + \frac{1}{2}C_{ijkl}\epsilon_{ij}\epsilon_{kl} - \beta_0\epsilon_{ij}\theta - \frac{1}{2}a\theta^2, \quad (23.2)$$

/54

where $c_0, c_1, c_{ij}, C_{ijkl}, \beta_{ij}$ are characteristic coefficients of the medium under consideration having the properties

$$c_{ij} = c_{ji}, C_{ijkl} = C_{jikl} = C_{jlik}, \beta_{ij} = \beta_{ji}.$$

In view of the fact that in the linear theory the tension tensor of Cauchy and the Piola-Kirchhoff tension tensors coincide, we have, according to (20.2) and (20.14)

$$t_{ij} = \frac{1}{2} \left(\frac{\partial \tilde{\sigma}}{\partial \varepsilon_{ij}} + \frac{\partial \tilde{\sigma}}{\partial \varepsilon_{ji}} \right) = C_{ijkl} \varepsilon_{kl} - \beta_{ij} \theta + c_{ij}, \quad (23.3)$$

$$\rho_0 \eta = - \frac{\partial \tilde{\sigma}}{\partial \theta} = \beta_{ij} \varepsilon_{ij} + a \theta + c_1.$$

As a general rule, the following assumptions are made in the linear theory of thermoelasticity:

- (α) when $\varepsilon_{ij} = 0$ and $\theta = 0$ we have $t_{ij} = 0$,
- (β) the absolute temperature T_0 is constant.

Without restricting the general application, we will assume that when $\varepsilon_{ij} = 0$ and $\theta = 0$ we have $\sigma = 0$ and $\eta = 0$. The state of the medium in which we have $\varepsilon_{ij} = 0, \theta = 0, t_{ij} = 0, \sigma = 0, \eta = 0, \theta_{,j} = 0$ is called the natural state. From the above assumptions it follows that $c_{ij} = c_0 = c_1 = 0$ so that the free energy takes the form

$$\tilde{\sigma} = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - \beta_{ij} \varepsilon_{ij} \theta - \frac{1}{2} a \theta^2. \quad (23.4)$$

The relations (23.3) become

$$t_{ij} = C_{ijkl} \varepsilon_{kl} - \beta_{ij} \theta, \quad (23.5)$$

$$\rho_0 \eta = \beta_{ij} \varepsilon_{ij} + a \theta.$$

If we consider the theorem 20.3 and (16.11) it follows

$$q_i = k_{ij} \theta_{,j}, \quad (23.6)$$

From (20.4) and (23.6) we obtain

$$k_{ij} \theta_{,i} \theta_{,j} > 0. \quad (23.7)$$

The quantities k_{ij} are components of a tensor called the thermal conductivity tensor.

In view of (23.1) and (23.5), (21.13) becomes in the linear theory

$$\rho_0 T_0 \dot{\eta} - q_{i,i} = \rho_0 r. \quad (23.8)$$

In conclusion, the equations of the linear theory of thermo-elasticity are

- equations of motion

$$t_{ji,j} + \rho_0 f_i = \rho_0 \ddot{u}_i, \quad (23.9)$$

- equations of energy

$$\rho_0 T_0 \dot{\eta} - q_{i,i} = \rho_0 r, \quad (23.10)$$

- constitutive equations

$$t_{ij} = C_{ijkl} \varepsilon_{kl} - \beta_{ij} \theta, \quad (23.11)$$

$$\rho_0 \eta = \beta_{ij} \varepsilon_{ij} + \alpha \theta,$$

$$q_i = k_{ij} \theta_{,j},$$

- geometric equations

$$2\varepsilon_{ij} = u_{i,j} + u_{j,i}. \quad (23.12)$$

In the above equations ρ_0 , C_{ijkl} , β_{ij} , a , k_{ij} characterize the properties of the medium under consideration. For inhomogeneous media these quantities are functions of x_r . In the case of homogeneous media they are constant.

The coefficients C_{ijkl} and β_{ij} satisfy the symmetry properties

$$C_{ijkl} = C_{klij} = C_{jilk}, \quad \beta_{ij} = \beta_{ji}. \quad (23.13)$$

In connection with the thermal conductivity tensor, there is no reason to consider it symmetrical (Truesdell [423]) as it is used in general. It is noted however that if this tensor is not symmetrical then the antisymmetric part does not contribute to the equation of energy (23.10). With respect to this question let us mention the works of Lessen [258], Day and Gurtin [79]. Müller [299] and Green and Lindsay [146] built a theory of thermoelasticity with a symmetrical conductivity tensor (see also Boschi and Iesan [36]).

If we consider (20.20) and (23.5), we can conclude that in this case

$$c^* = \frac{1}{\rho_0} T_0 a,$$

represents the specific heat of mass corresponding to the state of deformation in which $\epsilon_{ij} = 0$, $\theta = 0$. We will designate by c the specific heat per volume

$$c = \rho_0 c^* = T_0 a. \quad (23.14)$$

It should be noted that the functions C_{ijkl} , β_{ij} , k_{ij} , ρ_0 , a /56 (or c) depend in general on the reference temperature T_0 but do not depend on temperature θ . The dependence of θ will be incompatible with the linearization process.

If we consider (22.1) and (23.4) it follows that in the linear theory of isotropic media the function $\tilde{\sigma}$ is expressed as

$$\bar{\sigma} = \frac{1}{2} \lambda J_1^2 + \mu J_2 - \beta J_1 \theta - \frac{1}{2} a \theta^2, \quad (23.15)$$

where $J_1 = \epsilon_{11}$, $J_2 = \epsilon_{1j} \epsilon_{1j}$, and λ , μ , β , and a are characteristics of the material. Similarly we have

$$q_i = k \theta_{,i}, \quad (23.16)$$

where $k \geq 0$.

In this case we obtain

$$t_{ij} = \lambda \epsilon_{rr} \delta_{ij} + 2\mu \epsilon_{ij} - \beta \theta \delta_{ij}, \quad (23.17)$$

$$\rho_0 \eta = \beta \epsilon_{rr} + a \theta.$$

The coefficients λ and μ are called Lamé coefficients. If $\mu \neq 0$ and $3\lambda + 2\mu \neq 0$, the relations (23.17)₁ may be written differently

$$\epsilon_{ij} = \frac{1}{2\mu} t_{ij} - \frac{\lambda t_{rr} \delta_{ij}}{2\mu(3\lambda + 2\mu)} + \alpha \theta \delta_{ij}, \quad (23.18)$$

where

$$\alpha = \frac{\beta}{3\lambda + 2\mu}, \quad (23.19)$$

is called the coefficient of thermal expansion.

Introducing the designations

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}, \quad (23.20)$$

the equations (23.18) take the form of

$$\epsilon_{ij} = \frac{1+\nu}{E} t_{ij} - \frac{\nu}{E} t_{rr} \delta_{ij} + \alpha \theta \delta_{ij}. \quad (23.21)$$

The coefficients E and ν are known under the name of Young's modulus and Poisson's coefficient, respectively.

Thus the equations of the linear theory of thermoelasticity for isotropic media are

57

(23.22)

$$\begin{aligned} t_{ii,j} + \rho_0 f_i &= \rho_0 \dot{u}_{i,j} \\ \rho_0 T_0 \dot{\eta} - q_{i,i} &= \rho_0 r, \\ t_{ij} &= \lambda \varepsilon_{rr} \delta_{ij} + 2\mu \varepsilon_{ij} - \beta \theta \delta_{ij}, \\ \rho_0 \eta &= \beta \varepsilon_{rr} + \alpha \theta, \\ q_i &= k \theta_{,i}, \\ 2\varepsilon_{ij} &= u_{i,j} + u_{j,i}. \end{aligned}$$

These equations were postulated by Duhamel [91] and Neumann [305]. Providing a basis for them on the basis of thermodynamic principles was started with the works of Voigt [432], Lessen and Duke [257], Biot [20], and Lessen [260].

For the sake of simplicity, an agreement has been reached in the linear theory of thermoelasticity, to designate the density of the medium in the nondeformed state by ρ . The density of the medium in the deformed state will be $\rho(1 + \varepsilon_{11})$. Similarly we will use the specific force per initial unit volume, components of which are

$$f_i^0 = \rho_0 f_i, \quad (23.23)$$

and the flow of heat sources for initial unit volume

$$r^0 = \rho_0 r. \quad (23.24)$$

Often, in place of f_i^0 and r^0 , we will use the designations f_1 and r , respectively, with suitable mentions.

24. Curvilinear Coordinates

In the study of many problems of the mechanics of continuous media it is useful to employ curvilinear coordinates. We will derive here the equations of the theory of thermoelasticity in arbitrary curvilinear coordinates in the manner presented in the monographs of Green and Zerna [139] and Green and Adkins [141]. The results will be used also in other paragraphs of this work.

Let us assume that both the deformed and the nondeformed media are referred to the same system of rectangular cartesian coordinates.

The position vector of a generic point M from B is

$$\mathbf{R} = X_i \mathbf{a}_i, \quad (24.1)$$

where \mathbf{a}_i are versors of the coordinate axes. Let us assume that at the time t the point M reaches in $M^* \in \mathcal{M}$, and the position vector of point M^* is

$$\mathbf{r} = x_i \mathbf{a}_i. \quad (24.2)$$

The displacement vector may also be written in the following /58
manner:

$$\mathbf{u} = (x_i - X_i) \mathbf{a}_i. \quad (24.3)$$

The deformation of the medium is described by the functions $x_i = x_i(X_P, t)$. If we introduce a system of arbitrary curvilinear coordinates θ^1 by

$$X_i = X_i(\theta^1, \theta^2, \theta^3), \quad (24.4)$$

it follows that the deformation will be described by the relations

$$x_i = x_i(\theta^1, \theta^2, \theta^3, t). \quad (24.5)$$

Consequently we have

$$\mathbf{R} = \mathbf{R}(\theta^1, \theta^2, \theta^3), \mathbf{r} = \mathbf{r}(\theta^1, \theta^2, \theta^3, t), \mathbf{u} = \mathbf{u}(\theta^1, \theta^2, \theta^3, t). \quad (24.6)$$

Let us introduce the fundamental vectors $\underline{\mathbf{g}}_1, \underline{\mathbf{g}}^1$ and the metric tensors g_{1j}, g^{1j} into the nondeformed body corresponding to the coordinates θ^1

$$\underline{\mathbf{g}}_i = \frac{\partial \mathbf{R}}{\partial \theta^i}, \quad \underline{\mathbf{g}}^i \underline{\mathbf{g}}_i = \delta^i_j, \quad g_{ii} = \underline{\mathbf{g}}_i \underline{\mathbf{g}}_i = \frac{\partial X_i}{\partial \theta^i} \frac{\partial X_i}{\partial \theta^i}, \quad (24.7)$$

$$g^{ij} = \underline{\mathbf{g}}^i \underline{\mathbf{g}}^j = \frac{\partial \theta^i}{\partial X_i} \frac{\partial \theta^j}{\partial X_j}.$$

Similarly we introduce the fundamental vectors $\underline{\mathbf{G}}_1, \underline{\mathbf{G}}^1$ and the metric tensors G_{1j}, G^{1j} in the deformed body

$$\underline{\mathbf{G}}_i = \frac{\partial \mathbf{r}}{\partial \theta^i}, \quad \underline{\mathbf{G}}^i \underline{\mathbf{G}}_i = \delta^i_j, \quad G_{ii} = \underline{\mathbf{G}}_i \underline{\mathbf{G}}_i = \frac{\partial x_i}{\partial \theta^i} \frac{\partial x_i}{\partial \theta^i}, \quad (24.8)$$

$$G^{ij} = \underline{\mathbf{G}}^i \underline{\mathbf{G}}^j = \frac{\partial \theta^i}{\partial x_i} \frac{\partial \theta^j}{\partial x_j}.$$

The following relations take place

$$g^{ik} g_{kj} = \delta^i_j, \quad G^{ik} G_{kj} = \delta^i_j, \quad g^i = g^{ik} g_{kj}, \quad \underline{\mathbf{G}}^i = G^{ik} \underline{\mathbf{G}}_k. \quad (24.9)$$

We will designate

$$g = |g_{ij}|, \quad G = |G_{ij}|. \quad (24.10)$$

If we consider (2.14), (24.7), (24.8) we can write

$$dS^2 = g_{ij}d\theta^i d\theta^j, \quad ds^2 = G_{ij}d\theta^i d\theta^j. \quad (24.11)$$

In this case we have

$$ds^2 - dS^2 = 2\gamma_{ij}d\theta^i d\theta^j, \quad (24.12)$$

where

$$2\gamma_{ij} = G_{ij} - g_{ij}, \quad (24.13)$$

is the deformation tensor.

The displacement vector may be expressed in the forms

$$u = u^i g_i = u_{ij} \theta^j = U^i G_i = U_{ij} \theta^j. \quad (24.14)$$

If we designate by a comma the derivative with respect to θ^j , we can write

$$G_i = r_{,i} = R_{,i} + u_{,i} = g_i + u_{,i}, \quad (24.15)$$

so that from (24.7), (24.8), (24.13) we can conclude

$$2\gamma_{ij} = g_{ij} + g_{ji} + u_{,i} u_{,j} + u_{,j} u_{,i} = G_{ij} + G_{ji} - u_{,i} u_{,j}. \quad (24.16)$$

We will designate by a vertical line the covariant derivative with respect to θ using the metric tensors g_{ij} , g^{ij} and by two vertical lines the covariant derivative with respect to θ using the metric tensors G_{ij} , G^{ij} . We have

$$u_{,i} = u_{,i} g^j, \quad u_{,i} = U_{,i} G^j, \quad (24.17)$$

$$u_{,i} = u_{,i} - \tilde{\Gamma}_{ij}^k u_{,k}, \quad U_{,i} = U_{,i} - \tilde{\Gamma}_{ij}^k U_{,k}$$

$$g_{,i} = \tilde{\Gamma}_{ij}^k g^j = \tilde{\Gamma}_{ij}^k g_{,k}, \quad G_{,i} = \Gamma_{ij}^k G^j = \Gamma_{ij}^k G_{,k}$$

where $\tilde{\Gamma}_{1js}$, $\tilde{\Gamma}_{1j}^s$, and Γ_{1js} , Γ_{1j}^s are Christoffel's symbols corresponding to the metric tensors g_{1j} and G_{1j} , respectively. Thus, $2\tilde{\Gamma}_{irs} = g_{rs,1} + g_{s1,r} - g_{1r,s}$ etc.

The relations (24.16) become

$$2\gamma_{ij} = u_{i,j} + u_{j,i} + u^r_{,i}u_{r,j} = U_{i,j} + U_{j,i} + U^r_{,i}U_{r,j}. \quad (24.18)$$

If θ^1 coincides with the coordinate X_1 , then γ_{1j} coincides with E_{1j} ,

$$E_{ij} = \frac{\partial \theta^i}{\partial X_r} \frac{\partial \theta^j}{\partial X_s} \gamma_{rs} = \frac{1}{2} \left(\frac{\partial u_r}{\partial X_i} + \frac{\partial u_s}{\partial X_j} + \frac{\partial u_r}{\partial X_s} \frac{\partial u_s}{\partial X_i} \right). \quad (24.19)$$

Let us consider the mixed tensors

$$\gamma_j^i = g^{ir}\gamma_{rj} = \frac{1}{2} (g^{ir}G_{rj} - \delta_j^i), \quad \gamma_j^{*i} = G^{ir}\gamma_{rj} = \frac{1}{2} (\delta_j^i - G^{ir}g_{rj}). \quad (24.20)$$

We will take as direct invariants of the tensor γ_j^i the coefficients of the powers of μ from the development of the determinant

$$|2\gamma_j^i + (\mu + 1)\delta_j^i| = |g^{ir}G_{rj} + \mu\delta_j^i| = \mu^3 + I_1\mu^2 + I_2\mu + I_3. \quad (24.21)$$

We obtain

/60

$$I_1 = 3 + 2\gamma_j^j = g^{ij}G_{ij}, \quad (24.22)$$

$$I_2 = 3 + 4\gamma_j^j + 2(\gamma_j^i\gamma_i^j - \gamma_j^j\gamma_i^i),$$

$$I_3 = |\delta_j^i + 2\gamma_j^i| = |g^{ir}G_{rj}| = \frac{G}{g}.$$

Let us consider as the direct invariants of the tensor γ_j^{*i} the coefficients of the powers of μ from the development of the

determinant

$$|(\mu + 1) \delta_j^i - 2\gamma^{*ij}| = |G''g_{ij} + \mu \delta_j^i| = \mu^3 + J_1 \mu^2 + J_2 \mu + J_3. \quad (24.23)$$

It follows

$$\begin{aligned} J_1 &= 3 - 2\gamma^{*ij} = G''g_{ij}, \\ J_2 &= 3 - 4\gamma^{*ij} + 2(\gamma^{*ij}\gamma^{*kl} - \gamma^{*ik}\gamma^{*jl}), \\ J_3 &= |\delta_j^i - 2\gamma^{*ij}| = |G''g_{ij}| = \frac{g}{G}. \end{aligned} \quad (24.24)$$

In view of (24.21) and (24.23) we can establish the following relations between the invariants I_r and J_r .

$$J_1 = \frac{I_2}{I_3}, \quad J_2 = \frac{I_1}{I_3}, \quad J_3 = \frac{1}{I_3}. \quad (24.25)$$

From (24.25) we obtain

$$I_2 = I_3 J_1 = I_3 G''g_{ij}. \quad (24.26)$$

If we take into account (24.22) we can derive the following form for I_2

$$I_2 = \frac{1}{2} (I_1^2 - g''g''G_{,i}G_{,i}). \quad (24.27)$$

Let us consider a point P from \mathcal{Q} and the curves of coordinates which pass through P . Let $P_1 (1 = 1, 2, 3)$ be points situated on the curves θ^1 , respectively, so that $\overrightarrow{PP_1} \approx G_1 d\theta^1$, $\overrightarrow{PP_2} \approx G_2 d\theta^2$, $\overrightarrow{PP_3} \approx G_3 d\theta^3$.

Let us consider the curvilinear tetrahedron having angles $PP_1, P_2P_3, P_3P_1, P_1P_2$. Let \underline{n} be the versor of the normal exterior with respect to the surface $P_1P_2P_3$. Let us designate by $\frac{1}{2}dS_i$ the areas of the θ^i -constant surfaces. They can be represented vectorially in the following way

$$dS_i = \frac{G^i}{\sqrt{G^{ii}}} dS_i.$$

The area of the surface $P_1P_2P_3$ will be designated by $\frac{1}{2}dS$. In view of the /61
fact that the surface $P_1P_2P_3$ is vectorially equivalent to $\frac{1}{2} \sum_{i=1}^3 dS_i$, we have

$$ndS = \sum_{i=1}^3 \frac{G^i}{\sqrt{G^{ii}}} dS_i. \quad (24.28)$$

If we set

$$n = n_i G^i,$$

it follows from (24.28)

$$dS_i = n_i \sqrt{G^{ii}} dS. \quad (24.29)$$

Proceeding as in Paragraph 12 we obtain

$$tdS = t_i dS_i, \quad (24.30)$$

where $-t_i$ are tension vectors associated with the surfaces θ^i -constant and $\underline{t} = \underline{t}_{(n)}$. From (24.29) and (24.30) it follows

$$t = t_{ii} \sqrt{G^{ii}}, \quad (24.31)$$

this relation represents Cauchy's formula in curvilinear coordinates.

Taking (24.31) in account we can write

$$t_i \sqrt{G^{ii}} = \tau^{ij} G_j = \tau_j^i G^j, \quad (24.32)$$

where τ^{ij} is the contravariant tension tensor and τ_j^i is the mixed tension tensor. The covariant tension tensor is

$$\tau_{ij} = G_{ij} \tau_j^i.$$

From (24.31) and (24.32) we obtain

$$t = \frac{1}{\sqrt{G}} T_i n_i = \tau^{ij} n_i G_j = \tau_j n_i G^j, \quad (24.33)$$

in which

$$T_i = t_i \sqrt{G G^{ii}} = \sqrt{G} \tau^{ii} G_i = \sqrt{G} \tau_j G^j. \quad (24.34)$$

The tension vectors \underline{t}_i may be written also in the form

$$t_i = \sum_{j=1}^3 s_{ij} \frac{G_j}{\sqrt{G_{jj}}}, \quad (24.35)$$

where $G_j/\sqrt{G_{jj}}$ are versors of the \underline{G}_j vectors, and

$$s_{ij} = \sqrt{\frac{G_{jj}}{G^{ii}}} \tau^{ij}. \quad (24.36)$$

The quantities $S_{(ij)}$ are called physical components of the tension tensor. /62

If we take into account the theorem of divergence, we have

$$\int_{\partial V} t da = \int_V \frac{1}{\sqrt{G}} T_i n_i da = \int_V \frac{1}{\sqrt{G}} T_{i,i} dv, \quad (24.37)$$

so that from (13.3) it follows

$$T_{i,i} + \rho \sqrt{G} f = \rho \sqrt{G} a_i. \quad (24.38)$$

The equation (24.38) is the vectorial form of the equations of motion in **curvilinear coordinates**.

It follows from the principle of the kinetic momentum

$$\tau^{ij} = \tau^{ji}. \quad (24.39)$$

If we consider (24.34) and (24.38) the equations of motion

may also be written in the following way.

$$\tau''_{ij} + \rho f'_i = \rho a'_i, \quad (24.40)$$

where

$$t = f'_i G_i = f'_0 g_0, \quad a = a'_i G_i = a'_0 g_0. \quad (24.41)$$

It should be noted that the tension which acts on the surface element S_1 is $\underline{T}_1 d\theta^1 d\theta^2$. This follows easily on the basis of the fact that $dS_1 = \sqrt{g g''} d\theta^1 d\theta^2$. Let us introduce the tension vectors \underline{t}_1^0 by

$$\underline{T}_1 d\theta^1 d\theta^2 = t_1^0 \sqrt{g g''} d\theta^1 d\theta^2.$$

The vector \underline{t}_1^0 acts on the surface $\theta^1 = \text{constant}$ from \mathcal{B} and is measured on the unit area of the corresponding surface from the nondeformed body.

We will write

$$t_1^0 \sqrt{g g''} = T_1 = \sqrt{g} s'' G_1. \quad (24.42)$$

From (24.22), (24.34) and (24.42) it follows

$$s'' = \tau'' / I_3. \quad (24.43)$$

If S is the oriented surface from B , with a normal unit exterior \underline{N} , which by deformation becomes the surface \mathcal{S} from \mathcal{B} with a normal unit exterior \underline{n} , then the tension vector \underline{t}^0 which acts on \mathcal{S} and is measured on the unit surface of S , is given by

$$\underline{t}^0 = N_i s'' G_i = N_i s'_j G'_j = \frac{N_i T_i}{\sqrt{g}}, \quad (24.44)$$

where

$$N = N_i g^i = N^i g_i. \quad (24.45) \quad \text{163}$$

Obviously we have

$$G_i = g_i + u_{,i} = (\delta_i^j + u^j{}_{,i}) g_j. \quad (24.46)$$

The relations (24.34) and (24.42) may also be written in the following form

$$T_i = \sqrt{G} \pi^i{}_j g^j = \sqrt{g} \sigma^i{}_j g^j, \quad (24.47)$$

where

$$\pi^i{}_j = \tau^i{}_r (\delta_r^j + u^j{}_{,r}), \quad \sigma^i{}_j = s^i{}_r (\delta_r^j + u^j{}_{,r}). \quad (24.48)$$

The quantities π^{ij} and σ^{ij} are not symmetrical. From (24.34) and (24.42) we obtain

$$\pi^i{}_j g_j = \tau^i{}_r G_r, \quad \sigma^i{}_j g_j = s^i{}_r G_r, \quad (24.49)$$

and therefore the following relations take place

$$\pi^i{}_j g_j G^j = \pi^i{}_r g_j G^j, \quad \sigma^i{}_j g_j G^j = \sigma^i{}_r g_j G^j. \quad (24.50)$$

From (24.38), (24.41) and (24.47) another form of the equations of motion is derived

$$\sigma^i{}_j{}_{,i} + \rho_0 f^j_0 = \rho_0 a^j_0. \quad (24.51)$$

In the same way as in the derivation of relation (24.31), we obtain from (15.8)

$$h_{(a)} = h_i n_i \sqrt{G}^i, \quad (24.52)$$

where $-h_1$ is the flux through the surface $\theta^1 = \text{constant}$.

Let us designate

$$q^i = h_i \sqrt{g^{ii}}. \quad (24.53)$$

The heat flux vector is

$$q_i = q^i g_i = q_i g^i, \quad (24.54)$$

and thus

$$h_{(n)} = q^i n_i. \quad (24.55)$$

Repeating the procedure used in the study of the tension, the heat flux $H_{(N)}$ per unit area of the nondeformed body is introduced and we have

$$H_{(N)} = N_i h_i^0 \sqrt{g^{ii}}, \quad (24.56)$$

where the meaning of the new magnitudes is obvious. The heat flux vector per unit surface of the nondeformed body, \underline{Q} , is

$$Q = Q^i g_i = Q_i g^i, \quad (24.57)$$

where

$$Q^i = h_i^0 \sqrt{g^{ii}}. \quad (24.58) \quad \frac{164}{}$$

Therefore, $h_i^0 \sqrt{g^{ii}} = h_i \sqrt{G^{ii}}$ yields

$$Q^i = \sqrt{I_3} q^i. \quad (24.59)$$

Similarly we derive

$$Q^i_{||} = \sqrt{I_3} q^i_{||}. \quad (24.60)$$

The surface integrals from (15.7) are thus transformed

$$\begin{aligned} \int_{\partial \mathcal{V}} t v da &= \int_{\partial \mathcal{V}} \frac{\mathbf{T}_i v n_i}{\sqrt{G}} da = \int_{\mathcal{V}} \frac{1}{\sqrt{G}} (\mathbf{T}_i v_{,i} + \mathbf{T}_i v_{,i}) dv, \\ \int_{\partial \mathcal{V}} h_{(n)} da &= \int_{\partial \mathcal{V}} q^i n_i da = \int_{\mathcal{V}} q^i{}_{;i} dv. \end{aligned} \quad (24.61)$$

If we consider the relations (24.38), (24.39), (24.61) and also the fact that

$$\dot{\gamma}_{ij} = \frac{1}{2} \dot{G}_{ij} = \frac{1}{2} (\dot{\mathbf{G}}_i \mathbf{G}_j + \mathbf{G}_i \dot{\mathbf{G}}_j) = \frac{1}{2} (\mathbf{G}_i v_{,j} + \mathbf{G}_j v_{,i}),$$

we obtain from (15.7) the following form of the energy equation

$$\rho \dot{e} = s^{ij} \dot{\gamma}_{ij} + q^i{}_{;i} + \rho r. \quad (24.62)$$

The relation (24.62) may be written, also, in the following way

$$\rho_0 \dot{e} = s^{ij} \dot{\gamma}_{ij} + Q^i{}_{;i} + \rho_0 r, \quad (24.63)$$

on the basis of relations (24.43), (24.60).

The Clausius-Duhem inequality (18.3) is written as

$$\rho_0 T \dot{\eta} - \rho_0 r - Q^i{}_{;i} + \frac{1}{T} Q^i T_{,i} \geq 0. \quad (24.64)$$

From (18.4), (24.63) and (24.64) we obtain

$$-\rho_0 (\eta \dot{T} + \dot{\psi}) + s^{ij} \dot{\gamma}_{ij} + \frac{1}{T} Q^i T_{,i} \geq 0. \quad (24.65)$$

If we consider (20.14), the constitutive equations take the following form

$$s^{ij} = \frac{1}{2} \left(\frac{\partial \tilde{\psi}}{\partial \gamma_{ij}} + \frac{\partial \tilde{\psi}}{\partial \gamma_{ji}} \right), \quad \eta = - \frac{1}{\rho_0} \frac{\partial \tilde{\psi}}{\partial T}. \quad (24.66)$$

In view of relations (24.43) and (24.48) it follows that

/65

$$\tau'' = \frac{1}{2\sqrt{I_3}} \left(\frac{\partial \tilde{\sigma}}{\partial \gamma_{ij}} + \frac{\partial \tilde{\sigma}}{\partial \gamma_{ji}} \right), \quad (24.67)$$

$$\sigma'' = \frac{\partial \hat{\sigma}}{\partial u_{j|i}} = \pi'' \sqrt{I_3}.$$

The energy equation (21.3) is written

$$\rho_0 T \dot{\eta} - Q'_{|i} = \rho_0 r. \quad (24.68)$$

In the case of homogeneous and isotropic media we have

$$\tilde{\sigma} = \tilde{\sigma}(I_1, I_2, I_3, T), \quad (24.69)$$

where I_r are given by (24.22).

From (24.13), (24.10), (24.22) and (24.27) it follows that

$$\frac{\partial I_1}{\partial \gamma_{ij}} = 2g^{ij}, \quad \frac{\partial I_2}{\partial \gamma_{ij}} = 2(g^{ij}g^{rs} - g^{ir}g^{js})G_{rs}, \quad \frac{\partial I_3}{\partial \gamma_{ij}} = 2I_3 G^{ij}. \quad (24.70)$$

In view of relations (24.67), (24.69) and (24.70) we obtain

$$\tau^{ij} = \Phi g^{ij} + \Psi B^{ij} + p G^{ij}, \quad (24.71)$$

where

$$\begin{aligned} \Phi &= \frac{2}{\sqrt{I_3}} \frac{\partial \tilde{\sigma}}{\partial I_1}, \quad \Psi = \frac{2}{\sqrt{I_3}} \frac{\partial \tilde{\sigma}}{\partial I_2}, \quad p = 2\sqrt{I_3} \frac{\partial \tilde{\sigma}}{\partial I_3}, \\ B^{ij} &= I_1 g^{ij} - g^{ir}g^{js}G_{rs} = \frac{1}{g} e^{irm}e^{jsn}g_{rs}G_{mn}. \end{aligned} \quad (24.72)$$

In the case of incompressible media, p is an unknown function.

The constitutive relations for the heat flux component (22.2) become

(24.73)

$$Q^i = (\mathcal{C}_1 \delta_i^i + \mathcal{C}_2 \gamma_i^i + \mathcal{C}_3 \gamma_m^i \gamma_m^i) T^{||},$$

where

$$T^{||} = g^{rr} T_{||r} = g^{rr} \frac{\partial T}{\partial \theta^r},$$

and \mathcal{C}_i are functions of the invariants I_r , $I_4 = T_{||1} T^{||1}$, $I_5 = T_{||1} T_{||j} \gamma_1^j$, $I_6 = T_{||1} T_{||j} \gamma_s^j \gamma_1^s$, T . Similarly the following representation may be established [141].

$$Q^i = (\mathcal{C}'_1 \delta_i^i + \mathcal{C}'_2 \gamma_i^i + \mathcal{C}'_3 \gamma_m^i \gamma_m^i) T^{||}, \quad (24.74)$$

where

$$T^{||} = G^{rr} T_{||r} = G^{rr} \frac{\partial T}{\partial \theta^r},$$

/66

and \mathcal{C}'_i are functions of the invariant I_r , $T_{||}$, $T^{||}$, $T^{||} T_{||r} \gamma_1^r$, $T^{||} T_{||r} \gamma_s^r \gamma_1^s$, T .

In the linear theory, the constitutive equations in curved coordinates are

$$\tau^{ij} = C^{ijkl} \gamma_{kl} - b^{ij} \theta, \quad \rho_0 \eta = b^{ij} \gamma_{ij} + a \theta, \quad q^i = K^{ij} \theta_{,j}. \quad (24.75)$$

In the case of isotropic media, these equations become

$$\tau^{ij} = \lambda g^{ij} \gamma_r^r + 2\mu \gamma^{ij} - \beta \theta g^{ij}, \quad \rho_0 \eta = \beta \gamma_r^r + \alpha \theta, \quad q^i = k g^{ij} \theta_{,j}. \quad (24.76)$$

It is obvious that in (24.75) and (24.76) γ_{ij} is expressed as follows with the help of the components of the displacement vector

$$2\gamma_{ij} = u_{i,j} + u_{j,i}, \quad (24.77)$$

and $\gamma_j^i = g^{ik}\gamma_{kj}$, $\gamma^{ij} = g^{ik}g^{jl}\gamma_{kl}$.

In view of the fact that in the linear theory the tensors τ^{ij} , π^{ij} , σ^{ij} , s^{ij} coincide, the equations of motion may be written in the form

$$\tau^{ij}_{,j} + \rho_0 f^i = \rho_0 a^i. \quad (24.78)$$

In view of the fact that in this case $Q_1 = q^1$, the equation of energy becomes

$$\rho_0 T_0 \dot{\eta} - q^i_{,i} = \rho_0 r. \quad (24.79)$$

In general, the physical components of the vectors and tensors which appear in the above equations are used to study concrete problems.

For illustration, let us consider the equations of the linear theory of thermoelasticity for homogeneous and isotropic media in cylindrical coordinates. Let the cylindrical coordinates (r, ϕ, z) exist such that $X_1 = r \cos \phi$, $X_2 = r \sin \phi$, $X_3 = z$. In order to avoid any confusion, let us agree that in this case the flow rate of the heat sources per unit mass is indicated by r' . From (24.7), (24.9), (24.17) it follows that

$$\begin{aligned} g_{11} = g_{33} = 1, \quad g_{22} = r^2, \quad g^{11} = g^{33} = 1, \quad g^{22} = \frac{1}{r^2}, \quad g_{ij} = g^{ij} = 0 (i \neq j), \\ g = r^2, \quad g^1 = g_1, \quad g^3 = g_3, \quad g^2 = \frac{1}{r^2} g_2, \quad \tilde{\Gamma}_{122} = \tilde{\Gamma}_{212} = r, \quad \tilde{\Gamma}_{221} = -r, \\ \tilde{\Gamma}_{12}^1 = -r, \quad \tilde{\Gamma}_{12}^2 = \tilde{\Gamma}_{21}^2 = \frac{1}{r}, \end{aligned} \quad (24.80)$$

while the other symbols of Christoffel are zero. Let us designate

$$e_r = g_1, e_\varphi = \frac{1}{r} g_2, e_z = g_3, \quad (24.81)$$

$$u = u^i g_i = u_r g^r + u_\varphi g^\varphi + u_z g^z.$$

In (24.81), u_r , u_φ , u_z are the physical components of the displacement 67 vector. It follows that

$$u^1 = u_1 = u_r, u^2 = \frac{1}{r^2} u_2 = \frac{1}{r} u_\varphi, u^3 = u_3 = u_z. \quad (24.82)$$

The contravariant component of the acceleration vector may also be written

$$a^1 = \ddot{u}_r, a^2 = \frac{1}{r} \ddot{u}_\varphi, a^3 = \ddot{u}_z. \quad (24.83)$$

Taking into account (24.17), (24.77) and (24.80) we conclude

$$\begin{aligned} \gamma_{11} &= \frac{\partial u_1}{\partial r} = \frac{\partial u_r}{\partial r}, \quad \gamma_{22} = \frac{\partial u_2}{\partial \varphi} + r u_1 = r \frac{\partial u_\varphi}{\partial \varphi} + r u_r, \\ \gamma_{33} &= \frac{\partial u_3}{\partial z} = \frac{\partial u_z}{\partial z}, \quad \gamma_{12} = \frac{1}{2} \left(\frac{\partial u_1}{\partial \varphi} + \frac{\partial u_2}{\partial r} - \frac{2}{r} u_2 \right) = \\ &= \frac{1}{2} \left(\frac{\partial u_r}{\partial \varphi} + r \frac{\partial u_\varphi}{\partial r} - u_z \right), \quad \gamma_{23} = \frac{1}{2} \left(r \frac{\partial u_\varphi}{\partial z} + \frac{\partial u_z}{\partial \varphi} \right), \\ \gamma_{13} &= \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right). \end{aligned} \quad (24.84)$$

The physical components of the tensor γ_{ij} are expressed as

$$\varepsilon_{rr} = \gamma_{11}, \quad \varepsilon_{\varphi\varphi} = \frac{1}{r^2} \gamma_{22}, \quad \varepsilon_{zz} = \gamma_{33}, \quad (24.85)$$

$$\varepsilon_{r\varphi} = \frac{1}{r} \gamma_{12}, \quad \varepsilon_{\varphi z} = \frac{1}{r} \gamma_{23}, \quad \varepsilon_{rz} = \gamma_{13},$$

and, therefore, we obtain the following relations between the physical components of the deformation tensor and the physical components of the displacement vector

$$\begin{aligned} \epsilon_{rr} &= \frac{\partial u_r}{\partial r}, \quad \epsilon_{\varphi\varphi} = \frac{1}{r} \left(\frac{\partial u_\varphi}{\partial \varphi} + u_r \right), \quad \epsilon_{zz} = \frac{\partial u_z}{\partial z}, \\ \epsilon_{r\varphi} &= \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{1}{r} u_\varphi \right), \quad \epsilon_{rz} = \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \varphi} \right), \\ \epsilon_{\varphi z} &= \frac{1}{2} \left(\frac{\partial u_\varphi}{\partial z} + \frac{\partial u_z}{\partial r} \right). \end{aligned} \quad (24.86)$$

If we consider (24.80) it follows that the expressions

68

$$E^i = \tau^{ij}_{|j} = \tau^{ij}_{,j} + \Gamma^i_{jj} \tau^{jj} + \Gamma^j_{jj} \tau^{ii},$$

become

$$\begin{aligned} E^1 &= \frac{\partial \tau^{11}}{\partial r} + \frac{\partial \tau^{12}}{\partial \varphi} + \frac{\partial \tau^{13}}{\partial z} + \frac{1}{r} (\tau^{11} - r^2 \tau^{22}), \\ E^2 &= \frac{\partial \tau^{21}}{\partial r} + \frac{\partial \tau^{22}}{\partial \varphi} + \frac{\partial \tau^{23}}{\partial z} + \frac{3}{r} \tau^{12}, \\ E^3 &= \frac{\partial \tau^{31}}{\partial r} + \frac{\partial \tau^{32}}{\partial \varphi} + \frac{\partial \tau^{33}}{\partial z} + \frac{1}{r} \tau^{13}. \end{aligned}$$

The physical components of the tension tensor τ^{ij} are

$$\begin{aligned} \sigma_{rr} &= \tau^{11}, \quad \sigma_{\varphi\varphi} = r^2 \tau^{22}, \quad \sigma_{zz} = \tau^{33}, \\ \sigma_{r\varphi} &= r \tau^{12}, \quad \sigma_{\varphi z} = r \tau^{23}, \quad \sigma_{rz} = \tau^{13}, \end{aligned} \quad (24.87)$$

and thus we are able to write

$$\begin{aligned} E^1 &= \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\varphi}}{\partial \varphi} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r} (\sigma_{rr} - \sigma_{\varphi\varphi}), \\ E^2 &= \frac{1}{r} \left(\frac{\partial \sigma_{r\varphi}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\varphi\varphi}}{\partial \varphi} + \frac{\partial \sigma_{\varphi z}}{\partial z} + \frac{2}{r} \sigma_{r\varphi} \right), \\ E^3 &= \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\varphi z}}{\partial \varphi} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r} \sigma_{rz}. \end{aligned} \quad (24.88)$$

If we indicate by f_r, f_ϕ, f_z the physical components of the mass forces, then

$$f_\theta = f_r, f_\phi^2 = \frac{1}{r} f_\phi, f_z^2 = f_z,$$

and from (24.78), (24.83), (24.88) the following form of the equations of motion follows

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\phi}}{\partial \phi} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r} (\sigma_{rr} - \sigma_{\phi\phi}) + \rho_0 f_r = \rho_0 \frac{\partial^2 u_r}{\partial t^2}, \quad (24.89)$$

$$\frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{\partial \sigma_{\phi z}}{\partial z} + \frac{2}{r} \sigma_{r\phi} + \rho_0 f_\phi = \rho_0 \frac{\partial^2 u_\phi}{\partial t^2},$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\phi}}{\partial \phi} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r} \sigma_{rr} + \rho_0 f_z = \rho_0 \frac{\partial^2 u_z}{\partial t^2}.$$

In view of (24.76), (24.77), (24.80) and (24.82) the following 69 relations result

$$\mathbf{q} = q'_i \mathbf{e}_i = k \theta, \mathbf{q}' = k \text{grad } \theta = k \left(\frac{\partial \theta}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial \theta}{\partial \phi} \mathbf{e}_\phi + \frac{\partial \theta}{\partial z} \mathbf{e}_z \right), \quad (24.90)$$

$$\gamma'_i = u'^i_{,i} = \text{div } \mathbf{u} = \frac{1}{\sqrt{g}} (u^i \sqrt{g})_{,i} = \frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{1}{r} \frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z},$$

$$q'^i_{,i} = \text{div } \mathbf{q} = k \text{div grad } \theta = k \Delta \theta = k \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \theta}{\partial \phi^2} + \frac{\partial^2 \theta}{\partial z^2} \right],$$

which permit to write immediately the other equations of the linear theory of thermoelasticity of homogeneous and isotropic media.

Let us consider now the spherical coordinates (R, ϕ, ψ) in such a way that $X_1 = R \sin \phi \cos \psi$, $X_2 = R \sin \phi \sin \psi$, $X_3 = R \cos \phi$.

In this case we obtain the following relations between the deformations and displacements

$$\begin{aligned}
\epsilon_{RR} &= \frac{\partial u_R}{\partial R}, \quad \epsilon_{\varphi\varphi} = \frac{1}{R} \frac{\partial u_\varphi}{\partial \varphi} + \frac{1}{R} u_R, \\
\epsilon_{\psi\psi} &= \frac{1}{R \sin \varphi} \frac{\partial u_\psi}{\partial \psi} + \frac{1}{R} u_R + \frac{\operatorname{ctg} \varphi}{R} u_\varphi, \\
\epsilon_{\varphi\psi} &= \frac{1}{2} \left(\frac{1}{R} \frac{\partial u_\psi}{\partial \varphi} - \frac{\operatorname{ctg} \varphi}{R} u_\psi + \frac{1}{R \sin \varphi} \frac{\partial u_\varphi}{\partial \psi} \right), \\
\epsilon_{\psi R} &= \frac{1}{2} \left(\frac{1}{R \sin \varphi} \frac{\partial u_R}{\partial \psi} - \frac{1}{R} u_\psi + \frac{\partial u_\psi}{\partial R} \right), \\
\epsilon_{\varphi R} &= \frac{1}{2} \left(\frac{1}{R} \frac{\partial u_R}{\partial \varphi} - \frac{1}{R} u_\varphi + \frac{\partial u_\varphi}{\partial R} \right).
\end{aligned} \tag{24.91}$$

The meaning of the designations used is obvious. The equations of motion may be written in the form

$$\begin{aligned}
& \frac{\partial \sigma_{RR}}{\partial R} + \frac{1}{R \sin \varphi} \frac{\partial \sigma_{R\psi}}{\partial \psi} + \frac{1}{R} \frac{\partial \sigma_{R\varphi}}{\partial \varphi} + \\
& + \frac{1}{R} (2\sigma_{RR} - \sigma_{\varphi\varphi} - \sigma_{\psi\psi} + \sigma_{R\varphi} \operatorname{ctg} \varphi) + \rho_0 f_R = \rho_0 \frac{\partial^2 u_R}{\partial t^2}, \\
& \frac{\partial \sigma_{R\varphi}}{\partial R} + \frac{1}{R \sin \varphi} \frac{\partial \sigma_{\varphi\psi}}{\partial \psi} + \frac{1}{R} \frac{\partial \sigma_{\varphi\varphi}}{\partial \varphi} + \\
& + \frac{1}{R} [3\sigma_{R\varphi} + (\sigma_{\varphi\varphi} - \sigma_{\psi\psi}) \operatorname{ctg} \varphi] + \rho_0 f_\varphi = \rho_0 \frac{\partial^2 u_\varphi}{\partial t^2}, \\
& \frac{\partial \sigma_{R\psi}}{\partial R} + \frac{1}{R \sin \varphi} \frac{\partial \sigma_{\psi\psi}}{\partial \psi} + \frac{1}{R} \frac{\partial \sigma_{\varphi\psi}}{\partial \varphi} + \frac{1}{R} (3\sigma_{R\psi} + 2\sigma_{\varphi\psi} \operatorname{ctg} \varphi) + \rho_0 f_\psi = \rho_0 \frac{\partial^2 u_\psi}{\partial t^2}.
\end{aligned} \tag{24.92}$$

Similarly, we have

70

$$\mathbf{q} = k \operatorname{grad} \theta = k \left(\frac{\partial \theta}{\partial R} \mathbf{e}_R + \frac{1}{R} \frac{\partial \theta}{\partial \varphi} \mathbf{e}_\varphi + \frac{1}{R \sin \varphi} \frac{\partial \theta}{\partial \psi} \mathbf{e}_\psi \right), \quad (24.93)$$

$$\operatorname{div} \mathbf{u} = \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 u_R) + \frac{1}{R \sin \varphi} \frac{\partial}{\partial \varphi} (\sin \varphi u_\varphi) + \frac{1}{R \sin \varphi} \frac{\partial u_\psi}{\partial \psi},$$

$$\begin{aligned} \Delta \theta = \operatorname{div} \operatorname{grad} \theta &= \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial \theta}{\partial R} \right) + \\ &+ \frac{1}{R^2 \sin \varphi} \frac{\partial}{\partial \varphi} \left(\sin \varphi \frac{\partial \theta}{\partial \varphi} \right) + \frac{1}{R^2 \sin^2 \varphi} \frac{\partial^2 \theta}{\partial \psi^2}. \end{aligned}$$

c-2

25. Fundamental Equations. Statement of the Problems

This chapter is devoted to the dynamic theory of linear thermoelasticity. The fundamental equations of this theory are

- the equations of motion

$$t_{ij,j} + f_i = \rho \ddot{u}_i, \quad (25.1)$$

- the equations of energy

$$\rho T_0 \dot{\eta} - q_{i,i} = r, \quad (25.2)$$

- the constitutive equations

$$\begin{aligned} t_{ij} &= C_{ijkl} \epsilon_{kl} - \beta_{ij} \theta, \\ \rho \eta &= \beta_{ij} \epsilon_{ij} + a \theta, \\ q_i &= k_{ij} \theta_{,j}, \end{aligned} \quad (25.3)$$

- the geometric equations

$$2\epsilon_{ij} = u_{i,j} + u_{j,i}. \quad (25.4)$$

In the above relations in order to differentiate them from the notation from Paragraph 23, f_i represents the components of the specific volume forces, ρ is the density of the medium in the non-deformed state and r represents the flow rate of the heat sources per unit initial volume.

The coefficients C_{ijkl} and β_{ij} have the following symmetry properties

$$C_{ijmn} = C_{nmij}, \quad \beta_{ij} = \beta_{ji}. \quad (25.5)$$

Similarly the following relation takes place

$$k_i, 0, 0, \geq 0. \quad (25.6)$$

To the equations (25.1) -- (25.4) the initial conditions

$$u_i(x, 0) = a_i(x), \dot{u}_i(x, 0) = b_i(x), \eta(x, 0) = \eta_0(x), x \in \bar{B} \quad (25.7)$$

are added with the boundary conditions which, in case of a mixed problem, are

$$\begin{aligned} u_i &= \tilde{u}_i \text{ on } \Sigma_1 \times [0, t_0], t_i = t_i, n_i = \tilde{t}_i \text{ on } \Sigma_2 \times [0, t_0], \\ 0 &= \tilde{\theta} \text{ on } \Sigma_3 \times [0, t_0], q = q_i, n_i = \tilde{q} \text{ on } \Sigma_4 \times [0, t_0]. \end{aligned} \quad (25.8)$$

In the preceding relation, $a_i, b_i, \eta_0, \tilde{u}_i, \tilde{t}_i, \tilde{\theta}, \tilde{q}$ are prescribed functions, and Σ_r ($r = 1, 2, 3, 4$) are parts of ∂B , with $\Sigma_1 \cup \Sigma_2 = \Sigma_3 \cup \Sigma_4 = \partial B, \Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \emptyset$.

It is assumed that the functions $C_{ijkl}, \beta_{ij}, k_{ij}, \rho, a$, which characterize the properties of the medium are given. We will assume that $C_{ijkl}, \beta_{ij}, k_{ij}$, are of the C^1 on \bar{B} class and that they satisfy the relations (25.5), (25.6), and ρ and a are continuous on \bar{B} , ρ being strictly positive.

The preceding considerations refer to the thermoelastic properties of the medium. As far as the other prescribed functions (loads) are concerned we will assume that

- (α) f_i and r are continuous on $\bar{B} \times [0, t_0]$,
- (β) a_i, b_i, η_0 are continuous on \bar{B} ,
- (γ) \tilde{u}_i are continuous on $\Sigma_1 \times [0, t_0]$,
- (δ) \tilde{t}_i are regulated on sections on $\Sigma_2 \times [0, t_0]$ and are continuous with respect to t ,
- (ϵ) $\tilde{\theta}$ is continuous on $\Sigma_3 \times [0, t_0]$,
- (ζ) \tilde{q} is regulated on sections on $\Sigma_4 \times [0, t_0]$ and is continuous

with respect to t .

The expression admissible process is intended to mean an ordered set of functions $\Pi = \{u_i, \epsilon_{ij}, t_{ij}, \theta, \eta, q_i\}$ with the following properties (Carlson [46])

- (i) u_i are of the class of C^2 on $B \times [0, t_0)$,
- (ii) $u_i, \dot{u}_i, \ddot{u}_i, u_{i,j}, \dot{u}_{i,j}$ are continuous on $\bar{B} \times [0, t_0)$,
- (iii) ϵ_{ij} are components of a symmetric continuous tensor on $\bar{B} \times [0, t_0)$,
- (iv) t_{ij} are components of a symmetric tensor of the class $C^{1,0}$ on $B \times (0, t_0)$,
- (v) t_{ij} and $t_{ij,j}$ are continuous on $\bar{B} \times [0, t_0)$,
- (vi) θ is of the class $C^{2,1}$ on $B \times (0, t_0)$,
- (vii) $\theta, \theta_{,i}, \dot{\theta}$ are continuous on $\bar{B} \times [0, t_0)$,
- (viii) η is of the class $C^{0,1}$ on $B \times (0, t_0)$,
- (ix) $\eta, \dot{\eta}$ are continuous on $\bar{B} \times [0, t_0)$,
- (x) q_i are of the class $C^{1,0}$ on $B \times (0, t_0)$,
- (xi) q_i and $q_{j,j}$ are continuous on $\bar{B} \times [0, t_0)$.

If we define the addition of the admissible processes and the multiplication of a process with a scalar magnitude respectively by

$$\Pi + \Pi' = \{u_i + u'_i, \dots, q_i + q'_i\}, \quad \lambda \Pi = \{\lambda u_i, \dots, \lambda q_i\},$$

then the set of the admissible processes is a linear space.

We call the solution of a mixed problem an admissible process which satisfies the equations (25.1) -- (25.4) and conditions (25.7) and (25.8).

We call an admissible field of displacement temperature the ordered set of functions $U = (u_1, u_2, u_3, \theta)$ in which u_i satisfy the conditions (i) and (ii), and θ satisfies conditions (vi) and (vii). With the previous definitions the set of the admissible displacement temperature field is a linear space.

Obviously, the boundary conditions (25.8) suggest also other types of problems depending whether one or several of the parts of Σ_j are empty. Similarly, other kinds of boundary conditions may be considered, such as those corresponding to certain special problems (see, for example, Nowacki [315], Boley and Weiner [23]).

26. Characterization of a Mixed Problem with the Help of Displacement and Temperature

The problem of thermoelastic deformation of a continuous medium involves the determination of the components of the displacement vector and of temperature. The equations (25.1) -- (25.4) and the conditions (25.7), (25.8) may be formulated with the help of the unknowns $u_i(x, t)$ and $\theta(x, t)$.

Thus, in view of relations (25.4), (25.5) we can write the constitutive equations in the following form

$$t_{ij} = C_{ijkl} u_{k,l} - \beta_{ij} \theta, \quad (26.1)$$

$$\rho \eta = \beta_{ij} u_{i,j} + a \theta,$$

$$q_i = k_{ij} \theta_{,j}.$$

Substituting the previous expressions of functions t_{ij} , η and q_i in the equations (25.1), (25.2) we obtain the equations

$$(C_{ijkl} u_{k,l})_{,j} - (\beta_{ij} \theta)_{,j} - \rho \dot{u}_i = -f_i, \quad (26.2)$$

$$(k_{ij} \theta_{,j})_{,i} - T_0 \beta_{ij} \dot{u}_{i,j} - c \dot{\theta} = -r, \quad (26.3)$$

where in the place of T_0 we set the specific heat c , on the basis of relation (23.14).

The initial conditions may be written in the following form

$$u_i(x, 0) = a_i, \dot{u}_i(x, 0) = b_i, \theta(x, 0) = \theta_0, x \in \bar{B}, \quad (26.4)$$

where

$$\alpha \theta_0 = \rho \tau_0 - \beta_{ij} a_{i,j}.$$

In view of (26.1), the boundary conditions (25.8) become

$$\begin{aligned} u_i &= \bar{u}_i \text{ on } \Sigma_1 \times [0, t_0], (C_{ijkl} u_{k,l} - \beta_{ij} \theta) n_j = \bar{t}_i \text{ on } \Sigma_2 \times [0, t_0], \\ \theta &= \bar{\theta} \text{ on } \Sigma_3 \times [0, t_0], k_{ij} \theta_{,j} n_i = \bar{q} \text{ on } \Sigma_4 \times [0, t_0]. \end{aligned} \quad (26.5)$$

Thus, the mixed problem of thermoelasticity is characterized with the help of the displacement and temperature.

When the equations of thermoelasticity are expressed with the help of the components u_i of the displacement vector and the temperature change θ , we shall call a solution of the mixed problem an admissible field of temperature displacement which satisfies the equations (26.2), (26.3) and the conditions (26.4), (26.5). /74

The connection between this definition and that given in Paragraph 25 is obvious.

In the case of homogeneous media, the equations (26.2), (26.3) are written as

$$\begin{aligned} C_{ijkl} u_{k,l} - \beta_{ij} \theta_{,j} - \rho \ddot{u}_i &= -f_i, \\ k_{ij} \theta_{,j} - T_0 \beta_{ij} \dot{u}_{i,j} - c \dot{\theta} &= -r. \end{aligned} \quad (26.6)$$

If the continuous medium under consideration is homogeneous and isotropic, it takes the following form in view of (23.22),

equations (26.2), (26.3)

$$\mu u_{i,jj} + (\lambda + \mu) u_{r,r} - \beta \theta_{,i} - \rho \ddot{u}_i = -f_i \quad (26.7)$$

$$k \theta_{,ii} - T_0 \beta \dot{u}_{r,r} - c \dot{\theta} = -r.$$

In the study of certain materials it was noted that the second term of equation (26.3) may be neglected, as the results thus obtained agree with the corresponding experiment. In this case the equation (26.3) is written as

$$(k_{ij} \theta_{,j})_{,i} - c \dot{\theta} = -r, \quad (26.8)$$

and the problem is considerably simplified. Indeed, the equation (26.8), the initial condition for θ and the boundary conditions corresponding to the function θ represent a problem which can be separately studied. After determining the function θ , the problem is reduced to the study of the equations (26.2) with the initial conditions and with the boundary conditions corresponding to the u_i functions. This problem is a problem of elasticity in which the components of the specific volume force are

$$f_i = (\beta_{ij} \theta_{,j})_{,i}$$

and the tension prescribed on the surface Σ_2 is the components

$$\tilde{l}_i + \beta_{ij} \theta n_j.$$

The theory of thermoelasticity in which, instead of equation (26.3), the equation (26.8) is considered, is called the uncoupled theory of thermoelasticity. The designation of coupled thermoelasticity refers to the general theory described by equations (26.2), (26.3), equations which cannot be studied separately.

The equations of the coupled thermoelasticity describe the interaction between the thermal field and deformation. In the uncoupled theory the function θ is zero if $r, \theta_0, \tilde{\theta}$ and \tilde{q} are zero. In the case of

the coupled thermoelasticity this does not occur: there is a temperature variation due to mechanical deformation. This temperature variation causes in turn a deformation of the medium.

The coupling effect will be shown with the help of examples given in Paragraphs 34, 35, and 37.

In certain phenomena the inertial terms are negligible and the equation (25.1) is replaced by

75

$$l_{0,0} : f_0 = 0. \quad (26.9)$$

In this case we obtain the quasistatistical theory of thermoelasticity. Thus the fundamental equations of this theory are (26.9), (25.2) -- (25.4). In the following we will consider neither the uncoupled nor the quasistatistical theory. They have been studied in various treatises (for example by Parkus [334], Boley and Weiner [23], Nowacki [315]).

27. Another Formulation of the Problem

In this paragraph we will present another formulation of the problem of coupled thermoelasticity, in which the initial conditions are incorporated in the fundamental equations. The importance of this formulation will become obvious in the following paragraphs. It was given by Iesan [186] and it represents the generalization of a result from the theory of elastodynamics established by Ignaczak [207].

Let u, v be definite functions of $\bar{E} \times [0, t_0)$, which are continuous with respect to time on $[0, t_0)$, for any $x \in B$.

We will designate by $u * v$ the product of convolution of functions u and v

$$u * v := \int_0^t u(x, t - \tau) v(x, \tau) d\tau. \quad (27.1)$$

In the following we will use the following properties of the product of convolution

$$u * v = v * u, \quad u * (v * w) = (u * v) * w = u * v * w, \quad (27.2)$$

$$u * (v + w) = u * v + u * w,$$

$$u * v = 0 \text{ implies } u = 0 \text{ or } v = 0$$

Theorem 27.1. The functions u_1, t_{1j}, η satisfy the equations (25.1), (25.2) and the initial conditions (25.7) if and only if

$$g * u_{1i} + F_i = \rho u_{1i}, \quad (27.3)$$

$$\rho r_i = -\frac{1}{T_0} l * q_{1,i} + W, \quad \text{in } B \times [0, t_0), \quad (27.4)$$

where

$$g(t) \equiv t, \quad l(t) \equiv 1, \quad (27.5)$$

$$F_i = g * f_i + \rho(t b_i + a_i), \quad W = -\frac{1}{T_0} l * r_i + \rho \eta_0. \quad (27.6)$$

Proof. First of all let us observe the fact that

176

$$g * \ddot{u}_i = \int_0^t (t - \tau) \ddot{u}_i(x, \tau) d\tau = u_i(x, t) - t \dot{u}_i(x, 0) - u_i(x, 0), \quad (27.7)$$

$$l * \dot{\eta} = \int_0^t \dot{\eta}(x, \tau) d\tau = \eta(x, t) - \eta(x, 0).$$

If the functions u_1, t_{1j}, η satisfy the equations (25.1), (25.2) and the conditions (25.7), we can write

$$g * (t_{1j} + f_1) = \rho g * \ddot{u}_1 = \rho(u_1 - tb_1 - a_1),$$

$$\frac{1}{T_0} l * q_{1,t} + \frac{1}{T_0} l * r = \rho(\dot{\eta}_1 - \eta_0),$$

from which result the equations (27.3), (27.4) in which the notations of (27.6) were kept in mind.

Inversely, if u_1 , t_{1j} , η satisfy the equations (27.3) and (27.4) then, taking into account (27.6), (27.7), we can write

$$g * t_{1j}'' + g * f_1 + \rho(tb_1 + a_1) = \rho[g * \ddot{u}_1 + t\dot{u}_1(x, 0) + u_1(x, 0)], \quad (27.8)$$

$$\frac{1}{T_0} l * q_{1,t} + \frac{1}{T_0} l * r + \rho\eta_0 = \rho[l * \dot{\eta}_1 + \eta(x, 0)]. \quad (27.9)$$

If in these relations we make $t = 0$, we obtain

$$a_1 = u_1(x, 0), \quad \eta_0 = \eta(x, 0).$$

Deriving with respect to time, the relation (27.8) and then making $t = 0$, we conclude

$$\dot{u}_1(x, 0) = b_1.$$

On the basis of these results, the relations (27.8) and (27.9) are reduced to

$$g * (t_{1j,t} + f_1 - \rho\ddot{u}_1) = 0,$$

$$\frac{1}{T_0} l * (q_{1,t} + r - \rho T_0 \dot{\eta}) = 0.$$

In view of the last of the properties of (27.2) of the product of convolution, it results that the functions u_i , t_{ij} , and θ satisfy the equations (25.1)(25.2).

An immediate consequence of the previous theorem is

Theorem 27.2. An admissible process Γ is the solution of the mixed problem if and only if it satisfies the equations (27.3), (27.4), (25.3), (25.4) and the boundary conditions (25.8).

If we substitute the components of the displacement vector from (27.3) in the compatibility equations (3.21) and in equation (26.3) we can obtain an expression of the theories of the equations of thermoelasticity with the help of the components of the tension tensor and of the temperature variation (Iesan [186]).

177

28. The Theorem of Uniqueness

We will present the theorem of uniqueness of the solution of the problem of coupled thermoelasticity. In the case of isotropic media, this theorem was established by Weiner [440]. The theorem was extended to the case of anisotropic media by Ionescu-Cazimir [217], [218].

Theorem 28.1. If C_{ijkl} is a positive semidefined tensor and a is a strictly positive function, then the mixed problem has at least one solution.

Proof. Let us recall that we assumed already the fact that ρ and T_0 are strictly positive and k_{ij} is a positive semidefined tensor (see (25.6)). Obviously, if a is strictly positive, then the specific heat c is strictly positive and reciprocal.

Keeping in mind (25.3), we have

$$t_{ij} \dot{\epsilon}_{ij} = C_{ijkl} \epsilon_{kl} \dot{\epsilon}_{ij} - \beta_{ij} \theta \dot{\epsilon}_{ij},$$

$$\rho \theta \dot{\eta} = \beta_{ij} \dot{\epsilon}_{ij} \theta + a \dot{\theta} \theta,$$

whence it results that

$$t_{ij} \dot{z}_{ij} + \rho \dot{\theta} \dot{\gamma} = C_{ijkl} \varepsilon_{kl} \dot{z}_{ij} + a \dot{\theta} \dot{\gamma} = \frac{1}{2} \frac{\partial}{\partial t} (C_{ijkl} \varepsilon_{kl} \varepsilon_{ij} + a \dot{\theta}^2). \quad (28.1)$$

On the other hand in view of (25.2) and (25.4) we can write

$$\begin{aligned} t_{ij} \dot{z}_{ij} + \rho \dot{\theta} \dot{\gamma} &= t_{ij} v_{i,j} \dot{\gamma} + \frac{1}{T_0} \dot{\theta} (r + q_{i,i}) = \\ &= (t_{ij} v_i)_{,j} \dot{\gamma} - t_{i,j,j} v_i + \frac{1}{T_0} (\dot{\theta} q_{i,i}) - \frac{1}{T_0} q_{i,i} \dot{\theta} + \frac{1}{T_0} r \dot{\theta}, \end{aligned} \quad (28.2)$$

where $v_i = \dot{u}_i$.

Using (28.2), the formula of divergence and equations (25.1), we obtain

$$\begin{aligned} \int_B (t_{ij} \dot{z}_{ij} + \rho \dot{\theta} \dot{\gamma}) dv &= \int_{\partial B} \left(t_i v_i + \frac{1}{T_0} q \dot{\theta} \right) da + \\ &+ \int_B \left(f_i v_i + \frac{1}{T_0} r \dot{\theta} \right) dv - \int_B \rho \dot{t}_i v_i dv - \frac{1}{T_0} \int_B q_{i,i} \dot{\theta} dv. \end{aligned} \quad (28.3)$$

If we note

$$U(t) = \frac{1}{2} \int_B (\rho \dot{u}_i \dot{u}_i + C_{ijkl} \varepsilon_{kl} \varepsilon_{ij} + a \dot{\theta}^2) dv, \quad (28.4)$$

it follows from (28.1) -- (28.3), (25.6)

78

$$\begin{aligned} \dot{U} - \int_{\partial B} \left(t_i v_i + \frac{1}{T_0} q \dot{\theta} \right) da - \int_B \left(f_i v_i + \frac{1}{T_0} r \dot{\theta} \right) dv &= \\ &= - \frac{1}{T_0} \int_B q_{i,i} \dot{\theta} dv \leq 0. \end{aligned} \quad (28.5)$$

Let us assume that the problem under consideration would have

two solutions. Keeping in mind the linearity of the problem, the difference $\Pi^* = \{v_i^*, \epsilon_i^*, t_i^*, \theta^*, \eta^*, q_i^*\}$ of the two solutions is the solution of the mixed problem, in which

$$\begin{aligned} f_i = r = 0, \quad a_i = b_i = 0, \quad \gamma_0 = 0 \text{ (then } \theta_0 = 0), \\ \bar{a}_i = \bar{t}_i = 0, \quad \bar{q} = \bar{\theta} = 0. \end{aligned} \quad (28.6)$$

For Π^* the relation (28.5) implies

$$\dot{U}^* \leq 0, \quad t \in [0, t_0], \quad (28.7)$$

where U^* is the function U corresponding to the process of Π^* .

It follows from (28.7) that U^* is decreasing

$$U^*(t) \leq U^*(0), \quad t \in [0, t_0]. \quad (28.8)$$

Keeping in mind (28.4) and the initial conditions, it follows that

$$U^*(0) = 0. \quad (28.9)$$

Because of $\rho > 0, a > 0, C_{ijk} \epsilon_{ij} \epsilon_{kl} \geq 0$, we obtain

from (28.8) and (28.9)

$$U^*(t) = 0, \quad t \in [0, t_0]. \quad (28.10)$$

In view of (28.4) and (28.10) it follows

$$u_1^* = 0 \text{ and } \theta^* = 0 \text{ on } \bar{B} \times [0, t_0].$$

In view of the fact that the initial momentum u^* annuls itself, we have

$u_1^* = 0, \theta^* = 0$ on $\bar{B} \times [0, t_0)$,
and thus, $\Pi^* = \{0, 0, 0, \dots, 0\}$. This implies, thus, the uniqueness of the solution of the problem under consideration.

Theorems of uniqueness in which the assumption that C_{ijkl} is a positive semi-defined tensor is renounced have been advanced by Brun [41], Knops and Payne [235], Levine [261]. In reference [235] the continuous dependence of the solution of the initial data is established.

The problem of uniqueness was studied (see Paragraph 32) also /79
by Dafermos [73], who established also the theorems of existence and of asymptotic stability.

The theorem presented here of uniqueness may also be extended to other types of boundary conditions. Thus, it can easily be seen that it remains true for boundary conditions which imply

$$t_1^* u_1^* = 0, q^* \theta^* = 0 \text{ on } \partial B \times [0, t_0).$$

Similarly, the theorem is applied to the unbounded domains if the conditions of behavior at infinity are imposed

$$\lim q^\theta = \lim t_1 u_1 = 0.$$

In this case, the relation (28.5) of a sphere with a (suitably chosen) radius R is applied. If R tends toward infinity, the surface integrals from (28.5) are annulled according to the above conditions, and the theorem of uniqueness is then proven similarly as for finite domains (see Ionescu-Cazimir [218]).

29. The Theorem of Reciprocity

In the dynamic theory of linear elasticity, the theorem of

reciprocity was established by Graffi [136,137]. The result, obtained with the help of a Laplace transform, was extended to the case of coupled thermoelasticity of Ionescu-Cazimir [215, 216]. In another work, Graffi [138] derived a reciprocity theorem in the linear theory of elastodynamics without using Laplace transforms. The same result was obtained later by Wheeler [442]. However, this theorem yields a relation of reciprocity which implies both the displacement vector and the velocity vector. Let us mention also a result established by Wheeler and Sternberg [441] which is based on the assumption that the initial conditions are homogeneous.

Iesan developed a method for establishing the reciprocity theorems without using Laplace transforms whereby reciprocity relations which contain only the displacement vector components and loads [187] are obtained. This last fact is important in deriving the variational theorems (see Paragraph 31). The method was applied in various linear theories of the mechanics of continuous media (for example, Roschi and Mainardi [37], Iesan [194], [201]). We will present two theorems of reciprocity derived in [187], [201].

Let us consider an elastic medium subjected to two systems of loads

$$L^{(\alpha)} = \{f_i^{(\alpha)}, r^{(\alpha)}, \tilde{u}_i^{(\alpha)}, \tilde{t}_i^{(\alpha)}, \tilde{t}_j^{(\alpha)}, \tilde{q}^{(\alpha)}, a_i^{(\alpha)}, b_i^{(\alpha)}, \gamma_\theta^{(\alpha)}\}, (\alpha=1, 2). \quad (29.1)$$

Let $U^{(\alpha)} = \{u_i^{(\alpha)}, \dots, q^{(\alpha)}\}$ be the solution of the problem of thermoelasticity corresponding to the system of loads $L^{(\alpha)}$.

We will use the following notations

$$\begin{aligned} P_i^{(\alpha)} &= g * f_i^{(\alpha)} + \rho(t b_i^{(\alpha)} + a_i^{(\alpha)}), \quad t_i^{(\alpha)} = t_i^{(\alpha)} n_i, \\ W^{(\alpha)} &= \frac{1}{T_0} l * r^{(\alpha)} + \rho \gamma_\theta^{(\alpha)}, \quad q^{(\alpha)} = q^{(\alpha)} n_\theta, \end{aligned} \quad (29.2)$$

where the functions l and g are given by (27.5).

Theorem 29.1. If the conductivity tensor is symmetric and

180

$\Pi^{(1)}$ and $\Pi^{(2)}$ are solutions corresponding to the loading systems of $L^{(1)}$ and $L^{(2)}$, respectively, then the following reciprocity relations occur

$$\begin{aligned} & \int_n [F_i^{(1)} \cdot u_i^{(2)} - g \cdot W^{(1)} \cdot \theta^{(2)}] dv + \int_{on} g \cdot [t_i^{(1)} \cdot u_i^{(2)} - \\ & - \frac{1}{T_0} l \cdot q^{(1)} \cdot \theta^{(2)}] da = \int_n [F_i^{(2)} \cdot u_i^{(1)} - g \cdot W^{(2)} \cdot \theta^{(1)}] dv + \\ & + \int_{on} g \cdot [t_i^{(2)} \cdot u_i^{(1)} - \frac{1}{T_0} l \cdot q^{(2)} \cdot \theta^{(1)}] da. \end{aligned} \quad (29.3)$$

Proof. On the basis of relations (25.5), (27.2) we derive from (25.3) that

$$\begin{aligned} [t_{ij}^{(1)} + \beta_{ij} \theta^{(1)}] \cdot \varepsilon_{ij}^{(2)} &= [t_{ij}^{(2)} + \beta_{ij} \theta^{(2)}] \cdot \varepsilon_{ij}^{(1)}, \\ [\beta_{ij} \varepsilon_{ij}^{(1)} - \rho \gamma_{ij}^{(1)}] \cdot \theta^{(2)} &= [\beta_{ij} \varepsilon_{ij}^{(2)} - \rho \gamma_{ij}^{(2)}] \cdot \theta^{(1)}. \end{aligned} \quad (29.4)$$

Adding these relations we obtain

$$t_{ij}^{(1)} \cdot \varepsilon_{ij}^{(2)} - \rho \gamma_{ij}^{(1)} \cdot \theta^{(2)} = t_{ij}^{(2)} \cdot \varepsilon_{ij}^{(1)} - \rho \gamma_{ij}^{(2)} \cdot \theta^{(1)}. \quad (29.5)$$

If we introduce the notations

$$I_{ab} = \int_n g \cdot [t_{ij}^{(a)} \cdot \varepsilon_{ij}^{(b)} - \rho \gamma_{ij}^{(a)} \cdot \theta^{(b)}] dv, \quad (29.6)$$

it follows from (29.5) that

$$I_{12} = I_{21}. \quad (29.7)$$

In view of relations (25.3), (25.4), (27.3), (27.4) and using the theorem of divergence we can write

$$\begin{aligned}
I_{ab} &= \int_B g \cdot [t_i^{(a)} \cdot u_i^{(b)} - \left(\frac{1}{T_0} l \cdot q_i^{(a)} + W^{(a)} \right) \cdot \theta^{(b)}] dv = \\
&= \int_{\partial B} g \cdot \left[t_i^{(a)} \cdot u_i^{(b)} - \frac{1}{T_0} l \cdot q^{(a)} \cdot \theta^{(b)} \right] da + \\
&+ \int_B [F_i^{(a)} \cdot u_i^{(b)} - g \cdot W^{(a)} \cdot \theta^{(b)}] dv + \frac{1}{T_0} \int_B g \cdot l \cdot k_{ij} \theta^{(a)}_{,i} \cdot \theta^{(b)}_{,j} dv - \\
&- \int_B \rho u_i^{(a)} \cdot u_i^{(b)} dv.
\end{aligned} \tag{29.8}$$

If we keep in mind the properties of the product of convolution, /81
from (29.7) and (29.8) the reciprocity relation (29.3) follows.

In the case when the initial data are zero, we have

$$F_i^{(a)} = g \cdot f_i^{(a)}, \quad W^{(a)} = \frac{1}{T_0} l \cdot r^{(a)}, \tag{29.9}$$

and the relation (29.3) becomes

$$\begin{aligned}
&g \cdot \left\{ \int_B \left[f_i^{(1)} \cdot u_i^{(2)} - \frac{1}{T_0} l \cdot r^{(1)} \cdot \theta^{(2)} \right] dv + \int_{\partial B} \left[t_i^{(1)} \cdot u_i^{(2)} - \right. \right. \\
&\quad \left. \left. - \frac{1}{T_0} l \cdot q^{(1)} \cdot \theta^{(2)} \right] da - \int_B \left[f_i^{(2)} \cdot u_i^{(1)} - \right. \right. \\
&\quad \left. \left. - \frac{1}{T_0} l \cdot r^{(2)} \cdot \theta^{(1)} \right] dv - \int_{\partial B} \left[t_i^{(2)} \cdot u_i^{(1)} - \frac{1}{T_0} l \cdot q^{(2)} \cdot \theta^{(1)} \right] da \right\} = 0.
\end{aligned}$$

In view of the last property of (27.2) of the product of convolution, it follows that

$$\begin{aligned}
&\int_B \left[f_i^{(1)} \cdot u_i^{(2)} - \frac{1}{T_0} l \cdot r^{(1)} \cdot \theta^{(2)} \right] dv + \int_{\partial B} \left[t_i^{(1)} \cdot u_i^{(2)} - \right. \\
&\quad \left. - \frac{1}{T_0} l \cdot q^{(1)} \cdot \theta^{(2)} \right] da = \int_B \left[f_i^{(2)} \cdot u_i^{(1)} - \right. \\
&\quad \left. - \frac{1}{T_0} l \cdot r^{(2)} \cdot \theta^{(1)} \right] dv + \int_{\partial B} \left[t_i^{(2)} \cdot u_i^{(1)} - \frac{1}{T_0} l \cdot q^{(2)} \cdot \theta^{(1)} \right] da.
\end{aligned} \tag{29.10}$$

In view of the fact that in this case we have

$$u_i = l \cdot \dot{u}_i, \quad (29.11)$$

the relation (29.10) may be written in the following form

$$\begin{aligned} \int_B \left[f_i^{(1)} \cdot \dot{u}_i^{(2)} - \frac{1}{T_0} r^{(1)} \cdot \theta^{(2)} \right] dv + \int_{\partial B} \left[t_i^{(1)} \cdot \dot{u}_i^{(2)} - \right. \\ \left. - \frac{1}{T_0} q^{(1)} \cdot \theta^{(2)} \right] da = \int_B \left[f_i^{(2)} \cdot \dot{u}_i^{(1)} - \right. \\ \left. + \frac{1}{T_0} r^{(2)} \cdot \theta^{(1)} \right] dv + \int_{\partial B} \left[t_i^{(2)} \cdot \dot{u}_i^{(1)} - \frac{1}{T_0} q^{(2)} \cdot \theta^{(1)} \right] da. \end{aligned} \quad (29.12)$$

Relation (29.12) was derived by Ionescu-Cazimir [215], however, by using the Laplace transforms.

/82

Next, we will derive a reciprocity relation, assuming that $\Pi^{(1)}$ corresponds to a problem of coupled thermoelasticity, $P^{(1)}$ with loads of $L^{(1)}$, and $\Pi^{(2)}$ corresponds to a problem of uncoupled thermoelasticity, $P^{(2)}$, with loads of $L^{(2)}$.

In view of what has been stated in Paragraph 26, it follows that we have

$$\rho \gamma^{(2)} = \alpha \theta^{(2)}. \quad (29.13)$$

In this case the relations analogous to relations (29.4) are

$$\begin{aligned} [t_{ij}^{(1)} + \beta_{ij} \theta^{(1)}] \cdot \epsilon_{ij}^{(2)} &= [t_{ij}^{(2)} + \beta_{ij} \theta^{(2)}] \cdot \epsilon_{ij}^{(1)}, \\ [\beta_{ij} \epsilon_{ij}^{(1)} - \rho \gamma^{(1)}] \cdot \theta^{(2)} &= -\rho \gamma^{(2)} \cdot \theta^{(1)}. \end{aligned} \quad (29.14)$$

Adding these relations we obtain

$$t_{ij}^{(1)} * \epsilon_{ij}^{(2)} - \rho \gamma^{(1)} * \theta^{(2)} = t_{ij}^{(2)} * \epsilon_{ij}^{(1)} - \rho \gamma^{(2)} * \theta^{(1)} - \beta_{ij} \theta^{(1)} * \epsilon_{ij}^{(2)}. \quad (29.15)$$

By proceeding in the same way as in theorem 29.1, there follows from (29.15)

Theorem 29.2. If an elastic medium with a symmetric conductivity tensor is subjected to two systems of loads $L^{(\alpha)}$ ($\alpha = 1, 2$) then between the solutions $\Pi^{(\alpha)}$ corresponding to the problems $P^{(\alpha)}$ ($\alpha = 1, 2$) the following reciprocity relation prevails

$$\begin{aligned} \int_B [F_i^{(1)} * u_i^{(2)} - g * W^{(1)} * \theta^{(2)}] dv + \int_{\partial B} g * [t_i^{(1)} * u_i^{(2)} - \\ - \frac{1}{T_0} l * q^{(1)} * \theta^{(2)}] da = \int_B [F_i^{(2)} * u_i^{(1)} - \\ - g * W^{(2)} * \theta^{(1)}] dv + \int_{\partial B} g * [t_i^{(2)} * u_i^{(1)} - \\ - \frac{1}{T_0} l * q^{(2)} * \theta^{(1)}] da - \int_B \beta_{ij} g * \theta^{(1)} * \epsilon_{ij}^{(2)} dv. \end{aligned} \quad (29.16)$$

In the case in which the initial data are zero, relation (29.16) is reduced to

$$\begin{aligned} \int_B \left[f_i^{(1)} * u_i^{(2)} - \frac{1}{T_0} l * q^{(1)} * \theta^{(2)} \right] dv + \int_{\partial B} \left[t_i^{(1)} * u_i^{(2)} - \right. \\ \left. - \frac{1}{T_0} l * q^{(1)} * \theta^{(2)} \right] da = \int_B \left[f_i^{(2)} * u_i^{(1)} - \frac{1}{T_0} l * q^{(2)} * \theta^{(1)} \right] dv + \\ + \int_{\partial B} \left[t_i^{(2)} * u_i^{(1)} - \frac{1}{T_0} l * q^{(2)} * \theta^{(1)} \right] da - \int_B \beta_{ij} \theta^{(1)} * \epsilon_{ij}^{(2)} dv. \end{aligned} \quad (29.17)$$

If we consider (29.11) the previous relation will assume the /83
form of

$$\begin{aligned} & \int_V \left[f_i^{(1)} \cdot \dot{u}_i^{(2)} - \frac{1}{T_0} r^{(1)} \cdot \theta^{(2)} \right] dv + \int_{\partial B} \left[t_i^{(1)} \cdot \dot{u}_i^{(2)} - \right. \\ & \left. - \frac{1}{T_0} q^{(1)} \cdot \theta^{(2)} \right] da = \int_V \left[f_i^{(2)} \cdot \dot{u}_i^{(1)} - \frac{1}{T_0} r^{(2)} \cdot \theta^{(1)} \right] dv + \\ & + \int_{\partial B} \left[t_i^{(2)} \cdot \dot{u}_i^{(1)} - \frac{1}{T_0} q^{(2)} \cdot \theta^{(1)} \right] da - \int_B \beta_{ij} \theta^{(1)} \cdot \dot{\varepsilon}_{ij}^{(2)} dv. \end{aligned} \quad (29.18)$$

The relation (29.18) was derived by Ionescu-Cazimir [215], using Laplace transforms.

30. Applications of the Reciprocity Theorem

In order to simplify the exposition, we will consider the case in which the initial conditions are homogeneous and $\Sigma_1 = \Sigma_3 = \emptyset$.

Let us assume that the elastic medium under consideration is acted upon by the following system of loads.

$$f_i = \delta(x - \xi) \delta(t) \delta_{i1}, \quad r = 0, \quad \tilde{t}_i = 0, \quad \tilde{q} = 0, \quad a_i = b_i = \gamma_0 = 0, \quad (30.1)$$

where δ represents the Dirac distribution. From the mechanical viewpoint the situation corresponds to a force concentrated in point ξ directed according to axis Ox_1 , of unit magnitude and applied in the form of a shock at the initial moment (see for example, Courant [66], Sneddon [371], Kecs and Teodorescu [231], Teodorescu [411]), while the other loads are zero. We will designate by $U_i^{(1)}(x, \xi, t), \theta^{(1)}(x, \xi, t)$ the displacements and the temperature corresponding to the system of loads of (30.1).

In general, if

$$f_i = \delta(x - \xi) \delta(t) \delta_{i1}, \quad r = 0, \quad \tilde{t}_i = 0, \quad \tilde{q} = 0, \quad a_i = b_i = \gamma_0 = 0, \quad (30.2)$$

the solutions corresponding to the system of loads will be designated by $U_j^{(n)}(x, \xi, t), \Theta^{(n)}(x, \xi, t)$ ($j = 1, 2, 3$).

The load systems

$$f_i = 0, r = \delta(x - \xi) \delta(t), \tilde{t}_i = \tilde{q} = 0, a_i = b_i = \gamma_0 = 0, \quad (30.3)$$

correspond to a heat source concentrated in point ξ_x of unit magnitude and applied in the form of a shock at the initial moment. We will indicate by $U_1^{(4)}(x, \xi, t), \Theta^{(4)}(x, \xi, t)$ the displacements and the temperature corresponding to the system of loads (30.3)

The above load systems and the solutions which correspond to them do not lie within the framework of the assumptions from Paragraph 25. However, they are very important and the interpretation and mathematical handling follows the path known from the theory of elasticity (see for example, Sneddon[371], Gurtin [163], Kecs and Teodorescu [231]). The functions $U_j^{(n)}(x, \xi, t), \Theta^{(n)}(x, \xi, t)$ ($n = 1, 2, 3, 4$) constitute Green functions corresponding to the domain under consideration.

Let us apply the relation of reciprocity (29.10) to the system /84
for loads $L = \{f_i, r, \tilde{t}_i, \tilde{q}, a_i = b_i = \gamma_0 = 0\}$, to which correspond the displacement u_1 and the temperature θ and for the system of loads (30.1). In view of the fact that

$$\delta \cdot f = f, \quad (30.4)$$

from (29.10) we obtain

$$u_1(\xi, t) = \int_B \left[f_i \cdot U_i^{(1)} - \frac{1}{T_0} l \cdot r \cdot \Theta^{(1)} \right] dv + \int_{\partial B} \left[\tilde{t}_i \cdot U_i^{(1)} - \frac{1}{T_0} l \cdot \tilde{q} \cdot \Theta^{(1)} \right] da. \quad (30.5)$$

Similarly, using the same relation, but under the form of (29.12), to the system L and to the system (30.3) we derive

$$\begin{aligned} \theta(\xi, t) = & -T_0 \int_B \left[f_i \cdot \dot{U}_i^{(n)} - \frac{1}{T_0} r \cdot \Theta^{(n)} \right] dv - \\ & - T_0 \int_{\partial B} \left[\tilde{i}_i \cdot \dot{U}_i^{(n)} - \frac{1}{T_0} \tilde{q} \cdot \Theta^{(n)} \right] da. \end{aligned} \quad (30.6)$$

From the above follows

Theorem 30.1. If the conductivity tensor is symmetric, then the displacement u_1 and the temperature θ corresponding to the system of loads $L = \{f_i, r, \tilde{i}_i, \tilde{q}, a_i = b_i = \gamma_0 = 0\}$ are expressed by means of the Green functions corresponding to the domain under consideration by the relations

$$\begin{aligned} \varphi_s(\xi, t) = & \int_B \left[f_i \cdot U_i^{(s)} - \frac{1}{T_0} l \cdot r \cdot \Theta^{(s)} \right] dv + \\ & + \int_{\partial B} \left[\tilde{i}_i \cdot U_i^{(s)} - \frac{1}{T_0} l \cdot \tilde{q} \cdot \Theta^{(s)} \right] da, \quad (s = 1, 2, 3, 4), \end{aligned} \quad (30.7)$$

where $\phi_1 = u_1$, $-T_0 \phi_4 = 1 \cdot \theta$.

Let us designate by $V_i(x, \xi, t)$, $\mathcal{T}(x, \xi, t)$ the displacement and the temperature from the problem of uncoupled thermoelasticity, respectively, corresponding to the system of loads (30.3). If we apply the relation of reciprocity (29.18) to this problem of uncoupled thermoelasticity, and to the problem of thermoelasticity coupled with a system of loads $L = \{f_i, r, \tilde{i}_i, \tilde{q}, a_i = b_i = \gamma_0 = 0\}$, we obtain

$$\begin{aligned} \theta(\xi, t) = & -T_0 \int_B \left[f_i \cdot \dot{V}_i - \frac{1}{T_0} r \cdot \mathcal{T} \right] dv - \\ & - T_0 \int_{\partial B} \left[\tilde{i}_i \cdot \dot{V}_i - \frac{1}{T_0} \tilde{q} \cdot \mathcal{T} \right] da - T_0 \int_B \beta_{ij} \dot{V}_{i,j} \cdot \theta dv. \end{aligned} \quad (30.8)$$

Let $V_j^{(j)}(x, \xi, t), \mathcal{T}^{(j)}(x, \xi, t)$ ($j=1, 2, 3$) be the solutions of the problems of uncoupled thermoelasticity corresponding to the loads (30.2). Applying the relation of reciprocity (29.17) to the problem of coupled thermoelasticity corresponding to the system of loads $L = \{f_i, r, \bar{t}_i, \bar{q}, a_i = b_i = \gamma_0 = 0\}$, and the problem of uncoupled thermoelasticity corresponding to the system of loads (30.2) we obtain

$$u_i(\xi, t) = \int_B \left[f_i \cdot V_i^{(j)} - \frac{1}{T_0} l \cdot r \cdot \mathcal{T}^{(j)} \right] dv + \\ + \int_{\partial B} \left[\bar{t}_i \cdot V_i^{(j)} - \frac{1}{T_0} l \cdot \bar{q} \cdot \mathcal{T}^{(j)} \right] da + \int_B \beta_{r,i} V_{r,i}^{(j)} \cdot 0 dv. \quad (30.9)$$

Consequently, if we determine the function θ as a solution of the integral equation (30.8) and substitute it in (30.9) we obtain the component of the displacement vector. Thus, we have

Theorem 30.2. The problem of coupled thermoelasticity corresponding to the system of loads $L = \{f_i, r, \bar{t}_i, \bar{q}, a_i = b_i = \gamma_0 = 0\}$ is reduced to the resolution of the problem of uncoupled thermoelasticity corresponding to the system of loads (30.3) relative to the same medium, and to the integration of the equation (30.9).

These theorems may be useful to resolve certain concrete problems [128], [216]. They have been derived with the help of Laplace transforms of Ionescu-Cazimir [216] and later extended to other theories of thermoelasticity (see, for example, [194], [195]).

From theorem 30.1 the following relations follow

$$U_j^{(j)}(\xi, \eta, t) = U_j^{(j)}(\eta, \xi, t), \quad \Theta^{(j)}(\xi, \eta, t) = -T_0 \dot{U}_j^{(j)}(\eta, \xi, t), \\ \Theta^{(i)}(\xi, \eta, t) = \Theta^{(i)}(\eta, \xi, t), \quad (i, j = 1, 2, 3). \quad (30.10)$$

31. Variational Theorems

In reference [158] Gurtin presented the variational theorems which characterize completely the solution of a mixed problem from the linear theory of elastodynamics. In the theory of coupled thermoelasticity, this type of theorems was derived by Iesan [186], [187], [201]. A general method for deriving variational theorems in the case of linear theories of the dynamics of continuous media was presented in [201].

In the following we will consider the equations of coupled thermoelasticity for nonhomogeneous and anisotropic media with asymmetrical conductivity tensor.

If we consider the relations (26.1), then the equations (27.3) and (27.4) may be written in the following manner

$$\begin{aligned} g * (C_{ijkl} u_{k,l} - \beta_{ij} \theta)_{,j} + F_i &= \rho u_{i,t} \\ \beta_{ij} u_{i,j} + a\theta &= -\frac{1}{T_0} l * (k_{ij} \theta_{,j})_{,i} + W. \end{aligned} \quad (31.1)$$

Let us introduce the vectors

$$U = (u_1, u_2, u_3, \theta), \quad F = (F_1, F_2, F_3, -g * W),$$

and the operators

$$\begin{aligned} A_1 U &= -g * (C_{ijkl} u_{k,l} - \beta_{ij} \theta)_{,j} + \rho u_{i,t} \\ A_4 U &= g * \left[\frac{1}{T_0} l * (k_{ij} \theta_{,j})_{,i} - \beta_{ij} u_{i,j} - a\theta \right]. \end{aligned} \quad (31.2)$$

Keeping in mind (27.2) it follows that equations (31.1) are equivalent with the system

$$AU = F \quad (31.3)$$

where $AU = (A_1U, \dots, A_4U)$.

From this follows an immediate consequential of theorem 27.2.

Theorem 31.1. An admissible displacement-temperature field $U = (u_1, \theta)$ is a solution of the mixed problem if and only if it satisfies the equation (31.3) and the boundary conditions (26.5).

Let us consider the tension operators and the flux

$$\begin{aligned} t_1(U) &= (C_{ijkl} u_{k,l} - \beta_{ij} \theta) n_j, \\ q(U) &= k_{ij} \theta_{,j} n_i. \end{aligned} \quad (31.4)$$

We shall note

$$\mathcal{F}U = (t_1(U), t_2(U), t_3(U), -\frac{1}{T_0} t \cdot q(U)). \quad (31.5)$$

If we take into account (31.3) and (31.5), then the reciprocity relation (29.3) may be written in the following form

$$\int_B (V \cdot AU - U \cdot AV) dv = \int_{\partial B} q \cdot (U \cdot \mathcal{F}V - V \cdot \mathcal{F}U) da, \quad (31.6)$$

where

$$U = (u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, \theta^{(1)}), V = (u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, \theta^{(2)}).$$

In the case when the boundary conditions are homogeneous, the relation (31.6) will take the form

$$(V \otimes AU) = (U \otimes AV), \quad (31.7)$$

where by $f \otimes g$ we designate the scalar product in convolution (Hlavacek [178]) defined by

$$f \otimes g = \int_{\Omega} f \cdot g \, dv. \quad (31.8)$$

The relation (31.7) expresses the fact that in this case the linear operator A is symmetric in convolution. Let us consider the functional

$$\mathcal{F}(U) = (AU \otimes U) - 2(U \otimes P). \quad (31.9)$$

It is a known fact that (Hlavacek [178]) if operator A is symmetric in convolution on its domain of definition D_A , then

$$\delta \mathcal{F}(U) = 0, \quad (31.10)$$

if and only if $U \in D_A$ satisfies the equation (31.3).

It follows from (29.8) and (26.1), under the assumption that the boundary conditions are homogeneous, that

$$\begin{aligned} (AU \otimes U) = & \int_{\Omega} g \cdot (C_{ijkl} u_{i,j} \cdot u_{k,l} - 2\beta_{ij} u_{i,j} \cdot \theta - \\ & - \alpha \theta \cdot \theta) \, dv + \int_{\Omega} \rho u_i \cdot u_i \, dv - \frac{1}{T_0} \int_{\Omega} g \cdot l \cdot k_{ij} \theta_{,i} \cdot \theta_{,j} \, dv. \end{aligned} \quad (31.11)$$

Therefore, the following takes place

Theorem 31.2. Let \mathcal{X} be the set of admissible displacement-temperature fields which satisfy the conditions (26.5) in a homogeneous form. For $t \in [0, t_0)$ we define the functional $\Lambda_t\{\cdot\}$ on \mathcal{X} by

$$\begin{aligned} \Lambda_t(U) = & \int_B g \cdot (C_{001} u_{t,t} \cdot u_{t,t} - 2\beta_{01} u_{t,t} \cdot 0 - \alpha \cdot 0 \cdot 0) dv + \\ & + \int_B \mu u_t \cdot u_t dv - \frac{1}{T_0} \int_B g \cdot l \cdot k_{01} \theta_{t,t} \cdot \theta_{t,t} dv - 2 \int_B (F_t \cdot u_t - \\ & - g \cdot W \cdot \theta) dv, \end{aligned} \quad (31.12)$$

for any $U \in \mathcal{X}$. Then

$$\delta \Lambda_t\{U\} = 0 \quad (0 \leq t < t_0), \quad (31.13)$$

if and only if U is a solution of the mixed problem.

This may be proven directly without using the result (31.9), (31.10). For this the following notation is considered

$$\delta_\lambda \Lambda_t\{U\} = \frac{d}{d\lambda} \Lambda_t\{U + \lambda \tilde{U}\}|_{\lambda=0}, \quad (31.14)$$

where

$$(\alpha) \quad U + \lambda \tilde{U} \in \mathcal{X} \quad \text{for any scalar } \lambda.$$

As is known, we have

$$\delta \Lambda_t\{U\} = 0,$$

if $\delta_{\tilde{U}} \Lambda_t\{U\}$ exists and is equal to zero for any \tilde{U} which satisfies the condition (α) . Using this fact and also a generalization of the fundamental lemma from the calculus of variations (Gurtin [163]) results in the derived statement.

If we keep in mind that the following conditions

$$h_t(U) = \tilde{h}_t \text{ on } \Sigma_2 \times [0, t_0], q(U) = \tilde{q} \text{ on } \Sigma_1 \times [0, t_0],$$

are natural conditions (Mihlin [289]) this means that it is sufficient

as in theorem 31.2 that the admissible displacement-temperature field satisfy only the conditions $u_i = 0$ on $\Sigma_1 \times [0, t_0)$, $\theta = 0$ on $\Sigma_3 \times [0, t_0)$, and the other conditions will be satisfied.

Let us consider now the case of conditions at a nonhomogeneous boundary. Let $\Psi = \{\psi_i, \tau\}$ be an admissible displacement-temperature field which satisfies the conditions (26.5). If we introduce the vector W by the relation $W = U - \Psi$, then W satisfies the equation

$$AW = F - A\Psi, \quad (31.15)$$

and the conditions (26.5) in homogeneous form. In this case we are led to the functional

$$\Omega(W) = (AW \otimes W) - 2(W \otimes (F - A\Psi)),$$

which may be written in the form

$$\begin{aligned} \Omega(U - \Psi) &= (AU \otimes U) + (A\Psi \otimes U) - (AU \otimes \Psi) - \\ &- 2(U \otimes F) + 2(\Psi \otimes F) - (A\Psi \otimes \Psi). \end{aligned} \quad (31.16)$$

From (31.6), (29.8), (26.5) we have

$$\begin{aligned} (A\Psi \otimes U) - (AU \otimes \Psi) &= \int_{\Sigma_1} g * \tilde{u}_i * [t_i(U) - t_i(\Psi)] da + \\ &+ \int_{\Sigma_1} g * (\psi_i - u_i) * \tilde{l} da - \frac{1}{T_0} \int_{\Sigma_1} g * \tilde{\theta} * l * [q(U) - q(\Psi)] da - \\ &- \frac{1}{T_0} \int_{\Sigma_1} g * l * (\tau - \theta) * \tilde{q} da, \\ (AU \otimes U) &= \int_B g * (C_{ijkl} u_{i,j} * u_{k,l} - 2\beta_{ij} u_{i,j} * \theta - \\ &- \alpha \theta * \theta) dv + \int_B \rho u_i * u_i dv - \frac{1}{T_0} \int_B g * l * k_{ij} \theta_{,i} \theta_{,j} dv \\ &- \int_{\Sigma_1} g * \tilde{u}_i * t_i(U) da - \int_{\Sigma_1} g * u_i * \tilde{l}_i da + \frac{1}{T_0} \int_{\Sigma_1} g * l * \tilde{\theta} * q(U) da + \\ &+ \frac{1}{T_0} \int_{\Sigma_1} g * l * \theta * \tilde{q} da. \end{aligned} \quad (31.17)$$

If we keep in mind the relations (31.17) and give up the non-essential terms, then the functional (31.16) is replaced by

$$\begin{aligned} \mathfrak{M}_t(U) = & \int_{\Omega} g \cdot (C_{ijkl} u_{k,l} \cdot u_{i,j} - 2\beta_{ij} u_{i,j} \cdot \theta) - \\ & - a\theta \cdot \theta) dv + \int_{\Omega} \rho u_i \cdot u_i dv - \frac{1}{T_0} \int_{\Omega} g \cdot l \cdot k_{ij} \theta_{,i} \cdot \theta_{,j} dv - \\ & - 2 \int_{\Omega} (F_i \cdot u_i - g \cdot W \cdot \theta) dv - 2 \int_{\Sigma_1} g \cdot u_i \cdot \tilde{l}_i da + \\ & + \frac{2}{T_0} \int_{\Sigma_1} g \cdot l \cdot \theta \cdot \tilde{q} da. \end{aligned} \quad (31.18)$$

Thus, we have

789

Theorem 31.3. Let \mathcal{V} be the set of admissible displacement-temperature fields which satisfy the conditions $u_1 = \tilde{u}_1$ on $\Sigma_1 \times [0, t_0)$, $\theta = \tilde{\theta}$ on $\Sigma_3 \times [0, t_0)$. For $t \in [0, t_0)$ we define the functional $\mathfrak{M}_t(\cdot)$ on \mathcal{V} through (31.18), regardless of $U \in \mathcal{V}$. Then

$$\delta \mathfrak{M}_t(U) = 0, \quad (0 \leq t < t_0),$$

if and only if U is a solution of the mixed problem.

Now we will present a variational theorem (Iesan [186]), in which the admissible processes are not assumed to satisfy some equations or conditions of the problem. The theorem characterizes completely the solution of a mixed problem of coupled thermoelasticity. We will use the following notation

$$s_i = \frac{1}{T_0} q_i. \quad (31.19)$$

The relations $(25.3)_3$ may also be written in the form

$$s_i = - \frac{1}{T_0} k_{ij} \theta_{,j}. \quad (31.20)$$

or, equivalently

$$0_i = T_0 \lambda_{ij} s_j, \quad (31.21)$$

where the significance of the quantities λ_{ij} is obvious.

Theorem 31.4. Let \mathcal{J} be a set of admissible processes. For $t \in [0, t_0)$ we define the functional $\mathcal{D}_t(\cdot)$ on \mathcal{J} by

$$\begin{aligned} \mathcal{D}_t(\Pi) = & \int_{\Sigma} \left\{ \frac{1}{2} \left[C_{ijkl} g \cdot \varepsilon_{ij} \cdot \varepsilon_{kl} + \rho u_i \cdot u_i + \right. \right. \\ & + \frac{1}{a} g \cdot (\rho \eta - \beta_{ij} \varepsilon_{ij}) \cdot (\rho \tau - \beta_{ij} \varepsilon_{ij}) + T_0 \lambda_{ij} g \cdot l \cdot s_i \cdot s_j \left. \right] - \\ & - g \cdot l_{ij} \cdot \varepsilon_{ij} - \rho g \cdot \eta \cdot \theta - g \cdot l \cdot s_i \cdot \theta_i - \\ & - (g \cdot l_{ij} + P_i) \cdot u_i + g \cdot \theta \cdot W \left. \right\} dv + \int_{\Sigma_1} g \cdot l_i \cdot \tilde{u}_i da + \\ & + \frac{1}{T_0} \int_{\Sigma_1} g \cdot l \cdot \theta \cdot \tilde{q} da + \int_{\Sigma_1} g \cdot (l_i - \tilde{l}_i) \cdot u_i da + \\ & + \frac{1}{T_0} \int_{\Sigma_1} g \cdot l \cdot (\theta - \tilde{\theta}) \cdot q da, \end{aligned} \quad (31.22)$$

regardless of $\Pi = \{u_i, \varepsilon_{ij}, l_{ij}, \theta, \eta, q\} \in \mathcal{J}$.

Then

/90

$$\delta \mathcal{D}_t(\Pi) = 0, \quad (0 \leq t < t_0), \quad (31.23)$$

if and only if Π is a solution of the mixed problem.

Proof. Let $\Pi, \Pi' \in \mathcal{J}$, therefore, $\Pi + \lambda \Pi' \in \mathcal{J}$, for any scalar λ . If we take (31.14) into account we obtain

$$\begin{aligned}
\delta_{\Pi} \mathcal{D}_t(\Pi) = & \int_B \left\{ g * \left[C_{ijkl} \varepsilon_{kl} - \frac{1}{\alpha} \beta_{ij} (\rho \eta - \beta_{ij} \varepsilon_{ij}) - \right. \right. \\
& - t_{ij} \left. \right] * \varepsilon'_{ij} + g * \left[\frac{1}{\alpha} (\rho \eta - \beta_{ij} \varepsilon_{ij}) - 0 \right] * \rho \eta' - \\
& - (g * t_{ij,j} + F_i - \rho u_i) * u'_i + g * l * (T_0 \lambda_{ij} s_j - \theta_{,j}) * s'_j + \\
& + g * \left[\frac{1}{2} (u_{i,j} + u_{j,i}) - \varepsilon_{ij} \right] * t'_{ij} - g * (\rho \eta - l * s_{i,i} - W) * \theta' \left. \right\} dv - \\
& - \int_{\Sigma_1} g * (u_i - \tilde{u}_i) * t'_i da + \int_{\Sigma_1} g * (t_i - \tilde{t}_i) * u'_i da - \\
& - \frac{1}{T_0} \int_{\Sigma_1} g * l * (q - \tilde{q}) * \theta' da + \int_{\Sigma_1} g * l * (\theta - \tilde{\theta}) * s'_i n_i da.
\end{aligned} \tag{31.24}$$

If Π is a solution of the mixed problem, then, using theorem 27.2, it follows that

$$\delta_{\Pi} \mathcal{D}_t(\Pi) = 0, \quad t \in [0, t_0) \tag{31.25}$$

for any $\Pi \in \mathcal{J}$ and thus we have (31.23).

In a reciprocal way, let us assume that (31.25) occurs. Let $\Pi' = \{u'_1, 0, \dots, 0\}$, where u'_1 is annulled on ∂B . It follows from (31.24) and (31.25) that

$$\int_B (g * t_{ij,j} + F_i - \rho u_i) * u'_i dv = 0, \quad t \in [0, t_0). \tag{31.26}$$

Using a generalization of the fundamental lemma from the calculus of variations (Gurtin [163]), from (31.26) we obtain

$$g \cdot t_{ij} + F_i = \rho u_i$$

Similarly it is shown that Π satisfies the equations (27.3), (27.4), (25.3), (25.4) and the boundary conditions (25.8). On the basis of theorem 27.2, it follows that Π is a solution of the mixed problem.

Other variational theorems in the theory of coupled thermo-elasticity may be found in the works of Biot [20], Iesan [186], Nickell and Sackman [307], Rafalski [348], Carlson [46]. The importance of the variational theorems presented here was indicated by Nickell and Sackman [308], and Benthien, Gurtin and Ralston [19] who pointed out the potential applications of these theorems.

/91

32. Theorems of Existence

The existence of solutions of problems of thermoelasticity was studied in various papers, among which those written by Dafermos [73], Duvaut and Lions [93], Kupradze [248], etc., could be mentioned.

In this paragraph we will present some of the results derived by Dafermos [73], with reference to the mixed problem of thermo-elasticity formulated in Paragraph 26. Without restricting the generality, we will consider the conditions (26.5) in homogeneous form, indicating this fact by $(26.5)_0$. We assume that none of the parts Σ_r ($r = 1, 2, 3, 4$) of the boundary are empty.

We will consider the mixed problem in the formulation given in Paragraph 26, assuming that B is an open set, limited, connected and self-regulated in the sense of Fichera [111]. In order to demonstrate the existence of a solution, a method given by Vishik (see [431]) is used. First we will present some preparatory questions.

Let $C^m(\bar{B})$ be a set of scalar functions of the class of C^m on \bar{B} .

For $\varphi \in C^m(\bar{B})$ we define the norm

$$\|\varphi\|_{C^m(B)} = \sum_{k=0}^m \sum_{i_1, i_2, \dots, i_k} \max_B |\varphi_{i_1, i_2, \dots, i_k}|.$$

We designate by $\underline{C}^m(\bar{B})$ the set of vector fields, the components of which are from $C^m(\bar{B})$. We define the norm of $v \in \underline{C}^m(\bar{B})$ by

$$\|v\|_{\underline{C}^m(\bar{B})} = \sum_{i=1}^p \|v_i\|_{C^m(B)}.$$

Let $W_m(B)$ be the Hilbert space obtained by completing the set of $\underline{C}^m(\bar{B})$ in the norm of $\|\cdot\|_{W_m(B)}$ induced by the scalar product

$$(\varphi, \psi)_{W_m(B)} = \sum_{k=0}^m \int_B \varphi_{i_1, \dots, i_k} \psi_{i_1, \dots, i_k} dv.$$

We will designate by $\underline{W}_m(B)$ the Hilbert space obtained by completing the set of $\underline{C}^m(\bar{B})$ in the norm of $\|\cdot\|_{W_m(B)}$ induced by the scalar product

$$(v, \omega)_{W_m(B)} = \sum_{j=1}^q (v_j, \omega_j)_{W_m(B)} = \sum_{k=0}^m \int_B v_{j, i_1, \dots, i_k} \omega_{j, i_1, \dots, i_k} dv.$$

In this paragraph we will have occasion to use the cartesian product of normalized spaces. As a norm in the produced space the sum of the norms from the factor spaces will be used. Thus

$$\|(v, \varphi)\|_{W_l(B) \times W_m(B)} = \|v\|_{W_l(B)} + \|\varphi\|_{W_m(B)}.$$

Let H be a Banach space and $[0, t_0]$ a time interval. We will designate by $C^m([0, t_0]; H)$ the set of functions defined on $[0, t_0]$ with values in H , which have on $(0, t_0)$ derivatives with respect to time up to the order m , in H , continuously on $[0, t_0]$. /92

The meaning of the sets $L_1[0, t_0]; H$ and $L_2([0, t_0]; H)$ is derived in a similar manner. For the derivatives with respect to time of the function f , we will use the designation

$$\dot{f} = \frac{\partial f}{\partial t}, \quad \ddot{f} = \frac{\partial^2 f}{\partial t^2}, \quad \overset{(s)}{f} = \frac{\partial^s f}{\partial t^s}, \quad s \geq 3.$$

We are introducing the sets

$$\hat{C}^1(B) = \{\tau \in C^1(\bar{B}) \mid \tau = 0 \text{ on } \Sigma_3\},$$

$$\hat{C}^2(B) = \{v \in C^2(\bar{B}) \mid v = 0 \text{ on } \Sigma_1\},$$

$$\hat{W}_1(B) \equiv \text{completing the set } \hat{C}^1(B) \text{ in the norm } \|\cdot\|_{W_1(B)},$$

$$\hat{W}_2(B) \equiv \text{completing the set } \hat{C}^2(B) \text{ in the norm } \|\cdot\|_{W_2(B)}.$$

Let $\theta, \vartheta \in \hat{C}^1(B)$ and $u, v \in \hat{C}^2(B)$. We designate

$$\Phi(\theta, \vartheta) = \int_B k_{ij} \vartheta_{,j} \theta_{,i} dv, \quad (32.1)$$

$$\Psi(u, v) = \int_B C_{ijkl} v_{i,j} u_{k,l} dv. \quad (32.2)$$

We will assume that

- (i) the thermoconductivity tensor k_{ij} is symmetrical;
- (ii) the density ρ and the specific heat c satisfy the conditions

$$\text{ess inf}_B \rho(x) > 0, \quad \text{ess inf}_B c(x) > 0, \quad (32.3)$$

- (iii) there are positive constants M and N so that

$$\Phi(\vartheta, \vartheta) \geq M \int_B \vartheta_{,i} \vartheta_{,i} dv, \quad \forall \vartheta \in \hat{C}^1(B), \quad (32.4)$$

$$\Psi(v, v) \geq N \int_B \sum_{i,j=1}^3 a_{ij}(v) dv, \quad \forall v \in \hat{C}^1(B). \quad (32.5)$$

The relations (32.4) and (32.5) may be extended to $\hat{W}_1(B)$ and $\hat{W}(B)$, respectively. For $v \in \hat{W}_1(B)$, the Korn type inequality [177] takes place

$$\Psi(v, v) \geq \kappa \int_B (v_i v_i + v_{i,j} v_{i,j}) dv, \quad (32.6)$$

a relation which expresses the fact that $\Psi(\underline{v}, \underline{v})$ is coercive in $\hat{W}_1(B)$.

Let Q_{t_0} be the cylinder $B \times (0, t_0)$, $t_0 > 0$ and the set

93

$$\begin{aligned} \tilde{C}^0(B) &= \{(t, r) | (t, r) \in \\ &\in C^0(B) \times C^0(\bar{B})\}, \tilde{C}^1(B) = \{(u, \theta) | (u, \theta) \in \hat{C}^1(B) \times \hat{C}^1(B)\}, \mathcal{F}(Q_{t_0}) = C^\infty([0, t_0]; \\ &\tilde{C}^1(B)), \mathring{\mathcal{F}}(Q_{t_0}) = \{(v, \vartheta) | (v, \vartheta) \in \mathcal{F}(Q_{t_0}), v=0 \text{ on } B \times 0\}. \end{aligned}$$

Let us consider a certain element $\psi = (v, \vartheta) \in \mathring{\mathcal{F}}(Q_{t_0})$ and let $y = (\underline{u}, \theta)$ be the solution of the problem (26.2) -- (26.5)₀.

If we multiply (26.2) with $(t-t_0) \dot{v}_1$ and (26.3) with $(t-t_0) \frac{\vartheta}{T_0}$, summing up and integrating on Q_{t_0} it results that $y = (\underline{u}, \theta)$ satisfies the relation

$$\mathcal{L}(y, \psi) = \mathcal{D}(z, \psi) + \mathcal{E}(\chi, \psi), \quad \forall \psi \in \mathring{\mathcal{F}}(Q_{t_0}), \quad (32.7)$$

where $\mathcal{L}(y, \psi) = \int_0^{t_0} \int_B \left\{ (t-t_0) [\rho \dot{u}_i \dot{v}_i - C_{ijkl} u_{k,j} \dot{v}_{l,j} + \beta_{ij} \theta \dot{v}_{i,j} + a \theta \dot{\vartheta} + \beta_{ij} u_{i,j} \dot{\vartheta}] + \right.$

$$\left. + \rho \dot{u}_i \dot{v}_i + a \theta \dot{\vartheta} + \beta_{ij} u_{i,j} \dot{\vartheta} + \frac{1}{T_0} \int_0^t k_{ij} \theta_{,j} \vartheta_{,i} d\tau \right\} dv dt,$$

$$\mathcal{D}(z, \psi) = - \int_0^{t_0} \int_B (t-t_0) \left(f_i \dot{v}_i + \frac{1}{T_0} r \dot{\vartheta} \right) dv dt,$$

$$\mathcal{E}(\chi, \psi) = t_0 \int_B [\rho b_i \dot{v}_i + a \theta_0 \dot{\vartheta} + \beta_{ij} a_{i,j} \dot{\vartheta}]_{t=0} dv,$$

$$\chi = (a, b, \theta_0), \quad z = (f, r).$$

Let $y = (u, \theta) \in \mathcal{F}(Q_{t_0})$, $\psi = (v, \vartheta) \in \mathring{\mathcal{F}}(Q_{t_0})$, $z = (t, \tau) \in C^0([0, t_0]; \tilde{C}^0(B))$ and $\chi = (a, b, \theta_0)$ be so that $(a, \theta_0) \in \tilde{C}^1(B)$, $b \in \hat{C}^1(B)$. It is noted that \mathcal{L} may be defined for $y \in \mathcal{F}(Q_{t_0})$, $\psi \in \mathring{\mathcal{F}}(Q_{t_0})$; \mathcal{Q} may be defined for $z \in C^0([0, t_0]; \tilde{C}^0(B))$, $\psi \in \mathring{\mathcal{F}}(Q_{t_0})$, and \mathcal{E} for χ with the property that $(a, \theta_0) \in \tilde{C}^1(B)$, $b \in \hat{C}^1(B)$ and $\psi \in \mathring{\mathcal{F}}(Q_{t_0})$.

We assume that $C_{ijk}(x)$, $\beta_{ij}(x)$, $k_{ij}(x)$, $\rho(x)$, $a(x)$ are measurable functions essentially limited on B , satisfying the symmetry conditions mentioned in Paragraph 25.

Integrating in part the following identity may be easily verified

$$\begin{aligned} \mathcal{L}(\psi, \psi) &= \frac{1}{2} \int_0^{t_0} \int_B \left[\rho \dot{v}_i \dot{v}_i + C_{ijkl} v_{i,j} v_{k,l} + a \vartheta^2 + \right. \\ &\quad \left. + \frac{2}{T_0} \int_0^t k_{ij} \vartheta_{,i} \vartheta_{,j} d\tau \right] dv dt + \frac{1}{2} t_0 \int_B (\rho \dot{v}_i \dot{v}_i + a \vartheta^2) |_{t=0} dv, \\ &\quad \forall \psi \in \mathring{\mathcal{F}}(Q_{t_0}). \end{aligned} \quad (32.8)$$

By $V(Q_{t_0})$ we designate the Hilbert space obtained by completing $\mathcal{F}(Q_{t_0})$ in the norm of $|\cdot|$ induced by the scalar product

$$\langle (u, \theta), (v, \vartheta) \rangle = \int_0^{t_0} \int_B \left[u_i v_i + \dot{u}_i \dot{v}_i + u_{i,j} v_{i,j} + \theta \vartheta + \int_0^t C_{ijkl} \vartheta_{,i} d\tau \right] dv dt.$$

Let $\mathring{V}(Q_{t_0})$ be the closed linear variety of $V(Q_{t_0})$ obtained by completing $\mathring{\mathcal{F}}(Q_{t_0})$ in the norm of $|\cdot|$.

We will designate by $\mathring{U}(Q_{t_0})$ the Hilbert space obtained by completing $\mathring{\mathcal{F}}(Q_{t_0})$ in the norm induced by the scalar product

$$[(u, \theta), (v, \vartheta)] = \langle (u, \theta), (v, \vartheta) \rangle + \langle (\dot{u}, \dot{\theta}), (\dot{v}, \dot{\vartheta}) \rangle.$$

Let $G(B)$ be the completion of the set $C^0(B)$ in the norm of

$$\|\cdot\|_{W_0(B) \times W_0(B)}.$$

Let us designate by $H_0(B)$ the completion of the set of
 $\{(u, w, 0) | (u, 0) \in \tilde{C}^1(B), w \in \tilde{C}^1(B)\}$ in the norm

$$\|(u, w, 0)\|_0 = \left\{ \frac{1}{2} \int_B [\rho w_i w_i + C_{ijkl} u_{i,j} u_{k,l} + a u^2] dv \right\}^{1/2}.$$

In view of the introduced notations and using the inequality of Schwarz and the immersion theorem of Sobolev [381], it can easily be seen that \mathcal{L} and \mathcal{Q} may be extended by continuity in $V(Q_{t_0}) \times \dot{U}(Q_{t_0})$ and $L_1([0, t_0]; G(B)) \times \dot{U}(Q_{t_0})$, respectively. Similarly $\delta(\chi, \psi)$ has a meaning for $\chi \in H_0(B)$, $\psi \in \dot{U}(Q_{t_0})$.

From (32.8) it follows that

$$\begin{aligned} \mathcal{L}(\psi, \psi) &\geq \frac{1}{2} \int_0^t \int_B [\rho \dot{v}_i \dot{v}_i + C_{ijkl} v_{i,j} v_{k,l} + a v^2 + \\ &+ \frac{2}{T_0} \int_0^t k_{ij} v_{i,j} v_{i,j} d\tau] dv dt, \quad \forall \psi \in \dot{\mathcal{F}}(Q_{t_0}), \end{aligned} \quad (32.9)$$

an inequality which may be extended to $\dot{U}(Q_{t_0})$.

In view of (32.4) and (32.6), from (32.9) it follows that there is a constant $c_1 > 0$ which depends only on T_0 , $\text{ess inf}_B \rho(x)$, $\text{ess inf}_B a(r)$, M , N , κ so that

$$|\psi|^2 \leq c_1 \mathcal{L}(\psi, \psi), \quad \forall \psi \in \dot{U}(Q_{t_0}). \quad (32.10)$$

We call a solution with finite energy of the system (26.2), (26.3), (26.5)₀ in Q_{t_0} , with initial conditions of $\chi = (u, b, \theta_0) \in H_0(B)$ and mass loads $z = (t, r) \in L_1((0, t_0); G(B))$, the element $y = (u, \theta) \in V(Q_{t_0})$ which satisfies the following

$$\mathcal{L}(y, \psi) = \mathcal{D}(z, \psi) + \mathcal{E}(x, \psi), \quad \forall \psi \in \dot{V}(Q_{t_0}), \quad (32.11)$$

$$u(t) \xrightarrow{W_0(B)} u_0, \quad \text{for } t \rightarrow 0. \quad (32.12)$$

It follows from Sobolev's immersion theorem that if $y = (u, \theta) \in V(Q_{t_0})$, then $u \in C^0([0, t_0]; W_0(B))$ which is what (32.12) states. Obviously, any conventional solution is a solution with finite energy.

Let $y = (\underline{u}, \theta)$ be a conventional solution of the problem under consideration. If $y = (\tilde{u}, \tilde{\theta}) = \int_0^t y(x, \tau) d\tau$ then we have

$$\int_0^t \dot{u}_i d\tau = \dot{u}_i - a_i, \quad \int_0^t \ddot{u}_i d\tau = \ddot{u}_i - b_i, \quad \int_0^t \dot{\theta} d\tau = \dot{\theta} - \theta_0, \quad (32.13)$$

and therefore, \tilde{y} satisfies the system

/95

$$\begin{aligned} \rho \ddot{u}_i &= (C_{ijkl} \tilde{u}_{k,l})_{,j} - (\beta_{ij} \tilde{\theta})_{,j} + \int_0^t f_i d\tau + \rho b_i, \\ T_0(a \ddot{\theta} + \beta_{ij} \ddot{u}_{i,j}) &= (k_{ij} \tilde{\theta}_{,j})_{,i} + \int_0^t r d\tau + T_0(a \theta_0 + \beta_{ij} a_{i,j}). \end{aligned}$$

In general, the following takes place

Lema 32.1. Let $y \in V(Q_{t_0})$ be a solution with finite energy in Q_{t_0} with initial conditions $x = (\underline{a}, \underline{b}, \theta_0)$ and mass loads $z = (\underline{f}, r)$. Then $\tilde{y} = \int_0^t y(x, \tau) d\tau$ is similarly a solution with finite energy in Q_{t_0} with initial conditions $\tilde{x} = (0, \underline{a}, 0)$ and mass loads of

$$\tilde{z} = \int_0^t z(x, \tau) d\tau + (\rho b, T_0(a\theta_0 + \beta_{i,j}a_{i,j})).$$

Proof. In view of the above we must show that

$$\mathcal{L}(\tilde{y}, \psi) = \mathcal{L}(\tilde{z}, \psi) + \mathcal{L}(\tilde{x}, \psi), \quad \forall \psi = (v, \vartheta) \in \tilde{U}(Q_0) \quad (32.14)$$

Keeping in mind (32.13) and the definition of \mathcal{L} , we can write

$$\begin{aligned} I(\tilde{y}, \psi) &= \int_0^t \int_B \left\{ \rho \dot{u}_i \frac{d}{dt} [(t - t_0) \dot{v}_i] + a \dot{\theta} \frac{d}{dt} [(t - t_0) \vartheta] - \right. \\ &\quad - [C_{ijkl} \tilde{u}_{k,l} - \beta_{ij} \tilde{\theta}] (t - t_0) \dot{v}_{i,j} + \beta_{ij} \tilde{u}_{i,j} \frac{d}{dt} [(t - t_0) \vartheta] + \\ &\quad + \frac{1}{T_0} \int_0^t k_{ij} \tilde{\theta}_{,j} \vartheta_{,i} d\tau \Big\} dv dt = \int_0^t \int_B \left\{ \rho \dot{u}_i \frac{d}{dt} [(t - t_0) \dot{v}_i] + \right. \\ &\quad + a \dot{\theta} \frac{d}{dt} [(t - t_0) \tilde{\vartheta}] - (C_{ijkl} u_{k,l} - \beta_{ij} \theta) (t - t_0) \dot{v}_{i,j} + \\ &\quad + \beta_{ij} u_{i,j} \frac{d}{dt} [(t - t_0) \tilde{\vartheta}] + \frac{1}{T_0} \int_0^t k_{ij} \theta_{,j} \tilde{\theta}_{,i} d\tau \Big\} dv dt + \\ &\quad + t_0 \int_B \rho a_i \dot{v}_i \Big|_{t=0} dv = \mathcal{L}(y, \tilde{\psi}) + t_0 \int_B \rho a_i \dot{v}_i \Big|_{t=0} dv, \end{aligned}$$

where $\tilde{\psi} = (\tilde{v}, \tilde{\vartheta})$ satisfies

$$\frac{d}{dt} [(t - t_0) \dot{v}_i] = - (t - t_0) \dot{v}_{i,j}, \quad \frac{d}{dt} [(t - t_0) \tilde{\vartheta}] = - (t - t_0) \vartheta. \quad (32.15)$$

Integrating (32.15) and setting $\tilde{v} = 0$ for $t = 0$, we obtain

/96

$$\begin{aligned}\tilde{v}(v, t) &= - \int_0^t \frac{1}{\xi - t_0} \int_{\xi_0}^{\xi} (\tau - t_0) \dot{v}(x, \tau) d\tau d\xi, \\ \tilde{\theta}(x, t) &= - \frac{1}{t - t_0} \int_{t_0}^t (\tau - t_0) \theta(x, \tau) d\tau.\end{aligned}\quad (32.16)$$

In view of the fact $\psi \in \dot{U}(Q_0)$ it follows that $\tilde{\psi} \in \dot{U}(Q_0)$, as $\tilde{\psi}$ is defined by (32.16).

Keeping in mind the fact that

$$\begin{aligned}\mathcal{D}(\tilde{z}, \psi) &= \int_0^t \int_B \left[- \left(\int_0^t f_i d\tau + \rho b_i \right) (t - t_0) \dot{v}_i - \right. \\ &\quad \left. - \left(\frac{1}{T_0} \int_0^t r d\tau + a\theta_0 + \beta_{ij} a_{i,j} \right) (t - t_0) \theta \right] dv dt = \\ &= \int_0^t \int_B \left\{ \left(\int_0^t f_i d\tau + \rho b_i \right) \frac{d}{dt} [(t - t_0) \tilde{v}_i] + \left(\frac{1}{T_0} \int_0^t r d\tau + \right. \right. \\ &\quad \left. \left. + a\theta_0 + \beta_{ij} a_{i,j} \right) \frac{d}{dt} [(t - t_0) \tilde{\theta}] \right\} dv dt,\end{aligned}$$

and integrating partially, we obtain

$$\mathcal{D}(\tilde{z}, \psi) = \mathcal{D}(z, \tilde{\psi}) + \mathcal{E}(\chi, \tilde{\psi}).$$

Similarly we have

$$\mathcal{E}(\tilde{\chi}, \psi) = t_0 \int_B \rho a_i \dot{v}_i \Big|_{t=0} dv.$$

Substituting these results in (32.14), this is reduced to

$$\mathcal{L}(y, \tilde{\psi}) = \mathcal{D}(z, \tilde{\psi}) + \mathcal{E}(\chi, \tilde{\psi}),$$

a relation which is satisfied on the basis of the fact that

$\tilde{\psi} \in \dot{U}(Q_0)$ and $y \in V(Q_0)$ is a solution with finite energy.

Theorem 32.1. If there is a solution with finite energy of the mixed problem in Q_{t_0} , then the solution is unique.

Proof. We assume that there are two solutions with finite energy $y_1, y_2 \in V(Q_{t_0})$. In view of the linear nature of the problem it must be that $y_1 - y_2$ is a solution with a finite energy in Q_{t_0} of the problem which corresponds to zero loads. Moreover, $y_1 - y_2 \in \dot{V}(Q_{t_0})$.

According to lemma 32.1, it follows that

$$y = \int_0^t [y_1(x, \tau) - y_2(x, \tau)] d\tau,$$

is similarly a solution with finite energy with homogeneous data, so that

197

$$\mathcal{L}(y, \psi) = \mathcal{D}(0, \psi) + \mathcal{E}(0, \psi) = 0, \quad \forall \psi \in \dot{U}(Q_{t_0}). \quad (32.17)$$

In view of the fact that $y_1 - y_2 \in \dot{V}(Q_{t_0})$ it must be that $y \in \dot{U}(Q_{t_0})$, therefore we can take $\psi = y$ in (32.17) and therefore

$$\mathcal{L}(y, y) = 0.$$

The inequality (32.10) implies that $|y| = 0$ and therefore $y_1 - y_2 = 0$.

Theorem 32.2. If the initial conditions are zero, and $z = (f, r) \in L_2([0, t_0]; G(R))$, then there exists a solution with finite energy $y = (u, 0) \in \dot{V}(Q_{t_0})$. Moreover, there exists a constant $c' > 0$ which depends only on c_1, T_0, t_0 , so that

$$|y| \leq c' \|z\|_{L_2(Q_{t_0}) \times L_2(Q_{t_0})}. \quad (32.18)$$

Proof. We must show the existence of a $y \in \dot{V}(Q_{t_0})$ so that

$$\mathcal{L}(y, \psi) = \mathcal{D}(z, \psi), \quad \forall \psi \in \dot{U}(Q_{t_0}).$$

Let us consider $\psi \in \hat{U}(Q_i)$ fixed, and $\mathcal{L}(y, \psi)$ as a linear functional defined on $\hat{V}(Q_{t_0})$. According to the representation theorem of Riesz-Fréchet, there exists a correspondence of $S: \hat{U}(Q_i) \rightarrow \hat{V}(Q_i)$ so that

$$\mathcal{L}(y, \psi) = \langle y, S\psi \rangle, \quad \forall y \in \hat{V}(Q_i). \quad (32.19)$$

Let $R(S)$ be the set of the values of S in $\hat{V}(Q_{t_0})$. The application of S is biunivocal. Indeed, if there exists $\psi_0 \in \hat{U}(Q_i)$ so that $S\psi_0 = 0$, then (32.19) implies that $\mathcal{L}(y, \psi_0) = 0, \forall y \in \hat{V}(Q_i)$ and therefore $\hat{U}(Q_i) \subset \hat{V}(Q_i)$ results in particular that $\mathcal{L}(\psi_0, \psi_0) = 0$. The inequality (32.10) implies, then, $\psi_0 = 0$ and therefore S is biunivocal. From this it follows that there is an application of $S^{-1}: R(S) \rightarrow \hat{U}(Q_i)$. Using the inequality (32.10) and the inequality of Schwarz, we have

$$|\psi|^2 \leq c_1 \mathcal{L}(\psi, \psi) = c_1 \langle \psi, S\psi \rangle \leq c_1 |\psi| |S\psi|,$$

or

$$|\psi| \leq c_1 |S\psi|, \quad \forall \psi \in \hat{U}(Q_i). \quad (32.20)$$

Let us show that $R(S)$ is a dense set in $\hat{V}(Q_{t_0})$. If we assume the contrary, it means that there exists $y_0 \in \hat{V}(Q_i) \setminus R(S), y_0 \neq 0$, so that for any $\psi \in \hat{U}(Q_i)$ we have $\langle y_0, S\psi \rangle = 0$. According to (32.19), $\mathcal{L}(y_0, \psi) = 0$ for any $\psi \in \hat{U}(Q_i)$ which means that y_0 is a solution with finite energy for the problem with zero loads. According to theorem 32.1, it follows that $y_0 = 0$ and therefore $R(S)$ are dense in $\hat{V}(Q_{t_0})$. Consequently we can extend application of S^{-1} , by continuity, to the whole of $\hat{V}(Q_{t_0})$ so that

$$S^{-1}: \hat{V}(Q_i) \rightarrow \hat{U}(Q_i),$$

is a limited operator with a norm

/98

$$\|S^{-1}\| \leq c_1. \quad (32.21)$$

Let us consider the bilinear form $\mathcal{D}(z, \psi)$. Using (32.21) and the inequality of Schwarz, it follows that there is a constant c'' which depends only on T_0, t_0 , so that for every $\mu \in \dot{V}(Q_{t_0})$, we have

$$|\mathcal{D}(z, S^{-1}\mu)| \leq c'' \|S^{-1}\mu\| \|z\|_{L_2(Q_{t_0}) \times L_4(Q_{t_0})} \leq c'' c_1 \|\mu\| \|z\|_{L_2(Q_{t_0}) \times L_4(Q_{t_0})}.$$

This fact indicates that the functional $\mathcal{D}(z, S^{-1}\mu)$ is limited on $\dot{V}(Q_{t_0})$. According to the representational theorem of Riesz-Fréchet, there is $\tilde{y}(z) \in \dot{V}(Q_{t_0})$ so that

$$\mathcal{D}(z, S^{-1}\mu) = \langle \tilde{y}(z), \mu \rangle, \quad \forall \mu \in \dot{V}(Q_{t_0}).$$

In particular, for every $\psi \in \dot{V}(Q_{t_0})$, $S\psi \in \dot{V}(Q_{t_0})$ we have

$$\mathcal{D}(z, \psi) = \langle \tilde{y}(z), S\psi \rangle, \quad (32.22)$$

In view of (32.19) and (32.22), it follows that $\tilde{y}(z)$ is the solution sought. From (32.19) -- (32.22) we conclude that

$$\|\tilde{y}\|^2 = \langle \tilde{y}, \tilde{y} \rangle = \mathcal{D}(\tilde{y}, S^{-1}\tilde{y}) = \mathcal{D}(z, S^{-1}\tilde{y}) \leq c'' c_1 \|\tilde{y}\| \|z\|_{L_2(Q_{t_0}) \times L_4(Q_{t_0})}.$$

from which follows the evaluation from the theorem.

We will next study the existence of a solution in the case when the initial data are not zero.

Let $H(B) \equiv H_0(B) \cap \{\hat{W}_1(B) \times \hat{W}_1(B) \times \hat{W}_1(B)\}$ and $z = (l, r) \in G(B)$.

We define the application of $P(z): H_0(B) \rightarrow H(B)$ by

$$(u, v, \theta) = P(z)(v, w, \theta), \quad (32.23)$$

where $(v, w, \theta) \in H_0(B)$, and $(u, v, \theta) \in H(B)$ so that $(u, \theta) \in \hat{W}_1(B) \times \hat{W}(B)$ is the solution of the system

$$\begin{aligned}\Phi(0, D) &= - \int_B (\beta_{ij} T_0 v_{i,j} + T_0 a \theta - r) D \, dv, \quad \forall D \in \hat{W}_1(B), \\ \Psi(u, \omega) &= \int_B (\beta_{ij} \theta \omega_{i,j} - \rho w_i \omega_i + f_i \omega_i) \, dv, \quad \forall \omega \in \hat{W}_1(B).\end{aligned}\tag{32.24}$$

Therefore, if $(v, w, \theta) \in H_0(B)$ is given, we can determine with the help of the application of $P(z)$, the triplet $(u, v, \theta) \in H(B)$ namely $(u, \theta) \in \hat{W}_1(B) \times \hat{W}_1(B)$ is the weak solution of the system

$$\begin{aligned}(k_{ij} \theta_{,j})_{,i} &= T_0 (\beta_{ij} v_{i,j} + a \theta) - r, \\ (C_{ijkl} u_{k,l})_{,j} - (\beta_{ij} \theta)_{,j} &= \rho w_i - f_i,\end{aligned}\tag{32.25}$$

with boundary conditions (26.5)₀. In view of (32.4) -- (32.6) /99 it follows that the elliptic system (32.25) with boundary conditions under consideration determines uniquely (u, θ) . This fact indicates that the application of $P(z)$ is biunivocal and therefore there is an application of $P^{-1}(z)$. Similarly, we have (Fichera [111]).

$$\|P(z)\chi\|_{W_1(B) \times W_1(B) \times W_1(B)} \leq c_2 \{|\chi|_0 + \|z\|_{W_1(B) \times W_1(B)}\}.\tag{32.26}$$

Let $z_0, \dots, z_{m-1} \in G(B)$ and $P_m(z_0, \dots, z_{m-1}) = P(z_0) \circ \dots \circ P(z_{m-1})$.

It is obvious that $P_m(z_0, \dots, z_{m-1}): H_0(B) \rightarrow H(B)$. Let us designate by $H_m(B; z_0, \dots, z_{m-1})$ the set of the values of the application of $P_m(z_0, \dots, z_{m-1})$ from $H(B)$. In view of the fact that $P(z)$ is biunivocal, it will be the same way also with $P_m(z_0, \dots, z_{m-1})$ and therefore there exists $P_m^{-1}(z_0, \dots, z_{m-1}): H_m(B; z_0, \dots, z_{m-1}) \rightarrow H_0(B)$. Let $\chi \in H_m(B; 0, \dots, 0)$. We define the norm in $H_m(B) \equiv H_m(B; 0, \dots, 0)$ by

$$|\chi|_m = |P_m^{-1}(0, \dots, 0)\chi|_0.$$

From the definition and the properties of the application of $P(z)$ the following properties of the H_m spaces follow (see Dafermos [73]).

Lema 32.2. Let $z_i \in G(B)$, $(i = 0, \dots, m-1)$, be fixed elements in $G(B)$. Then

(i) $H_m(B; z_0, \dots, z_{m-1})$ is a complete metric space with the metrics defined by $d_m(x, x') = \|x - x'\|_m$. $H_m(B; z_0, \dots, z_{m-1})$ is a plane in $H(B)$, and $H_m(B)$ is a Banach space with a norm $\|\cdot\|_m$;

(ii) $H_0(B) \supset H(B) \supset H(B; z_0) \supset \dots \supset H_m(B; z_0, \dots, z_{m-1})$; moreover, $H_m(B; z_0, \dots, z_{m-1})$ is dense in $H_l(B; z_0, \dots, z_{l-1})$ for any $l < m$.

Theorem 32.3. If $z = (t, r) \in C^{m-1}([0, t_0]; G(B))$, $z \in L_1([0, t_0];$

$G(B)$), $\chi = (u_0, \dot{u}_0, \theta_0) \in H_m(B; z(0), \dots, z^{(m-1)}(0))$ ($m = 0, 1, \dots$)

then the mixed problem under consideration has a solution with finite energy $y \in V(Q_{t_0})$ in Q_{t_0} . This solution has the following properties

(a) $(u, \dot{u}, \theta)(t) \in C^m([0, t_0]; H_0(B))$ and $(u, \dot{u}, \theta)(0) = (u_0, \dot{u}_0, \theta_0)$,

(b) $(u^{(k)}, \dot{u}^{(k)}, \theta^{(k)})(t) \in H_{m-k}(B; z^{(k)}(t), \dots, z^{(m-1)}(t))$, $k = 0, \dots, m$,
 $t \in [0, t_0]$,

and

$$(u^{(k)}, \dot{u}^{(k)}, \theta^{(k)})(t) = P_{m-k}(z^{(k)}(t), \dots, z^{(m-1)}(t))(u, \dot{u}, \theta)(t), \quad (32.27)$$

(c) there is a constant $c_3 > 0$ which depends only on T_0 , so that

$$\begin{aligned} & \| (u^{(k)}, \dot{u}^{(k)}, \theta^{(k)})(t) \|_0^2 + \frac{1}{T_0} \int_0^t \Phi^{(k)}(u, \dot{u}, \theta)(\tau) d\tau \leq \\ & \leq \| P_{m-k}(z(0), \dots, z^{(m-1)}(0)) \chi \|_0^2 + c_3 \int_0^t \| z^{(k)}(\tau) \|_{W_0(B) \times W_0(B)}^2 d\tau, \end{aligned} \quad (32.28)$$

for $k = 0, 1, \dots, m$, $t \in [0, t_0]$. In (32.28) the equality takes place for $z \equiv 0$.

Proof. On the basis of lemma 32.2, $H_{m+1}(B; z(0), \dots, z^{(m+1)}(0))$ is dense in $H_m(B; z(0), \dots, z^{(m)}(0))$. Making use of this fact and of (32.28) it follows that it is sufficient to prove the theorem for $z \in C^\infty([0, t_0]; G(B))$, $\chi \in H_{m+1}(B; z(0), \dots, z^{(m+1)}(0))$.

We will use the following auxiliary construction. From (32.23) we have $(v, w, \theta) = P^{-1}(z)(u, v, \theta)$ where (w, θ) is the solution in the weak sense of the equations /100

$$\begin{aligned} T_0 a \theta &= (k_i \theta_{,i})_{,i} - T_0 \beta_{ij} v_{i,j} + r, \\ \rho w_i &= (C_{ijkl} u_{k,l})_{,j} - (\beta_{ij} \theta)_{,j} + f_i, \end{aligned} \quad (32.29)$$

with the boundary conditions (26.5)₀. We define

$$\begin{aligned} \chi^{(k)} &= (u_0^{(k)}, u_0^{(k+1)}, \theta_0^{(k)}) = P^{-1}(z^{(k)}(0))(u_0^{(k-1)}, u_0^{(k)}, \theta_0^{(k-1)}) = \\ &= P_k^{-1}(z(0), \dots, z^{(k-1)}(0))(u_0, \dot{u}_0, \theta_0), \quad (k = 1, 2, \dots, m+2). \end{aligned} \quad (32.30)$$

Keeping in mind (32.29), we have

$$\begin{aligned} T_0 a \theta_0^{(k)} &= (k_{ij} \theta_{0,j}^{(k-1)})_{,i} - T_0 \beta_{ij} u_{0i,j}^{(k-1)} + r^{(k-1)}(0), \\ \rho u_{0i}^{(k+1)} &= (C_{ijkl} u_{0r,s}^{(k-1)})_{,j} - (\beta_{ij} \theta_0^{(k-1)})_{,j} + f_i^{(k-1)}(0), \\ &\quad (k = 1, 2, \dots, m+2); \end{aligned} \quad (32.31)$$

in the weak sense.

By determining $\chi^{(k)}$ in this way we define

$$y^{(1)}(t) \equiv (u^{(1)}, \theta^{(1)})(t) = \left(\sum_{k=0}^{m+1} \frac{1}{k!} u_0^{(k)} t^k, \sum_{k=0}^{m+2} \frac{1}{k!} \theta_0^{(k)} t^k \right). \quad (32.32)$$

Let us introduce the notation

$$(\dot{u}^{(1)}, f^*, r^*)(t) = P^{-1}(z(t)) (u^{(1)}, \dot{u}^{(1)}, \theta^{(1)})(t). \quad (32.33)$$

On the basis of relation (32.29) we derive

$$\begin{aligned} T_0 ar^* &= (k_{ij} \dot{\theta}_j^{(1)})_i - T_0 \beta_{ij} \dot{u}_{i,j}^{(1)} + r(t), \\ f^* &= (C_{ijkl} u_{kl,i}^{(1)})_j - (\beta_{ij} \theta^{(1)})_j + f_i(t). \end{aligned} \quad (32.34)$$

Let

$$z^{(2)}(t) = ((f^* - \rho \ddot{u}^{(1)}), T_0 a(r^* - \dot{\theta}^{(1)}))(t). \quad (32.35)$$

In view of the fact that $z(t) \in C^x([0, t_0]; G(B))$, according to (32.32) -- (32.35) it follows that $z^{(2)}(t) \in C^x([0, t_0]; G(B))$ and

$$z^{(k)}(0) = 0, \quad (k = 0, 1, \dots, m+1). \quad (32.36)$$

Let $\hat{y}^{(2)} \in \dot{V}(Q_{t_0})$ be the solution with finite energy of the problem corresponding to zero initial conditions and mass loads of $z^{(m+2)}(t)$. The existence of $y^{(2)}(t)$ is ensured by theorem 32.2. Let

$$y^{(2)}(t) = (u^{(2)}, \theta^{(2)})(t) = \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \dots \int_0^{\tau_{m+1}} \hat{y}^{(2)}(\tau) d\tau. \quad (32.37)$$

Applying successively lemma 32.1 and making use of (32.36), /101
(32.37), we conclude that $y^{(k)}(t)$, $k = 0, 1, \dots, m+2$,
is a solution with finite energy of the problem with zero initial
conditions and mass loads of $z^{(k)}(t)$. Consequently

$$\mathcal{L}(y^{(k)}, \psi) = \mathcal{D}(z^{(k)}, \psi), \quad \forall \psi \in \dot{U}(Q_t), \quad (k = 0, 1, \dots, m+2). \quad (32.38)$$

In view of (32.30) -- (32.35) and the definition of \mathcal{D} we conclude

$$\mathcal{D}(z^{(2)}, \psi) = -\mathcal{L}(y^{(1)}, \psi) + \mathcal{D}(z, \psi) + \mathcal{E}(\chi, \psi), \quad \forall \psi \in \dot{U}(Q_t). \quad (32.39)$$

Keeping in mind the linearity of \mathcal{L} , it follows from (32.38) and (32.39) that

$$\begin{aligned} \mathcal{L}(y, \psi) &= \mathcal{D}(z, \psi) + \mathcal{E}(\chi, \psi), \quad \forall \psi \in \dot{U}(Q_t), \\ (k &= 0, 1, \dots, m+2), \end{aligned} \quad (32.40)$$

where

$$y(t) = y^{(1)}(t) + y^{(2)}(t). \quad (32.41)$$

Therefore $y^{(k)}(t)$, $k = 0, 1, \dots, m+2$, is a solution with finite energy in Q_{t_0} of the problem with initial conditions $\chi = (\overset{(k)}{u}_0, \overset{(k)}{u}_0, \overset{(k)}{\theta}_0)$ and mass loads $\overset{(k)}{z}$. From (32.32), (32.37) and (32.40) it follows that $y(t)$ given by (32.41) is the solution sought. Let us show that the properties (α) -- (γ) do take place. In view of the fact that $y^{(2)}(t) \in \dot{V}(Q_t)$, (32.27) implies the fact that $y^{(2)}(t) \in C^{m+1}([0, t_0]; H(B))$. Inasmuch as $y^{(1)}(t) \in C^\infty([0, t_0]; H_0(B))$ from (32.41) it follows that $y(t) \in C^{m+1}([0, t_0]; H_0(B))$. It is obvious that we have from (32.41) $(\underline{u}, \underline{u}, \theta)(0) = (\underline{u}_0, \underline{u}_0, \theta_0)$, which proves the affirmation of (α). In order to prove (β) it is sufficient to prove (32.27). The relation (32.40) takes place for any $\psi \in \dot{U}(Q_t)$ and any t_0 and therefore for any $t \in (0, t_0)$ and any $\psi \in U(Q_t)$, where $Q_t = B \times (0, t)$. Selecting $\psi = (0, T_0 D(x)) \in \dot{U}(Q_t)$ in (32.40), we find

$$\begin{aligned} & \int_0^t \int_B [T_0(a \overset{(h)}{\theta} D + \beta_{i,j} \overset{(h)}{u}_{i,j}, D) + \int_0^\tau k_{i,j} \overset{(h)}{\theta}_{i,j}, D, ds] dv d\tau = \\ & = - \int_0^t \int_B (\tau - t) \overset{(h)}{r} D dv d\tau + t \int_B (a \overset{(h)}{\theta}_0 + \beta_{i,j} \overset{(h)}{u}_{0i,j}) T_0 D dv, \end{aligned}$$

a relation that leads to

$$\int_B k_{i,j} \overset{(h)}{\theta}_{i,j}, D, dv = \int_B [\overset{(h)}{r} - T_0(a \overset{(h+1)}{\theta} + \beta_{i,j} \overset{(h+1)}{u}_{i,j})] D dv, \quad \forall D \in \hat{W}_1(B). \quad (32.42)$$

Taking $\psi = (\tau v(x), 0) \in \hat{U}(Q_t)$, $v(x) \in \hat{W}_1(B)$ in (32.40), we obtain

$$\begin{aligned} & \int_0^t \int_B [(\tau - t) (-C_{ijkl} \overset{(h)}{u}_{k,l} v_{i,j} + \beta_{i,j} \overset{(h)}{\theta}_{i,j} v_{i,j}) + \rho \overset{(h+1)}{u}_i v_i] dv d\tau = - \\ & - \int_0^t \int_B (\tau - t) f_i v_i dv d\tau + t \int_B \rho \overset{(h)}{u}_{0i} v_i dv. \end{aligned}$$

Similarly, we have

/102

$$\begin{aligned} \int_B C_{ijkl} \overset{(h)}{u}_{k,l} v_{i,j} dv &= \int_B (\beta_{i,j} \overset{(h)}{\theta}_{i,j} v_{i,j} - \rho \overset{(h+2)}{u}_i v_i + f_i v_i) dv, \\ &\forall v \in \hat{W}_1(B). \end{aligned} \quad (32.43)$$

It follows from (32.42), (32.43), (32.24) that

$$\begin{aligned} (\overset{(h)}{u}, \overset{(h+1)}{u}, \overset{(h)}{\theta})(t) &= P(z(t)) (\overset{(h+1)}{u}, \overset{(h+2)}{u}, \overset{(h+1)}{\theta})(t), \\ (k &= 0, 1, \dots, m-1). \end{aligned} \quad (32.44)$$

The relation (32.27) can easily be derived from (32.44) by induction.

Let us show that the statement made in (γ) does take place. Let us consider the relation (32.40) written for Q_t . In view of the fact that $\psi \equiv (\overset{(h)}{u} - \overset{(h)}{u}_0, \overset{(h)}{\theta}) \in \hat{U}(Q_t)$, we have

$$\mathcal{L}(y, \psi') = \mathcal{D}(z, \psi') + \mathcal{E}(\chi, \psi').$$

By deriving this relation with respect to t twice and by partial integration, we obtain

$$\begin{aligned} & \frac{1}{2} \int_B [\rho u_t^{(k+1)} u_t^{(k+1)} + C_{ijkl} u_{k,i}^{(k)} u_{l,j}^{(k)} + a \theta^2] (t) dr + \\ & + \int_0^t \int_B \frac{1}{T_0} k_{ij} \theta_{,j}^{(k)} \theta_{,i}^{(k)} d\tau dv + \frac{1}{2} \int_B (\rho u_{0t}^{(k+1)} u_{0t}^{(k+1)} - C_{ijkl} u_{0k,i}^{(k)} u_{0l,j}^{(k)} + \\ & + a \theta_0^2 + \beta_{ij} u_{0i,j}^{(k)} \theta_0) dr = \int_0^t \int_B (f_i^{(k)} u_i^{(k+1)} + \frac{1}{T_0} r \theta^{(k)}) dr d\tau + \\ & + \int_B (\rho u_{0t}^{(k+1)} u_{0t}^{(k+1)} + a \theta_0^2 + \beta_{ij} u_{0i,j}^{(k)} \theta_0) dr. \end{aligned}$$

By deriving the previous relation we obtain

$$\begin{aligned} \frac{d}{dt} \|\langle u, u, \theta \rangle(t)\|^2 + \frac{1}{T_0} \Phi(\theta, \theta)(t) &= \int_B (f_i^{(k)} u_i^{(k+1)} + \frac{1}{T_0} r \theta^{(k)})(t) dv \leq \\ &\leq 2c_3 \|\langle u, u, \theta \rangle(t)\|_0 \|z(t)\|_{W_0(B) \times W_0(E)}. \end{aligned}$$

We used the inequality of Schwarz in deriving the last inequality. Here, c_3 depends on T_0 . We can write

$$\begin{aligned} \frac{d}{dt} \left\{ \|\langle u, u, \theta \rangle(t)\|^2 + \frac{1}{T_0} \int_0^t \Phi(\theta, \theta)(\tau) d\tau \right\} &\leq \\ &\leq 2c_3 \left\{ \|\langle u, u, \theta \rangle(t)\|^2 + \frac{1}{T_0} \int_0^t \Phi(\theta, \theta)(\tau) d\tau \right\}^{1/2} \|z\|_{W_0(B) \times W_0(E)}. \end{aligned}$$

Dividing by $2\{\cdot\}^{1/2}$ and integrating on $(0, t)$ we

/103

conclude

$$\left\{ |(u, u, \theta)(t)|^2 + \frac{1}{T_0} \int_0^t \Phi(\theta, \theta)(\tau) d\tau \right\}^{1/2} \leq \quad (32.45)$$

$$\leq |(u, u, \theta)(0)| + c_3 \int_0^t \|z(\tau)\|_{W_0(B) \times W_0(B)} d\tau.$$

In view of the fact that

$$(u, u, \theta)(0) = (u_0, u_0, \theta_0) = P_k^{-1}(z(0), \dots, z^{(k)}(0))(u_0, u_0, \theta_0),$$

the relation (32.45) leads to (32.28).

The coefficients which characterize the properties of the medium satisfy the conditions of regularity of the order m if

- (i) $C_{ijkl}, k_{ij} \in C^{m-1}(\bar{B})$, and for $m = 1$ the generalized derivatives $C_{ijkl,j}$ and $k_{ij,i}$ do exist and are essentially limited on B ;
- (ii) $\beta_{ij} \in C^{m-2}(\bar{B})$, $m \geq 2$, and for $m = 1, 2$, the generalized derivatives $\beta_{ij,j}$ do exist and are essentially limited on B ;
- (iii) $\rho, a \in C^{m-2}(\bar{B})$ and have generalized derivatives of the order of $m - 1$ which are essentially limited on B . This can be proven (Dafermos [73]).

Theorem 32.4. If

$$(\alpha) \quad z = (f, r) \in C^k([0, t_0]; G(B) \cap (W_{m-k-1}(B') \times W_{m-k-1}(B')))$$

where B' is a certain compact subset of B and $z^{(m)} \in L_1([0, t_0]; G(B))$;

$$(\beta) \quad \chi \equiv (a, b, \theta_0) \in H_m(B; z(0), \dots, z^{(m-1)}(0));$$

(γ) $C_{ijkl}, \beta_{ij}, k_{ij}, \rho, a$ satisfy the regularity conditions of the order m , then the solution $y = (u, \theta)$ with finite energy in Q_{t_0} of the problem under consideration satisfies the conditions

$$(u, \dot{u}, \theta)(t) \in C^k([0, t_0]; C^{m-k-1}(B) \times C^{m-k-2}(B) \times C^{m-k-1}(B)),$$

for $k = 0, 1, \dots, m - 2$.

It follows from this theorem that if $y = (\underline{u}, \theta)$ is a solution with finite energy of the posed problem, and $m = 3$, then y is the conventional solution in the sense of the definition in Par. 26. These results were extended by Chirita [59] in the case of micropolar thermoelasticity.

In [73] Dafermos sets also the asymptotic behavior of the solution when the time tends toward infinity. Ericksen posed the problem of stability in the theory of thermoelasticity [104], [105]; this was studied also in various papers (see for example Koiter [238], Knops and Wilkes [236], Gurtin [166], Naghdi and Trapp [301]).

33. Homogeneous and Isotropic Media

/104

Throughout the rest of this chapter we will consider homogeneous and isotropic media. From the facts presented in Paragraph 26, it follows that in the case of homogeneous and isotropic media the equations of thermoelasticity are

$$\mu \Delta \underline{u} + (\lambda + \mu) \text{grad div } \underline{u} - \beta \text{grad } \theta - \rho \ddot{\underline{u}} = -\underline{f}, \quad (33.1)$$

$$k \Delta \theta - T_0 \beta \text{div } \dot{\underline{u}} - c \dot{\theta} = -r. \quad (33.2)$$

We will introduce the operators

$$\begin{aligned} \square_2 &= \Delta - \frac{1}{c_2^2} \frac{\partial^2}{\partial t^2}, & \Delta_1 &= \frac{k}{c} \Delta - \frac{\partial}{\partial t}, \\ \Delta_2 &= \Delta_1 - \frac{\lambda + 2\mu}{\lambda + \mu} \epsilon \frac{\partial}{\partial t}, \end{aligned} \quad (33.3)$$

where

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_2 = \sqrt{\frac{\mu}{\rho}}, \quad \epsilon = \frac{\beta^2 T_0}{c \rho c_1^2}. \quad (33.4)$$

Theorem 33.1. Let $u \in C^{3,2}$, $\theta \in C^{3,1}$, $f \in C^{1,0}$ on $B \times (0, t_0)$ which satisfies equations (33.1) Then \underline{u} , $\underline{\theta}$, \underline{f} satisfy the equations

$$\square_1 \operatorname{div} u = \frac{1}{\lambda + 2\mu} (\beta \Delta \theta - \operatorname{div} f), \quad (33.5)$$

$$\square_2 \operatorname{rot} u = - \frac{1}{\mu} \operatorname{rot} f \quad (33.6)$$

Proof. By applying to the equations (33.1) first the operator of divergence and then the rotor we obtain the equations (33.4) and (33.5), respectively.

It should be noted that the equation of the rotational wave is independent of temperature. The equation (33.5) was obtained by Voigt [432], and equation (33.6) by Cristea [68].

Theorem 33.2. If $u \in C^3$, $\theta \in C^{4,3}$, $f \in C^1$, $r \in C^2$ on $B \times (0, t_0)$ and equations (33.1) and (33.2) are satisfied, then we have

$$\begin{aligned} c_1^2 \Delta \Delta \theta - \left(\frac{T_0 \beta^2}{\rho} + c_1^2 \right) \Delta \theta - k \Delta \ddot{\theta} + c \ddot{\theta} = \\ = - \frac{T_0 \beta}{\rho} \operatorname{div} \dot{f} - c_1^2 \square_1 r. \end{aligned} \quad (33.7)$$

Proof. The operator \square_1 of the equation (33.2) is applied keeping in mind that (33.5). This result was derived by Cristea [68], Lessen and Duke [257], Chadwick and Sneddon [48] and Paria [332].

In a similar way it is shown that $\operatorname{div} \underline{u}$ also satisfies the equation (33.7) (Voigt [432]).

/105

34. Formal Representations of the Solutions

Let us assume that $f \in C^2$ on $B \times (0, t_0)$ and $f \in C^{2,1}$ on $B \times (0, t_0)$. In that case we can write

$$f = \text{grad } \theta + \text{rot } \gamma, \quad (34.1)$$

where $\theta \in C^{2,1}$, $\gamma \in C^{2,1}$ on $B \times (0, t_0)$ and $\text{div } \gamma = 0$.

We will assume that $\beta \neq 0$. If we set

$$u = \text{grad } \varphi + \text{rot } \psi, \quad (34.2)$$

we obtain from equation (33.1)

$$\text{grad } [(\lambda + 2\mu) \Delta \varphi - \rho \ddot{\varphi} - \beta \theta + \vartheta] + \text{rot } [\mu \Delta \psi - \rho \ddot{\psi} + \gamma] = 0.$$

Thus, we can satisfy the equations (33.1) by taking

$$\square_1 \varphi = \frac{1}{\lambda + 2\mu} (\beta \theta - \vartheta), \quad (34.3)$$

$$\square_2 \psi = -\frac{1}{\mu} \gamma. \quad (34.4)$$

It follows from (34.3)

$$\theta = \frac{1}{\beta} [(\lambda + 2\mu) \square_1 \varphi + \vartheta]. \quad (34.5)$$

Substituting (34.2) and (34.5) in equation (33.2), we obtain

$$\left(\square_1 \square_1 - c \frac{\partial}{\partial t} \Delta \right) \varphi = -\frac{\beta}{c(\lambda + 2\mu)} r - \frac{1}{\lambda + 2\mu} \Delta_1 \vartheta. \quad (34.6)$$

Thus, we have

Theorem 34.1. (Deresiewicz [82], Zorski [457]). If

$$\underline{u} = \text{grad } \varphi + \text{rot } \underline{\psi}, \quad \theta = \frac{1}{\beta} [(\lambda + 2\mu) \Delta_1 \varphi + \vartheta], \quad (34.7)$$

where $\varphi \in C^{4,2}$, $\underline{\psi} \in C^{3,2}$ on $B \times (0, t_0)$ and equations (34.6), (34.4) are satisfied, then \underline{u} and θ satisfy the equations (33.1), (33.2).

The following theorem establishes the completeness of solution (34.7).

Theorem 34.2. (Carlson [46]) Let $\underline{u} \in C^{M,N}$ on $B \times (0, t_0)$, ($M \geq 3$, $N \geq 2$), $\underline{u} \in C^0$ on $B \times (0, t_0)$ and $\theta \in C^{P,Q}$ on $B \times (0, t_0)$, ($P \geq 2$, $Q \geq 0$) which satisfy the equations (33.1) for $\underline{f} = \text{grad } \theta + \text{rot } \underline{\gamma}$ with θ and $\underline{\gamma}$ of the class $C^{P,Q}$ on $B \times (0, t_0)$ and $\text{div } \underline{\gamma} = 0$. In that case there is a function ϕ and a vector $\underline{\psi}$ of the class $C^{M,R}$ on $B \times (0, t_0)$ where $R = \min \{N + 2, Q + 2\}$, so that

/106

$$\underline{u} = \text{grad } \varphi + \text{rot } \underline{\psi}, \quad \theta = \frac{1}{\beta} [(\lambda + 2\mu) \Delta_1 \varphi + \vartheta],$$

$$\Delta_1 \underline{\psi} = -\frac{1}{\mu} \underline{\gamma}, \quad \text{div } \underline{\psi} = 0.$$

Moreover, if $M \geq 4$, $P \geq 2$, $Q \geq 1$, and \underline{u} and θ satisfy (33.2), then ϕ satisfies equation (34.6).

Proof. Keeping in mind (33.3) and (33.4), the equations (33.1) and (33.2) may be written, respectively, in the form

$$c_1^2 \Delta_1 \underline{u} + (c_1^2 - c_2^2) \text{grad div } \underline{u} - \frac{\beta}{\rho} \text{grad } \theta = -\frac{1}{\rho} \underline{f}, \quad (34.8)$$

$$\Delta_1 \theta - \frac{T_0 \beta}{c} \text{div } \dot{\underline{u}} = -\frac{1}{c} r. \quad (34.9)$$

If we keep in mind the fact that

$$\text{rot rot } u = \text{grad div } u - \Delta u,$$

the equation (34.8) may also be written in the following way

$$c_1^2 \text{grad div } u - c_2^2 \text{rot rot } u - \frac{\beta}{\rho} \text{grad } \theta + \frac{1}{\rho} f = \ddot{u}. \quad (34.10)$$

If we integrate this equation we can derive

$$\begin{aligned} u(t) = u(\tau) + \dot{u}(\tau)(t - \tau) + \text{grad} \int_{\tau}^t \int_{\tau}^t [c_1^2 \text{div } u - \kappa(\lambda)] d\lambda ds + \\ + \text{rot} \int_{\tau}^t \int_{\tau}^t [-c_2^2 \text{rot } u(\lambda) - \zeta(\lambda)] d\lambda ds, \end{aligned} \quad (34.11)$$

where $\kappa = (\beta\theta - \vartheta)/\rho$, $\zeta = -\gamma/\rho$.

Using for $\underline{u}(\tau)$ and $\dot{\underline{u}}(\tau)$ a decomposition of the form of (34.2), we can write

$$u(\tau) + \dot{u}(\tau)(t - \tau) = \text{grad } a(t) + \text{rot } b(t),$$

where a and \underline{b} are of the class C^2 and satisfy the conditions

$$\ddot{a} = 0, \quad \ddot{b} = 0, \quad \text{div } b = 0.$$

If we define the functions ϕ and $\underline{\psi}$ on $B \times (0, t_0)$ by

$$\begin{aligned} \varphi(t) = a + \int_{\tau}^t \int_{\tau}^t [c_1^2 \text{div } u(\lambda) - \kappa(\lambda)] d\lambda ds, \\ \psi(t) = b - \int_{\tau}^t \int_{\tau}^t [c_2^2 \text{rot } u(\lambda) + \zeta(\lambda)] d\lambda ds, \end{aligned}$$

then we can derive from (34.11)

$$u = \text{grad } \varphi + \text{rot } \psi$$

and in view of the fact that $\text{div } \gamma = 0$, we have $\text{div } \underline{\psi} = 0$.

It can easily be seen that the following relations take place

$$\begin{aligned}\ddot{\varphi} &= c_1^2 \operatorname{div} \underline{u} + \frac{1}{\rho} (\theta - \beta \theta), \\ \ddot{\psi} &= c_2^2 \operatorname{rot} \underline{u} + \frac{1}{\rho} \gamma, \\ \operatorname{div} \underline{u} &= \Delta \varphi, \quad \operatorname{rot} \underline{u} = -\Delta \psi,\end{aligned}$$

whence we obtain

$$\begin{aligned}0 &= \frac{1}{\beta} [(\lambda + 2\mu) \square_1 \varphi + \theta], \\ \square_2 \psi &= -\frac{1}{\mu} \gamma.\end{aligned}$$

Replacing \underline{u} and θ in equation (34.9) it follows that ϕ satisfies the equation (34.6).

Theorem 34.3. (Nowacki [316], Soos [385]). If

$$\begin{aligned}\underline{u} &= (\lambda + 2\mu) \left[\square_1 \underline{\Phi} - \varepsilon \Delta \frac{\partial}{\partial t} \right] \underline{\Phi} - (\lambda + \mu) \Delta_2 \operatorname{grad} \operatorname{div} \underline{\Phi} \\ &\quad + \frac{\beta}{\lambda + 2\mu} \operatorname{grad} \Psi, \\ \theta &= \frac{\beta T_0 \mu}{c} \square_2 \operatorname{div} \frac{\partial \underline{\Phi}}{\partial t} + \square_1 \Psi,\end{aligned}\tag{34.12}$$

where $\underline{\Phi}$ is a vector of the class C^6 on $B \times (0, t_0)$, and Ψ is a function of the class C^4 on $B \times (0, t_0)$ which satisfy the equations

$$\square_2 \left(\square_1 \Delta_1 - \epsilon \Delta \frac{\partial}{\partial t} \right) \Phi = \frac{1}{\mu(\lambda + 2\mu)} f, \quad (34.13)$$

$$\left(\square_1 \Delta_1 - \epsilon \Delta \frac{\partial}{\partial t} \right) \Psi = -\frac{1}{c} r, \quad (34.14)$$

then \underline{u} and θ satisfy the equations (33.1), (33.2), with the force \underline{f} and heat source r .

Proof. In view of (33.3) and (33.4), the relations (34.12) may be written in the following form

$$\begin{aligned} \underline{u} &= \rho c_1^2 \left[\square_1 \Delta_1 - \epsilon \Delta \frac{\partial}{\partial t} \right] \Phi - \rho(c_1^2 - c_2^2) \Delta_2 \text{grad div } \Phi + \frac{\beta}{\rho c_1^2} \text{grad } \Psi, \\ \theta &= -\frac{\epsilon \rho^2 c_1^2 c_2^2}{\beta} \square_2 \text{div } \frac{\partial \Phi}{\partial t} + \square_1 \Psi. \end{aligned} \quad (34.15)$$

If we replace (34.15) in (34.8), we obtain

/108

$$\begin{aligned} &c_2^2 \square_2 u + (c_1^2 - c_2^2) \text{grad div } u - \frac{\beta}{\rho} \text{grad } \theta = \\ &= \rho c_1^2 c_2^2 \square_2 \left(\square_1 \Delta_1 - \epsilon \Delta \frac{\partial}{\partial t} \right) \Phi - \left[\rho c_1^2 (c_2^2 - c_1^2) \left(\square_1 \Delta_1 - \epsilon \Delta \frac{\partial}{\partial t} \right) + \right. \\ &+ \rho c_2^2 (c_1^2 - c_2^2) \square_2 \Delta_2 + \rho (c_1^2 - c_2^2)^2 \Delta_2 + \epsilon \rho c_1^2 c_2^2 \square_2 \frac{\partial}{\partial t} \left. \right] \text{grad div } \Phi + \\ &+ \frac{\beta}{\rho} \left[\frac{c_2^2}{c_1^2} \square_2 + \frac{c_2^2 - c_1^2}{c_1^2} \Delta - \square_1 \right] \text{grad } \Psi. \end{aligned} \quad (34.16)$$

In view of (34.13) and the relations

$$\begin{aligned}
\rho c_1^2(c_1^2 - c_2^2) \square_2 \Delta_2 + \rho(c_1^2 - c_2^2) \Delta_2 \Delta + \epsilon \rho c_1^2 c_2^2 \square_2 \frac{\partial}{\partial t} = \\
= \rho c_1^2(c_1^2 - c_2^2) \left(\square_1 \Delta_1 - \epsilon \Delta \frac{\partial}{\partial t} \right), \\
\frac{c_2^2}{c_1^2} \square_2 + \frac{c_1^2 - c_2^2}{c_1^2} \Delta = \square_1,
\end{aligned}$$

the second member of (34.16) is reduced to $-\underline{f}/\rho$.

If we replace (34.15) in (34.9) and in consideration of (34.14) it follows that

$$\begin{aligned}
\Delta_1 \theta - \frac{T_0 \beta}{c} \operatorname{div} \dot{u} = \left(\square_1 \Delta_1 - \epsilon \Delta \frac{\partial}{\partial t} \right) \Psi + \\
+ \left[\frac{\epsilon \rho^2 c_1^2 c_2^2}{\beta} \square_1 \Delta_1 + \frac{T_0 \beta \rho}{c} (c_1^2 - c_2^2) \Delta \Delta_2 - \frac{T_0 \beta}{c} \rho c_1^2 (\square_1 \Delta_1 - \right. \\
\left. - \epsilon \Delta \frac{\partial}{\partial t}) \right] \operatorname{div} \Phi = -\frac{r}{c}.
\end{aligned}$$

In the quasistatic theory of thermoelasticity Ionescu-Cazimir [213] derived the Galerkin type representation (24.12). In the case of the dynamic theory of coupled thermoelasticity, the representation (34.12) was derived by Nowacki [316] and by de Soós [385] using the method of associated matrices (Moisil [292]).

De Soós gave, in [385], an expression of the equations of coupled thermoelasticity with the help of the functions t_{ij} and θ . This result extends the results derived by Ignaczak [208] in the uncoupled theory and by Ionescu-Cazimir [214] in the quasistatic case.

35. Thermoelastic Waves

/109

a) Plane harmonic waves. A plane thermoelastic wave is characterized by the fact that at a given moment on any plane

perpendicular with respect to a fixed direction, the components of the displacement vector and temperature are constant. The direction considered is called the direction of propagation of the wave. We will choose the system of coordinates in such a way that the axes Ox_1 coincide with the direction of propagation. In this case the plane wave is characterized by

$$u_i = u_i(x_1, t), \quad \theta = \theta(x_1, t). \quad (35.1)$$

In the absence of a mass force and a heat source the equations (33.1), (33.2) become

$$\begin{aligned} \left(\frac{\partial^2}{\partial x_1^2} - \frac{1}{c_1^2} \frac{\partial^2}{\partial t^2} \right) u_1 - \frac{\beta}{\rho c_1^2} \frac{\partial \theta}{\partial x_1} &= 0, \\ \left(\frac{\partial^2}{\partial x_1^2} - \frac{1}{c_2^2} \frac{\partial^2}{\partial t^2} \right) u_2 &= 0, \quad \left(\frac{\partial^2}{\partial x_1^2} - \frac{1}{c_2^2} \frac{\partial^2}{\partial t^2} \right) u_3 = 0 \\ \left(\frac{k}{c} \frac{\partial^2}{\partial x_1^2} - \frac{\partial}{\partial t} \right) \theta - \frac{T_0 \beta}{c} \frac{\partial^2 u_1}{\partial x_1 \partial t} &= 0. \end{aligned} \quad (35.2)$$

The component u_1 describes the longitudinal elastic wave, the components u_2 and u_3 transverse elastic waves, and θ the thermal wave. It is noted that the transversal waves propagate with a velocity c_2 and are not affected by the heat field. In the following we will not study them.

If

$$u_i = \operatorname{Re} [u_i^*(x_1, \omega) e^{-i\omega t}], \quad \theta = \operatorname{Re} [\theta^*(x_1, \omega) e^{-i\omega t}], \quad (i = \sqrt{-1}), \quad (35.3)$$

the plane wave under consideration is called harmonic. $\operatorname{Re} []$ was used to designate the real part of the expression from the parenthesis. In this paragraph we will analyze the thermomechanical interaction

studying the propagation of plane harmonic waves in space. The problem was studied by Chadwick and Sneddon [48] (see also Chadwick [49], Nowacki [319]).

It follows from (35.2) and (35.3) that u_1 and θ satisfy the equations

$$\begin{aligned} \left(\frac{d^2}{dx_1^2} + \sigma^2 \right) u_1^* - \frac{\beta}{\rho c_1^2} \frac{d\theta^*}{dx_1} &= 0, \\ \left(\frac{d^2}{dx_1^2} + q \right) \theta^* + \frac{i T_0 \beta \omega}{k} \frac{du_1^*}{dx_1} &= 0, \end{aligned} \quad (35.4)$$

where

$$\sigma^2 = \frac{\omega^2}{c_1^2}, \quad q = \frac{i \omega \rho}{k}. \quad (35.5)$$

In the following we will assume that ω is a real fixed constant.

Let us consider a fictive medium for which $\beta = 0$. In this /110
case the solution of the equations (35.4) has the form

$$\begin{aligned} \hat{u}_1^* &= C_1 e^{\lambda_1 x_1} + C_2 e^{-\lambda_1 x_1}, \\ \hat{\theta}^* &= D_1 e^{\lambda_1 x_1} + D_2 e^{-\lambda_1 x_1}, \end{aligned} \quad (35.6)$$

where

$$\lambda_1 = \sigma = \frac{\omega}{c_1}, \quad \lambda_2 = \sqrt{q} = (1 + i) \sqrt{\frac{\omega \rho}{2k}}, \quad (35.7)$$

and C_α and D_α are arbitrary constants.

The displacement \hat{u}_1 and the temperature $\hat{\theta}$ corresponding to the functions (35.6) have the expression

$$\{u_1 = \operatorname{Re} [C_1 e^{-i\omega(t-x_1/c)} + C_2 e^{-i\omega(t+x_1/c)}], \quad (35.8)$$

$$0 = \operatorname{Re} [D_1 e^{-\sqrt{\frac{\omega c}{2k}} x_1} e^{-i\omega(t-x_1/\sqrt{2k\omega})} + D_2 e^{\sqrt{\frac{\omega c}{2k}} x_1} e^{-i\omega(t+x_1/\sqrt{2k\omega})}],$$

The function \hat{u}_1 represents purely elastic waves which propagate with constant velocity c_1 in both directions of the axes Ox_1 . These waves do not present dispersion or damping effects. The function θ describes purely thermal waves. They are damped and are subject to dispersion in the sense that the phase velocity depends on the frequency. The damping coefficient is $\hat{\nu}_2 = \operatorname{Im} \lambda_2 = \sqrt{\frac{\omega c}{2k}}$,

and the dispersion results from the dependence of the phase velocity

$$\hat{c}_2 = \frac{\omega}{\operatorname{Re} \lambda_2} = \sqrt{\frac{2k\omega}{c}} \quad \text{on the frequency } \omega.$$

Let us study now the equations (35.4) in which β differs from zero. From (35.4) it follows that both u_1^* and θ^* satisfy the equation

$$\left[\left(\frac{d^2}{dx_1^2} + \sigma_2 \right) \left(\frac{d^2}{dx_1^2} + q \right) + \varepsilon q \frac{d^2}{dx_1^2} \right] y = 0, \quad (35.9)$$

where

$$\varepsilon = \frac{T_0 \beta^2}{c \rho c_1^2}. \quad (35.10)$$

Looking for the solution of the form $e^{i\eta x_1}$, we are led to the characteristic equation

$$\eta^4 - \eta^2(\sigma^2 + q + \varepsilon q) + q\sigma^2 = 0. \quad (35.11)$$

We obtain

$$\eta_{1,2}^2 = \frac{1}{2} \{ \sigma^2 + q \pm \epsilon q \pm [(\sigma^2 + q \pm \epsilon q)^2 - 4q\sigma^2]^{1/2} \}.$$

The coefficient β enters in equation (35.11) only through ϵ . Obviously, the roots of the equation depend on ϵ and for $\epsilon = 0$ we have

/111

$$\eta_1^2(0) = \sigma^2, \quad \eta_2^2(0) = q. \quad (35.12)$$

We will designate the roots so that

$$\eta_1(0) = \lambda_1 = \frac{\omega}{c_1}, \quad \eta_2(0) = \lambda_2 = (1 + i) \sqrt{\frac{\omega c}{2k}}. \quad (35.13)$$

If we ask that the functions u_1^* and θ^* given by (35.9) satisfy the equations (35.4) we obtain

$$u_1^* = A_1 e^{i\eta_1 x_1} + A_2 e^{-i\eta_1 x_1} + \frac{i\eta_2 \beta}{\rho c_1^2 (\sigma^2 - \eta_2^2)} [B_1 e^{i\eta_2 x_1} - B_2 e^{-i\eta_2 x_1}], \quad (35.14)$$

$$\theta^* = B_1 e^{i\eta_2 x_1} + B_2 e^{-i\eta_2 x_1} + \frac{T_0 \beta \omega \eta_1}{k(q - \eta_1^2)} [A_1 e^{i\eta_1 x_1} - A_2 e^{-i\eta_1 x_1}],$$

where A_α and B_α are arbitrary constants.

With the notation

$$\eta_\alpha = \frac{\omega}{v_\alpha} + i\vartheta_\alpha, \quad (35.15)$$

we can write

$$\begin{aligned}
u_1 = & \operatorname{Re} \left\{ A_1 e^{-\theta_1 r_1} e^{-i\omega \left(t - \frac{r_1}{v_1}\right)} + A_2 e^{\theta_1 r_1} e^{-i\omega \left(t + \frac{r_1}{v_1}\right)} + \right. \\
& + \frac{i\gamma_2 \beta}{\rho c_1^2 (\sigma^2 - \gamma_2^2)} \left[B_1 e^{-\theta_1 r_1} e^{-i\omega \left(t - \frac{r_1}{v_1}\right)} - B_2 e^{\theta_1 r_1} e^{-i\omega \left(t + \frac{r_1}{v_1}\right)} \right] \left. \right\}, \\
\theta = & \operatorname{Re} \left\{ B_1 e^{-\theta_1 r_1} e^{-i\omega \left(t - \frac{r_1}{v_1}\right)} + B_2 e^{\theta_1 r_1} e^{-i\omega \left(t + \frac{r_1}{v_1}\right)} + \right. \\
& + \frac{T_0 \beta \omega \gamma_1}{k(q - \gamma_1^2)} \left[A_1 e^{-\theta_1 r_1} e^{-i\omega \left(t - \frac{r_1}{v_1}\right)} - A_2 e^{\theta_1 r_1} e^{-i\omega \left(t + \frac{r_1}{v_1}\right)} \right] \left. \right\}.
\end{aligned} \tag{35.16}$$

The relations (35.16) contain on the one hand terms of the form

$$A_1 e^{-\theta_1 r_1} e^{-i\omega \left(t - \frac{r_1}{v_1}\right)}, \quad A_2 e^{\theta_1 r_1} e^{-i\omega \left(t + \frac{r_1}{v_1}\right)}, \tag{35.17}$$

called quasielastic ones, on the other hand terms of the form

$$B_1 e^{-\theta_1 r_1} e^{-i\omega \left(t - \frac{r_1}{v_1}\right)}, \quad B_2 e^{\theta_1 r_1} e^{-i\omega \left(t + \frac{r_1}{v_1}\right)}, \tag{35.18}$$

which are called quasithermal ones.

In order to interpret the result, we will consider the relations /112 (35.8) and the fact that for $\beta = 0$ ($\varepsilon = 0$) we have

$$\begin{aligned}
r_1(0) = c_1, \quad r_2(0) = \sqrt{\frac{2k\omega}{c}} = \hat{c}_2, \\
\theta_1(0) = 0, \quad \theta_2(0) = \sqrt{\frac{\omega c}{2k}} = \hat{\theta}_2.
\end{aligned} \tag{35.19}$$

The root $\eta_1(\epsilon)$ determines the quasielastic terms (35.17) and $\eta_1(0) = \lambda_1$ defines the purely elastic waves (35.8)₁. Similarly, $\eta_2(\epsilon)$ characterizes the quasithermal terms (35.18) and $\eta_2(0) = \lambda_2$ the purely thermal waves from (35.8).

From the above presented facts it follows that the effect of the interaction of the elastic deformation with the thermal field in the case of plane harmonic waves may be described by two aspects:

(i) Modification. The purely elastic and the purely thermal waves are modified. The quasielastic terms in contrast with purely elastic waves are subjected to damping and dispersion. The quasithermal terms and the purely thermal waves are subject to dispersion and damping, but with different phase velocities and damping coefficients.

(ii) Coupling. In the expression of function u_1 appear both quasielastic and also quasithermal terms. The same thing occurs also for the function θ . The presence of different type terms in the expression of the functions u_1 and θ characterizes the coupling effect.

In order to discuss the phenomenon from the purely physical viewpoint, it is convenient to introduce the following quantities

$$\xi = \frac{c_1}{\omega^*} \eta, \quad \chi = \frac{\omega}{\omega^*}, \quad \omega^* = \frac{cc_1^2}{k}. \quad (35.20)$$

The equation (35.11) takes the form

$$\xi^4 - \xi^2[\chi^2 + i\chi(1 + \epsilon)] + i\chi^3 = 0. \quad (35.21)$$

From (35.15) and (35.20) we have

$$\xi_a = \frac{c_1}{\omega^*} \eta_a = c_1 \left(\frac{\chi}{c_2} + i \frac{\eta_a}{\omega^*} \right). \quad (35.22)$$

Solving the equation (35.12) we conclude

$$\begin{aligned}\xi_1 &= \frac{1}{2} \sqrt{\chi} \{ [\chi + (1+i) \sqrt{2\chi} + i(1+\epsilon)]^{1/2} + \\ &\quad + [\chi - (1+i) \sqrt{2\chi} + i(1+\epsilon)]^{1/2} \}, \\ \xi_2 &= \frac{1}{2} \sqrt{\chi} \{ [\chi + (1+i) \sqrt{2\chi} + i(1+\epsilon)]^{1/2} - \\ &\quad - [\chi - (1+i) \sqrt{2\chi} + i(1+\epsilon)]^{1/2} \}.\end{aligned}\quad (35.23)$$

The expression of the phase velocities v_α and of the damping coefficient θ_α may be obtained from (35.22) and (35.23). /113

The data from table 1 indicate the fact that ϵ is a small number. Developing the powers in series, according to the parameter ϵ (the convergence is very rapid [49]) from (35.22) and (35.23) we obtain

$$\begin{aligned}v_1 &= c_1 \left[1 + \frac{\epsilon}{2(1+\chi^2)} - \frac{\epsilon^2(1-14\chi^2+\chi^4)}{8(1+\chi^2)^3} + O(\epsilon^3) \right], \\ \theta_1 &= \frac{\omega^*}{c_1} \left[\frac{\epsilon\chi^2}{2(1+\chi^2)} - \frac{\epsilon^2\chi^2(5-3\chi^2)}{4(1+\chi^2)^3} + O(\epsilon^3) \right], \\ v_2 &= c_1 \sqrt{2\chi} \left[1 - \frac{\epsilon(1+\chi)}{2(1+\chi^2)} + \frac{\epsilon^2(3+10\chi-8\chi^2-6\chi^3+5\chi^4)}{8(1+\chi^2)^3} + O(\epsilon^3) \right], \\ \theta_2 &= \frac{\omega^*}{c_1} \sqrt{\frac{\chi}{2}} \left[1 + \frac{\epsilon(1-\chi)}{2(1+\chi^2)} - \frac{\epsilon^2(1-6\chi-12\chi^2+10\chi^3+3\chi^4)}{8(1+\chi^2)^3} + O(\epsilon^3) \right].\end{aligned}$$

We will designate

$$\theta_1^{(\infty)} = \lim_{\chi \rightarrow \infty} \theta_1 = \frac{\epsilon \omega^*}{2c_1}.$$

The frequencies which will be reached in elastic media are limited at the top (Brillouin [40]) by the constant

$$\omega_c = 2\pi(c_1)_s \left(\frac{3\rho}{4\pi M} \right)^{1/3},$$

where M is the atomic mass of the medium and $(c_1)_s$ is the velocity of the longitudinal waves under adiabatic conditions. For solids the experimental data indicate that the following equations

$$\omega < \omega^* < \omega_e.$$

take place.

Information concerning the order of magnitude of the quantities which arise may be found in table 1, where the data were determined at 20°C.

Table 1

Quantity	Units of Measure- ment	Aluminum	Copper	Iron	Lead
$(c_1)_s$	cm sec ⁻¹	$6.32 \cdot 10^8$	$4.36 \cdot 10^8$	$5.80 \cdot 10^8$	$2.14 \cdot 10^8$
ϵ	-	$3.56 \cdot 10^{-8}$	$1.68 \cdot 10^{-8}$	$2.97 \cdot 10^{-8}$	$7.33 \cdot 10^{-8}$
ω^*	sec ⁻¹	$4.66 \cdot 10^{11}$	$1.73 \cdot 10^{11}$	$1.75 \cdot 10^{11}$	$1.91 \cdot 10^{11}$
$\partial_1^{(\infty)}$	cm ⁻¹	$1.31 \cdot 10^8$	$3.29 \cdot 10^8$	$4.48 \cdot 10^8$	$3.27 \cdot 10^8$
ω_e	sec ⁻¹	$9.80 \cdot 10^{12}$	$7.55 \cdot 10^{12}$	$9.95 \cdot 10^{12}$	$3.69 \cdot 10^{12}$

It follows from this that in agreement with experimental data, we can assume that $\chi \ll 1$. Developing in series according to the powers of χ , we obtain from (35.22) and (35.23)

/114

$$v_1 = c_1 \sqrt{1 + \epsilon} \left[1 - \frac{\chi^2 \epsilon (4 - 3 \epsilon)}{8(1 + \epsilon)^4} + O(\chi^4) \right],$$

$$\partial_1 = \frac{\omega^*}{c_1 \sqrt{1 + \epsilon}} \left[\frac{\chi^2 \epsilon}{2(1 + \epsilon)^2} + O(\chi^4) \right],$$

$$v_2 = c_1 \sqrt{\frac{2\chi}{1 + \epsilon}} \left[1 - \frac{\chi \epsilon}{2(1 + \epsilon)^2} + \frac{\chi^2 \epsilon (4 + \epsilon)}{8(1 + \epsilon)^4} + \frac{\chi^3 \epsilon (8 - 20\epsilon + \epsilon^2)}{16(1 + \epsilon)^6} + O(\chi^4) \right],$$

$$\partial_2 = \frac{\omega^*}{c_1} \sqrt{\frac{1}{2} \chi (1 + \epsilon)} \left[1 - \frac{\chi \epsilon}{2(1 + \epsilon)^2} - \frac{\chi^2 \epsilon (4 - \epsilon)}{8(1 + \epsilon)^4} + \frac{\chi^3 \epsilon (8 - 12\epsilon + \epsilon^2)}{16(1 + \epsilon)^6} + O(\chi^4) \right].$$

For frequencies reached in the laboratory and in nature (for which $\chi \ll 1$), we have $v_1 \approx c_1 \sqrt{1+\epsilon}$.

A similar study may be undertaken to analyze the case when the wavelength is given. For this, n is considered a real constant. In this case the equation (35.21) will be considered in the unknown χ (Chadwick [49]).

b) Rayleigh waves. Let us consider now an elastic medium which occupies the half-space $x_3 > 0$. We assume that it exchanges heat freely, by radiation, with the atmosphere $x_3 < 0$, the temperature of which is T_0 . We assume similarly that there are no mass forces and heat source and that the boundary of the half-space is free of tensions. Consequently, on the $x_3 = 0$ plane, we have the following conditions

$$t_{i3} = 0, \quad \frac{\partial \theta}{\partial x_3} + h\theta = 0, \quad \text{for } x_3 = 0, \quad (35.24)$$

where the transfer coefficient h is constant.

We will study the case of surface waves which propagate in the direction of the axes Ox_1 (Lockett [265]). We assume that the components of the displacement vector have the following form

$$u_1 = \varphi_{,1} - \psi_{,3}, \quad u_2 = 0, \quad u_3 = \varphi_{,3} + \psi_{,1}, \quad (35.25)$$

where ϕ and ψ are functions of only x_1 and x_3 . If these functions satisfy the equations

$$c_1^2 \Delta_2 \varphi - \ddot{\varphi} - \frac{\beta}{\rho} \theta = 0, \quad (35.26)$$

$$c_2^2 \Delta_2 \psi - \ddot{\psi} = 0, \quad (35.27)$$

$$k \Delta_2 \theta - T_0 \beta \Delta_2 \varphi - \epsilon \dot{\theta} = 0, \quad (35.28)$$

where

/115

$$\Delta_1 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2},$$

then the equations (33.1), (33.2) are satisfied in the absence of mass force and a heat source.

We must find solutions for the equations (35.26) -- (35.28) of the form

$$[\varphi, \psi, \theta] = \text{Re} \{ [\Phi(x_3), \Psi(x_3), \Lambda(x_3)] e^{i(\rho_1 t - \omega t)} \}. \quad (35.29)$$

If we substitute these expressions in the equations (35.26) - (35.28) we obtain for the functions of Φ , Ψ , Λ the following equations

$$\begin{aligned} \Phi'' - \left(p^2 - \frac{\omega^2}{c_1^2} \right) \Phi &= -\frac{\beta}{\rho c_1^2} \Lambda, \\ \Lambda'' - \left(p^2 - \frac{i\omega c}{k} \right) \Lambda &= -\frac{iT_0 \beta \omega}{k} (\Phi'' - p^2 \Phi), \\ \Psi'' - \left(p^2 - \frac{\omega^2}{c_2^2} \right) \Psi &= 0. \end{aligned} \quad (35.30)$$

In addition to the boundary conditions (35.24) we also have conditions which require that the solution should tend toward zero when x_3 tends toward infinity. Keeping this fact in mind, we will consider the following solutions for equations (35.30)

$$\begin{aligned} \Phi &= A e^{-z_1 \sqrt{p^2 - z_1^2}} + B e^{-z_2 \sqrt{p^2 - z_2^2}}, \\ \Lambda &= \frac{\rho c_1^2}{\beta} \left[A \left(\frac{\omega^2}{c_1^2} - z_1^2 \right) e^{-z_1 \sqrt{p^2 - z_1^2}} + B \left(\frac{\omega^2}{c_1^2} - z_2^2 \right) e^{-z_2 \sqrt{p^2 - z_2^2}} \right], \\ \Psi &= C e^{-z_3 \sqrt{p^2 - z_3^2}}, \end{aligned} \quad (35.31)$$

where A, B, C are arbitrary constants and z_1^2, z_2^2 are the roots of the equation

$$z^4 - \left[\frac{\omega^2}{c_1^2} + \frac{i\omega c}{k} (1 + \epsilon) \right] z^2 + \frac{i\omega^3 c}{k c_1^2} = 0, \quad (35.32)$$

and

$$z_2^2 = \frac{\omega^2}{c_2^2}. \quad (35.33)$$

Obviously, the equation (35.32) coincides with the equation (35.11). If we substitute the displacements (35.25) in relations (23.22), we obtain

$$\begin{aligned} t_{13} &= \rho c_2^2 (2 \varphi_{,13} + \psi_{,11} - \psi_{,33}), \quad t_{23} = 0, \\ t_{33} &= \rho c_1^2 \Delta_2 \varphi + 2 \rho c_2^2 (\psi_{,13} - \varphi_{,33}) = 0. \end{aligned} \quad (35.34)$$

In view of (35.31) and (35.35), we obtain from conditions /116
(35.24) the following relations among the constants A, B, and C.

$$(A + B) \left(2 - \frac{\omega^2}{p^2 c_2^2} \right) - 2 i b_3 C = 0, \quad (35.35)$$

$$2i(b_1 A + b_3 B) + \left(2 - \frac{\omega^2}{p^2 c_2^2} \right) C = 0,$$

$$A \left(\frac{h}{p} - b_1 \right) \left(b_1^2 - 1 + \frac{\omega^2}{p^2 c_1^2} \right) + B \left(\frac{h}{p} - b_2 \right) \left(b_2^2 - 1 + \frac{\omega^2}{p^2 c_1^2} \right) = 0,$$

where we have designated

$$b_i^2 = 1 - \frac{z_i^2}{p^2}. \quad (35.36)$$

Elimination of the constants A, B and C from equations (35.35) leads to the relation

$$\begin{aligned} & \left(2 - \frac{\omega^2}{p^2 c_1^2}\right)^2 \left(b_1^2 + b_1 b_2 + b_2^2 - 1 + \frac{\omega^2}{p^2 c_1^2}\right) - 4 b_1 b_2 b_3 (b_1 + b_2) = \\ & = \frac{p}{h} \left[\left(2 - \frac{\omega^2}{p^2 c_1^2}\right) (b_1 + b_2) - 4 b_3 \left(b_1 b_2 + 1 - \frac{\omega^2}{p^2 c_1^2}\right) \right], \end{aligned} \quad (35.37)$$

which connects the quantities p and ω .

If we consider in (35.29) ω as an independent variable and set

$$v = \frac{\omega}{p}, \quad (35.38)$$

then equation (35.37) becomes

$$\begin{aligned} & \left(2 - \frac{v^2}{c_1^2}\right)^2 \left(b_1^2 + b_1 b_2 + b_2^2 - 1 + \frac{v^2}{c_1^2}\right) - 4 b_1 b_2 b_3 (b_1 + b_2) = \\ & = \frac{h v}{\omega} \left[(b_1 + b_2) \left(2 - \frac{v^2}{c_1^2}\right) - 4 b_3 \left(b_1 b_2 + 1 - \frac{v^2}{c_1^2}\right) \right]. \end{aligned} \quad (35.39)$$

From (35.36) it follows that

$$b_1^2 + b_2^2 = 2 - \frac{1}{p^2} (z_1^2 + z_2^2), \quad b_1^2 b_2^2 = 1 - \frac{1}{p^2} (z_1^2 + z_2^2) + \frac{1}{p^4} z_1^2 z_2^2,$$

and from (35.32) we have

$$z_1^2 + z_2^2 = \frac{\omega^2}{c_1^2} + \frac{i \omega c}{k} (1 + \epsilon), \quad z_1^2 z_2^2 = \frac{i \omega^3 c}{k c_1^2}.$$

If we keep in mind relations (35.20) and (35.38), we obtain

/117

$$\frac{\omega c}{k p^2} = \frac{v^2}{\chi c_1^2}, \quad \frac{\omega^3 c}{k p^4 c_1^2} = \frac{v^3}{\chi c_1^4}.$$

with this we are able to write

$$\begin{aligned} & b_1^2 + b_2^2 = 2 - \left(\frac{v}{c_1}\right)^2 - \frac{i}{\chi} \left(\frac{v}{c_1}\right)^2 (1 + \epsilon), \\ & b_1^2 b_2^2 = 1 - \left(\frac{v}{c_1}\right)^2 - \frac{i}{\chi} \left(\frac{v}{c_1}\right)^2 (1 + \epsilon) + \frac{i}{\chi} \left(\frac{v}{c_1}\right)^4. \end{aligned} \quad (35.40)$$

Similarly, we have

$$b_3^2 = 1 - \left(\frac{v}{c_2}\right)^2 \quad (35.41)$$

Keeping in mind (35.40) and (35.41) we can derive from (35.37) an equation for v . It follows from (35.29) that $1/\operatorname{Re} v^{-1}$ is the phase velocity and $\omega \operatorname{Im} v^{-1}$ is a measure of the attenuation in the direction of the Ox_1 axis. The surface wave is subject to dispersion because these quantities depend on the frequency ω .

If we develop in series the powers of χ and neglect the terms of the order $\chi^{1/2}$ we can conclude that the velocity v ceases to depend on the frequency and on the constant h and we have

$$\left(2 - \frac{v^2}{c_2^2}\right)^2 = 4\left(1 - \frac{v^2}{c_2^2}\right)\left[1 - \frac{v^2}{(1 + \epsilon)c_1^2}\right]^{1/2}.$$

Setting $c_p^2 = (1 + \epsilon)c_1^2$, we obtain the known relation which connects the velocity of Rayleigh waves and the velocity of longitudinal and transversal elastic waves.

36. The Elastic Space Subjected to Concentrated Loads

Let us consider an elastic medium which occupies a whole space. We will study, in the case of harmonic waves, the effect of a concentrated mass force and of a concentrated heat source. The results presented here have been derived by Nowacki [316].

Let

$$f_j = \operatorname{Re} [f_j^*(x) e^{-i\omega t}], \quad r = \operatorname{Re} [r^*(x) e^{-i\omega t}]. \quad (36.1)$$

In this case, the unknown functions have the form

$$u_j = \operatorname{Re} [u_j^*(x, \omega) e^{-i\omega t}], \quad \theta = \operatorname{Re} [\theta^*(x, \omega) e^{-i\omega t}]. \quad (36.2)$$

The functions Φ and Ψ which realize the representation (34.12) may be taken in the following way

$$\Phi_j = \operatorname{Re} [\Phi_j^*(x, \omega) e^{-i\omega t}], \quad \Psi = \operatorname{Re} [\Psi^*(x, t) e^{-i\omega t}]. \quad (36.3)$$

It follows from (34.13), (34.14), (36.1) and (36.3) that the functions Φ_j^* and Ψ^* satisfy the equations

/118

$$\begin{aligned} (\Delta + \tau^2) [(\Delta + \sigma^2)(\Delta + q) + \varepsilon q \Delta] \Phi_j^* &= -\frac{\sigma}{k\mu(\lambda + 2\mu)} f_j^*, \\ [(\Delta + \sigma^2)(\Delta + q) + \varepsilon q \Delta] \Psi^* &= -\frac{1}{k} r^*, \end{aligned} \quad (36.4)$$

where we use the notations

$$\sigma^2 = \frac{\omega^2}{c_1^2}, \quad \tau^2 = \frac{\omega^2}{c_2^2}, \quad q = -\frac{i\omega c}{k}. \quad (36.5)$$

If

$$\begin{aligned} k_1^2 + k_2^2 &= \sigma^2 + q(1 + \varepsilon), \\ k_1^2 k_2^2 &= \sigma^2 q, \end{aligned} \quad (36.6)$$

then we are able to write

$$(\Delta + \sigma^2)(\Delta + q) + \varepsilon q \Delta = (\Delta + k_1^2)(\Delta + k_2^2). \quad (36.7)$$

Obviously, k_1^2 and k_2^2 are roots of the equation

$$z^2 - [\sigma^2 + q(1 + \varepsilon)]z + \sigma^2 q = 0. \quad (36.8)$$

In view of (36.7), the equations (36.4) become

$$\begin{aligned} (\Delta + \tau^2) (\Delta + k_1^2) (\Delta + k_2^2) \Phi_j^* &= - \frac{c}{k\mu(\lambda + 2\mu)} f_j^*, \\ (\Delta + k_1^2) (\Delta + k_2^2) \Psi^* &= - \frac{1}{k} r^*. \end{aligned} \quad (36.9)$$

From (34.12), (36.2) and (36.3) it follows that the functions u_j^* , θ^* are represented with the aid of the functions Φ_k^* , Ψ^* by the relations

$$\begin{aligned} u_j^* &= (\lambda + 2\mu) \frac{k}{c} (\Delta + k_1^2) (\Delta + k_2^2) \Phi_j^* - \\ &- (\lambda + \mu) \frac{k}{c} \left[(\Delta + q) + \frac{\lambda + 2\mu}{\lambda + \mu} q\varepsilon \right] \Phi_{j,\alpha}^* + \frac{\beta}{\lambda + 2\mu} \Psi_{,\alpha}^*, \\ \theta^* &= - \frac{\beta T_0 \mu i \omega}{c} (\Delta + \tau^2) \Phi_{j,\alpha}^* + (\Delta + \sigma^2) \Psi^*. \end{aligned} \quad (36.10)$$

If we assume that the mass force has the direction of the axis Ox_1 and we do not have a heat source, that is,

$$f_j^* = \delta_{1j} f^*, \quad \Psi^* = 0, \quad (36.11)$$

then we can consider $\Phi_2^* = 0$, $\Phi_3^* = 0$, $\Psi^* = 0$.

It follows from (36.9) that the function Φ_1^* satisfies the /119
equation

$$(\Delta + \tau^2) (\Delta + k_1^2) (\Delta + k_2^2) \Phi_1^* = - \frac{c}{k\mu(\lambda + 2\mu)} f^*. \quad (36.12)$$

Let G_j^* be functions which satisfy the equations

$$\begin{aligned} (\Delta + k_1^2) G_1^* &= - \frac{c}{k\mu(\lambda + 2\mu)} f^*, \\ (\Delta + k_2^2) G_2^* &= - \frac{c}{k\mu(\lambda + 2\mu)} f^*, \\ (\Delta + \tau^2) G_3^* &= - \frac{c}{k\mu(\lambda + 2\mu)} f^*. \end{aligned} \quad (36.13)$$

The solution of equation (36.12) may be put in the following form

$$\chi = a_1 G_1^* + a_2 G_2^* + a_3 G_3^*, \quad (36.14)$$

where a_j are constants. If we suppose that the function χ given by (36.14) satisfies equation (36.12) and if we consider the equations (36.13), then we obtain for the constants a_j the equations

$$\begin{aligned} a_1 + a_2 + a_3 &= 0, \\ (\tau^2 + k_2^2) a_1 + (\tau^2 + k_1^2) a_2 + (k_1^2 + k_2^2) a_3 &= 0, \\ k_2^2 \tau^2 a_1 + k_1^2 \tau^2 a_2 + k_1^2 k_2^2 a_3 &= 1. \end{aligned} \quad (35.15)$$

It follows from (36.15) that

$$\begin{aligned} a_1 &= \frac{1}{(k_1^2 - k_2^2)(k_1^2 - \tau^2)}, \quad a_2 = -\frac{1}{(k_1^2 - k_2^2)(k_2^2 - \tau^2)}, \\ a_3 &= \frac{1}{(k_1^2 - \tau^2)(k_2^2 - \tau^2)}. \end{aligned} \quad (36.16)$$

Here and in the following, we assume that the denominators of the previous fractions are different from zero, but the results which will be presented may also be derived in the contrary case.

If the mass force considered is concentrated in point y , then we have

$$f^* = \delta(x - y),$$

where δ is the Dirac measure.

In this case, the solutions of the equations (36.13) which satisfy at infinity the radiation conditions (Courant [66], Kupradze [247]) are

$$G_a^* = \frac{c}{4 \pi k \mu (\lambda + 2 \mu)} \frac{e^{i k_a R}}{R},$$

$$G_b^* = \frac{c}{4 \pi k \mu (\lambda + 2 \mu)} \frac{e^{i \tau R}}{R},$$
(36.17)

where

$$R = |x - y| = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2},$$
(36.18)

From (36.14) and (36.17) we derive

$$\chi(x, y; \omega) = \frac{c}{4 \pi k \mu (\lambda + 2 \mu)} \frac{1}{(k_1^2 - k_2^2) R} \left[\frac{e^{i k_1 R}}{k_1^2 - \tau^2} - \frac{e^{i k_2 R}}{k_2^2 - \tau^2} + \frac{(k_1^2 - k_2^2) e^{i \tau R}}{(k_1^2 - \tau^2)(k_2^2 - \tau^2)} \right].$$
(36.19)

The corresponding functions u_j^* and θ^* are obtained from (36.10) by setting $\Phi_1^* = \chi(x, y; \omega)$, $\Phi_2^* = \Phi_3^* = \Psi^* = 0$.

We will designate these functions by $u_j^{*(1)}(x, y; \omega)$ and $\theta^{*(1)}(x, y; \omega)$ respectively, thus indicating the fact that the mass force has the direction of the Ox_1 axis and is concentrated in point y .

If we keep in mind the relations

$$(k_a^2 - q)(k_a^2 - \sigma^2) = k_a^2 q^2 \varepsilon,$$

$$(k_1^2 - \tau^2)(k_2^2 - \tau^2) = -\sigma^2 \frac{\lambda + 2\mu}{\mu} \left(-\tau^2 + q + \frac{\lambda + 2\mu}{\lambda + \mu} \varepsilon q \right),$$

$$-k_a^2 + q + \frac{\lambda + 2\mu}{\lambda + \mu} \varepsilon q = \frac{\mu(k_a^2 - q)(k_a^2 - \tau^2)}{(\lambda + \mu)k_a^2}, \quad (x = 1, 2),$$

after calculation we obtain

$$u_j^{*(1)} = \frac{1}{4\pi\rho c_2^2} \frac{e^{ik_1 R}}{R} \delta_{1j} - \frac{1}{4\pi\rho\omega^2} \frac{\partial^2 E(R, \omega)}{\partial x_1 \partial x_j},$$

$$\theta^{*(1)} = \frac{q}{4\pi\beta(k_1^2 - k_2^2)} \frac{\partial}{\partial x_1} \left(\frac{e^{ik_1 R}}{R} - \frac{e^{ik_2 R}}{R} \right),$$
(36.20)

where

$$E(R, \omega) = \frac{\sigma^2(k_1^2 - q)}{k_1^2(k_1^2 - k_2^2)} \frac{e^{ik_1 R}}{R} - \frac{\sigma^2(k_2^2 - q)}{k_2^2(k_1^2 - k_2^2)} \frac{e^{ik_2 R}}{R} - \frac{e^{ik_1 R}}{R}.$$
(36.21)

Let us assume that

$$f_j^* = \delta_{j1} \delta(x - y), \quad r^* = 0.$$
(36.22)

In this case we will designate u_j^* and θ^* by $u_j^{*(1)}(x, y; \omega)$ and $\theta^{*(1)}(x, y; \omega)$, respectively. These functions are obtained from the relations (36.10), setting $\Phi_j^* = \delta_{j1} \chi(x, y; \omega)$ and $\Psi^* = 0$.

Thus, we derive

$$u_j^{*(1)}(x, y; \omega) = \frac{1}{4\pi\rho c_2^2} \frac{e^{ik_1 R}}{R} \delta_{1j} - \frac{1}{4\pi\rho\omega^2} \frac{\partial^2 E(R, \omega)}{\partial x_j \partial x_s},$$

$$\theta^{*(1)}(x, y; \omega) = \frac{q}{4\pi\beta(k_1^2 - k_2^2)} \frac{\partial}{\partial x_1} \left(\frac{e^{ik_1 R}}{R} - \frac{e^{ik_2 R}}{R} \right), \quad (s = 1, 2, 3).$$
(36.23)

Let us study now the effect of a concentrated heat source. /121
We will assume that

$$f_j^* = 0, \quad r^* \neq 0.$$

In this case we will take $\Phi_j^* = 0$. The function Ψ^* satisfies the equation

$$(\Delta + k_1^2)(\Delta + k_2^2)\Psi^* = -\frac{1}{k} r^*.$$
(36.24)

If we have

$$(\Delta + k_1^2) H_1^* = -\frac{1}{k} r^*, \quad (\Delta + k_2^2) H_2^* = -\frac{1}{k} r^*, \quad (36.25)$$

then we can write

$$\Psi^* = -\frac{1}{k_1^2 - k_2^2} (H_1^* - H_2^*). \quad (36.26)$$

If the heat source is concentrated in point y , then $r^* = \delta(x-y)$ and we obtain

$$\Psi^*(x, y; \omega) = -\frac{1}{4\pi k(k_1^2 - k_2^2)} \left(\frac{e^{ik_1 R}}{R} - \frac{e^{ik_2 R}}{R} \right). \quad (36.27)$$

In this case we designate the functions u_j^* and θ^* by $u_j^{*(4)}$ and $\theta^{*(4)}$, respectively. We derive from (36.10) and (36.27)

$$u_j^{*(4)}(x, y; \omega) = -\frac{\beta}{4\pi k \rho c_1^2(k_1^2 - k_2^2)} \frac{\partial}{\partial x} \left(\frac{e^{ik_1 R}}{R} - \frac{e^{ik_2 R}}{R} \right), \quad (36.28)$$

$$\theta^{*(4)}(x, y; \omega) = -\frac{1}{4\pi k(k_1^2 - k_2^2)} \left[(k_2^2 - \sigma^2) \frac{e^{ik_1 R}}{R} - (k_1^2 - \sigma^2) \frac{e^{ik_2 R}}{R} \right].$$

The functions $u_j^{*(p)}(x, y; \omega)$, $\theta^{*(p)}(x, y; \omega)$ ($p = 1, 2, 3, 4$) are Green functions for the problem under consideration.

Other problems referring to harmonic waves are presented in the monographs of Nowacki [315], [319].

37. The Method of the Potential in the Study of Vibrations

a) Preliminaries. In this paragraph we will present a study

of harmonic waves in homogeneous and isotropic media with the help of the method of the potential. Ignaczak and Nowacki [209], [210] derived the reduction of the problem to the limit of the integral equations in the case under consideration. Kupradze and Burciuladze [249] (see also [248]) studied these problems with the help of thermoelastic potentials.

Let us derive the equations which describe the thermoelastic process under consideration. We will assume that the loads have the form (36.1). We will use the designations from (36.2) for displacements and the temperature. Keeping in mind the relations (36.1) and (36.2) the following equations can be derived for the unknown functions u_j^* , θ^* from (33.1), (33.2)

$$\begin{aligned}\mu u_{j,rr}^* + (\lambda + \mu) u_{r,r}^* - \beta \theta_{,r}^* + \rho \omega^2 u_j^* &= -f_j^*, \\ k \theta_{,rr}^* + T_0 \beta i \omega u_{r,r}^* + i \omega c \theta^* &= -r^*.\end{aligned}\tag{37.1}$$

The constitutive equations imply the relations

$$\begin{aligned}t_{rs}^* &= \lambda u_{k,k}^* \delta_{rs} + \mu (u_{r,i}^* + u_{i,r}^*) - \beta \theta^* \delta_{rs}, \\ q_j^* &= k \theta_{,j}^*,\end{aligned}\tag{37.2}$$

where t_{rs}^* and q_j^* have the amplitudes of the functions t_{rs} and q_j , respectively.

If we designate by t_j^* the amplitude of the component t_j of the tension that acts in point x of the oriented surface S , then we have

$$t_j^* = t_{rs}^* n_s,\tag{37.3}$$

where n_s are components of the unitary exterior normal at S in x .

Similarly,

$$q^* = q_j^* n_j, \quad (37.4)$$

where q^* is the amplitude of the flux which acts on S.

The system of equations (37.1) may also be written in matrix form. We will consider the vector $v = (v_1, v_2, \dots, v_m)$ as a matrix column. Thus the product between the matrix $A = \|a_{ij}\|_{m \times m}$ and the vector v is an m -dimensional vector. The vector v multiplied with matrix A will designate the products between the matrix line $v = \|v_1, \dots, v_m\|$ and matrix A . We introduce the operator

$$D\left(-\frac{\partial}{\partial x}\right) = \|D_{mn}\left(-\frac{\partial}{\partial x}\right)\|_{m \times m}, \quad (37.5)$$

where

$$D_{jk} = \mu \delta_{jk} \Delta + (\lambda + \mu) \frac{\partial^2}{\partial x_j \partial x_k} + \rho \omega^2 \delta_{jk}, \quad (37.6)$$

$$D_{j4} = -\beta \frac{\partial}{\partial x_j}, \quad D_{4j} = T_0 \beta i \omega \frac{\partial}{\partial x_j},$$

$$D_{44} = k \Delta + i \omega c, \quad (j, k = 1, 2, 3).$$

If we designate $U = (u_1^*, u_2^*, u_3^*, \theta^*)$, $F = -(f_1^*, f_2^*, f_3^*, r^*)$ then the system (37.1) may be written in the following way

/123

$$\dot{D}\left(-\frac{\partial}{\partial x}\right) U = F. \quad (37.7)$$

Let us introduce the matrix operator

$$H\left(\frac{\partial}{\partial x}, n_x\right) = \left\| H_{jk}\left(\frac{\partial}{\partial x}, n_x\right) \right\|_{j,k}, \quad (37.8)$$

where

$$H_{11} = \mu \delta_{11} \frac{\partial}{\partial n} + \lambda n_1 \frac{\partial}{\partial x_1} + \mu n_1 \frac{\partial}{\partial x_1}, \quad (37.9)$$

$$H_{11} = -\beta n_1, H_{1j} = 0, H_{jk} = k \frac{\partial}{\partial n}, \quad (j, k = 1, 2, 3).$$

It follows from (37.2) -- (37.4) that the vector $(t_1^*, t_2^*, t_3^*, q^*) = P U$ may be written in the following form

$$P U = H\left(\frac{\partial}{\partial x}, n_x\right) U. \quad (37.10)$$

The reciprocity relation from Paragraph 29 becomes

$$\begin{aligned} & -i\omega T_0 \left[\int_H (f_j^{*(1)} u_j^{*(2)} - f_j^{*(2)} u_j^{*(1)}) dv + \int_{\partial H} (t_j^{*(1)} u_j^{*(2)} - \right. \\ & \left. - t_j^{*(2)} u_j^{*(1)}) da = \int_H (r^{*(1)} \theta^{*(2)} - r^{*(2)} \theta^{*(1)}) dv + \right. \\ & \left. + \int_{\partial H} (q^{*(1)} \theta^{*(2)} - q^{*(2)} \theta^{*(1)}) da. \right. \end{aligned} \quad (37.11)$$

Let us designate by

$$\tilde{D}\left(\frac{\partial}{\partial x}\right) = \left\| \tilde{D}_{jk}\left(\frac{\partial}{\partial x}\right) \right\|_{j,k}, \quad (37.12)$$

the adjunct operator of the operator $D\left(\frac{\partial}{\partial x}\right)$. We have

$$\begin{aligned} \hat{D}_{jk} &= D_{jk}, \quad \hat{D}_{jk} = -D_{kj}, \quad \hat{D}_{jj} = -D_{jj}, \\ \hat{D}_{jj} &= D_{jj}, \quad (j, k = 1, 2, 3). \end{aligned} \quad (37.13)$$

Let $V = (v_j^*, \vartheta^*), \quad \hat{F} = - (g_j, \gamma).$ The system /124

$$\hat{D} \left(\frac{\partial}{\partial x} \right) V = \hat{F}, \quad (37.14)$$

is written as

$$\begin{aligned} \mu v_{j,ss}^* + (\lambda + \mu) v_{s,jj}^* - T_0 i \beta \omega \vartheta_{,j}^* + \rho \omega^2 v_j^* &= -g_j, \\ k \vartheta_{,ss}^* + \beta v_{s,s}^* + i \omega c \vartheta^* &= -\gamma. \end{aligned} \quad (37.15)$$

Obviously, if we put the following in (37.15)

$$v_j^* = i \omega T_0 u_j^{*(2)}, \quad \vartheta^* = \vartheta^{*(2)}, \quad g_j = i \omega T_0 f_j^*, \quad \gamma = r^*, \quad (37.16)$$

then we obtain the system of equations (37.1).

$$\begin{aligned} \text{Let } i \omega T_0 f_j^{*(2)} &= g_j^{(2)}, \quad r^{*(2)} = \gamma^{(2)}, \quad i \omega T_0 u_j^{*(2)} = v_j^{*(2)}, \quad \vartheta^{*(2)} = \vartheta^{*(2)}, \quad \hat{F} = \\ &= - (g_j^{(2)}, \gamma^{(2)}), \quad \hat{F} = - (f_j^{*(1)}, r^{*(1)}), \quad V = (v_j^{*(2)}, \vartheta^{*(2)}), \quad U = (u_j^{*(1)}, \vartheta^{*(1)}). \end{aligned}$$

Let us designate by B_1 a finite domain from the three dimensional space limited by the closed Liapunov surface ∂B and by B_e its complementary $B_1 + \partial B$ with respect to the whole space.

From the relation (37.11) we conclude

$$\int_{B_1} (U \hat{D} V - V D U) dv = \int_{\partial B} (U \hat{H} V - V H U) da, \quad (37.17)$$

where

$$\begin{aligned} \dot{H}\left(\frac{\partial}{\partial x}, n_i\right) &= \left\| \dot{H}_m\left(\frac{\partial}{\partial x}, n_i\right) \right\|_{4 \times 4}, \\ \dot{H}_{12} &= H_{12}, \dot{H}_{14} = -i\omega T_{02} u_{11}, \\ \dot{H}_{21} &= 0, \dot{H}_{24} = H_{24}. \end{aligned} \quad (37.18)$$

b) Fundamental Solutions. Obviously the functions $u_j^{*(s)}(x, y; \omega)$, $\theta^{*(s)}(x, y; \omega)$ given by (36.23), (36.28) are fundamental solutions for the system (37.1). Let us consider the matrix as fundamental solution

$$\begin{aligned} \Gamma(x, y; \omega) &= \|\Gamma_{ms}\|_{4 \times 4}, \\ \Gamma_{11} &= u_1^{*(1)}, \quad \Gamma_{12} = \theta^{*(1)}. \end{aligned} \quad (37.19)$$

where

If $\tilde{\Gamma}(x, y; \omega)$ is the fundamental solution matrix of the system (37.14), then we have

$$\tilde{\Gamma}^*(y, x; \omega) = \Gamma(x, y; \omega). \quad (37.20)$$

We designate by A^* the transposal of matrix A . If $x \neq y$, each column of $\Gamma^{(s)}(x, y; \omega)$ ($s = 1, 2, 3, 4$) of matrix Γ satisfies, as a function of x , the system

$$D\left(\frac{\partial}{\partial x}\right) \Gamma^{(s)}(x, y; \omega) = 0. \quad (37.21)$$

If $\omega \rightarrow 0$, $\beta \rightarrow 0$ then matrix Γ becomes

$$\Pi(x, y) = \|\Pi_{jk}\|_{j,k=1,4}, \quad (37.22)$$

where $\Pi_{jk} = U_j^{(k)}(x, y)$ are fundamental solutions of the equations of elasticity [248], and $\Pi_{j4} = \Pi_{4j} = 0$. The matrix Π represents the fundamental solution matrix of the system

$$\mu v_{,ii} + (\lambda + \mu) v_{,ii} = -f_i,$$

$$k\Delta\theta = -r.$$

If we designate

$$G(x, y; \omega) = \Gamma(x, y; \omega) - \Pi(x, y),$$

it can be seen that the elements of matrix G remain limited when $x = y$, and the first order derivatives of these have in this point a pole of the first order.

We have

$$\Pi(x, y) = \Pi^*(x, y) = \Pi(y, x). \quad (37.23)$$

We will designate by $H_1^{(x)}$ the matrix line which has elements $H_{ij} \left(\frac{\partial}{\partial x}, n_i \right)$, $(i, j = 1, 2, 3, 4)$. We can write

$$t_i = H_i U, \quad q = H_4 U. \quad (37.24)$$

Let $\Pi^{(s)}(x, y)$ be the columns of matrix Π . As is known from the theory of elasticity (Kupradze [246]) we have

$$\int_{\partial B} H_i^{(s)} \Pi^{(s)}(x, y) da_s = -\eta_i(x) \delta_{ii}, \quad (37.25)$$

where

$$\eta(x) = \begin{cases} 1, & x \in B_i, \\ 1/2, & x \in \partial B, \\ 0, & x \in B_e. \end{cases}$$

Let there be a matrix

$$M(x, y) = \|M_{jk}\|_{j,k=1}^n, \quad (37.26)$$

with elements

/126

$$M_{jk} = H_j^{(k)} \Pi^{(k)}(x, y), \quad M_{jj} = M_{jj} = 0, \quad M_{kk} = H_k^{(k)} \Pi^{(k)}(x, y).$$

The matrix

$$L(x, y; \omega) = H\left(\frac{\partial}{\partial x}, n_x\right) \Gamma(x, y; \omega) - M(x, y), \quad (37.27)$$

has at point $x = y$ a pole of the first order. Let us designate by $\Lambda(x, y; \omega)$ the matrix obtained from $\tilde{H}\left(\frac{\partial}{\partial x}, n_x\right) \tilde{\Gamma}(x, y; \omega)$ by transposition and exchange of x with y . In view of (37.20) we can write

$$\begin{aligned} \Lambda(x, y; \omega) &= \left[\tilde{H}\left(\frac{\partial}{\partial y}, n_y\right) \tilde{\Gamma}(y, x; \omega) \right]^* = \\ &= \left[\tilde{H}\left(\frac{\partial}{\partial y}, n_y\right) \Gamma^*(x, y; \omega) \right]^*. \end{aligned} \quad (37.28)$$

From (37.10), (37.26) it follows that

$$\tilde{H}\left(\frac{\partial}{\partial x}, n_x\right) \Pi(x, y) = M(x, y) = H\left(\frac{\partial}{\partial x}, n_x\right) \Pi(x, y). \quad (37.29)$$

The matrix

$$N(x, y; \omega) = \Lambda(x, y; \omega) - M^*(y, x), \quad (37.30)$$

has for $x = y$ a pole of the first order.

Theorem 37.1. For $x \neq y$ each column of the matrix $\Lambda(x, y; \omega)$ satisfies as a function of x the system $D\left(\frac{\partial}{\partial x}\right) U = 0$, that is

$$D\left(\frac{\partial}{\partial x}\right) \Lambda(x, y; \omega) = 0, \quad x \neq y. \quad (37.31)$$

Proof. We have from (37.28)

$$\Lambda_{rs} = \tilde{H}_{rm} \left(\frac{\partial}{\partial y}, n_y \right) \Gamma_{rm}(x, y; \omega),$$

where Λ_{rs} are elements of the matrix $\Lambda(x, y; \omega)$. Thus, we can write

$$\begin{aligned} \left[D \left(\frac{\partial}{\partial x} \right) \Lambda(x, y; \omega) \right]_{rs} &= D_{rp} \left(\frac{\partial}{\partial x} \right) \tilde{H}_{rm} \left(\frac{\partial}{\partial y}, n_y \right) \Gamma_{pm}(x, y; \omega) = \\ &= \tilde{H}_{rm} \left(\frac{\partial}{\partial y}, n_y \right) D_{rp} \left(\frac{\partial}{\partial x} \right) \Gamma_{pm}(x, y; \omega) = 0. \end{aligned}$$

c) Formulas of representation. Conditions of radiation. /127

Let us say that $U = (u_j^*, u_l^*)$ is a regulated vector in B_1 if $U \in C^1(B_1) \cap C^2(B_1)$ and the functions $u_{r,j}^*$ ($r = 1, 2, 3, 4$) are integrable in B_1 . Let U be defined in B_e . Let us say that U is regulated in B_e if $U \in C^1(B_e) \cap C^2(B_e)$, and $u_{r,j}^*$ are integrable in $B_e \cap \tau(0, \delta)$, where $\tau(0, \delta)$ is a sphere with center in O and a radius δ regardless of δ .

Let us consider the relation (37.17). If we apply this relation for a regulated vector U and for the vector $V = \tilde{\Gamma}^{(s)}(y, x; \omega)$, ($s = 1, 2, 3, 4$), just as in Paragraph 30, we obtain

$$u_j^*(x) = \int_{\partial B} \tilde{\Gamma}^{(j)}(y, x; \omega) H U da_y - \quad (37.32)$$

$$\begin{aligned} - \int_{\partial B} U(y) \tilde{H} \left(\frac{\partial}{\partial y}, n_y \right) \times \tilde{\Gamma}^{(j)}(y, x; \omega) da_y - \int_{B_1} \tilde{\Gamma}^{(j)}(y, x; \omega) D \left(\frac{\partial}{\partial y} \right) U(y) dv_y, \\ (j = 1, 2, 3, 4), \end{aligned}$$

or in matrix form

$$\begin{aligned} U(x) &= \int_{\partial B} \tilde{\Gamma}^*(y, x; \omega) H U da_y - \int_{\partial B} (\tilde{H} \tilde{\Gamma}(y, x; \omega))^* U(y) da_y - \\ &- \int_{B_1} \tilde{\Gamma}^*(y, x; \omega) D \left(\frac{\partial}{\partial y} \right) U(y) dv_y. \end{aligned} \quad (37.33)$$

If U satisfies the equation

$$D\left(\frac{\partial}{\partial x}\right)U = 0, \quad (37.34)$$

then, using (37.20), (37.28), the relation (37.33) becomes

$$U(x) = \int_{\partial B} \Gamma(x, y; \omega) H U da_y - \int_{\partial B} \Lambda(x, y; \omega) U(y) da_y. \quad (37.35)$$

Any regulated solution of the equation (37.34) may be written in the form

$$U \equiv (u_j^*, u_i^*) = (u_j^{(1)} + u_j^{(2)}, u_i^*), \quad (37.36)$$

so that

$$\begin{aligned} (\Delta + k_1^2)(\Delta + k_2^2)u_j^{(1)} &= 0, \quad \text{rot } u^{(1)} = 0, \\ (\Delta + \tau^2)u_j^{(2)} &= 0, \quad \text{div } u^{(2)} = 0, \\ (\Delta + k_1^2)(\Delta + k_2^2)u_i^* &= 0. \end{aligned} \quad (37.37)$$

This statement can easily be proven, keeping in mind the results from Paragraph 34. Thus from (34.10) it follows that we can write

$$u^{(1)} = \frac{c_1^2}{\omega^2} \text{grad div } u^* - \frac{\beta}{\rho} \text{grad } \theta^*,$$

$$u^{(2)} = -\frac{c_2^2}{\omega^2} \text{rot rot } u^*,$$

$$u^* = u^{(1)} + u^{(2)}.$$

It can easily be seen that the relations (37.37) are satisfied.

/128

We say that the solution regulated in B_e of equation (37.34) satisfies at infinity the conditions of radiation if we have [248], [249]

$$\begin{aligned} u^{(1)} &= o(R^{-1}), & u_j^{(1)} &= O(R^{-2}), \\ u^{(2)} &= O(R^{-1}), & \frac{\partial u^{(2)}}{\partial R} - i\tau u^{(2)} &= o(R^{-1}), \\ \theta^* &= o(R^{-1}), & \theta_j^* &= O(R^{-2}), \end{aligned} \quad (37.38)$$

for any fixed y .

A formula of the type (37.35) occurs also in the domain B_e if the regulated vector U satisfies the radiation conditions [248].

d) Thermoelastic potentials. The formula (37.33) leads to the introduction of the potentials the simple layer potential

$$V(x; \psi) = \int_{\partial B} \Gamma(x, y; \omega) \psi(y) da_y, \quad (37.39)$$

the double layer potential

$$W(x; \psi) = \int_{\partial B} \Lambda(x, y; \omega) \psi(y) da_y, \quad (37.40)$$

the volume potential

$$U(x; \varphi) = \int_{B_1} \Gamma(x, y; \omega) \varphi(y) dv_y, \quad (37.41)$$

where the components of the vectors ψ and ϕ satisfy the condition of Hölder on ∂B and B_1 , respectively.

As in the case of elastic vibrations, we can state [247], [248]

Theorem 37.2. The single layer potential is continuous.

Theorem 37.3. The double layer potential tends toward /129
finite limits when $x \rightarrow z \in \partial B$, from the interior and from the exterior,
and these limits are equal to, respectively,

$$W_+(z; \psi) = -\frac{1}{2} \psi(z) + \int_{\partial B} \Lambda(z, y; \omega) \psi(y) da_y, \quad (37.42)$$

$$W_-(z; \psi) = \frac{1}{2} \psi(z) + \int_{\partial B} \Lambda(z, y; \omega) \psi(y) da_y,$$

the integrals from the right member being understood in the sense of the principal values.

The demonstration of this theorem is based on the representation of $W(x; \psi)$ in the form

$$\begin{aligned} W(x; \psi) &= \int_{\partial B} \Lambda(x, y; \omega) [\psi(y) - \psi(z)] da_y + \\ &+ \int_{\partial B} [\Lambda(x, y; \omega) - M^*(y, x)] \psi(z) da_y + \int_{\partial B} M^*(y, x) \psi(z) da_y. \end{aligned}$$

In view of the fact that $\psi(y)$ satisfies the conditions of Hölder and that the difference (37.30) has for $x = y$ a pole of the first order, it follows that the first two integrals above are continuous. If we keep in mind (37.25) we can conclude

$$W_+(z; \psi) + \psi(z) = W_-(z; \psi) + \frac{1}{2} \psi(z) = W_e(z),$$

and therefore (37.42).

Theorem 37.4. The operator $H\left(\frac{\partial}{\partial x}, n_x\right)$ applied to the potential of single layers is a vector which tends toward finite limits when $x \rightarrow z \in \partial B$, from the interior and from the exterior, and these limits are equal, respectively, to

$$\begin{aligned} \left[H \left(\frac{\partial}{\partial z}, n_i \right) V(z; \psi) \right]_i &= \frac{1}{2} \psi(z) + \\ &+ \int_{\partial B} \left[H \left(\frac{\partial}{\partial z}, n_i \right) \Gamma(z, y; \omega) \right] \psi(y) da_y, \end{aligned} \quad (37.43)$$

$$\begin{aligned} \left[H \left(\frac{\partial}{\partial z}, n_i \right) V(z; \psi) \right]_i &= - \frac{1}{2} \psi(z) + \\ &+ \int_{\partial B} \left[H \left(\frac{\partial}{\partial z}, n_i \right) \Gamma(z, y; \omega) \right] \psi(y) da_y. \end{aligned}$$

Similarly, as in the case of the conventional theory of /130
the potential, we have

$$D \left(\frac{\partial}{\partial x} \right) V(x; \psi) = 0, \quad D \left(\frac{\partial}{\partial x} \right) W(x; \psi) = 0, \quad x \notin \partial B, \quad (37.44)$$

$$D \left(\frac{\partial}{\partial x} \right) U(x; \varphi) = - \varphi(x), \quad x \in B_i. \quad (37.45)$$

In the following we will consider the equation (37.34) and the following problems at the boundary.

Interior problems: to find in B_1 the regulated solution of equation (37.34) which satisfies one of the conditions

$$(I_1) \quad \lim_{x \rightarrow z} U(x) = G(z),$$

$$(I_2) \quad \lim_{x \rightarrow z} H \left(\frac{\partial}{\partial x}, n_x \right) U(x) = G(z),$$

where $x \in B_i, z \in \partial B$, and $G(z)$ is a vector, the components of which satisfy the condition of Hölder;

Exterior problems: to find in B_e the regulated solution of

the equation (37.34) which would satisfy the conditions of radiation (37.38) and one of the conditions

$$(E_1) \quad \lim_{z \rightarrow \infty} U(x) = G(z),$$

$$(E_2) \quad \lim_{z \rightarrow \infty} H\left(\frac{\partial}{\partial x}, n_x\right) U(x) = G(z),$$

where $x \in B$, $z \in \partial B$, and $G(z)$ is a vector, the components of which satisfy the conditions of Hölder.

Homogeneous problems, corresponding to the problems (I_1) , (I_2) , (E_1) , (E_2) for $G(z) = 0$, will be designated by (I_1^0) , (I_2^0) , (E_1^0) , (E_2^0) .

The following can be proven [248]

Theorem 37.5. The solutions to problems (E_1^0) , (E_2^0) which satisfy the conditions of radiation are zero in B_e .

Theorem 37.6. For ω^2 different from the eigenvalues of the problem $(\Delta + \tau^2) u^{(2)}(x) = 0$, $\operatorname{div} u^{(2)}(x) = 0$, $x \in B$, $u^{(2)} = 0$ on ∂B , the problem (I_1^0) has banal solutions.

Theorem 37.7. For ω^2 different from the eigenvalues of the problem

$$(\Delta + \tau^2) u^{(2)}(x) = 0, \operatorname{div} u^{(2)}(x) = 0, x \in B,$$

$$2 \frac{\partial u^{(2)}}{\partial n} + n \times \operatorname{rot} u^{(2)} = 0, \text{ on } \partial B,$$

the problem (I_2^0) has banal solutions.

e) Reduction of the boundary problems to integral equations. /131

Let us look for the solution of the problems (I_1) and (E_1) in the form of a double layer potential and for the solution of the problems (I_2) and (E_2) in the form of a single layer potential. In view of the theorems 37.3, 37.4, we obtain, for the unknown density, the following singular integral equations.

$$-\frac{1}{2}\psi(z) + \int_{\partial B} \Lambda(z, y) \psi(y) da_y = G(z), \quad (37.46)$$

$$\frac{1}{2}\psi(z) + \int_{\partial B} H\left(\frac{\partial}{\partial z}, n_z\right) \Gamma(z, y; \omega) \psi(z) da_y = G(z), \quad (37.47)$$

$$\frac{1}{2}\psi(z) + \int_{\partial B} \Lambda(z, y) \psi(y) da_y = G(z), \quad (37.48)$$

$$-\frac{1}{2}\psi(z) + \int_{\partial B} H\left(\frac{\partial}{\partial z}, n_z\right) \Gamma(z, y; \omega) \psi(z) da_y = G(z). \quad (37.49)$$

These equations are integral singular bidimensional equations. Following the procedure used in the case of elastic vibrations, the following can be proven [248].

Theorem 37.8. If $2\pi(\lambda + 2\mu) \neq \pm\mu$, then, for the equations (37.46) -- (37.49) the theorems and the alternative of Fredholm may be applied.

Similarly the existence and uniqueness of the solutions of problems (E_1) and (E_2) may be proven.

We limited ourselves here to the presentation of the method and of certain results obtained with their help. A more detailed exposition of these questions would exceed the frame of this work. This has been done in the monograph written by Kupradze, Gegelia, Basileisvili and Burciuladze [248], where other boundary problems are also examined. Problems of plane deformation in the case of stationary vibrations were also studied by the method of the potential [189], [197], [227].

38. Short-Time Solutions. Lessen's Method

In Paragraphs 35 -- 37, the problems concerning the particular case of harmonic waves were considered, in which the equations of the theory are reduced to elliptic equations. Let us consider now the general case of non-periodic waves. One of the most widely used methods in the study of the problems of coupled thermoelasticity is that of integral transformations. However, serious difficulties appear when these transformations are inverted. In general, developments in series according to the powers of the parameter ϵ are used (Lessen [259]) or solutions which are valid for a short time are determined. We will consider some problems which will illustrate these methods.

a) Elastic half-space. Let us consider a homogeneous and isotropic medium which occupies the half-space $x_1 > 0$. We assume that there are no mass forces and heat sources and the initial data are zero. We will study the case when the boundary $x_1 = 0$ is free of tensions and is subjected to a thermal field independent of the point. It is obvious that the solution of the problem must become /132 zero at infinity. In this case we have a one-dimensional problem so that

$$u_1 = u_1(x_1, t), \quad \theta = \theta(x_1, t), \quad u_2 = u_3 = 0. \quad (38.1)$$

The equations (33.1), (33.2) are reduced to

$$(\lambda + 2\mu) \frac{\partial^2 u_1}{\partial x_1^2} - \rho \frac{\partial^2 u_1}{\partial t^2} - \beta \frac{\partial \theta}{\partial x_1} = 0, \quad (38.2)$$

$$k \frac{\partial^2 \theta}{\partial x_1^2} - c \frac{\partial \theta}{\partial t} - \beta T_0 \frac{\partial^2 u_1}{\partial x_1 \partial t} = 0.$$

The components of the tension tensor which differ from zero have the expressions

$$t_{11} = (\lambda + 2\mu) \frac{\partial u_1}{\partial x_1} - \beta \theta, \quad t_{22} = t_{33} = \lambda \frac{\partial u_1}{\partial x_1} - \beta \theta. \quad (38.3)$$

If we introduce the non-dimensional variables

$$x = \frac{c_1 c}{k} x_1, \quad \tau = \frac{c_1^2 c}{k} t, \quad u = \frac{c_1 c}{k} u_1, \quad T = \frac{\theta}{T_0}, \quad (38.4)$$

then the equations (38.2) become

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial \tau^2} \right) u - A \frac{\partial T}{\partial x} = 0, \quad (38.5)$$

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial \tau} \right) T - \frac{\beta}{c} \frac{\partial^2 u}{\partial x \partial \tau} = 0,$$

where

$$A = \frac{\beta T_0}{\lambda + 2\mu}. \quad (38.6)$$

We assume that at the boundary we have the conditions

$$t_{11}(0, \tau) = 0, \quad \theta(0, \tau) = T^* H(\tau), \quad (38.7)$$

where T^* is a constant and $H(\tau)$ is Heaviside's function, defined in the following way

$$H(\tau) = \begin{cases} 0 & , \tau < 0, \\ 1 & , \tau > 0. \end{cases} \quad (38.8)$$

Let us consider the following initial conditions

/133

$$u(x, 0) = 0, \quad \dot{u}(x, 0) = 0, \quad T(x, 0) = 0. \quad (38.9)$$

In the theory of uncoupled thermoelasticity the problem posed was resolved by Danilovskaia [75]. In the theory of coupled

thermoelasticity this problem was studied in various papers. Let us mention in this sense the works of Hetnarski [172], Boley and Tolins [24], Muki and Breuer [294], Boley and Hetnarski [25].

We will present first the solution of the problem in the uncoupled case (problem resolved by Danilovskaia) and afterwards we will study the problem within the framework of the theory of coupled thermoelasticity.

In the uncoupled theory the equations (38.5) are substituted by

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial \tau^2} \right) u - A \frac{\partial T}{\partial x} &= 0, \\ \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial \tau} \right) T &= 0. \end{aligned} \quad (38.10)$$

We will designate by \bar{f} the Laplace transform with respect to τ of the function f , that is

$$\bar{f}(x, p) = \mathcal{L}[f(x, \tau)] = \int_0^\infty e^{-p\tau} f(x, \tau) d\tau. \quad (38.11)$$

If we consider the conditions (38.9), we obtain from (38.10)

$$\begin{aligned} \left(\frac{d^2}{dx^2} - p^2 \right) \bar{u} - A \frac{d\bar{T}}{dx} &= 0, \\ \left(\frac{d^2}{dx^2} - p \right) \bar{T} &= 0. \end{aligned} \quad (38.12)$$

Applying the Laplace transform to the boundary conditions, it follows that

$$\bar{u}_n(0, p) = 0, \quad \bar{T}(0, p) = \frac{T^*}{pT_0}. \quad (38.13)$$

The solution of the equation (38.12)₂ which becomes zero at infinity, is

$$\bar{T} = C_1 e^{-V\bar{r}}, \quad (38.14)$$

where C_1 is an arbitrary constant. Imposing the condition (37.13)₂, it follows that

$$C_1 = \frac{T^*}{p T_0},$$

and therefore

$$\bar{T} = \frac{T^*}{T_0} \frac{e^{-V\bar{r}}}{p}. \quad (38.15)$$

In view of the fact that we have (Carslaw and Jaeger [47])

/134

$$\mathcal{L}^{-1} \left(\frac{e^{-V\bar{r}}}{p} \right) = \operatorname{erfc} \frac{x}{2\sqrt{\tau}},$$

where

$$\operatorname{erfc} y = \frac{2}{\sqrt{\pi}} \int_y^\infty e^{-\xi^2} d\xi, \quad (38.16)$$

it follows from (38.4) and (38.15) that

$$\theta(x, \tau) = T^* \operatorname{erfc} \frac{x}{2\sqrt{\tau}}. \quad (38.17)$$

We obtain from (38.15) and (38.12)₁

$$\left(\frac{d^2}{dx^2} - p^2 \right) \bar{u} + \frac{A T^*}{T_0} \frac{e^{-V\bar{r}}}{p} = 0. \quad (38.18)$$

The solution of the equation (38.18) which becomes zero at infinity has the form

$$u = C_2 e^{-\rho x} - \frac{T A^* e^{-V \bar{\rho} x}}{T_0(1-p) p \sqrt{p}}, \quad (38.19)$$

where C_2 is an arbitrary constant. We will determine this constant, imposing the condition (38.13)₁. We have from (38.3)

$$t_{11} = (\lambda + 2\mu) \frac{d\bar{u}}{dx} - \beta_0 T_0 T,$$

and therefore

$$t_{11} = (\lambda + 2\mu) \left[-p C_2 e^{-\rho x} + \frac{T^* A e^{-V \bar{\rho} x}}{T_0(1-p) p \sqrt{p}} \right] - \beta T^* \frac{e^{-V \bar{\rho} x}}{p}. \quad (38.20)$$

If we impose the condition (38.13)₁, we obtain

$$C_2 = \frac{A T^*}{T_0 p(1-p)},$$

so that we can write

$$t_{11} = -\frac{\beta T^*}{1-p} (e^{-\rho x} - e^{-V \bar{\rho} x}), \quad (38.21)$$

$$u = \frac{\beta T^*}{(\lambda + 2\mu)(1-p)p} \left(e^{-\rho x} - \frac{e^{-V \bar{\rho} x}}{\sqrt{p}} \right).$$

From (38.21) we can derive the expressions of functions t_{11} /135
and u . Thus, in view of the relations (Carslaw and Jaeger [47])

$$\mathcal{L}^{-1} \left(\frac{e^{-\rho x}}{p-1} \right) = e^{x-\tau} H(\tau-x),$$

$$\mathcal{L}^{-1} \left(\frac{e^{-V \bar{\rho} x}}{p-1} \right) = \frac{1}{2} e^x \left\{ e^{-x} \operatorname{erfc} \left(\frac{x}{2\sqrt{\tau}} - \sqrt{\tau} \right) + e^x \operatorname{erfc} \left(\frac{x}{2\sqrt{\tau}} + \sqrt{\tau} \right) \right\},$$

we can write

$$t_{11} = -\beta T^* [S(x, \tau) - e^{x-\tau} H(\tau-x)], \quad (38.22)$$

where

$$S(x, \tau) = \frac{1}{2} e^x \left[e^{-x} \operatorname{erfc} \left(\frac{x}{2\sqrt{\tau}} - \sqrt{\tau} \right) + e^x \operatorname{erfc} \left(\frac{x}{2\sqrt{\tau}} + \sqrt{\tau} \right) \right]. \quad (38.23)$$

The term from (38.22) which contains $H(\tau - x)$ has the character of an elastic wave, while the other term characterizes the diffusion. It is obvious that $x = \tau$, that is, $x_1 = c_1 t$ characterizes the wave front. Let us take any given point of the medium under consideration with coordinates x_1^* . Before the moment $t^* = x_1^*/c_1$ the tension t_{11} in this point assumes the expression $-\beta T^* S(x^*, \tau)$. At the moment t^* in the point under consideration an elastic wave arrives. After the moment t^* the expression of the tension is $-\beta T^* [\dot{S}(x^*, \tau) - e^{-x^*}]$. The tension is discontinuous on the wave front and the magnitude of the discontinuity (of the jump) is

$$[t_{11}]_{x_1=c_1 t^*} = \beta T^*. \quad (38.24)$$

It should be noted that the jump of the function t_{11} on the wave front does not depend on the distance to plane $x_1 = 0$.

Let us now study the problem within the framework of the theory of coupled thermoelasticity. If we express the functions u and T in the form

$$u = \frac{\partial \varphi}{\partial x}, \quad T = \frac{1}{A} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial \tau^2} \right) \varphi, \quad (38.25)$$

then the first equation from (38.5) is satisfied. From the second equation from (38.5) we conclude that the function φ must satisfy the equation

$$\left\{ \frac{\partial^4}{\partial x^4} - \left[\frac{\partial}{\partial \tau} + (1 + \varepsilon) \right] \frac{\partial^3}{\partial x^2 \partial \tau} + \frac{\partial^3}{\partial \tau^3} \right\} \varphi = 0, \quad (38.26)$$

where ε is the coupling constant defined in (33.4).

In view of (38.25), we conclude from (38.3)

$$t_{11} = (\lambda + 2\mu) \frac{\partial^2 \varphi}{\partial \tau^2}. \quad (38.27)$$

C - 3

The initial conditions are satisfied if

$$\varphi(x, 0) = 0, \quad \frac{\partial \varphi(x, 0)}{\partial \tau} = 0, \quad \frac{\partial^2 \varphi(x, 0)}{\partial \tau^2} = 0. \quad (38.28)$$

Applying the Laplace transform to equation (38.26), we obtain /136

$$\left\{ \frac{d^4}{dx^4} - p[p + (1 + \epsilon)] \frac{d^2}{dx^2} + p^3 \right\} \varphi = 0. \quad (38.29)$$

The general solution of equation (38.29) which becomes zero at infinity is

$$\varphi = B_1 e^{-k_1 x} + B_2 e^{-k_2 x}, \quad (38.30)$$

where B_α are arbitrary constants and

$$k_{1,2}^2 = \frac{p}{2} \{ p + 1 + \epsilon \pm \sqrt{p^2 - 2p(1 + \epsilon) + (1 + \epsilon)^2} \}. \quad (38.31)$$

In view of

$$i_{11} = (\lambda + 2\mu) p^2 \varphi, \quad T = \frac{1}{A} \left(\frac{d^2}{dx^2} - p^2 \right) \varphi, \quad (38.32)$$

B_α is determined from the conditions (38.13). For the function φ we obtain the expression

$$\varphi = \frac{A T^*}{T_0 p (k_1^2 - k_2^2)} (e^{-k_1 x} - e^{-k_2 x}). \quad (38.33)$$

It follows from (38.25), (38.32) and (38.33) that

$$\begin{aligned} \bar{u} &= - \frac{\beta T^*}{(\lambda + 2\mu) p (k_1^2 - k_2^2)} (k_1 e^{-k_1 x} - k_2 e^{-k_2 x}), \\ 0 &= \frac{T^*}{p (k_1^2 - k_2^2)} [(k_1^2 - p^2) e^{-k_1 x} - (k_2^2 - p^2) e^{-k_2 x}], \\ i_{11} &= \frac{\beta T^* p}{(k_1^2 - k_2^2)} (e^{-k_1 x} - e^{-k_2 x}). \end{aligned} \quad (38.34)$$

The problem of inverting the transforms (38.34) is difficult. According to Abel's [426] theorem, it follows that if \bar{f} is the Laplace transform of function f , then we have

$$\lim_{\tau \rightarrow 0} f(\tau) = \lim_{p \rightarrow \infty} p f(p),$$

and thus, large values of p correspond to low values of τ . This situation permits the inversion of the Laplace transforms for a short time. The method permits, however, carrying out an exact determination for any time of the magnitude of the discontinuities on the wave front. If we develop in series the powers of $1/p$, we can write

/137

$$\begin{aligned} k_1 &= p + \frac{\epsilon}{2} + \frac{\epsilon(4-\epsilon)}{8p} + O\left(\frac{1}{p^2}\right), \\ k_2 &= \sqrt{p} - \frac{\epsilon}{2\sqrt{p}} + \frac{\epsilon(3\epsilon-4)}{8p\sqrt{p}} + O\left(\frac{1}{p^{3/2}}\right), \\ \frac{1}{k_1^2 - k_2^2} &= \frac{1}{p^2} + \frac{1-\epsilon}{p^3} + \frac{1-4\epsilon+\epsilon^2}{p^4}, \\ \frac{k_1^2 - p^2}{k_1^2 - k_2^2} &= \frac{\epsilon}{p} + \frac{\epsilon(2-\epsilon)}{p^2} + \frac{\epsilon(3+\epsilon^2-6\epsilon)}{p^3} + \frac{2\epsilon(1+\epsilon^2-3\epsilon)}{p^4}, \\ \frac{k_2^2 - p^2}{k_1^2 - k_2^2} &= -1 + \frac{\epsilon}{p} + \frac{\epsilon(2-\epsilon)}{p^2} + \frac{1-6\epsilon+3\epsilon^2}{p^3} - \frac{2\epsilon(1-3\epsilon+\epsilon^2)}{p^4}, \\ \frac{k_1}{k_1^2 - k_2^2} &= \frac{1}{p} + \frac{2-\epsilon}{2p^2} + \frac{8-24\epsilon+3\epsilon^2}{8p^3} + \frac{\epsilon(5\epsilon^2-21\epsilon)}{8p^4}, \\ \frac{k_2}{k_1^2 - k_2^2} &= \frac{1}{p^{3/2}} + \frac{2-3\epsilon}{2p^{5/2}} + \frac{8-40\epsilon+15\epsilon^2}{8p^{7/2}}. \end{aligned} \quad (38.35)$$

Similarly, we have

$$\begin{aligned} e^{-k_1 x} &\approx e^{-px} e^{-\frac{\epsilon}{2}x} e^{-\frac{\epsilon(4-\epsilon)}{8p}x}, \\ e^{-k_2 x} &\approx e^{-\sqrt{p}x} e^{-\frac{\epsilon}{2\sqrt{p}}x} e^{-\frac{\epsilon(3\epsilon-4)}{8p\sqrt{p}}x}. \end{aligned}$$

In the following we will assume that

$$\begin{aligned} e^{-\frac{\epsilon(4-\epsilon)}{8p}x} &\approx 1 - \frac{\epsilon(4-\epsilon)}{8p}x, & e^{-\frac{\epsilon}{2\sqrt{p}}x} &\approx 1 + \frac{\epsilon}{2\sqrt{p}}x, \\ e^{-\frac{\epsilon(3\epsilon-4)}{8p\sqrt{p}}x} &\approx 1 + \frac{\epsilon(4-3\epsilon)}{8p\sqrt{p}}x. \end{aligned}$$

We obtain from (38.34)

$$\begin{aligned} \frac{i_{11}}{\beta T^*} &= e^{-\frac{\epsilon}{2}x} e^{-\frac{\epsilon}{2}p^2} \left[\frac{1}{p} - \frac{\epsilon(4-\epsilon)x - 8(1-\epsilon)}{8p^3} + o\left(\frac{1}{p^3}\right) \right] - \\ &= e^{-\frac{\epsilon}{2}p^2} \left[\frac{1}{p} + \frac{1-\epsilon}{p^3} + \frac{\epsilon x}{2p\sqrt{p}} + o\left(\frac{1}{p^{3/2}}\right) \right], \\ \frac{0}{T^*} &= e^{-\frac{\epsilon}{2}x} \left[\frac{e^{-\frac{\epsilon}{2}p^2}}{p^3} \epsilon + o\left(\frac{1}{p^3}\right) \right] - \\ &= e^{-\frac{\epsilon}{2}p^2} \left[-\frac{1}{p} - \frac{\epsilon x}{2p\sqrt{p}} + \frac{\epsilon}{p^3} + o\left(\frac{1}{p^{3/2}}\right) \right]. \end{aligned} \quad (38.36)$$

In view of the relations (Carslaw and Jaeger [47])

/138

$$\mathcal{L}^{-1}\left(\frac{e^{-ps}}{p}\right) = H(\tau - x), \quad \mathcal{L}^{-1}\left(\frac{e^{-ps}}{p^2}\right) = (\tau - x) H(\tau - x),$$

$$\mathcal{L}^{-1}\left(\frac{e^{-V\bar{p}^2}}{p}\right) = \operatorname{erfc} \frac{x}{2\sqrt{\tau}},$$

$$\mathcal{L}^{-1}\left(\frac{e^{-V\bar{p}^2}}{p^2}\right) = \left(\tau + \frac{1}{2}x^2\right) \operatorname{erfc} \frac{x}{2\sqrt{\tau}} - x\left(\frac{\tau}{\pi}\right)^{1/2} e^{-\frac{x^2}{4\tau}},$$

$$\mathcal{L}^{-1}\left(\frac{e^{-V\bar{p}^2}}{p\sqrt{p}}\right) = 2\left(\frac{\tau}{\pi}\right)^{1/2} e^{-\frac{x^2}{4\tau}} - x \operatorname{erfc} \frac{x}{2\sqrt{\tau}},$$

we conclude

$$\begin{aligned} i_{11} &= \beta T^* e^{-\frac{\epsilon}{2}x} \left\{ 1 - \frac{1}{8} [\epsilon(4-\epsilon)x - 8(1-\epsilon)] (\tau - x) \right\} H(\tau - x) - \\ &= \beta T^* \left\{ \operatorname{erfc} \frac{x}{2\sqrt{\tau}} + (1-\epsilon) \left[\left(\tau + \frac{1}{2}x^2\right) \operatorname{erfc} \frac{x}{2\sqrt{\tau}} - \right. \right. \\ &\quad \left. \left. - x\left(\frac{\tau}{\pi}\right)^{1/2} e^{-\frac{x^2}{4\tau}} \right] + \frac{\epsilon x}{2} \left[2\left(\frac{\tau}{\pi}\right)^{1/2} e^{-\frac{x^2}{4\tau}} - x \operatorname{erfc} \frac{x}{2\sqrt{\tau}} \right] \right\}, \\ 0 &= T^* \left\{ \operatorname{erfc} \frac{x}{2\sqrt{\tau}} + \epsilon e^{-\frac{\epsilon}{2}x} (\tau - x) H(\tau - x) + \right. \\ &\quad \left. + \frac{\epsilon x}{2} \left[2\left(\frac{\tau}{\pi}\right)^{1/2} e^{-\frac{x^2}{4\tau}} - x \operatorname{erfc} \frac{x}{2\sqrt{\tau}} \right] - \right. \\ &\quad \left. - \epsilon \left[\left(\tau + \frac{1}{2}x^2\right) \operatorname{erfc} \frac{x}{2\sqrt{\tau}} - x\left(\frac{\tau}{\pi}\right)^{1/2} e^{-\frac{x^2}{4\tau}} \right] \right\}. \end{aligned} \quad (38.37)$$

It should be noted that in this case we have

$$[t_{11}] = \beta T^* e^{-\frac{x}{2\sqrt{\tau}}}. \quad (38.38)$$

Similarly, it follows from (38.37) that

$$[\dot{\theta}] = \epsilon T^* e^{-\frac{x}{2\sqrt{\tau}}}. \quad (38.39)$$

Let us note the fact that the magnitudes of the discontinuities are exact for any time.

For the function u we derive the expression

/139

$$u = - \frac{\beta T^*}{\lambda + 2\mu} \left[e^{-\frac{x}{2\sqrt{\tau}}} (\tau - x) H(\tau - x) - (4\tau)^{3/2} i^3 \operatorname{erfc} \frac{x}{2\sqrt{\tau}} \right],$$

where

$$i^n \operatorname{erfc} x = \int_1^\infty i^{n-1} \operatorname{erfc} \xi d\xi, \quad i^0 \operatorname{erfc} x = \operatorname{erfc} x.$$

The expressions of the tensions t_{22} and t_{33} may be obtained in a similar manner.

It can easily be seen that the solution obtained goes to zero at infinity. Within the theory of coupled thermoelasticity the problem under consideration above has been first studied by Hetnarski [172] who gives a solution for short time, neglecting the powers of ϵ which are larger than or equal to two. In another study, Hetnarski [173] presents a solution developed according to the powers of ϵ . The problem was studied by Boley and Tolins [24] and by Muki and Breuer [294].

Boley and Hetnarski [25] studied systematically the discontinuities in the case of the half-space. As has been seen, the magnitude of the discontinuities may be studied independently of the derivation

of a complete solution of the problem. In [25] the Laplace transform is used to study 16 problems referring to the half-space, considering various boundary conditions. The discontinuities are classified and their magnitude was determined for every partial problem.

b) Concentrated loads. Let us consider an elastic medium which occupies a whole space. We will illustrate the method given by Lessen who studied the action of a concentrated heat source. In general, if a solution in the form of (34.7) is sought from (34.3) and from (34.6), it follows that the Laplace transforms of the functions ϕ and ψ_1 with reference to the variable t satisfy the following equations

$$\begin{aligned} \left[\left(\Delta - \frac{p^2}{c_1^2} \right) \left(\Delta - \frac{cp}{k} \right) - \frac{cp^2}{k} \Delta \right] \phi = \\ \frac{\beta}{k(\lambda + 2\mu)} r - \frac{1}{\lambda + 2\mu} \left(\Delta - \frac{c}{k} p \right) \psi, \\ \left(\Delta - \frac{p^2}{c_1^2} \right) \psi = \frac{1}{\mu} \dot{r}. \end{aligned} \quad (38.40)$$

We assumed that the initial conditions were homogeneous. If ζ_1 and ζ_2 satisfy the equation

$$\zeta^4 - \left[\frac{p^2}{c_1^2} + \frac{cp}{k} (1 + \varepsilon) \right] \zeta^2 + \frac{cp^3}{kc_1^3} = 0, \quad (38.41)$$

then the equation which satisfies the function $\bar{\phi}$ may be written in the following manner

$$(\Delta - \zeta_1^2)(\Delta - \zeta_2^2) \bar{\phi} = \frac{\beta}{k(\lambda + 2\mu)} r - \frac{1}{\lambda + 2\mu} \left(\Delta - \frac{c}{k} p \right) \bar{\psi}. \quad (38.42)$$

Obviously, the functions ψ_1 do not depend on the coefficient ε . It follows from (38.41) that

/140

$$\zeta_{i,2} = \frac{1}{2} \left\{ \frac{p^2}{c_1^2} + \frac{pc}{k} (1 + \epsilon) \pm \right. \\ \left. \pm \left[\left(\frac{p^2}{c_1^2} + \frac{pc}{k} (1 + \epsilon) \right)^2 - \frac{4cp^3}{kc_1^2} \right]^{1/2} \right\} \quad (38.43)$$

In view of the fact that, in general, the coefficient ϵ is a small number ($\epsilon \ll 1$) Lessen suggested a development of the following form for determining the function $\bar{\phi}$

$$\bar{\phi} = \bar{\phi}_0 + \epsilon \bar{\phi}_1 + \epsilon^2 \bar{\phi}_2 + \dots \quad (38.44)$$

In this case there follow from (38.40), or (38.42) for the functions $\bar{\phi}_0, \bar{\phi}_1, \dots$, the equations

$$\left(\Delta - \frac{p^2}{c_1^2} \right) \left(\Delta - \frac{cp}{k} \right) \bar{\phi}_0 = - \frac{\beta}{k(\lambda + 2\mu)} \bar{r} - \\ - \frac{1}{(\lambda + 2\mu)} \left(\Delta - \frac{c}{k} p \right) \bar{\theta}, \\ \left(\Delta - \frac{p^2}{c_1^2} \right) \left(\Delta - \frac{cp}{k} \right) \bar{\phi}_1 = \frac{pc}{k} \Delta \bar{\phi}_0, \quad (38.45)$$

It can easily be seen that the function $\bar{\phi}_0$ corresponds to the problem of uncoupled thermoelasticity. The function $\bar{\phi}_1$ is the solution of a problem of the same type in which, however, the loads are determined by the function ϕ_0 , etc.

Let us assume that (Nowacki [319])

$$f_1 = 0, \quad r = Q\delta(x-y)\delta(t), \quad (38.46)$$

where Q is constant. In this case let us take $\psi_1 = 0$ and from (38.45) we obtain (see Paragraph 36)

$$\bar{\varphi}_0 = \frac{\beta c_1^2 Q}{4\pi k(\lambda + 2\mu) R p \left(p^2 - \frac{c_1^2 c}{k} \right)} \left(e^{-\frac{R}{c_1} p} - e^{-\kappa \sqrt{\frac{p^2}{k}}} \right). \quad (38.47)$$

From (38.47) and (34.7) the corresponding temperature variation is determined

$$\theta_0 = \frac{cQ}{4\pi k R} e^{-\kappa \sqrt{\frac{p^2}{k}}}. \quad (38.48)$$

If we designate

$$\rho = \frac{c_1^2}{k} R, \quad \tau = \frac{c_1^2}{k} t,$$

then the original functions have the expression

/141

$$\varphi_0(\rho, \tau) = -\frac{\beta Q}{4\pi c(\lambda + 2\mu)} [E_1(\rho, \tau) - E_2(\rho, \tau) H(\tau - \rho)], \quad (38.49)$$

$$\theta_0(\rho, \tau) = \frac{c_1^2 c^3 \beta Q}{8k^3 (\pi \tau)^{3/2}} e^{-\frac{\rho^2}{4\tau}},$$

where

$$\begin{aligned} E_1(\rho, \tau) &= \operatorname{erfc} \left(\frac{\rho}{2\sqrt{\tau}} \right) - \frac{1}{2} e^{-\tau} \left[e^{\rho} \operatorname{erfc} \left(\frac{\rho}{2\sqrt{\tau}} + \sqrt{\tau} \right) + \right. \\ &\quad \left. + e^{-\rho} \operatorname{erfc} \left(\frac{\rho}{2\sqrt{\tau}} - \sqrt{\tau} \right) \right], \\ E_2(\rho, \tau) &= e^{\tau - \rho} - 1. \end{aligned} \quad (38.50)$$

The functions $\bar{\phi}_1$ etc. are determined afterwards. The solution of this problem was studied by Hetnarski [174]. Similarly, Hetnarski [175] considered the problem of the elastic space when

$$f_i = 0, \quad r = Q\delta(x - y)H(\tau).$$

In this case we take $\psi_1 = 0$ and equation (38.42) becomes

$$(\Delta - \zeta_1^2)(\Delta - \zeta_2^2)\bar{\phi} = -\frac{Q}{kp(\lambda + 2\mu)}\delta(x - y).$$

As in the case of the procedure used for equation (36.24) we derive for $\bar{\phi}$ the expression

$$\bar{\phi} = \frac{\beta Q}{4\pi kp(\lambda + 2\mu)R(\zeta_1^2 - \zeta_2^2)}(e^{-\zeta_1 R} - e^{-\zeta_2 R}).$$

In order to obtain the function ϕ , Hetnarski uses both approximation procedures: he neglects the powers of ϵ which are larger than or equal to 2 and determines the solution for a short time.

The problem of the elastic space acted upon by concentrated mass forces was studied by de Soós [386] with the aid of representation (34.12). In this case the Laplace transforms of the functions ϕ_1 satisfy the equations

$$\left(\Delta - \frac{p^2}{c_i^2}\right)(\Delta - \zeta_1^2)(\Delta - \zeta_2^2)\bar{\phi}_i = -\frac{c}{k\mu(\lambda + 2\mu)}f_i. \quad (38.51)$$

It can easily be seen that if, in the equation (38.41), we replace ζ and p by iz and $-i\omega$ respectively, we obtain equation (36.8). Let us assume that

$$f_i = \delta_{ii}\delta(x - y)\delta(t), \quad r = 0.$$

In this case we will take $\phi_2 = \phi_3 = \psi = 0$. The function $\bar{\phi}_1$ satisfies the equation

/142

$$(\Delta - \zeta_1)(\Delta - \zeta_2)(\Delta - \zeta_3)\bar{\Phi}_1 = -\frac{c}{k\mu(\lambda + 2\mu)}\delta(R), \quad (38.52)$$

where $\zeta_3^2 = p^2/c_2^2$.

If we consider the results from Paragraph 36, it follows that the solution of the equation (38.52) is

$$\begin{aligned} \Phi_1 = \frac{c}{4\pi k\mu(\lambda + 2\mu)(\zeta_3^2 - \zeta_1^2)} R \left[\frac{e^{-\zeta_1 R}}{\zeta_3^2 - \zeta_1^2} - \right. \\ \left. - \frac{e^{-\zeta_3 R}}{\zeta_3^2 - \zeta_2^2} + \frac{(\zeta_3^2 - \zeta_1^2)e^{-\zeta_3 R}}{(\zeta_3^2 - \zeta_1^2)(\zeta_3^2 - \zeta_2^2)} \right]. \end{aligned} \quad (38.53)$$

From (34.12) and (38.53) the Laplace transforms of the components of the displacement vector and of the temperature variation can be determined. The functions were originally determined for a short time and neglecting the powers of ε^n ($n \gg 2$), in [386]. Solutions both for concentrated forces directed according to other axes and for a concentrated heat source can be similarly obtained.

Lessen [259], Chadwick and Sneddon [48], Paria [333], Sneddon [373], Lockett and Sneddon [259], Eason and Sneddon [99], Nariboli [302], Galka [128] and others also studied problems referring to the space and half-space. Investigation of the coupling effect between the mechanical field and the thermal field formed the subject of many studies (see, for example, [1], [14], [15], [92], [303], [308], [383], [446]).

39. Propagation of the Discontinuity Surfaces

The mathematical method used for investigating the discontinuity surfaces in the mechanics of continuous media, outlined at the end of the last century, was developed by Hadamard in [167]. In this

paragraph we will study the discontinuity surfaces in the theory of coupled thermoelasticity as given by Chadwick and Powdrill [51]. Other studies concerning the propagation of discontinuity surfaces in the theory of thermoelasticity were carried out by Chen [56], Truesdell [424] and others.

a) Geometric preliminaries. We will express the position vector of a point in the form

$$\mathbf{R} = \mathbf{R}_0 + \theta^3 \mathbf{a}_3, \quad (39.1)$$

where \mathbf{R}_0 depends only on θ^1, θ^2 and \mathbf{a}_3 is a unit vector which depends, similarly, only on θ^1 and θ^2 . The equation $\theta^3 = 0$ determines a surface

$$\mathbf{R}_0 = \mathbf{R}_0(\theta^1, \theta^2). \quad (39.2)$$

The vector \mathbf{a}_3 is normal to this surface. If we designate

$$\mathbf{a}_\alpha = \mathbf{R}_{0,\alpha}, \quad (39.3)$$

then the covariant vectors from (24.7) may be written in the following way

$$\mathbf{g}_\alpha = \mathbf{a}_\alpha + \theta^3 \mathbf{a}_{3,\alpha}, \quad \mathbf{g}_3 = \mathbf{a}_3. \quad (39.4)$$

It is obvious that

/143

$$\mathbf{a}_3 \mathbf{a}_\alpha = 0, \quad \mathbf{a}_3 \mathbf{a}_3 = 1, \quad \mathbf{a}_3 \mathbf{a}_{3,\alpha} = 0. \quad (39.5)$$

For $\theta^3 = 0$ we have $\mathbf{g}_r = \mathbf{a}_r$. We will designate by \mathbf{a}^r , \mathbf{a}_{rs} , \mathbf{a}^{rs} , \mathbf{a} , the values which \mathbf{g}^r , \mathbf{g}_{rs} , \mathbf{g}^{rs} , \mathbf{g} given in (24.7)-(24.10) respectively take for $\theta^3 = 0$. Therefore

$$\begin{aligned}
 a_{\alpha\beta} &= a_{\beta\alpha}, a_{3i} = \delta_{ij}, a_{\alpha\beta} a^{\beta\gamma} = \delta_{\alpha}^{\gamma}, a^{3i} = \delta^{ij}, \\
 a^2 &= a^{\alpha\beta} a_{\beta} = a_{\alpha\beta} a^{\beta}, a^{\alpha} a^{\beta} = a^{\alpha\beta} a^{\gamma} = a_{\gamma}^{\gamma}.
 \end{aligned}
 \tag{39.6}$$

In view of the fact that

$$dR_0 = \kappa_{\alpha} d\theta^{\alpha} = a^{\alpha} d\theta_{\alpha},$$

the arc element from the surface (39.2) has the form

$$ds^2 = a_{\alpha\beta} d\theta^{\alpha} d\theta^{\beta} = a^{\alpha\beta} d\theta_{\alpha} d\theta_{\beta}. \tag{39.7}$$

The expression (39.7) is the first fundamental form of the surface. The scalar product

$$dR_0 da_3 = -b_{\alpha\beta} d\theta^{\alpha} d\theta^{\beta} = -b^{\alpha\beta} d\theta_{\alpha} d\theta_{\beta},$$

where

$$b_{\alpha\beta} = -n_{\alpha} n_{3,\beta} = -n_{\beta} n_{3,\alpha} = n_{\beta} n_{\alpha,\beta} = n_{\alpha} n_{\beta,\alpha}, \tag{39.8}$$

is the second fundamental form of the surface. The coefficients $b_{\alpha\beta}$, $b^{\alpha\beta}$ are symmetrical surface tensors connected by the relations

$$\begin{aligned}
 b_{\beta}^{\alpha} &= a^{\alpha\lambda} b_{\beta\lambda} = a_{\beta\lambda} b^{\alpha\lambda}, \\
 b^{\alpha\beta} &= a^{\alpha\lambda} b_{\lambda}^{\beta}, b_{\alpha\beta} = a_{\alpha\lambda} b_{\beta}^{\lambda}.
 \end{aligned}
 \tag{39.9}$$

The invariant

$$2H = b_{\alpha}^{\alpha}, \tag{39.10}$$

is the average curvature of the surface, and

$$K = \frac{|b_{\alpha\beta}|}{a}, \tag{39.11}$$

is the gaussian curvature of the surface.

The symbols of Christoffel with respect to the surface $\theta^3 = 0$ are obtained from (24.17) setting $\theta^3 = 0$. If we designate these symbols by Γ_{jk}^1 , keeping in mind (39.4), we have

$$\begin{aligned}\Gamma_{\beta\gamma}^1 &= a^\alpha a_{\alpha,\gamma}, \quad \Gamma_{\beta 3}^1 = a^\alpha a_{\alpha,\beta} = a^{\alpha\beta} a_{\alpha 3,\beta} = -a^{\alpha\beta} b_{\alpha\beta} = -b_\beta^\beta, \\ \Gamma_{\alpha\beta}^2 &= a^\alpha a_{\alpha,\beta} = -a_\beta a^{\alpha\beta}_{,\alpha} = b_{\alpha\beta}, \\ \Gamma_{\alpha\beta}^3 &= a^\alpha a_{\alpha,\beta} = 0, \quad \Gamma_{33}^3 = 0.\end{aligned}\tag{39.12}$$

It follows from (24.17) and (39.12) that

/144

$$\begin{aligned}u_{\alpha,\beta} &= \Gamma_{\alpha\beta}^\lambda a_\lambda + b_{\alpha\beta} a_3, \quad u_{\beta 3} = -\Gamma_{\beta\lambda}^\alpha a^\lambda + b_\beta^\alpha a_3, \\ u_{3,\alpha} &= -b_\alpha^\lambda a_\lambda.\end{aligned}\tag{39.13}$$

If we consider (39.13)₂ we can write

$$u_{3,\alpha} u_{\alpha,\beta} = b_{\alpha\lambda} b_\beta^\lambda,\tag{39.14}$$

and therefore

$$da_3 da_3 = b_{\alpha\beta} b_\gamma^\beta d\theta^\alpha d\theta^\gamma,\tag{39.15}$$

an expression which represents the third fundamental form of the surface.

Let the vector be

$$v = v^\alpha a_\alpha + v^3 a_3 = r_\alpha a^\alpha + v^3 a_3.\tag{39.16}$$

Taking into account (39.13) we obtain

$$\begin{aligned} v_{,a} &= (v_{\lambda;a} - b_{a\lambda} v_3) a^\lambda + (v_{3,a} + b_a^\lambda v_\lambda) a_3 = \\ &= (v_{,a}^\lambda - b_a^\lambda v^3) a_\lambda + (v_{,a}^3 + b_{\lambda a} v^\lambda) a_3, \end{aligned} \quad (39.17)$$

where

$$v_{\lambda;a} = v_{\lambda,a} - \Gamma_{\lambda a}^\mu v_\mu, \quad v_{,a}^\lambda = v_{,a}^\lambda + \Gamma_{\mu a}^\lambda v^\mu. \quad (39.18)$$

b) Singular surfaces. Let I be an interval of the real axis $R, G = I \times R^3$ and Σ a hypersurface from G which allows the representation

$$x_i = \varphi_i(\theta_1, \theta_2, t), \quad \theta_1, \theta_2 \in U \subset R^2, \quad t \in I. \quad (39.19)$$

For a fixed value of t , the equations (39.19) define a surface Σ_t . Thus, the relations (39.19) describe the movement of the surface Σ_t when $t \in I$.

We say that Σ is a smooth hypersurface if the functions ϕ_i are biunivocal and of the class of C^2 on $U \times I$ and the matrix

$\left[\frac{\partial \varphi_i}{\partial \theta^a} \right]$ has the rank of 2 in any point of Σ . The variables θ_1, θ_2 define a system of curved coordinates on the surface of Σ_t .

We will designate by \underline{n} the normal to Σ_t and therefore in the relations from (a) we have

$$a_1 = \underline{n}. \quad (39.20)$$

It follows from (39.3) and (39.5) that

$$a_a = \varphi_{i,a} a_i, \quad n = \frac{1}{a} a_1 \wedge a_2, \quad (39.21)$$

and thus

$$n_i = \frac{1}{a} \varphi_{i,a} \varphi_{1,a} \varphi_{2,a}. \quad (39.22)$$

Similarly we have

/145

$$n_i n_i = 1, \quad n_i \varphi_{i,a} = 0. \quad (39.23)$$

Keeping in mind (39.6), (39.8), (39.21) we conclude

$$a_{ab} = \varphi_{i,a} \varphi_{i,b}, \quad h_{ab} = \varphi_{i,ab} n_i. \quad (39.24)$$

From (39.13) we obtain the relations

$$\begin{aligned} \varphi_{i,ab} &= h_{ab} n_i + \Gamma_{ab}^\lambda \varphi_{i,\lambda}, \\ n_{i,a} &= -h_a^\lambda \varphi_{i,\lambda} = -a^{\lambda\rho} h_{\rho a} \varphi_{i,\lambda} = -a^{\lambda\rho} \varphi_{j,\rho a} \varphi_{i,\lambda} n_j. \end{aligned} \quad (39.25)$$

Using the covariant derivative

$$\varphi_{i,ab} = \varphi_{i,a,b} - \Gamma_{ab}^\lambda \varphi_{i,\lambda}, \quad (39.26)$$

and the relation (39.23), it follows that (39.24) may be written in the following form

$$h_{ab} = \varphi_{i,ab} n_i. \quad (39.27)$$

Similarly we have from (39.25)

$$\begin{aligned} \varphi_{i,ab} &= h_{ab} n_i, \\ n_{i,a} &= -a^{\lambda\rho} \varphi_{j,\rho a} \varphi_{i,\lambda} n_j. \end{aligned} \quad (39.28)$$

If $R = y_1 \alpha_1$, then, on the basis of relations (24.7) it follows that

$$g^{ij} \frac{\partial y_i}{\partial \theta_j} \frac{\partial y_j}{\partial \theta_i} = \delta_{ij}. \quad (39.29)$$

Keeping in mind (39.1) and (39.14) we conclude

$$y_i = \varphi_i + 0^3 n_i,$$

and the relation (39.29) takes the form

$$g^{ab}(\varphi_{i,a} + 0^3 n_{i,a})(\varphi_{j,b} + 0^3 n_{j,b}) + n_i n_j = \delta_{ij}.$$

Thus we have on the surface of Σ_t

$$g^{ab} \varphi_{i,a} \varphi_{j,b} = \delta_{ij} - n_i n_j. \quad (39.30)$$

The velocity of a point from the mobile surface Σ_t at the moment t , has the component

$$V_i = \frac{\partial \varphi_i}{\partial t}, \quad (39.31)$$

and the velocity of Σ in the direction of the unit normal at Σ_t is

$$V = V_i n_i. \quad (39.32)$$

If the surface is locally represented by

$$\Phi(t, x_i) = 0, \quad (39.33)$$

where Φ is of the class C^2 , then

$$n_i = (\Phi_i, \Phi_{,i})^{-1/2} \frac{\partial \Phi}{\partial t} + \Phi_{,i} V_i = 0, \quad (39.34)$$

$$V = - (\Phi_{,i} \Phi_{,i})^{-1/2} \frac{\partial \Phi}{\partial t}.$$

Let us calculate the first and second order derivatives of a function on a hypersurface which intersects each domain of definition.

Let f be a real function: $\Omega \rightarrow \mathbb{R}$ and $\Sigma \subset \mathcal{G}$, a hypersurface. Let us assume that $\Omega = D \times I$ where D is an open set from E^3 and $\Omega \cap \Sigma_t \neq \emptyset$ for any $t \in I$. In view of the fact that the results which will be derived refer only to that part of Σ which is included in Ω we will replace $\Omega \cap \Sigma$ by Σ_t , and $\Omega \cap \Sigma_t$ by Σ_t .

We will assume for the beginning that $f \in C^1(\Omega)$. On Σ we have $f = f(t, \phi_1(\theta_1, \theta_2))$ and thus

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i} \frac{\partial \phi_i}{\partial t}, \quad f_{,a} = \frac{\partial f}{\partial x_i} \varphi_{i,a}. \quad (39.35)$$

From (39.35)₂ and (39.30) we obtain

$$\frac{\partial f}{\partial x_i} = a^{ab} f_{,a} \varphi_{i,b} + \frac{\partial f}{\partial n} n_i. \quad (39.36)$$

In view of (39.36) it follows from (39.35)₁ that

$$\frac{\partial f}{\partial t} = \frac{df}{dt} - a^{ab} f_{,a} \varphi_{i,b} V_i = V \frac{\partial f}{\partial n}. \quad (39.37)$$

We define the δ -derivative with respect to time (Thomas [413]) by

$$\frac{\delta}{\delta t} = \frac{\partial}{\partial t} + V \frac{\partial}{\partial n}. \quad (39.38)$$

This refers to the derivative with respect to time, following the motion of Σ along the normal at Σ_t .

We conclude from (39.37)

$$\frac{\delta f}{\delta t} = \frac{df}{dt} - a^{ab} \varphi_{i,b} V_i f_{,a}, \quad (39.39)$$

and

$$\frac{\partial f}{\partial t} = \frac{\delta f}{\delta t} - V \frac{\partial f}{\partial n}. \quad (39.40)$$

If f is a known function of t and θ_α on Σ then the derivatives df/dt , $f_{,\alpha}$ can be calculated. The derivatives which are determined in this sense will be called interior derivatives on Σ . From (39.39) it follows that $\delta f/\delta t$ is an interior derivative on Σ .

The relations (39.36) and (39.40) express connection between /147 the first order derivatives of f on Σ and are called conditions of compatibility for the partial derivatives of the first order of the function f on Σ .

If f and $\partial f/\partial n$ are given on Σ then the relations (39.36) and (39.40) determine the first order derivatives of its f functions on Σ .

If we consider (39.39) and (39.31) we can write

$$\frac{\delta \varphi_{i,a}}{\delta t} = V'_{i,a} - a^{\alpha\beta} \varphi_{j,\beta} V'_j \varphi_{i;\alpha\beta},$$

whence, on the basis of relation (39.28)₂ we obtain

$$\frac{\delta \varphi_{i,a}}{\delta t} n_i = V'_{i,a} n_i + V'_j n_{j,a} - V'_{i,a} \quad (39.41)$$

From (39.23) and (39.41) we can derive

$$n_i \frac{\delta n_i}{\delta t} = 0, \quad \varphi_{i,a} \frac{\delta n_i}{\delta t} = -V'_{i,a}. \quad (39.42)$$

If we multiply both members of the relation (39.42)₂ by $a^{\alpha\beta} \varphi_{j,\beta}$ and sum up after the index α , then, on the basis of relations (39.30) and (39.42)₁, it follows that

$$\frac{\delta n_i}{\delta t} = -a^{\alpha\beta} \varphi_{i,\alpha} V'_{\beta}. \quad (39.43)$$

Let us assume that $f \in C^2(\Omega)$. In this case the relations (39.36) and (39.40) can be applied to the functions $\partial f / \partial x_i$ and $\partial f / \partial t$. Thus, we have

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = a^{\alpha\beta} \left(\frac{\partial f}{\partial x_j} \right)_{;\alpha} \varphi_{i,\beta} + \frac{\partial}{\partial n} \left(\frac{\partial f}{\partial x_j} \right) n_i, \quad (39.44)$$

whence we obtain

$$\frac{\partial}{\partial n} \left(\frac{\partial f}{\partial x_i} \right) = a^{\alpha\beta} \left(\frac{\partial f}{\partial x_i} \right)_{;\alpha} \varphi_{i,\beta} n_j + \frac{\partial^2 f}{\partial n^2} n_i, \quad (39.45)$$

in which we indicated

$$\frac{\partial^2 f}{\partial n^2} = \frac{\partial}{\partial n} \left(\frac{\partial f}{\partial x_j} \right) n_j = \frac{\partial^2 f}{\partial x_i \partial x_j} n_i n_j. \quad (39.46)$$

In view of (39.23), (39.27), (39.28), (39.26) and (39.45) we find from (39.44) that

$$\begin{aligned} \frac{\partial^2 f}{\partial x_i \partial x_j} &= a^{\alpha\beta} \left\{ \left(\frac{\partial f}{\partial n} \right) + a^{\alpha\gamma} b_{\alpha\gamma} f_{;\rho} \right\} (n_i \varphi_{j,\beta} + n_j \varphi_{i,\beta}) + \\ &+ a^{\alpha\beta} a^{\gamma\delta} \left(f_{;\alpha\gamma} - b_{\alpha\gamma} \frac{\partial f}{\partial n} \right) \varphi_{i,\beta} \varphi_{j,\delta} + \frac{\partial^2 f}{\partial n^2} n_i n_j. \end{aligned} \quad (39.47)$$

From (39.47) on the basis of the relations (39.23), (39.10) /148
we obtain

$$\frac{\partial^2 f}{\partial x_i \partial x_i} = a^{\alpha\beta} f_{;\alpha\beta} - 2H \frac{\partial f}{\partial n} + \frac{\partial^2 f}{\partial n^2}. \quad (39.48)$$

If we apply the formula (39.36) to the function $\partial f / \partial t$ we can derive

$$\frac{\partial^2 f}{\partial x_i \partial t} = a^{\alpha\beta} \left(\frac{\partial f}{\partial t} \right)_{;\alpha} \varphi_{i,\beta} + \frac{\partial}{\partial n} \left(\frac{\partial f}{\partial t} \right) n_i. \quad (39.49)$$

In view of (39.40), (39.36), (39.43), (39.46) it follows that

$$\begin{aligned}\frac{\delta}{\delta t} \left(\frac{\partial f}{\partial n} \right) &= n_i \frac{\delta}{\delta t} \left(\frac{\partial f}{\partial x_i} \right) + \frac{\partial f}{\partial x_i} \frac{\delta n_i}{\delta t} = \\ &= \frac{\partial}{\partial n} \left(\frac{\partial f}{\partial t} \right) + V \frac{\partial^2 f}{\partial n^2} - a^{\alpha\beta} f_{,\alpha} V_{,\beta}.\end{aligned}\quad (39.50)$$

If we keep in mind relation (39.50), we can derive from (39.49)

$$\begin{aligned}\frac{\partial^2 f}{\partial x_i \partial t} &= a^{\alpha\beta} \left(\frac{\delta f}{\delta t} - V \frac{\partial f}{\partial n} \right)_{, \alpha} \varphi_{i, \beta} + \left\{ \frac{\delta}{\delta t} \left(\frac{\partial f}{\partial n} \right) + \right. \\ &\quad \left. + a^{\alpha\beta} f_{,\alpha} V_{,\beta} - V \frac{\partial^2 f}{\partial n^2} \right\} n_i.\end{aligned}\quad (39.51)$$

In view of (39.40), (39.50) we obtain

$$\frac{\partial^2 f}{\partial t^2} = \frac{\delta}{\delta t} \left(\frac{\delta f}{\delta t} - V \frac{\partial f}{\partial n} \right) - V \left\{ \frac{\delta}{\delta t} \left(\frac{\partial f}{\partial n} \right) + a^{\alpha\beta} f_{,\alpha} V_{,\beta} - V \frac{\partial^2 f}{\partial n^2} \right\}.\quad (39.52)$$

The relations (39.47), (39.51) and (39.52) are the conditions of compatibility for the partial derivatives of the second order of the function f on Σ .

Using the conditions of compatibility we will derive the corresponding conditions for the discontinuities of the partial derivatives of a function which appear on a smooth hypersurface.

Let us designate by D_t^+ and by D_t^- the open and disjunct subsets of D , the reunion of which is $D \setminus \Sigma$. We assume that the normal \underline{n} at Σ_t is directed toward D_t^+ . Let

$$\Omega^+ = \{(x, t) | x \in D_t^+, t \in I\}, \Omega^- = \{(x, t) | x \in D_t^-, t \in I\}.$$

We assume that f is continuous on $\Omega^+ \cup \Sigma$ and $\Omega^- \cup \Sigma$ but it is not continuous on Ω . Let f^+ and f^- be the extensions of f on the open subsets which contain $\Omega^+ \cup \Sigma$ and $\Omega^- \cup \Sigma$ respectively. The jump of f through Σ is

$$[f] = f^+ - f^-.$$

If $[f]$ is different from zero throughout nearly the whole of Σ , then we can say that Σ is a singular absolute hypersurface for the function f .

Let us assume that $f \in C^{r-1}(\Omega)$ ($r \geq 1$) and $f \in C^r(\Omega^+ \cup \Sigma)$, $f \in C^r(\Omega^- \cup \Sigma)$. We can thus calculate the jumps of the derivatives of order r of f on Σ . If at least one of these is different from zero throughout nearly the whole of Σ , then Σ is called a singular hypersurface of the order r for the function f . /149

Let Σ be a singular absolute hypersurface for the function f . We assume that f is of the class of C^1 on $\Omega^+ \cup \Sigma$ and on $\Omega^- \cup \Sigma$. The conditions of compatibility (39.36) and (39.40) are applicable on Σ both for f^+ and for f^- . In view of the fact that

$$[f]_{,a} = (f^+ - f^-)_{,a} = f^+_{,a} - f^-_{,a}, \quad \frac{\delta[f]}{\delta t} = \frac{\delta f^+}{\delta t} - \frac{\delta f^-}{\delta t}, \quad (39.53)$$

we can derive the relations

$$\left[\frac{\partial f}{\partial x_i} \right] = n^{\alpha\beta} [f]_{,a} \varphi_{i,\alpha\beta} + \left[\frac{\partial f}{\partial n} \right] n_i, \quad \left[\frac{\partial f}{\partial t} \right] = \frac{\delta[f]}{\delta t} - v \left[\frac{\partial f}{\partial n} \right], \quad (39.54)$$

which represent the conditions of compatibility for the jumps of the partial derivatives of the first order of the function f through Σ .

If Σ is a singular hypersurface of the first order for f , then $[f] = 0$ and we have

$$\left[\frac{\partial f}{\partial x_i} \right] = \left[\frac{\partial f}{\partial n} \right] n_i, \quad \left[\frac{\partial f}{\partial t} \right] = -V \left[\frac{\partial f}{\partial n} \right]. \quad (39.55)$$

Let us assume that f is of the class C^2 on $\Omega^+ \cup \Sigma$ and on $\Omega^- \cup \Sigma$. In view of (39.47), (39.51), (39.52) and (39.53) we obtain the following conditions of compatibility for the jumps of the partial derivatives of the second order of function f through Σ .

$$\begin{aligned} \left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right] &= a^{\alpha\beta} \left\{ \left[\frac{\partial f}{\partial n} \right]_{,\alpha} + a^{\gamma\delta} b_{\alpha\gamma} [f]_{,\delta} \right\} (n_i \varphi_{j,\beta} + n_j \varphi_{i,\beta}) + \\ &\quad + a^{\alpha\beta} a^{\gamma\delta} \left\{ [f]_{,\alpha\gamma} - b_{\alpha\gamma} \left[\frac{\partial f}{\partial n} \right] \right\} \varphi_{i,\beta} \varphi_{j,\delta} + \left[\frac{\partial^2 f}{\partial n^2} \right] n_i n_j, \\ \left[\frac{\partial^2 f}{\partial x_i \partial t} \right] &= a^{\alpha\beta} \left\{ \frac{\delta [f]}{\delta t} - V \left[\frac{\partial f}{\partial n} \right] \right\}_{,\alpha} \varphi_{i,\beta} + \\ &\quad + \left\{ \frac{\delta}{\delta t} \left[\frac{\partial f}{\partial n} \right] + a^{\alpha\beta} [f]_{,\alpha} V_{,\beta} - V \left[\frac{\partial^2 f}{\partial n^2} \right] \right\} n_i, \\ \left[\frac{\partial^2 f}{\partial t^2} \right] &= \frac{\delta}{\delta t} \left\{ \frac{\delta [f]}{\delta t} - V \left[\frac{\partial f}{\partial n} \right] \right\} - V \left\{ \frac{\delta}{\delta t} \left[\frac{\partial f}{\partial n} \right] + \right. \\ &\quad \left. + a^{\alpha\beta} [f]_{,\alpha} V_{,\beta} - V \left[\frac{\partial^2 f}{\partial n^2} \right] \right\}. \end{aligned} \quad (39.56)$$

From (39.56)₁ we obtain

$$\left[\frac{\partial^2 f}{\partial x_i \partial x_i} \right] = a^{\alpha\beta} [f]_{,\alpha\beta} - 2H \left[\frac{\partial f}{\partial n} \right] + \left[\frac{\partial^2 f}{\partial n^2} \right]. \quad (39.57)$$

/150

If Σ is a singular hypersurface of the first order for f , then $[f] = 0$. If Σ is a singular hypersurface of the second order, we have $\left[\frac{\partial f}{\partial n}\right] = 0$ and therefore

$$\begin{aligned}\left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right] &= \left[\frac{\partial^2 f}{\partial n^2}\right] n_i n_j, \left[\frac{\partial^2 f}{\partial x_i \partial t}\right] = -V \left[\frac{\partial^2 f}{\partial n^2}\right] n_i, \\ \left[\frac{\partial^2 f}{\partial t^2}\right] &= V^2 \left[\frac{\partial^2 f}{\partial n^2}\right].\end{aligned}\tag{39.58}$$

The relations (39.54), (39.56) and (39.57) have been derived by Thomas [413] and the relations (39.55) and (39.58) are attributed to Hadamard [167].

Let us assume that Σ is a singular hypersurface of the order $r \geq 2$ for the function f . In this case the relations (39.55) are applicable for the jumps of the partial derivatives of the order $r-1$ of the function f . Applying these relations successively, we can write

$$\left[\frac{\partial^r f}{\partial x_1 \partial x_j \dots \partial x_m \partial t^{r-1}}\right] = -V \left[\frac{\partial^r f}{\partial x_j \dots \partial x_m \partial t^{r-1} \partial x_p \partial x_q}\right] n_p n_q n_t,$$

and step by step we reach the following conditions of compatibility

$$\left[\frac{\partial^r f}{\partial x_1 \partial x_j \dots \partial x_m \partial t^{r-1}}\right] = (-V)^{r-1} \left[\frac{\partial^r f}{\partial n^r}\right] n_1 n_j \dots n_m, \quad (0 \leq s \leq r),\tag{39.59}$$

where (Hadamard [167])

$$\frac{\partial^r f}{\partial n^r} = \frac{\partial^r f}{\partial x_p \dots \partial x_s} n_p \dots n_s.\tag{39.60}$$

If f is of the class C^{r+1} on $\Omega^+ \cup \Sigma$ and on $\Omega^- \cup \Sigma$ the conditions of compatibility (39.56), (39.57), in which $[f] = 0$ are applicable to the partial derivatives of the order of $(r-1)$ of the function f . In these conditions, substituting f by $\frac{\partial^{r+1}f}{\partial x_m \dots \partial x_p}$ and using (39.59), we obtain

$$\begin{aligned} & \left[\frac{\partial^{r+1}f}{\partial x_p \dots \partial x_s \partial x_t \partial x_l} \right] = a^{ab} \left\{ \left[\frac{\partial^r f}{\partial n^r} \right] n_p \dots n_s \right\}_{,a} (n_t \varphi_{l,b} + \\ & + n_l \varphi_{t,b}) - a^{ab} a^{cd} b_{ab} \left[\frac{\partial^r f}{\partial n^r} \right] n_p \dots n_s \varphi_{t,b} \varphi_{l,d} + \\ & + \left[\frac{\partial^{r+1}f}{\partial x_p \dots \partial x_s \partial x_t \partial x_l} \right] n_k n_l n_t, \\ & \left[\frac{\partial^{r+1}f}{\partial x_p \dots \partial x_s \partial x_t \partial t} \right] = a^{ab} \left\{ V \left[\frac{\partial^r f}{\partial n^r} \right] n_p \dots n_s \right\}_{,a} \varphi_{t,b} + \\ & + \frac{\delta}{\delta t} \left\{ \left[\frac{\partial^r f}{\partial n^r} \right] n_p \dots n_s \right\} n_t - V \left[\frac{\partial^{r+1}f}{\partial x_p \dots \partial x_s \partial x_t \partial x_l} \right] n_k n_l n_t, \\ & \left[\frac{\partial^{r+1}f}{\partial x_p \dots \partial x_s \partial t^2} \right] = \frac{\delta V}{\delta t} \left[\frac{\partial^r f}{\partial n^r} \right] n_p \dots n_s - 2V \frac{\delta}{\delta t} \left\{ \left[\frac{\partial^r f}{\partial n^r} \right] n_p \dots n_s \right\} + \\ & + V^2 \left[\frac{\partial^{r+1}f}{\partial x_p \dots \partial x_s \partial x_t \partial x_l} \right] n_k n_l, \\ & \left[\frac{\partial^{r+1}f}{\partial x_p \dots \partial x_s \partial x_t \partial x_k} \right] = -2H \left[\frac{\partial^r f}{\partial n^r} \right] n_p \dots n_s + \\ & + \left[\frac{\partial^{r+1}f}{\partial x_p \dots \partial x_s \partial x_t \partial x_l} \right] n_k n_l. \end{aligned} \tag{39.61}$$

/151

c) Characteristic hypersurfaces and wave fronts. We will assume that the elastic medium which occupies the domain B is homogeneous and isotropic. In the following we will indicate $\mathcal{B} = \bar{B} \times I$. As shown in Paragraph 34, the equations of thermoelasticity for such media may be written in the following form

$$\begin{aligned}\frac{\partial^2 u_i}{\partial t^2} &= c_i^2 \frac{\partial^2 u_i}{\partial x_j \partial x_j} + (c_i^2 - c_i^2) \frac{\partial^2 u_i}{\partial x_i \partial x_i} - \beta \frac{\partial \theta}{\partial x_i}, \\ c \frac{\partial \theta}{\partial t} + T_0 \beta \frac{\partial^2 u_i}{\partial x_i \partial t} &= k \frac{\partial^2 \theta}{\partial x_i \partial x_i},\end{aligned}\tag{39.62}$$

if there are no mass forces and heat sources.

These equations were derived under the assumption that the functions u_i and θ are of the class C^2 on \mathcal{M} . If the functions u_i, θ of the class C^2 on \mathcal{M} satisfy the equations (39.62) in \mathcal{M} , we say that (u_i, θ) represents a strict solution of these equations.

Let us determine the characteristic types of the equations (39.62). The problem will be formulated in the following manner. Let S be a finite union of smooth hypersurfaces which has a non-empty intersection with \mathcal{M} . From now on we will designate $S \cap \mathcal{M}$ by S . Let us assume that on the hypersurface S from S are prescribed $u_i, \theta, \frac{\partial u_i}{\partial n}, \frac{\partial \theta}{\partial n}$, as analytical functions of x_1, x_2 and t . These functions represent the data from the corresponding Cauchy problem. The problem of the determination is placed into a certain neighborhood of S of a strict solution of the equations (39.62) which should satisfy on S the conditions of the Cauchy problem.

In order to extend the solution of the S variety by means of series of powers, we must determine on S the partial derivatives of u_i and θ of any order with respect to x_i and t . If, in the relations (39.36) and (39.40) f is substituted by u_i and θ , it follows that we can determine on S the partial derivatives of the first order of the function of u_i and θ , with the help of the data of the problem. In view of (39.47), (39.51) and (39.52), it follows that in order to determine the second order partial derivatives, the derivatives $\frac{\partial^2 u_i}{\partial n^2}$ and $\frac{\partial^2 \theta}{\partial n^2}$ on S are needed. In order to determine

these derivatives, we will use the equations (39.62). If we evaluate the terms from (39.62) on S with the help of the relations (39.47), (39.48), (39.51), (39.52) we obtain

$$\begin{aligned} (V^2 - c_2^2) \frac{\partial^2 u_1}{\partial n^2} - (c_1^2 - c_2^2) \frac{\partial^2 u_1}{\partial n^2} n_1 n_1 &= A_1, \\ T_{03} V \frac{\partial^2 u_2}{\partial n^2} n_1 + k \frac{\partial^2 \theta}{\partial n^2} &= A_4, \end{aligned} \quad (39.63)$$

where the functions A_r ($r = 1, 2, 3, 4$) are determined by the Cauchy data. From the system (39.63) we can determine the derivatives

$$\frac{\partial^2 u_1}{\partial n^2}, \frac{\partial^2 \theta}{\partial n^2} \quad \text{on } S \text{ if, and only if}$$

$$k \det \{ (V^2 - c_2^2) \delta_{ij} - (c_1^2 - c_2^2) n_i n_j \} \neq 0. \quad (39.64)$$

If, in the point of S we have

$$k \det \{ (V^2 - c_2^2) \delta_{ij} - (c_1^2 - c_2^2) n_i n_j \} = 0, \quad (39.65)$$

then we cannot determine the derivatives sought, and the method cannot be used. If the relation (39.65) occurs throughout the whole of S , then S is called a characteristic variety of equations (39.62). We say that (39.65) is the characteristic equation of the equations (39.62). Carrying out the calculations, the equation (39.65) is reduced to

$$k(V^2 - c_1^2)(V^2 - c_2^2)^2 = 0. \quad (39.66)$$

If the hypersurface is given in the implicit form (39.33), then

in view of (39.34), the equation (39.66) may be written in the following way

$$k \left\{ \left(\frac{\partial \Phi}{\partial t} \right)^2 - c_1^2 \frac{\partial \Phi}{\partial x_1} \frac{\partial \Phi}{\partial x_1} \right\} \left\{ \left(\frac{\partial \Phi}{\partial t} \right)^2 - c_2^2 \frac{\partial \Phi}{\partial x_1} \frac{\partial \Phi}{\partial x_1} \right\} = 0.$$

It follows from this that the characteristic variety of the equations (39.62) consists of two hypersurfaces, given by

$$\Phi = 0, \left(\frac{\partial \Phi}{\partial t} \right)^2 = c_1^2 \frac{\partial \Phi}{\partial x_1} \frac{\partial \Phi}{\partial x_1}, \quad (39.67)$$

$$\Phi = 0, \left(\frac{\partial \Phi}{\partial t} \right)^2 = c_2^2 \frac{\partial \Phi}{\partial x_1} \frac{\partial \Phi}{\partial x_1}. \quad (39.68)$$

As we have seen, a hypersurface Σ may be interpreted as a mobile surface in the three dimensional euclidian space. In the case of a characteristic hypersurface Σ , we say that the surface Σ_t is the wave front and for its motion we will use the term propagation of the wave front.

/153

It follows from (39.66) that the wave front $\Sigma_t (t \in I)$ represents families of parallel surfaces; the propagation of a wave front takes place with one of the constant velocities c_1, c_2 .

d) Weak thermoelastic waves. Inasmuch as equations (39.62) are equations with second order partial derivatives, the singular hypersurfaces for u_1 and θ of the order more than or equal to two will be designated as weak singular hypersurfaces. We will show that weak singular thermoelastic hypersurfaces are characteristic. We will refer to these hypersurfaces by the expression of waves.

We assume that Σ is a weak singular hypersurface of the order r .

If $r > 2$, then the equations (39.62) occur on Ω ; if $r = 2$, then (39.62) takes place on $\Omega \setminus \Sigma$. Thus, for any $r \geq 2$, equations obtained by the application of the equations (39.62) of the operator $\frac{\partial^{r-1}}{\partial x_m \dots \partial x_p}$ take place on $\Omega \setminus \Sigma$. Inasmuch as Σ is a singular hypersurface of the order r for the functions u_i and θ , the jumps through Σ of the derivatives of the order $(r-1)$ are identically zero, and the jumps through Σ of the partial derivatives of order r are defined.

Making the jump through Σ of each term from the equations obtained, we find

$$\begin{aligned} \left[\frac{\partial^r u_i}{\partial x_m \dots \partial x_p \partial t^2} \right] &= c_1^2 \left[\frac{\partial^r u_i}{\partial x_m \dots \partial x_p \partial x_j \partial x_j} \right] + \\ &+ (c_1^2 - c_2^2) \left[\frac{\partial^r u_j}{\partial x_m \dots \partial x_p \partial x_i \partial x_j} \right], \\ T_0 \beta \left[\frac{\partial^r u_j}{\partial x_m \dots \partial x_p \partial x_j \partial t} \right] &= k \left[\frac{\partial^r \theta}{\partial x_m \dots \partial x_p \partial x_j \partial x_j} \right]. \end{aligned} \quad (39.69)$$

If we keep in mind the relations (39.59) we can give another form to the equations (39.69), expressing the jumps from (39.69) by the right members from (39.59). Then, multiplying with $n_m \dots n_p$ and summing up, we obtain

$$\begin{aligned} (V^2 - c_2^2) \lambda_i - (c_1^2 - c_2^2) \lambda_j n_j n_i &= 0, \\ k \xi + T_0 \beta V \lambda_i n_i &= 0, \end{aligned} \quad (39.70)$$

where

$$\lambda_i = \left[\frac{\partial^r u_i}{\partial n^r} \right], \quad \xi = \left[\frac{\partial^r \theta}{\partial n^r} \right].$$

Keeping in mind (39.59) it follows that the jumps λ_1, ξ determine the jumps through Σ of the partial derivatives of the order r of the functions u_i and θ and therefore on the basis of the hypothesis all of them cannot be zero. Therefore, the determinant of the system (39.70) must be zero. This condition coincides with the characteristic equation (39.66). Thus, the singular weak hypersurfaces in thermoelasticity are characteristic hypersurfaces which propagate with constant velocities c_1 and c_2 . They will be designated as weak thermoelastic waves. /154

If we consider the relations

$$v_i = \frac{\partial u_i}{\partial t}, \quad \varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad \omega_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right),$$

then on the basis of the equations (39.59) we conclude

$$\left[\frac{\partial^{r-1} v_i}{\partial t^{r-1}} \right] = (-V)^r \lambda_i, \quad \left[\frac{\partial^{r-1} \varepsilon_{ij}}{\partial t^{r-1}} \right] = (-V)^{r-1} \lambda_i n_j, \quad (39.71)$$

$$\left[\frac{\partial^{r-1} \omega_{ij}}{\partial t^{r-1}} \right] = -\frac{1}{2} (-V)^{r-1} (\lambda_j n_i - \lambda_i n_j).$$

In the case when the wave is propagated with a velocity c_1 it follows from (39.70)₁ that $\lambda_1 = \lambda n$ where $\lambda = \lambda_p n_p$ and

$$\left[\frac{\partial^{r-1} v_i}{\partial t^{r-1}} \right] = (-V)^r \lambda n_i \neq 0, \quad \left[\frac{\partial^{r-1} \varepsilon_{ii}}{\partial t^{r-1}} \right] = (-V)^{r-1} \lambda \neq 0, \quad (39.72)$$

$$\left[\frac{\partial^{r-1} \omega_{ij}}{\partial t^{r-1}} \right] = 0.$$

If the velocity of the wave is c_2 , it follows from (39.70) that $\lambda = 0$ and therefore we have

$$\left[\frac{\partial^{r-1} v_i}{\partial t^{r-1}} \right] n_i = 0, \quad \left[\frac{\partial^{r-1} \epsilon_{ij}}{\partial t^{r-1}} \right] = 0, \quad \left[\frac{\partial^{r-1} \omega_{ij}}{\partial t^{r-1}} \right] \neq 0. \quad (39.73)$$

In view of the above we will say that a weak thermoelastic wave is dilative if it propagates with a velocity of c_1 and rotational if it propagates with a velocity of c_2 . For a weak dilative wave of the order r , the discontinuity in the derivative of the order $(r-1)$ with respect to time of the velocity follows the direction of the normal to the wave front. For a rotational weak wave of the order r , this discontinuity is tangential to the wave fronts.

It follows from (39.70) that

$$\xi = - \frac{T_0 \beta \lambda c_1}{k}, \quad \text{for dilative waves,} \quad (39.74)$$

$$\xi = 0, \quad \text{for rotational waves.}$$

Thus, the jumps of the partial derivatives of the order r of the functions u_i and θ through a weak thermoelastic wave Σ of the order r are determined by the vector of singularity $\underline{\lambda}$ of components λ_i . The magnitude of this vector is a measure of the "force" of Σ .

Let us apply the operator $\frac{\partial^{r-1}}{\partial x_m \dots \partial x_p}$ to equations (39.62). Then, we form the jump through Σ to each term from the equations obtained and express it with the help of relations (39.59), (39.61). Multiplying with $n_m \dots n_p$ and summing up, we obtain

$$\begin{aligned}
 & (V^2 - c_2^2) \mu_t - (c_1^2 - c_2^2) \{ \mu_j n_j n_t + a^{\alpha\beta} \lambda_{j;\alpha} \varphi_{j;\beta} n_t + \\
 & + a^{\alpha\beta} \varphi_{t;\beta} (\lambda_{j;\alpha} n_j - \lambda_j a^{\alpha\beta} h_{\alpha\gamma} \varphi_{j;\beta}) \} - \\
 & - \frac{\delta V}{\delta t} \lambda_t - 2V \frac{\delta \lambda_t}{\delta t} + 2H c_2^2 \lambda_t + \frac{\beta}{\rho} \xi n_t = 0, \\
 & T_0 \beta \left(V \mu_j n_j - \frac{\delta \lambda_j}{\delta t} n_j + V a^{\alpha\beta} \lambda_{j;\alpha} \varphi_{j;\beta} + a^{\alpha\beta} V_{;\alpha} \lambda_j \varphi_{t;\beta} \right) + \\
 & + k \gamma + (T_0 a V - 2kH) \xi = 0,
 \end{aligned} \tag{39.75}$$

where

$$\mu_t = \left[\frac{\partial^{r+1} u_t}{\partial n^{r+1}} \right], \quad \gamma = \left[\frac{\partial^{r+1} \theta}{\partial n^{r+1}} \right].$$

In deriving the relations (39.75) we considered (39.23). In view of (39.43) and the fact that for weak thermoelastic waves V is a constant, it follows that

$$\frac{\delta \lambda}{\delta t} = - \frac{\delta}{\delta t} (\lambda_j n_j) = n_j \frac{\delta \lambda_j}{\delta t}. \tag{39.76}$$

If we consider (39.10), (39.28)₁, we obtain

$$\begin{aligned}
 a^{\alpha\beta} \lambda_{j;\alpha} \varphi_{j;\beta} &= a^{\alpha\beta} (\lambda_j \varphi_{t;\beta})_{;\alpha} - 2H \lambda_j, \\
 \lambda_{j;\alpha} n_j &= \lambda_j a^{\alpha\beta} h_{\alpha\gamma} \varphi_{j;\beta} = \lambda_{j\alpha}.
 \end{aligned} \tag{39.77}$$

The relations (39.76), (39.77) allow us to write the equations (39.75) in the form

$$\begin{aligned} (V^2 - c_2^2) \mu_i - (c_1^2 - c_2^2) \{ \mu_j n_j n_i + a^{ab} (\lambda_j \varphi_{j,b})_{;a} n_i + a^{ab} \lambda_{j;a} \varphi_{i,b} \} - \\ - 2V \frac{\delta \lambda_i}{\delta t} + 2H \{ c_2^2 \lambda_i + (c_1^2 - c_2^2) \lambda n_i \} + \frac{\beta}{\rho} \xi n_i = 0, \end{aligned} \quad (39.78)$$

$$\begin{aligned} T_{0\beta} V \mu_j n_j + k\gamma - T_{0\beta} \left\{ \frac{\delta \lambda}{\delta t} - V a^{ab} (\lambda_j \varphi_{j,b})_{;a} + 2V H \lambda \right\} + \\ + (T_0 a V - 2kH) \xi = 0. \end{aligned}$$

If we multiply (39.78)₁ with n_1 and add, we obtain

/156

$$\begin{aligned} (V^2 - c_1^2) \mu_j n_j - (c_1^2 - c_2^2) a^{ab} (\lambda_j \varphi_{j,b})_{;a} - \\ - 2V \frac{\delta \lambda}{\delta t} + 2H c_1^2 \lambda + \frac{\beta}{\rho} \xi = 0. \end{aligned} \quad (39.79)$$

(1) Weak dilative waves. For weak dilative waves we have $V = c_1$, $\lambda_1 = \lambda n_1$, $\xi = -T_0 \beta c_1 \lambda / k$, and the relation (39.79) leads to the following equation of transport

$$\frac{1}{c_1} \frac{\delta \lambda}{\delta t} = \frac{d\lambda}{dn} = \lambda \left(H - \frac{1}{2c_1} \varepsilon \omega^* \right), \quad (39.80)$$

where ε and ω^* are given by (33.4) and (35.20) respectively.

Keeping in mind (39.80), the equations (39.78) become

$$\mu_i - \mu_j n_j n_i = \alpha^{ab} \lambda_{,a} \varphi_{i,b},$$

$$T_0 \beta c_i \mu_j n_j + k\gamma = T_0 \beta c_i \lambda \left[H + \frac{\omega^*}{c_i} \left(T_0 - \frac{1}{2} \epsilon \right) \right]. \quad (39.81)$$

Let us assume that at the moment $t = t^0$ the average curvature and the gaussian curvature of the wave front Σ_{t^0} are H_0 and K_0 , respectively. The mean curvature of the surface Σ_t may be expressed in the form (Thomas [414])

$$H = \frac{H_0 - nK_0}{1 - 2nH_0 + n^2K_0}, \quad (39.82)$$

where n is the distance measured on the normal from Σ_{t^0} . With the help of relation (39.82), the equation (39.80) may be integrated and we obtain

$$\lambda = \lambda_0 (1 - 2nH_0 + n^2K_0)^{-\frac{1}{2}} e^{-\frac{\omega^*}{2c_i} n}, \quad (39.83)$$

where $\lambda = \lambda_0$ on Σ_{t^0} .

It follows from (39.74)₁ and (39.83) that the jumps of the partial derivatives of the order r of the functions u_1 and θ through a weak dilative wave of the order r are determined by the values of the latter on an initial wave front. More precisely, it is enough to know λ on the initial wave front Σ_{t^0} . It may be noted from (39.81) that the jumps of the partial derivatives of the order $r = 1$ of the functions u_1 and θ are not determined except when the jump of one of the partial derivatives of the order $r = 1$ are known ahead of time. This occurs because equations (39.81)₁ are not independent; by multiplying them with n_1 and summing, we obtain an identity.

(11) Rotational weak waves. In this case $V = c_2$, $\lambda = 0$ and $\xi = 0$. The equation (39.79) is reduced to

$$\mu_1 n_1 = -a^{ab}(\lambda, \varphi, \theta)_{,a}, \quad (39.84)$$

and from (39.78) we obtain the following transport equation /157

$$\frac{1}{c_2} \frac{\delta \lambda_1}{\delta t} = \frac{d\lambda_1}{dn} = \lambda_1 H, \quad (39.85)$$

and $\gamma = 0$.

In view of (39.82), equation (39.85) may be integrated and we obtain

$$\lambda_1 = \lambda_1^0 (1 - 2nH_0 + n^2 K_0)^{-\frac{1}{2}}, \quad (39.86)$$

where $\lambda_1 = \lambda_1^0$ on Σ_t^0 . This indicates that the jumps of the partial derivatives of the order r of the functions u_1 through a weak rotational wave front of the order r are determined by the values of the latter on an initial wave front. The partial derivatives of the order r and $r + 1$ of the function θ are continuous on the hypersurfaces of the type under consideration. In order to determine the jumps of the partial derivatives of the order $r + 1$ through Σ_t it is necessary that we know two of the quantities of μ_1 . Let us note the fact that in the relation (39.83) for weak dilative waves, the factor $\exp(-\epsilon \omega n / 2c_1)$ appears; this indicates that the values of the jumps of the partial derivatives of the order r of the functions u_1 and θ tend towards zero when the time interval $t - t^0$ tends toward infinity.

e) Strong thermoelastic waves. To complete the study of singular surfaces we study the case of hypersurfaces of zero and first order. These hypersurfaces are called strong singular hypersurfaces. The presence of discontinuities implied by these hypersurfaces contradicts the

hypothesis of regularity in which the equations of thermoelasticity were derived. Therefore it is necessary to establish beforehand a theory that would permit us to study the strong singular hypersurfaces.

Let $\mathfrak{M}_1(\mathcal{D})$ be a set of the regulated regions contained in \mathcal{D} . If $P \in \mathfrak{M}_1(\mathcal{D})$ we will designate by \mathcal{D}^0 the interior of \mathcal{D} , and by $\partial\mathcal{D}$ the boundary of \mathcal{D} . Let $\mathcal{L}(G)$ be the set of functions with compact support on G .

If the functions $u_i, \theta, \varepsilon_{ij}, v_i, \eta, t_{ij}, q_i$ satisfy the relations

$$\begin{aligned} v_i &= -\frac{\partial u_i}{\partial t}, \quad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \\ t_{ij} &= \lambda \varepsilon_{rr} \delta_{ij} + 2\mu \varepsilon_{ij} - \beta \theta \delta_{ij}, \\ \rho \eta &= \beta \varepsilon_{rr} + \alpha \theta, \\ q_i &= k \theta_{,i} \end{aligned} \tag{39.87}$$

and for any $\mathcal{D} \in \mathfrak{M}_1(\mathcal{D})$ and any $(\zeta_0, \zeta) \in \mathcal{L}(\mathcal{D}^0)$ we have

$$\int_{\mathcal{D}} \left(\rho v_i \frac{\partial \zeta_i}{\partial t} - t_{ij} \frac{\partial \zeta_j}{\partial x_i} \right) dv = 0, \quad \int_{\mathcal{D}} \left(\rho T_0 \eta \frac{\partial \zeta}{\partial t} - q_i \frac{\partial \zeta}{\partial x_i} \right) dv = 0, \tag{39.88}$$

then we will say that (u_1, θ) represents a weak solution of the equation (39.62) on \mathcal{D} .

It can be shown [51] that if the functions u_1, θ represent a weak solution of the equation (39.62) on \mathcal{D} and are of the class C^2 on \mathcal{D} then (u_1, θ) is a strict solution of these equations on \mathcal{D} and vice versa.

/158

Let us assume that (u_1, θ) is a weak solution of equations (39.62) on \mathcal{D} . Let Σ be a strong singular hypersurface, and D an open set from B so that $\Omega = D \times I$ intersects Σ the same way as before. We assume that u_1 and θ are functions of the class C^2 on Ω^+ and on Ω^- . Let $\mathcal{P} \subset \Omega$ be an arbitrary regulated region that intersects Σ and $\mathcal{P}^+ = \mathcal{P} \cap \Omega^+$, $\mathcal{P}^- = \mathcal{P} \cap \Omega^-$ be non-empty sets. In view of the fact that (u_1, θ) is a strict solution of the equations (39.62) on Ω^+ and on Ω^- , it follows that the equations

$$t_{0,i} = \rho \dot{v}_i, \quad \rho T_{0i} \dot{\eta} - q_{i,i} = 0,$$

are satisfied in \mathcal{P}^+ and in \mathcal{P}^- . Therefore, we have

$$\begin{aligned} \int_{\mathcal{P} \cup \mathcal{P}^-} \left\{ \frac{\partial}{\partial t} (\rho v_i \zeta_i) - \frac{\partial}{\partial x_j} (t_{ij} \zeta_i) \right\} dv &= 0, \\ \int_{\mathcal{P} \cup \mathcal{P}^-} \left\{ \frac{\partial}{\partial t} (\rho T_{0i} \eta \zeta_i) - \frac{\partial}{\partial x_i} (\zeta_i q_i) \right\} dv &= 0. \end{aligned} \quad (39.89)$$

Let (v_0, v_1, v_2, v_3) be the components of the unitary normal at $\partial \mathcal{P}$. In view of (39.34) we can write

$$v_0 = -A V, \quad v_i = A n_i \text{ on } \Sigma,$$

using the theorem of divergence and the fact that the functions (ζ_1, ζ) become zero on $\partial \mathcal{P}$, we obtain from (39.89)

$$\begin{aligned} \int_{\Sigma \cap \mathcal{P}} A \zeta_i \{ \rho V [v_i] + [t_{ij}] n_j \} da &= 0, \\ \int_{\Sigma \cap \mathcal{P}} A \zeta \{ \rho T_{0i} V [\eta] + [q_i] n_i \} da &= 0. \end{aligned}$$

In view of the fact that the quantities below the integral are continuous functions on Σ , it follows from the above relations that

$$\rho V [v_i] + [l_i] n_i = 0, \quad \rho T_0 V [\eta] + [q_i] n_i = 0 \text{ on } \Sigma. \quad (39.90)$$

These relations are called dynamic discontinuity relations.

If we use (39.87) the relations (39.90) may be written in the form

$$\begin{aligned} V \left[\frac{\partial u_i}{\partial t} \right] + (c_i^2 - 2c_i^2) \left[\frac{\partial u_i}{\partial x_j} \right] n_i + c_i^2 \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] n_j - \\ - \frac{\beta}{\rho} [0] n_i = 0, \\ T_0 V \left[\frac{\partial u_j}{\partial x_j} \right] + T_0 V [0] + k \left[\frac{\partial \theta}{\partial x_j} \right] n_j = 0 \text{ on } \Sigma. \end{aligned} \quad (39.91)$$

The equations (39.91) are applicable both to the absolute singular /159
hypersurfaces and to the first order singular hypersurfaces.

In the following we will assume that the functions u_i are continuous on \mathcal{Q} . The conditions of compatibility for the first order partial derivatives of the functions u_i are thus those given in (39.55)

$$\left[\frac{\partial u_i}{\partial x_j} \right] = \lambda_i n_j, \quad \left[\frac{\partial u_i}{\partial t} \right] = -V \lambda_i, \quad (39.92)$$

where $\lambda_i = \left[\frac{\partial u_i}{\partial n} \right]$.

It can be shown that the law of the conservation of mass imposes a restriction on the jumps of the functions u_i and in our assumptions this has been identically satisfied [51]. It follows, similarly, that the discontinuity relation corresponding to the second law of thermodynamics is satisfied.

We will study the basic properties of the strong singular

surfaces on the basis of the discontinuity relations (39.91) and of equations (39.62). Substituting in (39.54) θ for f and keeping in mind the relations obtained and also (39.92) it follows from (39.91) that

$$\begin{aligned} (V^2 - c_2^2) \lambda_i - (c_1^2 - c_2^2) \lambda_i n_i + \frac{\beta}{\rho} [0] n_i &= 0, \\ T_0 \beta V \lambda + k \xi + T_0 \alpha V [0] &= 0, \end{aligned} \quad (39.93)$$

where $\xi = \left[\frac{\partial \theta}{\partial n} \right]$, $\lambda = \lambda_i n_i$. Forming the jump through Σ to each term

of (39.62) and applying the conditions of compatibility (39.56) to the functions u_i (with $[u_i] = 0$) and (39.36), (39.37) (39.57) to the function θ we obtain

$$\begin{aligned} (V^2 - c_2^2) \mu_i - (c_1^2 - c_2^2) \{ \mu_j n_j n_i + a^{ab} (\lambda_j \varphi_{j,b})_{,a} n_i + a^{ab} \lambda_{,a} \varphi_{i,b} \} - \\ - 2V \frac{\delta \lambda_i}{\delta t} - \lambda_i \frac{\delta V}{\delta t} + 2H \{ c_2^2 \lambda_i + (c_1^2 - c_2^2) \lambda_i n_i \} + \\ + \frac{\beta}{\rho} \{ \xi n_i + a^{ab} [0]_{,a} \varphi_{i,b} \} = 0, \end{aligned} \quad (39.94)$$

$$\begin{aligned} T_0 \beta V \mu_j n_j + k \gamma - T_0 \beta \left\{ \frac{\delta \lambda}{\delta t} - V a^{ab} (\lambda_j \varphi_{j,b})_{,a} + 2 V H \lambda \right\} + \\ + (T_0 \alpha V - 2kH) \xi - T_0 \alpha \frac{\delta [0]}{\delta t} + k a^{ab} [0]_{,ab} + \\ + T_0 \beta \lambda_j \left[a^{ab} \varphi_{j,b} V_{,a} + \frac{\delta n_j}{\delta t} \right] = 0, \end{aligned}$$

where

$$\mu_i = \left[\frac{\partial^2 u_i}{\partial n^2} \right], \quad \gamma = \left[\frac{\partial^2 \theta}{\partial n^2} \right].$$

In deriving the relations (39.94), the relations (39.43), (39.77) have also been taken into account.

The equations (39.93) and (39.94) represent the basis of the study of strong singular hypersurfaces. They differ from the similar equations (39.70), (39.78) for weak waves by the presence of the terms in $[\theta]$ and $\delta V/\delta t$. /160

(i) Shock waves. If Σ is a first order singular hypersurface for u_1 and θ then $[\theta] = 0$ on Σ and equations (39.70) and (39.93) are formally identical. As has been shown before, these types of hypersurfaces are characteristic surfaces and thus propagate with one of the constant velocities c_1, c_2 . Therefore $\delta V/\delta t = 0$ and equations (39.78) and (39.94) coincide. Therefore, the relations (39.83) and (39.86) are applied also to the strong dilative and rotational first order thermoelastic waves, respectively.

If $V = c_1$, it follows from (39.93) that $\lambda_1 = \lambda n_1$ and

$$[v_i] = -V \lambda n_i \neq 0 \quad [e_{ij}] = \lambda \neq 0, \quad [\omega_{ij}] = 0. \quad (39.95)$$

The discontinuity of the velocity is directed according to the normal to Σ_t . The wave Σ is called a shock wave. The velocity of the shock wave c_1 is the displacement velocity of Σ .

If $V = c_2$, we obtain from (39.93)₁ $\lambda = 0$ and

$$[v_i] n_i = 0, \quad [e_{ii}] = 0, \quad [\omega_{ij}] \neq 0.$$

(ii) Absolute singular hypersurfaces. If Σ is an absolute singular hypersurface for u_1, θ , then $[\theta]$ does not become zero on Σ . Multiplying (39.93)₁ and (39.94)₁ with n_1 and summing, we obtain

$$\begin{aligned} (V^2 - c_1^2) \lambda + \frac{\beta}{\rho} [\theta] &= 0, \\ (V^2 - c_1^2) \mu_1 n_1 - (c_1^2 - c_2^2) a^{ab} (\lambda_1 \varphi_{t;ab})_{;a} - \\ - 2 V \frac{\delta \lambda_1}{\delta t} n_1 - \lambda \frac{\delta V}{\delta t} + 2H c_1^2 \lambda + \frac{\beta}{\rho} \xi &= 0. \end{aligned} \quad (39.96)$$

From $(39.96)_1$, in view of the fact that $[\theta] \neq 0$ it follows that $V \neq c_1$. Therefore, in order that an absolute singular hypersurface be characteristic, V must take the value of c_2 . If $V = c_2$ then $\delta V / \delta t = 0$ and (39.76) takes place. Eliminating $\mu_j n_j$ and $[\theta]$ between the equations $(39.94)_1$ and (39.66) we obtain the transport equation

$$\frac{1}{c_2} \frac{\delta}{\delta t} (\lambda_i - \lambda n_i) = \frac{d}{dn} (\lambda_i - \lambda n_i) = (\lambda_i - \lambda n_i) H.$$

If we replace H by (39.82) and integrate, it follows that

$$\lambda_i - \lambda n_i = (\lambda_i^0 - \lambda^0 n_i) (1 - 2 n H_0 + n^2 K_0)^{-1/2},$$

where $\lambda_i = \lambda_i^0$, $\lambda = \lambda^0$ on Σ^0 .

Consequently the tangential component to Σ_t of the vector $\underline{\lambda}$ satisfy the same relation that the component of $\underline{\lambda}$ satisfies in the case of weak rotational waves

/161

It follows from $(39.96)_1$ that $\underline{\lambda}$ may be determined only when $[\theta]$ is known on Σ .

If $V \neq c_1, c_2$, the equations $(39.93)_1$ and $(39.96)_1$ imply that

$$\begin{aligned} \lambda &= - \frac{\beta [\theta]}{\rho(V^2 - c_1^2)}, \quad \lambda_i = \lambda n_i, \\ \xi &= - \frac{T_0 V}{k} \left[a - \frac{\beta^2}{\rho(V^2 - c_1^2)} \right] [\theta]. \end{aligned} \quad (39.97)$$

Substituting (39.97) in $(39.96)_2$ we obtain the transport equation

$$\begin{aligned} 2 \frac{\delta \lambda}{\delta t} + \lambda \left\{ - \frac{T_0}{\rho k} [a \rho(V^2 - c_1^2) - \beta^2] + \frac{1}{V} \frac{\delta V}{\delta t} - \right. \\ \left. - \frac{2H}{V} c_1^2 \right\} = \frac{V^2 - c_1^2}{V} \mu_j n_j. \end{aligned}$$

The relations (39.93) indicate that the jumps through Σ of

the partial derivatives of the functions u_1 and θ may be determined only when L is a free surface ($V \neq c_1, c_2$), and $[\theta]$ and V are known.

If these requirements are satisfied and λ and ξ are determined, then equations (39.94) determine μ_1, γ and therefore we can determine the jumps of the second order partial derivatives of the functions u_1 and θ .

40. Equations of Thermoelastic Equilibrium

Let us consider the case of equilibrium in the linear theory of thermoelasticity. In view of the fact that now the functions involved do not depend on time, it follows that the fundamental equations (25.1) -- (25.3) are reduced to

- equilibrium equations

$$t_{ij,j} + f_i = 0, \quad (40.1)$$

- energy equations

$$q_{i,i} = -r, \quad (40.2)$$

- constitutive equations

$$\begin{aligned} t_{ij} &= C_{ijkl} \varepsilon_{kl} - \beta_{ij} \theta, \\ q_i &= k_{ij} \theta_{,j}, \end{aligned} \quad (40.3)$$

- geometric equations

$$2\varepsilon_{ij} = u_{i,j} + u_{j,i}. \quad (40.4)$$

In (40.1) -- (40.4) we used the notations from Paragraph 25. The coefficients C_{ijkl} , β_{ij} have the symmetry properties (25.5).

To the above equations the boundary conditions are added which, in the case of a mixed problem, have the form

$$\begin{aligned} u_i &= \tilde{u}_i \text{ pe } \Sigma_1, \quad t_i \equiv t_{ij} n_j = \tilde{t}_i \text{ pe } \Sigma_2, \\ \theta &= \tilde{\theta} \text{ pe } \Sigma_3, \quad q_i n_i = \tilde{q} \text{ pe } \Sigma_4, \end{aligned} \quad (40.5)$$

where Σ_i ($i = 1, 2, 3, 4$) are parts of ∂B so that $\Sigma_1 \cup \Sigma_2 = \Sigma_3 \cup \Sigma_4 = \partial B$,
 $\Sigma_i \cap \Sigma_j = \Sigma_i \cap \Sigma_j = \emptyset$.

Let us observe that, in the case of equilibrium, the problem is uncoupled, in the sense that the determination of the temperature is independent of the deformation of the medium.

We obtain from (40.2), (40.3)₂ and (40.5)₂ the following boundary problem for the temperature variation

$$\begin{aligned} (k_{ij}\theta_{,j})_{,i} &= -r \text{ in } B, \\ 0 &= \tilde{\theta} \text{ on } \Sigma_3, \quad k_{ij}\theta_{,j}n_i = \tilde{q} \text{ on } \Sigma_1. \end{aligned} \quad (40.6)$$

In the following we will assume, in general, that the function θ is determined.

We call an admissible state the ordered set of functions $\{u_i, \varepsilon_{ij}, t_{ij}\}$ with the properties

/163

- (i) $u_i \in C^2(B), u_i \in C^1(\bar{B}),$
- (ii) $\varepsilon_{ij} := \varepsilon_{ji}, \varepsilon_{ij} \in C^0(\bar{B}),$
- (iii) $t_{ij} := t_{ji}, t_{ij} \in C^1(B); t_{ij}, t_{ij,j} \in C^0(\bar{B}).$

If we define the addition of admissible states and the multiplication of an admissible state with a scalar by

$$\begin{aligned} \{u_i, \varepsilon_{ij}, t_{ij}\} + \{u'_i, \varepsilon'_{ij}, t'_{ij}\} &= \{u_i + u'_i, \varepsilon_{ij} + \varepsilon'_{ij}, t_{ij} + t'_{ij}\}, \\ \lambda\{u_i, \varepsilon_{ij}, t_{ij}\} &= \{\lambda u_i, \lambda \varepsilon_{ij}, \lambda t_{ij}\}, \end{aligned}$$

then the set of the admissible states is a linear space.

By the term thermoelastic state, corresponding to a specific force of volume f_1 and to the temperature variation θ , we understand an admissible state which satisfies the equations (40.1), (40.3)₁, (40.4).

The determination of the thermoelastic equilibrium is reduced to the resolution of two problems. The first of them is the boundary problem (40.6).

The second problem consists in the determination of a thermoelastic state corresponding to the functions f_1 and θ which satisfy the boundary conditions $(40.5)_1$. If there is such a state, it is called the solution of the problem. The problem of the determination of the function θ is considered as an auxiliary problem.

Therefore, the study of a thermoelastic equilibrium problem amounts to the resolution of boundary problem for equation $(40.6)_1$ and the integration of the equations

$$t_{ij,j} + f_i = 0, \quad (40.7)$$

$$t_{ij} = C_{ijkl}\epsilon_{kl} - \beta_{ij}\theta, \quad (40.8)$$

$$2\epsilon_{ij} = u_{i,j} + u_{j,i}, \quad (40.9)$$

in which the function θ is known, with the boundary conditions $(40.5)_1$.

41. Analogy with Problems from the Theory of Elasticity

Considering the function θ known, we can obtain an expression of the mixed problem with the help of the components of the displacement vector. Thus, proceeding as in the case of Paragraph 26, we obtain the following form of the equilibrium equations.

$$(C_{ijkl}u_{k,l})_{,j} + f_i - (\beta_{ij}\theta)_{,j} = 0. \quad (41.1)$$

The boundary conditions obtain the form

$$u_i = \bar{u}_i \text{ on } \Sigma_1, \quad C_{ijkl}u_{k,l}n_j = \bar{t}_i + \beta_{ij}\theta n_j \text{ on } \Sigma_2. \quad (41.2)$$

The problem (41.1), (41.2) is a problem of elastostatics corresponding to the specific forces of volume $f_1 - (\beta_{1j}\theta)_{,j}$, the displacement \tilde{u}_1 on Σ_1 and the surface forces $\tilde{t}_1 + \beta_{1j}\theta n_j$ on Σ_2 . Consequently the effect of the temperature in this case is shown by the appearance of the term $-(\beta_{1j}\theta)_{,j}$ in the equilibrium equations as a specific volume force and of term $\beta_{1j}\theta n_j$ in the boundary conditions as a surface force.

On the basis of this observation (due to Duhamel [91]) it is possible to transpose a series of results from the theory of elastostatics to the theory of the thermoelastic equilibrium.

It can easily be seen that on the basis of the linearity of the theory, the solution of the problem may be written in the following way

$$\{u_i, \epsilon_{ij}, t_{ij}\} = \{u_i^{(1)}, \epsilon_{ij}^{(1)}, t_{ij}^{(1)}\} + \{u_i^{(2)}, \epsilon_{ij}^{(2)}, t_{ij}^{(2)}\},$$

where $\{u_i^{(1)}, \epsilon_{ij}^{(1)}, t_{ij}^{(1)}\}$ is the solution of the corresponding problem of the load system $I^{(1)} = \{f_1, \tilde{u}_1, \tilde{t}_1\}$ and $\{u_i^{(2)}, \epsilon_{ij}^{(2)}, t_{ij}^{(2)}\}$ is the solution of the corresponding problem of the load system

$$I^{(2)} = \{-(\beta_{ij}\theta)_{,j}, 0, \beta_{ij}\theta n_j\}.$$

In general, in thermoelastic equilibrium problems it is assumed that the functions f_1 , \tilde{u}_1 and \tilde{t}_1 are zero, thus only the effect of the temperature variation is studied. In the contrary case, an auxiliary problem of elasticity corresponding to the mechanical loads must be sought.

Strictly speaking, thermoelastic equilibrium problems are considered, which are, on the one hand, problems in which the function θ and then the corresponding functions u_i are determined effectively, and on the other hand, problems in which the solution is based essentially on the equation which satisfies the function θ or the special mode

of the loading system $I^{(2)}$. If $\Sigma_1 = \partial B$, the problem is called the first boundary problem or the problem with displacement given at the boundary. If $\Sigma_2 = \partial B$, we say that we have the second boundary problem.

Further on we will present a characterization of the second boundary problem with the help of the tension tensor components.

We assume that the tensor C_{ijkl} is inversable. In this case we conclude from the constitutive equations (40.3)

$$\varepsilon_{ij} = A_{ijkl} t_{kl} + \alpha_{ij} \theta, \quad (41.3)$$

where

$$\alpha_{ij} = A_{ijkl} \beta_{kl}. \quad (41.4)$$

We will consider the case when the tensions at the boundary are prescribed. We assume that A_{ijkl} and α_{ij} are functions of the class C^2 on \bar{B} and that the domain B is simply connected. Keeping in mind the compatibility conditions (3.21) it follows that the symmetric tensor t_{ij} of the class C^2 on \bar{B} is the tension tensor corresponding to a solution of the second boundary problem from the theory of thermoelasticity if, and only if

$$t_{ij,j} + f_i = 0, \quad (41.5)$$

$$c_{pim} c_{ejn} (A_{ijkl} t_{kl} + \alpha_{ij} \theta)_{,mn} = 0, \text{ in } B, \quad (41.6)$$

$$t_{ij} n_j = \bar{t}_i \text{ on } \partial B. \quad (41.7)$$

The mean value of the function f , continuous on B , is defined by

$$f^*(B) = \frac{1}{V(B)} \int_B f \, dv, \quad (42.1)$$

where $V(B)$ is the volume of the domain B .

Theorem 42.1. If u_i and ϵ_{ij} are continuous functions on \bar{B} which satisfy the equations (40.4), then the mean values of the components of the deformation tensor depend only on the boundary values of the displacement and we have

$$\epsilon_{ij}^*(B) = \frac{1}{2V(B)} \int_{\partial B} (u_i n_j + u_j n_i) \, da. \quad (42.2)$$

Moreover, if the medium is homogeneous, and θ is a continuous function on \bar{B} , then the mean values of the components of the tension tensor depend only on the boundary values of the displacement and the mean value of the temperature variation $\theta^*(B)$ and we have

$$t_{ij}^*(B) = \frac{1}{V(B)} C_{ijkl} \int_{\partial B} u_k n_l \, da - \beta_{ij} \theta^*(B). \quad (42.3)$$

Proof. The relation (42.2) follows immediately by applying the theorem of divergence. It follows from (40.8) and (40.9) that

$$t_{ij} = C_{ijkl} u_k n_l - \beta_{ij} \theta,$$

and consequently, using (42.1) and the theorem of divergence we can derive (42.3).

If the boundary conditions are

$$u_i = 0 \text{ on } \partial B, \quad (42.4)$$

then it follows from (42.2) and (42.3) that

$$\epsilon_{ij}^*(B) = 0, \quad t_{ij}^*(B) = -\beta_{ij}\theta^*(B). \quad (42.5)$$

In view of the relation (5.9) we conclude that in this case the variation of the volume is zero.

Theorem 42.2. If $\{u_i, \epsilon_{ij}, t_{ij}\}$ is a thermoelastic state which corresponds to a specific volume force f_i and the temperature variation of θ , then the mean values of the components of the tension tensor depend only on the boundary values of the tension and of the forces f_i and we have

$$t_{ij}^*(B) = \frac{1}{V(B)} \left(\int_{\partial B} x_i t_j da + \int_B x_i f_j dv \right). \quad (42.6)$$

Moreover, if the medium is homogeneous and the tensor C_{ijkl} is inversable, then the mean values of the components of the deformation tensor depend only on the specific volume force, the boundary values of the tension and the mean value of the temperature variation and we have /166

$$\epsilon_{ij}^*(B) = \frac{1}{V(B)} A_{ijkl} \left[\int_{\partial B} x_k t_l da + \int_B x_k (t_{lk,k} + f_l) dv + \int_B t_{ij} dv \right] + \alpha_{ij} \theta^*(B). \quad (42.7)$$

Proof. In view of (42.1) and the relation

$$\int_{\partial B} x_i t_{jk} n_k da + \int_B x_i f_j dv = \int_B x_i (t_{jk,k} + f_j) dv + \int_B t_{ij} dv,$$

we obtain (42.6). In view of the fact that the medium is homogeneous, the relations (41.3) imply

$$\epsilon_{ij}^*(B) = A_{ijkl} t_{kl}^*(B) + \alpha_{ij} \theta^*(B),$$

and, in view of (42.6), (42.7) follows.

If $f_1 = 0$ and the medium is free of tensions, that is

$$t_{ij}n_j = 0 \text{ on } \partial B,$$

then we obtain from (42.6)

$$t_{ij}^*(B) = 0. \quad (42.8)$$

Similarly, in this case the relation (42.7) becomes (Nowacki [315])

$$\varepsilon_{ij}^*(B) = \alpha_{ij} \theta^*(B). \quad (42.9)$$

From (5.9) and (42.9), it follows (Nowacki [314], Hieke [175]) that

$$\delta v = V(B) \theta^*(B) \alpha_{ii}. \quad (42.10)$$

43. The Theorem of Reciprocity

Although the theorems of the linear theory of thermoelasticity may be derived from the theorems of the theory of elasticity, using the analogy from Paragraph 41, we prefer to derive some of them directly in order to facilitate the reading. Similarly, we agree also to take mechanical loads into consideration.

In the case of a thermoelastic equilibrium, we will call the set $\{f_i, \tilde{u}_i, \tilde{t}_i, \theta\}$ a thermomechanical load system (it is assumed that the function θ is known).

Let an elastic medium be subjected to two systems of thermo-mechanical loads

$$L^{(\alpha)} = \{f_i^{(\alpha)}, \tilde{u}_i^{(\alpha)}, \tilde{t}_i^{(\alpha)}, \theta^{(\alpha)}\}, \quad (\alpha = 1, 2).$$

We will designate by $\Pi^{(\alpha)} = \{u_i^{(\alpha)}, \epsilon_{ij}^{(\alpha)}, t_{ij}^{(\alpha)}\}$ the solution of the thermoelastic equilibrium problem (40.7) -- (40.9) corresponding to the loading system $L^{(\alpha)}$.

Theorem 43.1. If an elastic medium is subjected to two systems of loading $L^{(\alpha)}$ ($\alpha = 1, 2$), then between the corresponding solutions $\Pi^{(\alpha)}$ the following reciprocity relation prevails

/167

$$\begin{aligned} \int_V f_i^{(1)} u_i^{(2)} dv - \int_B (\beta_{,j} \theta^{(1)})_{,j} u_i^{(2)} dv + \int_{\partial B} t_{ij}^{(1)} u_i^{(2)} da + \int_{\partial B} \beta_{,j} \theta^{(1)} n_j u_i^{(2)} da = \\ = \int_V f_i^{(2)} u_i^{(1)} dv - \int_B (\beta_{,j} \theta^{(2)})_{,j} u_i^{(1)} dv + \int_{\partial B} t_{ij}^{(2)} u_i^{(1)} da + \int_{\partial B} \beta_{,j} \theta^{(2)} n_j u_i^{(1)} da. \end{aligned} \quad (43.1)$$

Proof. Let

$$2W_{\alpha\beta} = (t_{ij}^{(\alpha)} + \beta_{,j} \theta^{(\alpha)}) \epsilon_{ij}^{(\beta)}. \quad (43.2)$$

In view of the constitutive equations, we can write

$$2W_{\alpha\beta} = C_{ijkl} \epsilon_{ij}^{(\beta)} \epsilon_{kl}^{(\alpha)}. \quad (43.3)$$

From (43.3), on the basis of relations (25.5) we derive

$$W_{12} = W_{21}. \quad (43.4)$$

Using the theorem of divergence and the equations of equilibrium

we have

$$\begin{aligned} 2 \int_{\Omega} W_{, \alpha} d\Omega = & \int_{\Omega} t_{ij}^{(\alpha)} u_{i,j}^{(\beta)} dv + \int_{\Omega} \beta_{ij} \theta_{,i}^{(\alpha)} u_{j,j}^{(\beta)} da = \int_{\partial \Omega} t_{ij}^{(\alpha)} u_{i,j}^{(\beta)} da + \\ & + \int_{\partial \Omega} \beta_{ij} \theta_{,i}^{(\alpha)} n_j u_{j,j}^{(\beta)} da + \int_{\Omega} f_i^{(\alpha)} u_i^{(\beta)} dv - \int_{\Omega} (\beta_{ij} \theta_{,i}^{(\alpha)})_{,j} u_j^{(\beta)} dv. \end{aligned} \quad (43.5)$$

(43.1) follows from (43.4) and (43.5).

Let us give another form to the established reciprocity relation. The equations (41.1) may be written in the vectorial form in the following manner

$$\underline{A} \underline{u} = \underline{p}, \quad (43.6)$$

where the vectors $\underline{A} \underline{u} = (A_i u)$ and $\underline{p} = (p_i)$ have the components

$$A_i u = - (C_{ijkl} u_{k,l})_{,j}, \quad (43.7)$$

and

$$p_i = f_i - (\beta_{ij} \theta)_{,j}, \quad (43.8)$$

respectively. If we introduce the vector $\underline{T} \underline{u}$ with the components

$$T_i u = C_{ijkl} u_{k,l} n_j, \quad (43.9)$$

then the problem (41.1), (41.2) consists in the integration of equation (43.6) with the conditions

$$u = \bar{u} \text{ on } \Sigma_1, \quad \underline{T} \underline{u} = \bar{\pi} \text{ on } \Sigma_2, \quad (43.10)$$

where $\pi_1 = t_1 + \beta_{1j} \theta n_j$. Let \underline{u} and \underline{v} be displacement vectors with the components $u_i^{(1)}$ and $u_i^{(2)}$ respectively. We will designate

$$2W(u, v) = 2W_{12} = C_{ijkl} \varepsilon_{ij}(u) \varepsilon_{kl}(v). \quad (43.11)$$

From the relations (43.5) we derive

/168

$$\int_{\partial B} v T u \, da + \int_B v A u \, dv = 2 \int_B W(u, v) \, dv, \quad (43.12)$$

$$\int_{\partial B} u T v \, da + \int_B u A v \, dv = 2 \int_B W(u) \, dv, \quad (43.13)$$

where $\underline{W}(\underline{u}) = W(\underline{u}, \underline{u})$ is the elastic potential corresponding to the displacement \underline{u} . The reciprocity relation (43.1) may be written in the following way

$$\int_B (u A v - v A u) \, dv = \int_{\partial B} (v T u - u T v) \, da. \quad (43.14)$$

Let us observe similarly that the relation (43.1) may also be written in the following form (Maysel [281])

$$\begin{aligned} \int_B f_i^{(1)} u_i^{(2)} \, dv + \int_{\partial B} t_i^{(1)} u_i^{(2)} \, da + \int_B \beta_{ij} \theta^{(1)} \varepsilon_{ij}^{(2)} \, dv = \\ = \int_B f_i^{(2)} u_i^{(1)} \, dv + \int_{\partial B} t_i^{(2)} u_i^{(1)} \, da + \int_B \beta_{ij} \theta^{(2)} \varepsilon_{ij}^{(1)} \, dv. \end{aligned} \quad (43.15)$$

The reciprocity relations may be used to establish certain formulas of representation of the type of those given in Paragraph 30 (Maysel [280]). It is possible to introduce potentials which are similar to the simple layer potentials, double layer potentials and mass potentials, from the conventional theory of the potential. By means of this potential the boundary problems are reduced to the study of certain integral equations (Kupradze [246]).

44. The Theorem of Uniqueness

The uniqueness of the solutions of the problems under consideration

is derived from the study of the uniqueness of the solutions of problems from the theory of elasticity. For the unity of the treatment we will mention the following theorem

Theorem 44.1. If C_{ijkl} is a definite positive tensor, then
 (α) the first boundary problem has at most one solution;
 (β) any two solutions of the second boundary problem differ by a rigid displacement;
 (γ) the mixed problem has at most one solution.

Proof. Let $\{u_i^0, \epsilon_{ij}^0, t_{ij}^0\}$ be the difference of two solutions of one of the boundary problems under consideration. On the basis of the linearity of the theory, $\{u_i^0, \epsilon_{ij}^0, t_{ij}^0\}$ corresponds to zero loads and it follows from (43.13) that

$$\int_{\Omega} W(u^0) dv = 0, \quad (44.1)$$

where $\underline{u}^0 = (u_i^0)$. In view of the fact that W is a square positively defined form, it follows that $\epsilon_{ij}^0 = 0$ and therefore /169

$$u_i^0 = a_i + e_{ijk} b_j x_k,$$

in which a_i and b_i are arbitrary constants. For the first boundary problem and the mixed problem it follows that $u_i^0 = 0$. The solution of the second boundary problem is determined, however, by the rigid displacement.

If the following conditions are imposed

$$\int_{\Omega} u_i dv = 0, \quad \int_{\Omega} e_{ijk} x_j u_k dv = 0, \quad (44.2)$$

it can easily be seen that the solution of the second boundary problem, which satisfies the conditions (44.2), is unique. The first of the conditions of (44.2) eliminates the translation and the second one the arbitrary rigid rotation.

45. Variational Theorems

We will present two variational theorems from the theory of elasticity with reference to the equations of the thermoelastic equilibrium.

We call an admissible kinematic state an admissible state which satisfies the relations (40.8), (40.9) and the conditions imposed at the boundary portion Σ_1 .

Let $s = \{u_i, \varepsilon_{ij}, t_{ij}\}$ be an admissible kinematic and functional state

$$\begin{aligned} \Psi(s) = & \frac{1}{2} \int_B C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} dv + \int_B (\beta_{ij} \theta)_{,j} u_i dv - \int_B f_i u_i dv - \\ & - \int_{\Sigma_1} \beta_{ij} \theta n_j u_i da - \int_{\Sigma_1} \bar{t}_{ij} u_i da, \end{aligned} \quad (45.1)$$

defined on the set K of the admissible kinematic states.

Theorem 45.1. (Theorem of the minimum potential energy).
If C_{ijkl} is a symmetric and positively defined tensor, Ψ is the functional defined on K by (45.1) and s is the solution of the mixed problem, then

$$\Psi(s) \leq \Psi(s^*), \quad (45.2)$$

for any $s^* \in K$.

Proof. Let $s, s^* \in K$ and $s' = s^* - s$. Obviously s' is an admissible state and we have

$$2\varepsilon'_{ij} = u'_{i,j} + u'_{j,i}, \quad t'_{ij} = C_{ijkl} \varepsilon'_{kl} \quad \text{in } B, \quad u'_i = 0 \quad \text{on } \Sigma_1. \quad (45.3)$$

We can write

$$\begin{aligned}
 \Psi(s^*) - \Psi(s + s') - \Psi(s) &= \frac{1}{2} \int_B C_{ijkl} \epsilon'_{ij} \epsilon'_{kl} dv + \\
 &+ \int_B (\beta_{ij} \theta)_{,j} u'_i dv - \int_B f_i u'_i dv - \int_{\Sigma_1} \beta_{ij} \theta n_j u'_i da - \int_{\Sigma_1} \tilde{l}_i u'_i da + \\
 &+ \int_B C_{ijkl} \epsilon_{ij} \epsilon'_{kl} dv = \Psi(s) + \frac{1}{2} \int_B C_{ijkl} \epsilon'_{ij} \epsilon'_{kl} dv + \\
 &+ \int_B (\beta_{ij} \theta)_{,j} u'_i dv - \int_B f_i u'_i dv - \int_{\Sigma_1} \beta_{ij} \theta n_j u'_i da - \int_{\Sigma_1} \tilde{l}_i u'_i da + \\
 &+ \int_B (t_{ij} + \beta_{ij} \theta) u'_{i,j} dv = \Psi(s) + \frac{1}{2} \int_B C_{ijkl} \epsilon'_{ij} \epsilon'_{kl} dv - \\
 &- \int_B (t_{ij,j} + f_i) u'_i dv + \int_{\Sigma_1} (t_{ij} n_j - \tilde{l}_i) u'_i dv.
 \end{aligned}
 \tag{45.4}$$

In view of the fact that s is a solution of the mixed problem, it follows from (45.4) that

$$\Psi(s^*) - \Psi(s) = \frac{1}{2} \int_B C_{ijkl} \epsilon'_{ij} \epsilon'_{kl} dv. \tag{45.5}$$

In view of the fact that C_{ijkl} is a positive defined tensor, we can write

$$\begin{aligned}
 \Psi(s) &\leq \Psi(s^*), \\
 \Psi(s) = \Psi(s^*) &\Leftrightarrow \epsilon'_{ij} = \epsilon^*_{ij} - \epsilon_{ij} = 0.
 \end{aligned}$$

The functional (45.1) may be substituted by the functional

$$\Gamma(s) = \frac{1}{2} \int_B C_{ijkl} \epsilon_{ij} \epsilon_{kl} dv - \int_B \beta_{ij} \epsilon_{ij} \theta dv - \int_B f_i u_i dv - \int_{\Sigma_1} \tilde{l}_i u_i da. \tag{45.6}$$

This is due to the fact that the second term of the right member of the relation (45.1) may be written in the following form

$$\int_B (\beta_{ij,0})_{,i} u_i dv = \int_{\Sigma_1} \beta_{ij,0} n_j \bar{u}_i da + \int_{\Sigma_2} \beta_{ij,0} n_j u_i da - \int_B \beta_{ij,0} \varepsilon_{ij} dv,$$

and the integral extended at Σ_1 may be eliminated because the function θ is assumed to be known.

Remark. Let us consider the mixed problem of the thermoelastic equilibrium written in the form (43.6), (43.10). Theorem 45.1 may be derived also with the help of the results mentioned in Paragraph 43. Let us consider first the equation

$$Au = p, \quad (45.7)$$

where A is a linear operator in a real Hilbert space, and u and f are elements of the space. It is known (Mihlin [288]) that if the operator A is positively defined, then the equation (45.7) has a generalized solution which minimizes the functional

$$F(u) = (Au, u) - 2(u, f). \quad (45.8)$$

The domain of this functional is a new Hilbert space which is the closure of the domain D_A of the operator A in the metric generated by the scalar product (Au, u) .

/171

The operator A , defined by the lineal D_A which is dense in the Hilbert space under consideration, is called positive if

$$\begin{aligned} (Au, v) &= (u, Av) \quad \forall u, v \in D_A, \\ (Au, u) &\geq 0, \quad \forall u \in D_A, \quad (Au, u) = 0 \Rightarrow u = 0. \end{aligned} \quad (45.9)$$

The positive operator A is called positively defined if for every $u \in D_A$ the following inequality takes place

$$(Au, u) \geq \gamma^2 \|u\|^2, \quad (45.10)$$

where γ is a positive constant.

Let operator \underline{A} be definite through (43.7) and the conditions (43.10) in a homogeneous form. Let us consider the real Hilbert space $L_2(\Omega)$ the elements of which are square vectorial functions summable in B ; the scalar product of this space is defined by

$$(u, v) = \int_B uv \, dv = \int_B \sum_{i=1}^3 u_i v_i \, dv.$$

The symmetry of the operator \underline{A} follows from (43.14), keeping in mind the fact that we have homogeneous conditions on ∂B . Inasmuch as C_{ijkl} is a definite positive tensor, it follows from (43.13) and theorem 44.1 that the operator \underline{A} satisfies the conditions (45.9). It follows that the solution \underline{u}_0 of the problem under consideration minimizes the functional (45.8). In view of (43.13), (43.8) we can write

$$F(u) = 2 \int_B W(u) \, dv - 2 \int_B [f_i - (\beta_{ij} 0)_{,j}] u_i \, dv.$$

For nonhomogeneous boundary conditions it can easily be shown that we are led to consider the functional

$$\Phi(u) = \int_B W(u) \, dv - \int_B [f_i - (\beta_{ij} 0)_{,j}] u_i \, dv - \int_{\Sigma} (\bar{t}_i + \beta_{ij} 0 n_j) u_i \, da. \quad (45.11)$$

In this case we can state that in the set of the displacement vectors corresponding to the admissible kinematic state, the solution of the problem minimizes the functional (45.11).

It can be shown that the operator \underline{A} is positively defined (Mihlin [288]). From this fact follows the existence of a generalized solution and the applicability of the variational method stated in [288].

We will now present a known variational theorem from the theory

of elasticity (see Gurtin [163]) with reference to thermoelastostatics.

Theorem 45.2. (The Hu-Washizu Theorem). Let \mathcal{A} be the set of admissible states and Λ a functional defined on \mathcal{A} by

$$\begin{aligned}\Lambda(s) = & \frac{1}{2} \int_B C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} dv - \int_B (t_{ij} + \beta_{ij} \theta) \varepsilon_{ij} dv - \\ & - \int_B (t_{i,j} + f_i) u_i dv + \int_{\Sigma_1} t_i \bar{u}_i da + \int_{\Sigma_2} (t_i - \bar{t}_i) u_i da,\end{aligned}\quad (45.12)$$

for every $s = (u_i, \varepsilon_{ij}, t_{ij}) \in \mathcal{A}$.

Then

/172

$$\delta \Lambda(s) = 0, \quad (45.13)$$

if and only if s is the solution of a mixed problem.

Proof. Let $s = \{u_i, \varepsilon_{ij}, t_{ij}\}$ and $s' = \{u'_i, \varepsilon'_{ij}, t'_{ij}\}$ be two admissible states. Then $s + \lambda s' \in \mathcal{A}$ for any scalar λ . Then in view of (31.14) we will form the expression

$$\delta_s \Lambda(s) = \frac{d}{d\lambda} \Lambda(s + \lambda s')|_{\lambda=0}.$$

It can easily be seen that

$$\begin{aligned}\delta_s \Lambda(s) = & \int_B [(C_{ijkl} \varepsilon_{kl} - t_{ij} - \beta_{ij} \theta) \varepsilon'_{ij} - (t_{i,j} + f_i) u'_i - \\ & - t'_{ij} \varepsilon_{ij} - t'_{i,j} u_i] dv + \int_{\Sigma_1} t'_i u_i da + \int_{\Sigma_2} [t'_i u_i + (t_i - \bar{t}_i) u'_i] da.\end{aligned}\quad (45.14)$$

If we keep in mind

$$\int_B t'_{i,j} u_i dv = \int_{\partial B} t'_{ij} n_j u_i da - \int_B u_{i,j} t'_{ij} dv,$$

then (45.14) takes the form

$$\begin{aligned}\delta_s \Lambda(s) = & \int_B (C_{ijkl} \varepsilon_{kl} - \beta_{ij} \theta - t_{ij}) \varepsilon'_{ij} dv - \\ & - \int_B (t_{i,j} + f_i) u'_i dv + \int_B \left[\frac{1}{2} (u_{i,j} + u_{j,i}) - \varepsilon_{ij} \right] t'_{ij} dv + \\ & + \int_{\Sigma_1} (\bar{u}_i - u_i) t'_i da + \int_{\Sigma_2} (t_i - \bar{t}_i) u'_i da.\end{aligned}\quad (45.15)$$

If s is a solution of the mixed problem, then it follows from (45.15) that

$$\delta_{i'} \Lambda(s) = 0 \text{ for any } s' \in \mathcal{A}, \quad (45.16)$$

which implies (45.13).

In order to prove the reciprocal, let us assume that relation (45.13) takes place. This implies (45.16). If we select $s' = \{u_1', 0, 0\}$ where u_1' becomes zero on ∂B , then from (45.15) and (45.16) we have

$$\int_{\Omega} (t_{ij} + f_i) u_i' dv = 0.$$

On the basis of the fundamental lemma from the calculus of variations it follows that $t_{ij,j} + f_i = 0$. Let us select $s' = \{u_1', 0, 0\}$ where u_1' becomes zero on Σ_1 . We obtain from (45.15) and (45.14)

$$\int_{\Sigma_1} (t_i - \tilde{t}_i) u_i' da = 0,$$

whence, on the basis of a generalization of the fundamental lemma (Gurtin [163]) it follows that $t_i = \tilde{t}_i$ on Σ_2 . Let $s' = \{0, \varepsilon_{ij}', 0\}$ where ε_{ij}' becomes zero on ∂B . We derive from (45.15) and (45.16)

$$\int_B (C_{ijkl} \varepsilon_{kl} - \beta_{ij} \theta - t_{ij}) \varepsilon_{ij}' dv = 0,$$

whence follows $t_{ij} = C_{ijkl} \varepsilon_{kl} - \beta_{ij} \theta$. Taking $s' = \{0, 0, t_{ij}'\}$ where t_{ij}' becomes zero on ∂B , we obtain $2\varepsilon_{ij} = u_{i,j} + u_{j,i}$. If we do not request that t_{ij}' become zero on ∂B , we obtain from (45.15) and (45.16)

$$\int_{\Sigma_1} (\tilde{u}_i - u_i) t_{ij,n} da = 0,$$

whence, using an extension of the fundamental lemma (Gurtin [163]), we derive that $u_1 = \tilde{u}_1$ on Σ_1 . Consequently $s = \{u_{ij}, \varepsilon_{ij}, t_{ij}\}$ is a solution of the mixed problem.

/173

46. The Existence of the Solution. Approximation Method

The existence of solutions of the boundary problems of the linear theory of thermoelasticity results from the theorems of existence established in the theory of linear elastostatics (for example see Fichera [112]). We will limit ourselves in the following to the presentation of an approximation method of the solutions and to the study of the existence of approximation solutions (Mihlin [288], Gurtin [163]).

The standard method for determining a approximate solution of the mixed problem consists in minimizing the functional (45.1) or (45.11) in a particular class of functions. It is attempted to find an approximate solution $u_i^{(N)}$ in the form

$$u_i^{(N)} = \bar{u}_i^{(N)} + \sum_{n=1}^N \alpha_n \phi_i^{(n)}, \quad (46.1)$$

where $\phi_1^{(1)}, \dots, \phi_1^{(N)}$ are given functions which become zero on Σ_1 , and $\bar{u}_1^{(N)}$ is a function which approximates \bar{u}_1 on Σ_1 . The term $\bar{u}_1^{(N)}$ will be omitted in the case of the second boundary problem. The constants $\alpha_1, \alpha_2, \dots, \alpha_N$ are so selected that the $\Phi(u^{(N)})$ is minimum.

If we set

$$\Phi(\alpha_1, \alpha_2, \dots, \alpha_N) = \Phi(u^{(N)}), \quad (46.2)$$

then we have

$$\Phi(\alpha_1, \alpha_2, \dots, \alpha_N) = A + \sum_{n=1}^N A_n \alpha_n + \frac{1}{2} \sum_{m,n=1}^N A_{mn} \alpha_m \alpha_n, \quad (46.3)$$

where

$$\begin{aligned} A &= \frac{1}{2} \int_{\Omega} C_{ijkl} \bar{u}_{i,j}^{(N)} \bar{u}_{k,l}^{(N)} dv - \int_B [f_i - (\beta_{ij} \theta)_{,j}] \bar{u}_i^{(N)} dv - \\ &\quad - \int_{\Sigma_2} (\bar{l}_i + \beta_{ij} \theta n_j) \bar{u}_i^{(N)} da, \\ A_n &= \int_{\Omega} C_{ijkl} \bar{u}_{i,j}^{(N)} \phi_{k,l}^{(n)} dv - \int_B [f_i - (\beta_{ij} \theta)_{,j}] \phi_i^{(n)} dv - \int_{\Sigma_2} (\bar{l}_i + \beta_{ij} \theta n_j) \phi_i^{(n)} da, \\ A_{mn} &= \int_{\Omega} C_{ijkl} \phi_{i,j}^{(n)} \phi_{k,l}^{(m)} dv. \end{aligned} \quad (46.4)$$

When Σ_1 is empty, we will have $\bar{u}_1^{(N)} = 0$ in the above relations and therefore

/174

$$A = 0, \quad A_n = - \int_B [f_i - (\beta_i, 0),_{,j}] \varphi_i^{(n)} dv - \int_{\partial B} (\tilde{t}_i + \beta_{ij} \theta n_j) \varphi_i^{(n)} da. \quad (46.5)$$

If C_{ijkl} are components of a semidefinite positive tensor, then the matrix (A_{mn}) will be positively semidefinite and $\Phi(\alpha_1, \alpha_2, \dots, \alpha_N)$ will have a minimum in $\alpha_1 = \alpha_1^0$ if, and only if α_1^0 is a solution of the system

$$\sum_{i=1}^N A_{ri} \alpha_i = -A_r, \quad (r = 1, 2, \dots, N). \quad (46.6)$$

Next we will establish the conditions in which the system (46.6) has solutions and in which the approximate solution $u_1^{(N)}$ converges in energy for $N \rightarrow \infty$ on the solution of the problem.

In the following we will assume that C_{ijkl} is positively defined. We will designate by M_0 the set of continuous vectorial functions with continuous first order derivatives on portions of \bar{B} . Let G_N be an N -dimensional subspace of M_0 with the property that elements of G_N become zero on Σ_1 . If $\Sigma_1 \neq \emptyset$, we designate by $\bar{u}^{(N)}$ a given element of M_0 , and by G_N^* the set of the vectorial functions of the form

$$g^* = g + \bar{u}^{(N)}, \quad g \in G_N.$$

If $\Sigma_1 = \emptyset$, then $G_N^* = G_N$.

Theorem 46.1. There is $u^{(N)} \in G_N$ so that for every $v \in G_N^*$ we have

$$\Phi(u^{(N)}) \leq \Phi(v). \quad (46.7)$$

If $\Sigma_1 \neq \emptyset$ then $\underline{u}^{(N)}$ is unique; if $\Sigma_1 = \emptyset$ then $\underline{u}^{(N)}$ is determined up to a rigid displacement. Moreover, if \underline{u} is the displacement vector which corresponds to the solution of the mixed problem, then

$$\int_{\Omega} W(\underline{u} - \underline{u}^{(N)}) dv = \inf_{\underline{w} \in G_N^0(\Omega)} \int_{\Omega} W(\underline{u} - \underline{w}) dv. \quad (46.8)$$

The relation (46.8) specifies the sense in which the vector $\underline{u}^{(N)}$ is optimum.

Proof. Let $\underline{\phi}^{(1)}, \underline{\phi}^{(2)}, \dots, \underline{\phi}^{(N)}$ be a base in G_N . In order to establish the existence of a solution $\underline{u}^{(N)}$ for (46.7) it is sufficient to establish the existence of a solution α_1^0 of the system (46.6).

Inasmuch as C_{ijkl} is positively defined, it follows from (46.4) that

$$\sum_{r,s=1}^N A_{rs} \alpha_r \alpha_s > 0. \quad (46.9)$$

Let us assume that

$$\sum_{r,s=1}^N A_{rs} \alpha_r \alpha_s = 0, \quad (46.10)$$

and let

$$h_i = - \sum_{r=1}^N \alpha_r \phi_i^{(r)}. \quad (46.11)$$

From (46.4) and (46.10) we obtain

/175

$$\int_{\Omega} C_{ijkl} h_{i,j} h_{k,l} dv = 0,$$

and in view of the fact that C_{ijkl} is positively defined, it follows that \underline{h} represents a rigid displacement. If $\Sigma_i \neq \emptyset$ then $\underline{h} = 0$.

In view of the fact that $\underline{\phi}^{(1)}, \dots, \underline{\phi}^{(N)}$ is a base in G_N it follows that

$$\alpha_1 = \alpha_2 = \dots = \alpha_N = 0. \quad (46.12)$$

Therefore if $\Sigma_i = \emptyset$ then (46.10) implies (46.12) and accordingly it follows from (46.9) that the matrix (A_{rs}) is positively defined. This state of affairs indicates that the system (46.6) has, in this case, a unique solution.

Let us consider the case when $\Sigma_i \neq \emptyset$. If

$$\sum_{r=1}^N A_r \alpha_r = 0, \quad (46.13)$$

then (46.10) takes place, and therefore \underline{h} , defined by (46.11) is a rigid displacement. From (46.5) and (46.11) we obtain

$$\sum_{r=1}^N A_r \alpha_r = - \int_B [f_i - (\beta_{ij} \theta)_{,j}] h_i \, dv - \int_{\partial B} (\tilde{t}_i + \beta_{ij} \theta n_j) h_i \, da.$$

Inasmuch as \underline{h} is a rigid displacement and the forces f_1 and \tilde{t}_1 are in equilibrium (which implies that $f_1 - (\beta_{1j} \theta)_{,j}$ and $\tilde{t}_1 + \beta_{1j} \theta n_j$ are in equilibrium) it follows that

$$\sum_{r=1}^N A_r \alpha_r = 0.$$

Therefore (A_1, A_2, \dots, A_N) is the orthogonal with any solution $(\alpha_1, \alpha_2, \dots, \alpha_N)$ of the homogeneous equation (46.13). Using Fredholm's alternative, it follows that the nonhomogeneous equation (46.6) has a solution.

In order to demonstrate (46.8) we will show first that (Tong and Pian [417]) if $\{u_1, \epsilon_{1j}, t_{1j}\}$ is a solution of the mixed problem, then for any $w \in M_0$ we have

$$\Phi(w) - \Phi(u) = \int_B W(w - u) dv + \int_{\Sigma_1} (t_{ij} + \beta_{ij}\theta) n_j (w_i - \tilde{u}_i) da. \quad (46.14)$$

Let us designate $w_1 - u_1 = u'_1$. Obviously

$$\begin{aligned} \int_B W(w) dv &= \int_B W(u') dv + \int_B W(u) dv + \int_B C_{ijkl} u_{k,i} u'_{l,j} dv = \\ &= \int_B W(u') dv + \int_B W(u) dv + \int_B (t_{ij} + \beta_{ij}\theta) u'_{i,j} dv = \\ &= \int_B W(w - u) dv + \int_B W(u) dv + \int_{\partial B} (t_{ij} + \beta_{ij}\theta) n_j u'_i da - \\ &\quad - \int_B (t_{ij} + \beta_{ij}\theta)_{,j} u'_i dv. \end{aligned} \quad (46.15)$$

We obtain from (45.11) and (46.15)

/176

$$\begin{aligned} \Phi(w) - \Phi(u) &= \int_B W(w) dv - \int_B W(u) dv - \\ &\quad - \int_B [f_i - (\beta_{ij}\theta)_{,j}] (w_i - u_i) dv - \int_{\Sigma_1} (\tilde{t}_{ij} + \beta_{ij}\theta n_j) (w_i - u_i) da = \\ &= \int_B W(w - u) dv - \int_B (t_{ij,j} + f_i) (w_i - u_i) dv + \\ &\quad + \int_{\Sigma_1} (t_{ij} + \beta_{ij}\theta) n_j (w_i - \tilde{u}_i) da. \end{aligned}$$

Keeping in mind the fact that t_{1j} satisfies the equilibrium equations, we derive the relation (46.14)

If $w \in G_N^*$, then $\underline{w} = \underline{u}^{(N)}$ on Σ_1 .

Inasmuch as $u^{(N)} \in G_N^*$, it follows from (46.7) and (46.14) that

$$\begin{aligned} \int_B W(u^{(N)} - u) dv &= \Phi(u^{(N)}) - \Phi(u) - \int_{\Sigma_1} (t_j + \beta_{ij} \theta) n_j (u_i^{(N)} - \tilde{u}_i) da \leq \\ &\leq \Phi(w) - \Phi(u) - \int_{\Sigma_1} (t_j + \beta_{ij} \theta) n_j (w_i - u_i) da = \\ &= \int_B W(w - u) dv, \end{aligned}$$

for any $w \in G^*$. The same procedure is used when $\Sigma_1 = \emptyset$.

Let us study the convergence of the approximate solution. This has been done in detail by Mihlin [288], [289].

We will assume that

(1) for any displacement vector \underline{u} corresponding to a admissible kinematic state, there is a series $\{w^{(N)}\}$, $w^{(N)} \in G_N^*$ such that

$$\int_B W(u - w^{(N)}) dv \rightarrow 0 \text{ for } N \rightarrow \infty. \quad (46.16)$$

Theorem 46.2. Let us assume that (1) takes place and $\underline{u}^{(N)}$ and G_N^* satisfy for every $N = 1, 2, \dots$ the conditions imposed by theorem 46.1. If $\underline{u}^{(N)}$ is a series of approximate solutions (solutions for (46.7)), and \underline{u} is a displacement vector corresponding to a solution of the mixed problem, then

$$\int_B W(u - u^{(N)}) dv \rightarrow 0 \text{ for } N \rightarrow \infty. \quad (46.17)$$

Proof. Taking into consideration (1), it follows that there is a series $\{w^{(N)}\}$, $w^{(N)} \in G_R^*$ such that (46.16) takes place. The relation (46.9) and the fact that C_{ijkl} is positively defined imply

$$0 \leq \int_B W(u - u^{(N)}) dv \leq \int_B W(u - v^{(N)}) dv.$$

In view of (46.16), (46.17) follows.

The above theorem includes the method of the finite element (see for example Tong and Pian [417], Zienkiewicz [456], Oden [325]).

47. Homogeneous and Isotropic Media

Let us consider now the case of the thermoelastic equilibrium of homogeneous and isotropic media. It follows from the statements presented in Paragraph 23 that in this case the constitutive equations (40.3) have the form

$$\begin{aligned} t_{ij} &= \lambda \varepsilon_{rr} \delta_{ij} + 2 \mu \varepsilon_{ij} - \beta \theta \delta_{ij}, \\ q_i &= k \theta_{,i}. \end{aligned} \quad (47.1)$$

Inasmuch as the medium under consideration is assumed to be homogeneous, the coefficients λ , μ , β , k are constant.

The boundary problem (40.6) becomes

$$\begin{aligned} k \Delta \theta &= -r \text{ in } B, \\ \theta &= \bar{\theta} \text{ on } \Sigma_3, \quad k \frac{\partial \theta}{\partial n} = \bar{q} \text{ on } \Sigma_1. \end{aligned} \quad (47.2)$$

If this problem is resolved, the following equations must be studied

$$t_{ij} + f_i = 0, \quad (47.3)$$

$$t_{ij} = \lambda \varepsilon_{ij} + 2\mu \varepsilon_{ij} - \beta \theta \delta_{ij}, \quad (47.4)$$

$$2\varepsilon_{ij} = u_{i,j} + u_{j,i} \text{ in } R, \quad (47.5)$$

with the boundary problems (40.5)₁.

From (47.3) -- (47.4) results the following expression of the equilibrium equations with the help of the components of the displacement vector

$$\mu \Delta u_i + (\lambda + \mu) u_{r,r} + f_i - \beta \theta_{,i} = 0. \quad (47.6)$$

The boundary conditions assume the form

$$u_i = \tilde{u}_i \text{ on } \Sigma_1, \quad [\lambda u_{r,r} \delta_{ij} + \mu(u_{i,j} + u_{j,i})] n_j = \tilde{t}_i + \beta \theta n_i \text{ on } \Sigma_2. \quad (47.7)$$

If we assume that $k \neq 0$, we conclude from (47.6) and (47.2)₁ /178

$$\begin{aligned} (\lambda + 2\mu) \Delta \operatorname{div} u &= -\operatorname{div} f - \frac{\beta}{k} r, \\ \mu \Delta \operatorname{rot} u &= -\operatorname{rot} f. \end{aligned} \quad (47.8)$$

If $f_1 = r = 0$, it follows from (47.6) and (47.8)₁ that

$$\Delta \Delta u_i = 0.$$

We have thus (Gurtin [163], Carlson [46])

Theorem 47.1. If the functions u_i , of the class C^4 on B and θ of the class C^2 on B satisfy the equations

$$\mu \Delta u_i + (\lambda + \mu) u_{i,jj} - \beta \theta_{,i} = 0, \quad (47.9)$$

$$\Delta \theta = 0, \quad (47.10)$$

then

$$\Delta \operatorname{div} u = 0, \quad \Delta \operatorname{rot} u = 0, \quad \Delta \Delta u_i = 0.$$

Let us assume that in (47.4) the function θ is known. We will present the method due to Goodier [135], to determine a particular solution for equations (47.9) which allows the reduction of the thermoelasticity problem to a problem of elasticity, without mass forces and with loads at the boundary. A particular solution of equations (47.9) is sought in the form

$$u_i^* = \Phi_{,i}. \quad (47.11)$$

The function Φ , assumed to be of the class C^3 in B, is known under the name of thermoelastic potential of the displacement.

Substituting (47.11) in (47.9), we obtain

$$[(\lambda + 2\mu) \Delta \Phi - \beta \theta]_{,i} = 0, \quad (47.12)$$

and thus the equations (47.9) are satisfied if the function Φ is

the solution of the equation

$$\Delta \Phi = \frac{\beta}{\lambda + 2\mu} 0 \ln B. \quad (47.13)$$

The components of the tension tensor corresponding to the displacement (47.11) have the form

$$t_{ij}^* = \lambda \Phi_{,rr} \delta_{ij} + 2\mu \Phi_{,ij} - \beta 0 \delta_{ij}.$$

In view of (47.13) it follows that

$$t_{ij}^* = 2\mu(\Phi_{,ij} - \Phi_{,rr} \delta_{ij}). \quad (47.14)$$

Let us mention also the use in some cases of a Boussinesq-Papkovic-Neuber representation for the direct study of equations (47.6). Thus, it can easily be verified that the functions

/179

$$u_i = \psi_i - \frac{1}{4(1-\nu)} (x_r \psi_{,r} + \varphi)_{,i}, \quad (47.15)$$

where ϕ and ψ_r are functions of class C^3 on B which satisfy the equations

$$\Delta \psi_i = -\frac{1}{\mu} (f_i - \beta 0_{,i}), \quad \Delta \varphi = \frac{1}{\mu} x_j (f_j - \beta 0_{,j}), \quad (47.16)$$

are solutions of equations (47.6).

It is shown (see for example Gurtin [163]) that any solution of equations (47.6) may be written in the form (47.15).

Often it is convenient to characterize the fundamental equations by means of the components of the tension tensor. It should be recalled (see Paragraph 23) that if $\mu \neq 0$ and $\nu \neq -1$, then the relations (47.4) may be written in the following way

$$\varepsilon_{ij} = \frac{1+\nu}{E} t_{ij} - \frac{\nu}{E} t_{rr} \delta_{ij} + \alpha \theta \delta_{ij}. \quad (47.17)$$

Making use of the analogy from Paragraph 41, we can state (Boley and Weiner [23], Carlson [46])

Theorem 47.2. Let there be tensor ε_{ij} of the class C^2 and the tension tensor t_{ij} which satisfy the relations (47.17) where θ is a function of the class C^2 which satisfies the equation (47.2)₁. If also $k \neq 0$ and $\nu \neq 1$, then the equilibrium equations (47.3) and the compatibility equations (3.4) imply the following compatibility equations for the tensions

$$\begin{aligned} t_{ij,rs} + \frac{1}{1+\nu} t_{rs,ij} + E\alpha \left[\frac{1}{1+\nu} \theta_{,ij} - \frac{1}{k(1-\nu)} r \delta_{ij} \right] + \\ + \frac{\nu}{1-\nu} \delta_{ij} f_{r,rs} + f_{r,i} + f_{r,s} = 0. \end{aligned} \quad (47.18)$$

Inversely, if t_{ij} is a symmetric tensor of the class C^2 on B , which satisfies the equilibrium equations and the equations (47.18), then the tensor ε_{ij} defined by (47.17) satisfies the compatibility equations (3.4). If the domain B is simply connected, then there is a displacement vector which satisfies the relations (47.5).

In the case of a domain which is multiply connected, in order to

ensure the uniformity of the displacement, the following conditions are derived from (3.27) and (47.17)

$$\begin{aligned} \frac{1}{E} \int_{L_n} (t_{ik} - c_{ikr} c_{rjm} \xi_j t_{kk,m}) d\xi_k - \frac{\nu}{E} \int_{L_n} (t_{rr} \delta_{ik} - c_{ikr} c_{rjm} \xi_j t_{rr,m}) d\xi_k + \\ + \alpha \int_{L_n} (0 \delta_{ik} - c_{ikr} c_{rjm} \xi_j 0_{,m}) d\xi_k = 0, \\ \frac{1}{E} \int_{L_n} c_{ikm} t_{kk,m} d\xi_k - \frac{\nu}{E} \int_{L_n} c_{ikm} t_{rr,m} d\xi_k + \alpha \int_{L_n} c_{ikm} 0_{,m} d\xi_k = 0, \end{aligned} \quad (47.19)$$

(n = 1, 2, ..., N).

In conclusion, let us derive the expression of the thermal field which does not produce tensions in homogeneous and isotropic media. /180

We assume that we are dealing with a second case of the boundary problems, and that $t_{ij} = 0$ on ∂B . We consider also that there are no mass forces and thermal sources. If we set $t_{ij} = 0$ in B , then it follows from (47.18) that

$$\theta_{,ij} = 0,$$

and therefore

$$\theta = a_1 x_1 + b, \quad (47.20)$$

where a_1 and b are arbitrary constants.

It follows from (47.17) and (47.20) that

$$\varepsilon_{ij} = \alpha(a_1 x_1 + b) \delta_{ij}. \quad (47.21)$$

The displacements corresponding to the deformations (47.21) are

$$u_i = \alpha \left[(a_j x_j + b) x_i - \frac{1}{2} a_i x_j x_j \right] + u_i^0, \quad (47.22)$$

where u_i^0 are components of a rigid displacement.

48. Special Problems

In this paragraph we will study some of the special problems of thermoelastic equilibrium for homogeneous and isotropic media in the absence of mass forces. We will assume, with the exception of Paragraph 48 (f), that there are no heat sources. Other problems of this type are found in the monograph of Nowacki [315].

a) The problem of the half space. Let us assume the medium under consideration occupies the half space $x_3 \geq 0$. We will study the problem of thermoelastic equilibrium for the case in which, at the boundary, $x_3 = 0$ is the prescribed temperature variation

$$\theta(x_1, x_2, 0) = \tilde{\theta}(x_1, x_2). \quad (48.1)$$

In addition, we assume that the boundary $x_3 = 0$ is free of tensions. This last case implies the following boundary conditions

$$t_{i3}(x_1, x_2, 0) = 0. \quad (48.2)$$

We will also impose the conditions

$$u_i = 0, \quad \theta = 0, \quad x_3 \rightarrow \infty. \quad (48.3)$$

This problem has been studied by many investigators (Melan and Parkus [286], Lurye [271], Sternberg and McDowell [393], Sneddon and Lockett [375]).

In the following we will present the solution given by Sneddon and Lockett [375].

If we designate (Sneddon and Berry [372])

$$a^2 - 1 = \frac{\lambda + \mu}{\mu}, \quad b = \frac{\beta}{\mu}, \quad (48.4)$$

then the relations (47.1) are written

$$\frac{1}{\mu} t_{ij} = u_{i,j} + u_{j,i} + [(a^2 - 2) \epsilon_{rr} - b\theta] \delta_{ij}. \quad (48.5)$$

Equations (47.9) become

$$u_{i,rr} + (a^2 - 1) u_{r,ri} = b\theta_{,i}. \quad (48.6)$$

In order to integrate the equations of thermoelasticity, we will use the Fourier transform. Let

$$u_r^*(\alpha_1, \alpha_2, x_3) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_r(x_1, x_2, x_3) e^{i\alpha_1 x_1} e^{i\alpha_2 x_2} dx_1 dx_2, \quad (48.7)$$

$$\theta^*(\alpha_1, \alpha_2, x_3) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta(x_1, x_2, x_3) e^{i\alpha_1 x_1} e^{i\alpha_2 x_2} dx_1 dx_2.$$

The equations (47.10) and (48.6) imply

$$\left(\frac{d^2}{dx_3^2} - \alpha^2\right) \theta^* = 0, \quad (48.8)$$

$$\left(\frac{d^2}{dx_3^2} - \alpha^2\right) u_i^* - (a^2 - 1) \alpha_i \alpha_j u_j^* = -ib \alpha_i \theta^*, \quad (48.9)$$

where

$$\alpha^2 = \alpha_1^2 + \alpha_2^2, \quad \alpha_3 = i \frac{d}{dx_3}, \quad i^2 = -1. \quad (48.10)$$

The solution of the equation (48.8) has the form

$$\theta^* = A e^{-\alpha x_3} + B e^{\alpha x_3}, \quad (48.11)$$

where A and B are arbitrary constants.

We derive from (48.9) and (48.11)

/182

$$u_1^* = (C_\beta + D x_\beta x_3) e^{-\alpha x_3} + (C'_\beta + D' x_\beta x_3) e^{\alpha x_3}, \quad (48.12)$$

$$u_3^* = (C_3 - ix D x_3) e^{-\alpha x_3} + (C'_3 + ix D' x_3) e^{\alpha x_3}.$$

In (48.12) C_r and C'_r are arbitrary constants, and D and D' are defined by

$$D = -\frac{a^2 - 1}{(a^2 + 1) \alpha} \left(x_\beta C'_\beta - ix C_3 - \frac{ibA}{a^2 - 1} \right), \quad (48.13)$$

$$D' = \frac{a^2 - 1}{(a^2 + 1) \alpha} \left(x_\beta C_\beta + ix C_3 - \frac{ibB}{a^2 - 1} \right).$$

If we consider the conditions (48.3) it follows that

$$\begin{aligned} 0^* &= A e^{-\alpha x_1}, \\ u_\beta^* &= (C_\beta + D x_\beta x_3) e^{-\alpha x_1}, \\ u_3^* &= (C_3 - i \alpha D x_3) e^{-\alpha x_1}. \end{aligned} \quad (48.14)$$

Applying the Fourier transform to the relations (48.5) and keeping in mind (48.14) we conclude

$$\begin{aligned} t_{\beta 3}^* &= -\mu(\alpha C_\beta + \alpha x_\beta D x_3 - x_\beta D + i x_\beta C_3 + \alpha_\beta \alpha D x_3) e^{-\alpha x_1}, \\ t_{33}^* &= \mu[i(2 - \alpha^2)(\alpha_\beta C_\beta + x^2 D x_3) + \alpha^2(i x^2 D x_3 - \alpha C_3 - \\ &\quad - i \alpha D) - b A] e^{-\alpha x_1}. \end{aligned} \quad (48.15)$$

From (48.1) and (48.2) we have

$$0^*(\alpha_1, \alpha_2, 0) = \tilde{0}^*(\alpha_1, \alpha_2), \quad t_{\alpha 1}^*(\alpha_1, \alpha_2, 0) = 0. \quad (48.16)$$

If we impose the condition (48.16)₁, it follows that

$$A = \tilde{0}^*(\alpha_1, \alpha_2). \quad (48.17)$$

Similarly, we obtain from (48.15) and (48.16)₂

$$C_\beta = \frac{i b \tilde{0}^* \alpha_\beta}{2(\alpha^2 - 1) \alpha^2}, \quad C_3 = - \frac{b \tilde{0}^*}{2(\alpha^2 - 1) \alpha}. \quad (48.18)$$

In view of (48.17) and (48.18), it follows from (48.13)₁ that /183
 $D = 0$. Consequently, the relations (48.14) take the form

$$\begin{aligned} \theta^* &= \tilde{\theta}^*(\alpha_1, \alpha_2) e^{-\alpha_1 z}, \\ u_\beta^* &= \frac{ib\tilde{\theta}^*\alpha_\beta}{2(a^2-1)\alpha^2} e^{-\alpha_1 z}, \quad u_3^* = -\frac{b\tilde{\theta}^*}{2(a^2-1)\alpha} e^{-\alpha_1 z}. \end{aligned} \quad (48.19)$$

If we replace A and C_β given by (48.17) and (48.18) in (48.15), and we keep in mind that $D = 0$, it follows that

$$t_{i3}^* = 0, \quad x_3 \geq 0. \quad (48.20)$$

Therefore, a state of plane tension, parallel with the boundary $x_3 = 0$ takes place in the body.

Inversing the transforms (48.19), θ and u_1 are determined. Sternberg and McDowell [393] considered the case of an axially symmetrical problem in which

$$\tilde{\theta} = \theta_0 H(c - r),$$

where θ_0 and c are constant, $r^2 = x_1^2 + x_2^2$ and H is the Heaviside function. In this case, using cylindrical coordinates (r, ϕ, z) , it follows from (48.19) that

$$\begin{aligned} u_r &= \frac{bc\theta_0}{2(a^2-1)} J(1, 1; -1), \\ u_z &= -\frac{bc\theta_0}{2(a^2-1)} J(1, 0; -1), \end{aligned}$$

where

$$J(\mu, \nu; \lambda) = \int_0^\infty J_\mu(\alpha r) J_\nu(\alpha r) e^{-\alpha z} \alpha^\lambda d\alpha.$$

$J_\mu(x)$ was used to designate the first species and μ order Bessel function. For the nonzero components of the tension tensor, the following relations result

$$\begin{aligned}\sigma_{rr} + \sigma_{\varphi\varphi} &= -\frac{cb\theta_0\mu}{a^2 - 1} J(1, 0; 0), \\ \sigma_{rr} - \sigma_{\varphi\varphi} &= \frac{cb\theta_0\mu}{a^2 - 1} \left[J(1, 0; 0) - \frac{2}{r} J(1, 1; -1) \right].\end{aligned}$$

The integrals $J(1, 1; -1)$, $J(1, 0; -1)$, $J(1, 0; 0)$ have been tabulated for various values of r/c and z/c by Eason, Noble and Sneddon [98].

b) Elastic layer placed on a rigid foundation. Let us consider an elastic layer which occupies the region $0 \leq x_3 \leq d$ and which is placed on a rigid, frictionless foundation. Let us assume that the surface $x_3 = 0$ is free of tensions and the surface $x_3 = d$ is in /184 in contact with the foundation. On the surface $x_3 = 0$, it is assumed that the temperature is given, while on the surface in contact with the foundation, a thermal flux is assumed.

Thus, the boundary conditions are

$$\begin{aligned}t_{13} &= 0, \quad \theta = \bar{\theta} \quad \text{for } x_3 = 0, \\ u_3 &= 0, \quad t_{13} = 0, \quad \frac{\partial \theta}{\partial x_3} = Q \quad \text{for } x_3 = d.\end{aligned}\tag{48.21}$$

This problem was studied by Sneddon and Lockett [376].

In the previous problem, it was shown that the solutions of the equations (47.9), (47.10) have the form (48.11), (48.12). Obviously, these solutions may also be written in the following way

$$\theta^* = E \operatorname{ch} \alpha x_3 + E' \operatorname{sh} \alpha x_3,\tag{48.22}$$

$$\begin{aligned}
 u_3^* &= (A_\beta + P' \alpha_\beta x_3) \operatorname{ch} \alpha x_3 + (A'_\beta + P \alpha_\beta x_3) \operatorname{sh} \alpha x_3, \\
 u_3^* &= (A_\beta + i \alpha P' x_3) \operatorname{sh} \alpha x_3 + (A'_\beta + i \alpha P x_3) \operatorname{ch} \alpha x_3,
 \end{aligned}
 \tag{48.23}$$

where A_1, A'_1, E, E' are integration constants and P and P' are defined by

$$\begin{aligned}
 P &= \frac{(a^2 - 1)}{(a^2 + 1) \alpha} \left(A_\beta \alpha_\beta + i z A_3 - \frac{i b E}{a^2 - 1} \right), \\
 P' &= \frac{(a^2 - 1)}{(a^2 + 1) \alpha} \left(A'_\beta \alpha_\beta + i z A'_3 - \frac{i b E'}{a^2 - 1} \right).
 \end{aligned}
 \tag{48.24}$$

For the components t_{13} of the tension tensor, the following expressions are found

$$\begin{aligned}
 \frac{1}{\mu} t_{\beta 3}^* &= (\alpha A_\beta - i \alpha_\beta A_3 + P \alpha_\beta) \operatorname{sh} \alpha x_3 + 2 P \alpha_\beta x_3 \operatorname{ch} \alpha x_3 + \\
 &+ (\alpha A'_\beta - i \alpha_\beta A'_3 + P' \alpha_\beta) \operatorname{ch} \alpha x_3 + 2 P' \alpha_\beta x_3 \operatorname{sh} \alpha x_3, \\
 \frac{1}{\mu} t_{33}^* &= -i(a^2 - 2) (A_\beta \alpha_\beta \operatorname{ch} \alpha x_3 + P \alpha^2 x_3 \operatorname{sh} \alpha x_3) - \\
 &- b E \operatorname{ch} \alpha x_3 + a^2 \alpha (A_3 \operatorname{ch} \alpha x_3 + i P' \operatorname{ch} \alpha x_3 + \\
 &+ i \alpha x_3 P \operatorname{sh} \alpha x_3) - i(a^2 - 2) (A'_\beta \alpha_\beta \operatorname{sh} \alpha x_3 + \\
 &+ P' \alpha^2 x_3 \operatorname{ch} \alpha x_3) - b E' \operatorname{sh} \alpha x_3 + \\
 &+ a^2 \alpha (A'_3 \operatorname{sh} \alpha x_3 + i P' \operatorname{sh} \alpha x_3 + i \alpha x_3 P' \operatorname{ch} \alpha x_3).
 \end{aligned}
 \tag{48.25}$$

The thermal conditions imposed at the boundary imply

/185

$$E = \tilde{0}^*, \quad E' = \frac{Q^*}{\alpha C} - \frac{\tilde{0}^* S}{C}, \quad (48.26)$$

where $S = \text{sh } \alpha d$, $C = \text{ch } \alpha d$.

The tension conditions imply

$$\begin{aligned} -i(a^2 - 2) A_\beta \alpha_\beta - bE + a^2 A_3 \alpha + a^2 i \alpha P &= 0, \\ \alpha A'_\beta - i \alpha_\beta A'_3 + P' \alpha_\beta &= 0, \\ (\alpha A_\beta - i \alpha_\beta A_3 + P \alpha_\beta) S + 2P \alpha \alpha_\beta dC + 2P' \alpha \alpha_\beta dS &= 0, \\ (A_3 + iP' \alpha d) S + (A'_3 + iP \alpha d) C &= 0. \end{aligned} \quad (48.27)$$

If we designate

$$D = -\frac{b\tilde{0}^*}{2ix^2(a^2-1)} - \frac{bdSC^{-1}Q^*}{2ix^2(a^2-1)(S+\alpha dC^{-1})},$$

then we have

$$\begin{aligned} A_\beta &= \alpha_\beta D, \quad A_3 = \frac{1}{2a^2\alpha} (bE - 2ix^2 D), \\ A'_\beta &= \frac{1}{2a^2\alpha^2} (ibE' - 2x^2 DSC^{-1}) \alpha_\beta, \quad A'_3 = ix DSC, \\ P &= \frac{1}{2a^2x} [2x^2 D(a^2-1) - ibE], \\ P' &= -\frac{1}{2a^2x} [2x^2 ASC^{-1}(a^2-1) + ibE']. \end{aligned} \quad (48.28)$$

In the case when the foundation consists of a material which does not conduct heat, the elastic layer is thermally insulated from the foundation and we have $Q = 0$, and therefore

$$D = \frac{ib\tilde{\theta}^*}{2\alpha^2(\alpha^2 - 1)}.$$

We obtain $P = P' = 0$ and therefore

$$A'_2 = -\frac{ib\tilde{\theta}^* SC^{-1}}{2\alpha^2(\alpha^2 - 1)} \alpha_2, \quad A_3 = \frac{b\tilde{\theta}^*}{2\alpha(\alpha^2 - 1)}, \quad (48.29)$$

$$A'_3 = -\frac{b\tilde{\theta}^* SC^{-1}}{2\alpha(\alpha^2 - 1)}.$$

If we consider the above relations, it follows that in this case /186
the Fourier transforms of the temperature and of the displacement components are expressed as

$$\begin{aligned} \theta^* &= \tilde{\theta}^* (\operatorname{ch} \alpha x_3 - SC^{-1} \operatorname{sh} \alpha x_3), \\ u_2^* &= \frac{ib\tilde{\theta}^* \alpha_2}{2\alpha^2(\alpha^2 - 1)} (\operatorname{ch} \alpha x_3 - SC^{-1} \operatorname{sh} \alpha x_3), \\ u_3^* &= \frac{b\tilde{\theta}^*}{2\alpha(\alpha^2 - 1)} (\operatorname{sh} \alpha x_3 - SC^{-1} \operatorname{ch} \alpha x_3). \end{aligned} \quad (48.30)$$

In general the inversion of the transforms θ^* , u_1^* implies serious difficulties. For some particular cases, the functions were originally determined, and the results were interpreted (Sneddon and Lockett [375], [376]). Other problems referring to the elastic layer are presented in [315].

c) Elastic space with a circular crack. At the start, let us mention the fact that if we take

$$\begin{aligned} u_x &= \Phi_{,x} + (\alpha^2 - 1)x_3 \Phi_{,x3} + x_3 \psi_{,x}, \\ u_3 &= -\alpha^2 \Phi_{,3} + (\alpha^2 - 1)x_3 \Phi_{,33} + x_3 \psi_{,3} - \psi, \\ \theta &= \frac{2}{b} \psi_{,3}, \end{aligned} \quad (48.31)$$

where $\Phi(x_1, x_2, x_3)$ and $\psi(x_1, x_2, x_3)$ are harmonic functions

$$\Delta\Phi = 0, \quad \Delta\psi = 0, \quad (48.32)$$

then the functions u_1, θ satisfy the thermoelastic equilibrium equations in the absence of mass forces and heat source. This can easily be derived by substituting (48.31) in the equations (48.6) and keeping in mind (48.32). A representation (48.31) is due to Sneddon [377] and is useful for studying some problems referring to the half space. We obtain from (48.31) and (48.5)

$$\begin{aligned} t_{\alpha 3} &= 2\mu [(a^2 - 1)x_3 \Phi_{,\alpha 33} + x_3 \psi_{,\alpha 3}], \\ t_{33} &= 2\mu [(a^2 - 1)x_3 \Phi_{,333} - (a^2 - 1)\Phi_{,33} + x_3 \psi_{,33} - \psi_{,3}], \end{aligned} \quad (48.33)$$

whence it follows that for $x_3 = 0$, we have $t_{\alpha 3} = 0$.

Let us consider the problem of determining the thermoelastic equilibrium in the case when, in an elastic medium which fully occupies the space, there is a crack (Olesiak and Sneddon [326]).

We assume that the crack is circular, and that it is located in the plane $x_1 O x_2$ and has its center at the origin. If the radius of the crack is equal to unity, and we use cylindrical coordinates (r, ϕ, z) , then the crack is described by the disk

$$r \leq 1, \quad z = 0. \quad (48.34)$$

Let us consider the case when the crack is free of tensions. We assume that the temperature, as a function of r , is prescribed on the surface of the crack. In this case we have the boundary conditions

/187

$$\sigma_{rr} = \sigma_{\theta\theta} = 0, \quad z = 0, \quad (48.35)$$

$$\sigma_r = 0, \quad z = 0, \quad 0 \leq r \leq 1, \quad (48.36)$$

$$u_r = 0, \quad z = 0, \quad r > 1, \quad (48.37)$$

$$\theta = g(r), \quad z = 0, \quad 0 \leq r \leq 1, \quad (48.38)$$

$$\theta_{,z} = 0, \quad z = 0, \quad r > 1, \quad (48.39)$$

by means of which we are able to determine the displacement and the temperature of the half space $z \geq 0$ and therefore, by symmetry, the space in its entirety. Obviously, we also have the condition that the displacement, temperature, tension and the flux should tend towards zero at infinity.

In order to resolve this problem, we will use the representation (48.31). With that, the conditions (48.35) will be similarly satisfied. The problem under consideration is a problem with axial symmetry. Keeping in mind (48.31), (48.32) and the conditions (48.38), (48.39), it follows that in order to find the temperature the function $\psi(r, z)$ must be determined, satisfying the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \psi = 0, \quad (48.40)$$

and the conditions

$$\left[\frac{\partial \psi}{\partial z} \right]_{z=0} = \frac{b}{2} g(r), \quad 0 \leq r \leq 1, \quad (48.41)$$

$$\left[\frac{\partial^2 \psi}{\partial z^2} \right]_{z=0} = 0, \quad r > 1.$$

Similarly, it follows from (48.31), (48.32), (48.33), (48.36) and (48.37) that the function $\Phi(r, z)$ must satisfy the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \Phi = 0, \quad (48.42)$$

and the conditions

$$\left[(a^2 - 1) \frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial \Phi}{\partial z} \right]_{z=0} = 0, \quad 0 \leq r \leq 1, \quad (48.43)$$

$$\left[a^2 \frac{\partial \Phi}{\partial z} + \Phi \right]_{z=0} = 0, \quad r > 1.$$

We will designate by \hat{f} the zero order Hankel transform of the function $f(r, z)$. We have (Sneddon [371])

/188

$$\hat{f}(\xi, z) = \int_0^\infty r f(r, z) J_0(\xi r) dr, \quad (48.44)$$

$$f(r, z) = \int_0^\infty \xi \hat{f}(\xi, z) J_0(\xi r) d\xi.$$

Applying the zero order Hankel transform, the following equation follows from (48.40)

$$\left(\frac{d^2}{dz^2} - \xi^2 \right) \hat{\Phi} = 0. \quad (48.45)$$

This has the solution

$$\hat{\Phi} = A(\xi) e^{-\xi z}, \quad (48.46)$$

where A is an unknown function. If we designate

$$q(\xi) = \frac{2}{a^2} \xi A(\xi), \quad (48.47)$$

then the function ψ may be written in the form

$$\psi = \frac{a^2}{2} \int_0^\infty q(\xi) e^{-\xi r} J_0(\xi r) d\xi, \quad (48.48)$$

and we conclude from (48.31)

$$0 = -\frac{a^2}{b} \int_0^\infty \xi q(\xi) e^{-\xi r} J_0(\xi r) d\xi. \quad (48.49)$$

Similarly, it follows from (48.42) that

$$\hat{\Phi} = B(\xi) e^{-\xi r}. \quad (48.50)$$

Setting

$$f(\xi) = (a^2 - 1) [2\xi^2 B(\xi) - q(\xi)], \quad (48.51)$$

we obtain

$$\Phi = \frac{1}{2} \int_0^\infty \frac{1}{\xi} \left[\frac{1}{a^2 - 1} f(\xi) + q(\xi) \right] e^{-\xi r} J_0(\xi r) d\xi. \quad (48.52)$$

It follows from (48.31), (48.33), (48.48) and (48.51) that

$$u_z(r, 0) = \frac{a^2}{2(a^2 - 1)} \int_0^\infty f(\xi) J_0(\xi r) d\xi, \quad (48.53)$$

$$\sigma_{zz}(r, 0) = \mu \int_0^\infty \xi [q(\xi) - f(\xi)] J_0(\xi r) d\xi.$$

The conditions (48.41) lead to the following integral equations /189 for the function q

$$\int_0^{\infty} \xi q(\xi) J_0(\xi r) d\xi = -\frac{b}{a^2} g(r), \quad 0 \leq r \leq 1, \quad (48.54)$$

$$\int_0^{\infty} \xi^2 q(\xi) J_0(\xi r) d\xi = 0, \quad r > 1. \quad (48.55)$$

In view of (48.53), the conditions (48.43) imply the equations

$$\int_0^{\infty} \xi [q(\xi) - f(\xi)] J_0(\xi r) d\xi = 0, \quad 0 \leq r \leq 1, \quad (48.56)$$

$$\int_0^{\infty} f(\xi) J_0(\xi r) d\xi = 0, \quad r > 1. \quad (48.57)$$

If we keep in mind (48.54), then the equation (48.56) is replaced by

$$\int_0^{\infty} \xi f(\xi) J_0(\xi r) d\xi = -\frac{b}{a^2} g(r), \quad 0 \leq r \leq 1. \quad (48.58)$$

The solution of the dual integral equations (48.58) and (48.57) is (Sneddon [371])

$$f(\xi) = -\frac{2b}{\pi a^2} \int_0^1 \sin \xi t dt \int_0^1 \frac{r g(r) dr}{\sqrt{t^2 - r^2}}. \quad (48.59)$$

Similarly the solution of the dual integral equations (48.54), (48.55) is given by

$$\begin{aligned} \xi q(\xi) = & -\frac{2b}{\pi a^2} \cos \xi \int_0^1 \frac{q(y) dy}{\sqrt{1-y^2}} - \\ & -\frac{2b\xi}{\pi a^2} \int_0^1 \frac{y dy}{\sqrt{1-y^2}} \int_0^1 x g(xy) \sin(\xi x) dx. \end{aligned} \quad (48.60)$$

If the faces of the crack are kept at a constant temperature, $g(r) = -T^*$, then the following is obtained

$$q(\xi) = \frac{2bT^*}{\pi a^2} \frac{\sin \xi}{\xi^2}, \quad f(\xi) = \frac{2bT^*}{\pi a^2 \xi} (\sin \xi - \xi \cos \xi).$$

In this case we have

$$\sigma_{xx}(r, 0) = \frac{2bT^*}{\pi a^2} (r^2 - 1)^{-1/2}, \quad r > 1,$$

$$u_r(r, 0) = \frac{bT^*}{\pi (a^2 - 1)} (1 - r^2)^{1/2}, \quad 0 \leq r \leq 1.$$

A similar study can be made also in the case when the thermal flux is prescribed on the faces of the crack (Olesiak and Sneddon [326], Sneddon [378]). Other problems referring to media with cracks were studied by Sneddon [374], Das [78] and others.

/190

d) Boussinesq's problem for a heated die. Let us consider the problem of the thermoelastic equilibrium of a semi-infinite medium which is in contact with a rigid heated die (Sneddon [379]). The presence of the die produces a temperature variation in the medium under consideration. We assume that the boundary of the die is a revolution surface and that the axis of the die is Ox_3 . Similarly, we assume that the nondeformed boundary of the medium is the plane $x_3 = 0$ and that the tip of the die starts to enter into contact with the original medium. In view of the fact that the die is pressed normally at the plane $x_3 = 0$, then, when the equilibrium is established, it will have, in contact with the plane $x_3 = 0$, a circle with a radius a . We will consider $a = 1$.

We will assume that the portion $0 \leq r \leq 1$, $z = 0$ is the prescribed displacement u_z and on $r > 1$, $z = 0$, the normal tension is zero. We assume, similarly, that the surfaces in contact are smooth and that the temperature of the die is a function of r and z . These assumptions

imply the following conditions

$$\begin{aligned} \sigma_{rz}(r, 0) &= 0, \quad r > 0, \quad \sigma_{rz}(r, 0) = 0, \quad r > 1, \\ u_r(r, 0) &= g(r), \quad 0 \leq r \leq 1, \end{aligned} \quad (48.61)$$

We will consider the thermal conditions of the form

$$\theta(r, 0) = \theta_0(r) H(1-r). \quad (48.62)$$

In order to study the problem we use the representation (48.31). On this occasion we introduce the functions $q(\xi)$ and $f(\xi)$ by the relations

$$q(\xi) = -\frac{1}{a^2-1} \xi^2 A(\xi), \quad f(\xi) = a^2 [\xi^2 B(\xi) + q(\xi)]. \quad (48.63)$$

In this case the functions Φ and ψ have the form

$$\begin{aligned} \Phi &= \int_0^\infty \xi^{-2} [a^{-2} f(\xi) - q(\xi)] e^{-\xi r} J_0(\xi r) d\xi, \\ \psi &= -(a^2-1) \int_0^\infty \xi^{-1} q(\xi) e^{-\xi r} J_0(\xi r) d\xi, \end{aligned} \quad (48.64)$$

and therefore we have

$$\begin{aligned} u_r(r, 0) &= \int_0^\infty \xi^{-1} [f(\xi) - q(\xi)] J_0(\xi r) d\xi, \\ \sigma_{rz}(r, 0) &= -2h(1-a^2) \int_0^\infty f(\xi) J_0(\xi r) d\xi, \quad \sigma_{rz}(r, 0) = 0, \\ \theta(r, 0) &= 2h^{-1}(a^2-1) \int_0^\infty q(\xi) J_0(\xi r) d\xi. \end{aligned} \quad (48.65)$$

From (48.31), (48.46), (48.63) it follows that

/191

$$q(\xi) = \frac{b\xi}{2(a^2 - 1)} \hat{u}(\xi, 0) = \frac{b\xi}{2(a^2 - 1)} \int_0^\infty r \theta(r, 0) J_0(\xi r) dr. \quad (48.66)$$

If we designate

$$D(r) = \int_0^\infty \xi^{-1} q(\xi) J_0(\xi r) d\xi, \quad (48.67)$$

then from (48.65) and (48.61) the dual integral equations follow for the function $f(\xi)$

$$\int_0^\infty \xi^{-1} f(\xi) J_0(\xi r) d\xi = D(r) + g(r), \quad 0 \leq r \leq 1, \quad (48.68)$$

$$\int_0^\infty f(\xi) J_0(\xi r) d\xi = 0, \quad r > 1.$$

We will write a solution of these equations in the form $f = f_1 + f_2$, where f_1 and f_2 satisfy the equations

$$\int_0^\infty \xi^{-1} f_1(\xi) J_0(\xi r) d\xi = D(r), \quad 0 \leq r \leq 1, \quad (48.69)$$

$$\int_0^\infty f_1(\xi) J_0(\xi r) d\xi = 0, \quad r > 1,$$

$$\int_0^\infty \xi^{-1} f_2(\xi) J_0(\xi r) d\xi = g(r), \quad 0 \leq r \leq 1, \quad (48.70)$$

$$\int_0^\infty f_2(\xi) J_0(\xi r) d\xi = 0, \quad r > 1.$$

The solution of the equations (48.70) does not depend on the thermal field and it can be written as in (48.59).

If we keep in mind (48.52) it follows that

$$\hat{\theta}(\xi, 0) = \int_0^\infty r \theta_0(r) J_0(\xi r) dr.$$

In this case we derive

$$f_1(\xi) = \frac{h\xi}{2(a^2 - 1)} \hat{\theta}(\xi, 0).$$

Solutions may be obtained also for other thermal conditions on the plane $z = 0$ (Sneddon [374]).

49. Plane Thermoelastic Deformation

/192

In this paragraph we will assume that B is a straight cylinder with a length l , the bases of which are perpendicular with respect to the generators. We select a reference in such a way that the axes Ox_3 are parallel with the generators of the cylinder. Let Σ be the domain of the transversal section and L its boundary.

a) Statement of the problem. Let us assume that the domain B is occupied by an isotropic and homogeneous medium.

Let us define the state of plane thermoelastic deformation parallel with the plane x_1Ox_2 as being that state of deformation in which

$$u_\alpha = u_\alpha(x_1, x_2), \quad \theta = \theta(x_1, x_2), \quad u_3 = 0. \quad (49.1)$$

Keeping in mind (49.1) we conclude from (47.5)

$$2\varepsilon_{\alpha\beta} = u_{\alpha,\beta} + u_{\beta,\alpha}, \quad \varepsilon_{\alpha 3} = 0, \quad \varepsilon_{33} = 0, \quad (49.2)$$

and therefore

$$t_{\alpha\gamma} = \lambda \varepsilon_{\alpha\gamma} + 2\mu \varepsilon_{\alpha\gamma} - \beta \theta \delta_{\alpha\gamma}, \quad (49.3)$$

$$t_{33} = \lambda \varepsilon_{\gamma\gamma} - \beta \theta, \quad t_{\alpha 3} = 0. \quad (49.4)$$

The previous relations indicate that in this case $t_{ij} = t_{ij}(x_1, x_2)$.

It follows from the third equilibrium equation that the state of plane deformation parallel with the plane $x_1 O x_2$ is possible only if $f_3 = 0$. The other equilibrium equations are reduced to

$$t_{\alpha\beta,\beta} + f_\alpha = 0, \quad (49.5)$$

whence it follows that the state of plane deformation under consideration demands that the functions f_α do not depend on x_3 .

The tension on the lateral surface of the cylinder is given by

$$t_i = t_{i\alpha} n_\alpha,$$

and therefore, on the basis of relations (49.4) it follows that $t_3 = 0$, and t_α does not depend on x_3 . Therefore, if a tension is prescribed on the lateral surface, then, in order for the state of plane deformation under consideration to be possible, this tension must be parallel with the plane $x_1 O x_2$ and it should not depend on x_3 .

The surface force which is applied along a portion of the cylinder S , contained between two right sections at unit distance one phase from the other, and between two generators which project in points P_1, P_2 from the plane $x_1 O x_2$ is given by the relations

$$R_i = \int_S t_i da = \int_{P_1, P_2} t_i ds.$$

In our case we have

$$R_\alpha = \int_{P_1, P_2} t_{\alpha\beta} n_\beta ds, \quad R_3 = 0.$$

We will consider $t_{\alpha\beta} n_\beta$ as a tension on an arc element. We obtain the expressions for the components of the moment resulting from the tension

$$M_\alpha = t_{\alpha\beta} R_\beta x_3, \quad M_3 = \int_{P_1 P_2} (x_1 t_{30} - x_2 t_{10}) n_0 ds.$$

Equation (47.2) is reduced to

$$\theta_{,\alpha\alpha} = -\frac{1}{k} r, \quad (49.6)$$

which indicates that the plane deformation under consideration requires that r should not depend on x_3 . Proceeding as before, it follows that the expression $k\theta_{,\alpha} n_\alpha$ has the interpretation of a flux per unit of arc.

/193

Consequently, in order to determine the thermal field, the equation (49.6) must be integrated in the domain Σ , with certain boundary conditions L . These conditions may take the form

$$\theta = \tilde{\theta} \text{ on } L_1, \quad k\theta_{,\alpha} n_\alpha = \tilde{q} \text{ on } L_2, \quad (49.7)$$

where $L_1 \cup L_2 = L$, $L_1 \cap L_2 = \emptyset$ and $\tilde{\theta}$ and \tilde{q} are prescribed functions.

After the determination of the functions θ the system of equations

$$t_{\alpha\beta,\beta} + f_\alpha = 0, \quad (49.8)$$

$$t_{\alpha\gamma} = \lambda \varepsilon_{\gamma\gamma} \delta_{\alpha\gamma} + 2\mu \varepsilon_{\alpha\gamma} - \beta \theta \delta_{\alpha\gamma}, \quad (49.9)$$

$$2\varepsilon_{\alpha\beta} = u_{\alpha,\beta} + u_{\beta,\alpha}, \quad (49.10)$$

must be integrated in the domain Σ with certain boundary conditions. The equations (49.6), (49.8) -- (49.10) are plane thermoelastic deformation equations. In the case when the tensions are given

at the boundary these conditions have the form

$$t_{\alpha\beta} n_\beta = \tilde{t}_\alpha(s), \quad s \in L, \quad (49.11)$$

where \tilde{t}_α are prescribed functions, n_α are components of the external versor normal to L .

The conditions necessary and sufficient for the existence of the boundary problem (49.8) -- (49.11) are (Mushelisvili [296], Fichera [122])

$$\int_\Sigma f_\alpha da + \int_L \tilde{t}_\alpha ds = 0, \quad \int_\Sigma e_{\alpha\beta\gamma} x_\alpha f_\beta da + \int_L e_{\alpha\beta\gamma} x_\alpha \tilde{t}_\beta ds = 0. \quad (49.12)$$

If the functions u_α and θ have been determined, we obtain from (49.4) the expression of the component t_{33} and thus are able to calculate the tensions which act on the bases of the cylinder in order to maintain the state of deformation under consideration.

Assuming that the function θ is known, we can express the mixed problem by means of the displacement vector. Thus, it follows from (49.8) -- (49.10) that the functions u_i must satisfy the equations

$$\mu u_{\alpha,\rho\rho} + (\lambda + \mu) u_{\rho,\rho\alpha} + f_\alpha - \beta \theta_{,\alpha} = 0, \quad (49.13)$$

and the boundary conditions

$$u_\alpha = \tilde{u}_\alpha \text{ on } L', \quad [\lambda u_{\rho,\rho} \delta_{\alpha\beta} + \mu(u_{\alpha,\beta} + u_{\beta,\alpha})] n_\beta = \tilde{t}_\alpha + \beta \theta n_\alpha \text{ on } L'', \quad (49.14)$$

where $L' \cup L'' = L$, $L' \cap L'' = O$.

The plane deformation problem may be studied in general by methods known from the theory of elasticity (Muskhelishvili [296], Melan and Parkus [286], Boley and Weiner [23], Nowacki [315], Grindei [152], Kovalenko [242], Green and Zerna [139], Kupradze [246], Teodorescu [409], Fichera [112]).

As we have shown in the problems of thermoelasticity, it is assumed that the mechanical loads are zero. Strictly speaking, those problems are considered, rightly, problems of plane thermoelasticity; on the one hand, those whose solution is based essentially on the equation which is satisfied by the function θ , or on the special character of the loading system, and, on the other hand, concrete problems in which the function θ and then the corresponding displacements are effectively determined. /194

b) Reduction of the problem to the isothermal case. Let us consider first a case in which there are no mass forces and thermal sources. Consequently the function θ satisfies the equation

$$\theta_{,aa} = 0. \quad (49.15)$$

Let us designate by $F(z)$ the analytical function of the complex variable $z = x_1 + ix_2$, which has the function θ as real part

$$F(z) = \theta + i\psi. \quad (49.16)$$

Let us introduce the function u_α^* by the relation

$$u_1^* + iu_2^* = \int F(z) dz. \quad (49.17)$$

It is obvious that we have

$$u_{1,1}^* = u_{2,2}^* = 0, \quad u_{1,2}^* = -u_{2,1}^*. \quad (49.18)$$

If we consider the unknown functions u'_α such that

$$u_\alpha = u'_\alpha + \frac{\beta}{2(\lambda + \mu)} u_\alpha^*, \quad (49.19)$$

then we obtain from (49.2) and (49.18)

$$\varepsilon_{\alpha\beta} = \varepsilon'_{\alpha\beta} + \frac{\beta\theta}{2(\lambda + \mu)} \delta_{\alpha\beta}, \quad (49.20)$$

where

$$2\varepsilon'_{\alpha\beta} = u'_{\alpha,\beta} + u'_{\beta,\alpha}. \quad (49.21)$$

In view of (49.20) and (49.3) it follows that

$$t_{\alpha\beta} = \lambda \varepsilon'_{\gamma\gamma} \delta_{\alpha\beta} + 2\mu \varepsilon'_{\alpha\beta}. \quad (49.22)$$

Consequently the problem of thermoelasticity was reduced to a problem of elasticity in the displacements u'_α , the deformations $\varepsilon'_{\alpha\beta}$ and the tensions $t_{\alpha\beta}$ referring to the same domain. It should be noted that the tensions $t_{\alpha\beta}$ from the new problem, called the auxiliary problem, coincide with those from the initial problem.

Let us assume that the domain Σ is simply connected. If zero mechanical loads are prescribed at the boundary, then the solution of the auxiliary problem is

$$t'_{\alpha\beta} = 0, \quad u'_\alpha = 0.$$

Thus, in a simple connected medium which is in a plane thermo-elastic deformation state, in the absence of heat sources and mechanical loads, no $t_{\alpha\beta}$ tensions appear and the displacements are given by

$$u_\alpha = \frac{\beta}{2(\lambda + \mu)} u_\alpha^0. \quad (49.23)$$

The tension t_{33} is expressed as

/195

$$t_{33} = - \frac{\beta\mu\theta}{\lambda + \mu}. \quad (49.24)$$

Obviously, in this case the deformation of the medium is due to the temperature variation caused by the thermal loads at the boundary.

Let us consider the case of a multiple connected domain. We assume that the boundary L of the domain Σ consists of the external outline L_0 and of n internal outlines L_r ($r = 1, 2, \dots, n$) which do not have common points. The function $F(z)$ whose real part is the uniform function θ , can have many forms

$$F(z) = \sum_{k=1}^n B_k \log(z - z_k) + F_0(z), \quad (49.25)$$

where z_k are points situated inside the outlines L_k , B_k are real constants and $F_0(z)$ is a holomorphic function in the domain under consideration.

In this case, we derive the relation

$$u_1^0 + iu_2^0 = z \sum_{k=1}^n B_k \log(z - z_k) + \sum_{k=1}^n (a_k + ib_k) \log(z - z_k) + f_0(z), \quad (49.26)$$

in which a_k, b_k are real constants, and $f_0(z)$ is a holomorphic function in Σ .

We obtain from (49.26)

$$[u_1^* + iu_2^*]_{L_k} = 2\pi i(zB_k + a_k + ib_k). \quad (49.27)$$

We designate by $[f]_{L_k}$ the variation of f when it follows a positive direction, an outline which contains the outline L_k .

In view of the fact that the functions u_α are assumed to be uniform, it follows that the functions u'_α have many forms.

$$\begin{aligned} [u'_1]_{L_k} &= -\frac{\pi\beta}{\lambda + \mu} (B_k x_2 + b_k), \\ [u'_2]_{L_k} &= -\frac{\pi\beta}{\lambda + \mu} (B_k x_1 + a_k). \end{aligned} \quad (49.28)$$

In this case in the auxiliary elasticity problem the displacements u'_α must have the multiform nature given by (49.28). It is said that the auxiliary problem is a problem with dislocations. The quantities

$$\epsilon_k = -\frac{\pi\beta}{\lambda + \mu} B_k, \quad \alpha_k = -\frac{\pi\beta b_k}{\lambda + \mu}, \quad \beta_k = \frac{\pi\beta a_k}{\lambda + \mu}, \quad (49.29)$$

are called the characteristics of the dislocation. The results presented are due to Mushelisvili [296].

Let us consider now the case when only the mechanical loads are zero. In order to reduce the problem of thermoelasticity to a problem of elasticity, let us look for the solution in the form

$$u_\alpha = u'_\alpha + \Phi_\alpha, \quad (49.30)$$

where the function $\Phi(x_1, x_2)$ satisfies in the domain Σ the equation

$$\Phi_{,aa} = \frac{\beta}{\lambda + 2\mu} \theta. \quad (49.31)$$

If we replace (49.30) in the equations (49.13) (in which $f_1 = 0$) and we keep in mind (49.31), it follows that the functions u'_α satisfy the equations

$$\mu u'_{\alpha,\beta\beta} + (\lambda + \mu) u'_{\beta,\alpha\alpha} = 0. \quad (49.32)$$

We obtain from (49.9), (49.4) the expressions for the tensions

$$\begin{aligned} t_{\alpha\beta} &= t'_{\alpha\beta} + 2\mu(\Phi_{,\alpha\beta} - \delta_{\alpha\beta} \Phi_{,\gamma\gamma}), \\ t_{33} &= \lambda \varepsilon'_{\beta\beta} - 2\mu \Phi_{,aa}, \end{aligned} \quad (49.33)$$

where

$$\begin{aligned} t'_{\alpha\beta} &= \lambda \varepsilon'_{\beta\beta} \delta_{\alpha\beta} + 2\mu \varepsilon'_{\alpha\beta}, \\ 2\varepsilon'_{\alpha\beta} &= u'_{\alpha,\beta} + u'_{\beta,\alpha}. \end{aligned} \quad (49.34)$$

It is obvious that the equations (49.32) are equivalent with

$$t'_{\alpha\beta,\beta} = 0. \quad (49.35)$$

If we have the boundary conditions

$$t_{\alpha\beta} n_\beta = 0,$$

then we derive the following conditions for the functions u'_α

$$t'_\alpha = t'_{\alpha\beta} n_\beta = -2\mu (\Phi_{,\alpha\beta} n_\beta - \Phi_{,rr} n_\alpha) \text{ on } L. \quad (49.36)$$

Consequently, the problem of thermoelasticity is reduced to the determination of a function Φ which satisfies equation (49.31) and to the integration of the problem of plane elastic deformation (49.34), (49.35), (49.36).

As is known from the theory of elasticity (see for example, Muskhelishvili [296], Sokolnikoff [382], Gurtin [163]) the solution of the problem of plane elasticity may be determined with the help of the Airy function. In this case we have

$$t'_{\alpha\beta} = \delta_{\alpha\beta} A_{,rr} - A_{,\alpha\beta}, \quad (49.37)$$

where the function $A(x_1, x_2)$ is biharmonic

$$\Delta \Delta A = 0. \quad (49.38)$$

It follows from (49.37) that

/197

$$t'_1 = \frac{d}{ds} (A_{,2}), \quad t'_2 = -\frac{d}{ds} (A_{,1}). \quad (49.39)$$

With this, the conditions (49.36) become

$$\frac{d}{ds} (A_{,2}) = 2\mu \frac{d}{ds} (\Phi_{,2}), \quad -\frac{d}{ds} (A_{,1}) = 2\mu \frac{d}{ds} (\Phi_{,1}) \text{ on } L. \quad (49.40)$$

The plane deformations may also be studied with the help of complex variables. On the basis of the theorem of Goursat (for example, Muskhelishvili [296], Gurtin [163]) the function A may be expressed (see also Paragraph 55) by two analytical functions of complex variable $z = x_1 + ix_2$. Thus

$$2A = \bar{z} \psi(z) + z \bar{\psi}(z) + \chi(z) + \bar{\chi}(z). \quad (49.41)$$

C-4

The displacements u'_α and the tensions $t'_{\alpha\beta}$ are represented in the form

$$\begin{aligned} 2\mu (u'_1 + iu'_2) &= \frac{\lambda + 3\mu}{\lambda + \mu} \psi(z) - z\overline{\psi'(z)} - \overline{\chi'(z)}, \\ t'_{11} + t'_{22} &= 4 \operatorname{Re} [\psi'(z)], \\ t'_{22} - t'_{11} + 2it'_{12} &= 2[\bar{z}\psi''(z) + \chi'(z)]. \end{aligned} \quad (49.42)$$

Similarly, we have

$$t'_1 + it'_2 = -i \frac{d}{ds} (A_{,1} + iA_{,2}) = -i \frac{d}{ds} [\psi(z) + z\overline{\psi'(z)} + \overline{\chi'(z)}]. \quad (49.43)$$

In view of (49.30), (49.33) and (49.52) we obtain, for the solution of the problem of thermoelasticity, the relations

$$\begin{aligned} 2\mu(u_1 + iu_2) &= \frac{\lambda + 3\mu}{\lambda + \mu} \psi(z) - z\overline{\psi'(z)} - \overline{\chi'(z)} + 2\mu(\Phi_{,1} + i\Phi_{,2}), \\ t_{11} + t_{22} &= 4 \operatorname{Re} [\psi'(z)] - \frac{2\mu\beta}{\lambda + 2\mu} 0, \\ t_{22} - t_{11} + 2it_{12} &= 2[\bar{z}\psi''(z) + \chi'(z)] + 2\mu(\Phi_{,22} - \Phi_{,11} + 2i\Phi_{,12}). \end{aligned} \quad (49.44)$$

A detailed study of the problem of plane elastic deformation with the help of complex variables may be found in the monographs of Muskhelishvili [296] and Green and Zerna [139].

c) Thermal tensions in a tube. Let us consider the case when Σ is the domain contained between two concentric circles with radii R_1 and R_2 ($R_1 < R_2$) and a center at the origin. We will use polar coordinates (r, ϕ) . In this system the equations of thermoelasticity

are derived easily from (24.86) -- (24.90).

We assume that there are no mass forces and that the boundary of the domain is free of tensions. We will consider the case where the temperature θ is a function which depends only on r . Let $u_r = u(r)$, $u_\phi = 0$. It follows from (24.86) that

/198

$$\varepsilon_{rr} = \frac{du}{dr}, \quad \varepsilon_{\phi\phi} = -\frac{1}{r} u, \quad \varepsilon_{r\phi} = 0. \quad (49.45)$$

If we consider the relations (24.76), (24.80), (24.85), (24.86) then the equations (49.9) are substituted by

$$\begin{aligned} \sigma_{rr} &= \lambda (\varepsilon_{rr} + \varepsilon_{\phi\phi}) + 2\mu \varepsilon_{rr} - \beta\theta, \\ \sigma_{\phi\phi} &= \lambda (\varepsilon_{rr} + \varepsilon_{\phi\phi}) + 2\mu \varepsilon_{\phi\phi} - \beta\theta, \\ \sigma_{r\phi} &= 2\mu \varepsilon_{r\phi}. \end{aligned} \quad (49.46)$$

We obtain from (49.45) and (49.46)

$$\begin{aligned} \sigma_{rr} &= \frac{1}{r} \lambda \frac{d}{dr} (ru) + 2\mu \frac{du}{dr} - \beta\theta, \\ \sigma_{\phi\phi} &= \frac{1}{r} \lambda \frac{d}{dr} (ru) + \frac{2\mu}{r} u - \beta\theta, \\ \sigma_{r\phi} &= 0. \end{aligned} \quad (49.47)$$

The equilibrium equations are reduced to

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r} (\sigma_{rr} - \sigma_{\phi\phi}) = 0. \quad (49.48)$$

The boundary conditions may be written in the form

$$\sigma_{rr}(R_1) = 0, \quad \sigma_{rr}(R_2) = 0. \quad (49.49)$$

If we replace the tensions from (49.47) in the equation (49.48) the following equation may be derived for the function u

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (ru) \right] = \frac{\beta}{\lambda + 2\mu} \frac{d\theta}{dr}. \quad (49.50)$$

The solution of this equation is

$$u = A_1 r + \frac{1}{r} A_2 + \frac{\beta}{(\lambda + 2\mu)r} \int_{R_1}^r r \theta dr, \quad (49.51)$$

where A_α are integration constants. We obtain from (49.51) and (49.47)

$$\begin{aligned} \sigma_{rr} &= 2(\lambda + \mu) A_1 - \frac{2\mu}{r^2} A_2 - \frac{2\mu\beta}{(\lambda + 2\mu)r^2} \int_{R_1}^r r \theta dr, \\ \sigma_{\phi\phi} &= 2(\lambda + \mu) A_1 + \frac{2\mu}{r^2} A_2 - \frac{2\mu\beta}{\lambda + 2\mu} + \frac{2\mu\beta}{(\lambda + 2\mu)r^2} \int_{R_1}^r r \theta dr. \end{aligned} \quad (49.52)$$

Similarly, it follows from (49.4) and (49.51) that

/199

$$t_{33} = \sigma_{zz} = \lambda(\epsilon_{rr} + \epsilon_{\phi\phi}) - \beta\theta = 2\lambda A_1 - \frac{2\mu\beta}{\lambda + 2\mu} \theta. \quad (49.53)$$

If we impose the conditions (49.49) we derive the following values for the integration constants

$$\begin{aligned} (\lambda + \mu) A_1 &= \frac{\mu\beta}{(\lambda + 2\mu)(R_2^2 - R_1^2)} \int_{R_1}^{R_2} r \theta dr, \\ A_2 &= \frac{\beta R_1^2}{(\lambda + 2\mu)(R_2^2 - R_1^2)} \int_{R_1}^{R_2} r \theta dr. \end{aligned} \quad (49.54)$$

Replacing these values in (49.52) we obtain for σ_{rr} and $\sigma_{\phi\phi}$ the expressions

$$\begin{aligned}\sigma_{rr} &= \frac{2\mu\beta}{(\lambda + 2\mu)r^2} \left[\frac{r^2 - R_1^2}{R_2^2 - R_1^2} \int_{R_1}^{R_2} r \theta dr - \int_{R_1}^r r \theta dr \right], \\ \sigma_{\theta\theta} &= \frac{2\mu\beta}{(\lambda + 2\mu)r^2} \left[\frac{r^2 + R_1^2}{R_2^2 - R_1^2} \int_{R_1}^{R_2} r \theta dr + \int_{R_1}^r r \theta dr - r^2 \theta \right].\end{aligned}\quad (49.55)$$

The radial component of the displacement is given by

$$u = \frac{\beta}{(\lambda + 2\mu)(R_2^2 - R_1^2)} \left(\frac{\mu r}{\lambda + \mu} + \frac{R_1^2}{r} \right) \int_{R_1}^{R_2} r \theta dr + \frac{\beta}{(\lambda + 2\mu)r} \int_{R_1}^r r \theta dr. \quad (49.56)$$

If there are no heat sources and at the boundary of the domain Σ we have the conditions

$$\theta(R_1) = T_1, \quad \theta(R_2) = T_2, \quad (49.57)$$

where T_1 and T_2 are constant, then the solution of equation (49.15) which satisfies (49.57) is

$$\theta = C_1 \ln r + C_2, \quad (49.58)$$

in which we designated

$$C_1 = \frac{T_2 - T_1}{\ln R_2 - \ln R_1}, \quad C_2 = \frac{T_1 \ln R_2 - T_2 \ln R_1}{\ln R_2 - \ln R_1}. \quad (49.59)$$

In this case we can easily determine from (49.55) and (49.56) the solution of the problem of plane deformation. If $T_2 = 0$, then we obtain for the tensions the expressions

$$\begin{aligned}\sigma_{rr} &= \frac{\mu\beta T_1}{(\lambda + 2\mu) \ln R_2/R_1} \left[\ln \frac{r}{R_2} - \frac{R_1^2}{R_2^2 - R_1^2} \left(1 - \frac{R_2^2}{r^2} \right) \ln \frac{R_2}{R_1} \right], \\ \sigma_{\theta\theta} &= \frac{\mu\beta T_1}{(\lambda + 2\mu) \ln R_2/R_1} \left[1 - \ln \frac{R_2}{r} - \frac{R_1^2}{R_2^2 - R_1^2} \left(1 + \frac{R_2^2}{r^2} \right) \ln \frac{R_2}{R_1} \right].\end{aligned}\quad (49.60)$$

The method of Mushelisvili presented in Paragraph 52b may also /200
be used to study this problem. Thus, in the case of the temperature
distribution (49.58) the function $F(z)$ introduced in (49.16) is

$$F(z) = C_1 \log z + C_2. \quad (49.61)$$

We obtain from (49.17) and (49.61)

$$u_1^* + iu_2^* = z(C_1 \log z + C_2 - C_1). \quad (49.62)$$

The constants B_k, a_k, b_k from (49.26) are reduced to $B_1 = C_1$,
 $a_1 = b_1 = 0$. The solution of the auxiliary problem of elasticity
which corresponds to the characteristics of dislocation

$$\varepsilon_1 = -\frac{\pi\beta C_1}{\lambda + \mu}, \quad \alpha_1 = \beta_1 = 0,$$

was established by Mushelisvili [296]. This has the form

$$2\mu(u_r' + iu_\theta') = e^{-i\varphi} \left[\frac{\lambda + 3\mu}{\lambda + \mu} \psi(z) - z\bar{\psi}'(\bar{z}) - \gamma_1(z) \right],$$

$$\psi(z) = \int \Psi(z) dz, \quad \gamma_1(z) = \int \Lambda(z) dz, \quad (49.63)$$

$$\Psi(z) = \frac{\varepsilon_1 \mu (\lambda + \mu)}{2\pi (\lambda + 2\mu)} \left\{ \frac{1}{2} - \frac{R_2^2 \ln R_2 - R_1^2 \ln R_1}{R_2^2 - R_1^2} \right\} + \frac{\varepsilon_1 \mu (\lambda + \mu)}{2\pi (\lambda + 2\mu)} \log z,$$

$$\Lambda(z) = -\frac{\varepsilon_1 \mu (\lambda + \mu) R_1^2 R_2^2}{\pi z^2 (\lambda + 2\mu) (R_2^2 - R_1^2)} \ln \frac{R_2}{R_1},$$

$$\sigma_{rr}' = \frac{\varepsilon_1 \mu (\lambda + \mu)}{\pi (\lambda + 2\mu)} \left(\ln r + \frac{R_1^2 R_2^2}{r^2 (R_2^2 - R_1^2)} \ln \frac{R_2}{R_1} - \frac{R_2^2 \ln R_2 - R_1^2 \ln R_1}{R_2^2 - R_1^2} \right),$$

$$\sigma_{\theta\theta}' = \frac{\varepsilon_1 \mu (\lambda + \mu)}{\pi (\lambda + 2\mu)} \left(\ln r - \frac{R_1^2 R_2^2}{r^2 (R_2^2 - R_1^2)} \ln \frac{R_2}{R_1} - \frac{R_2^2 \ln R_2 - R_1^2 \ln R_1}{R_2^2 - R_1^2} + 1 \right),$$

$$\sigma_{r\theta}' = 0.$$

The displacements are determined from (49.19), (49.62) and (49.63).

Other problems involving the state of plane thermoelastic deformation will be found in the monographs written by Melan and Parkus [286], Boley and Weiner [23], Nowacki [315], Pors [35], Grindel [152], Teodorescu [409], where there is also a large bibliography of works in this area.

50. Generalized Plane Deformation

a) Statement of the problem. We assume that the cylinder under consideration in the previous paragraph consists of a nonhomogeneous and anisotropic medium for which the coefficients C_{ijkl} , β_{ij} , k_{ij} do not depend on the axial coordinate x_3 .

Let us define the state of generalized plane thermoelastic deformation parallel to the plane x_1Ox_2 of the medium under consideration as being the state of deformation in which we have

/201

$$u_i = u_i(x_1, x_2), \quad \theta = \theta(x_1, x_2). \quad (50.1)$$

These restrictions imply the case that $\varepsilon_{ij} = \varepsilon_{ij}(x_1, x_2)$, $t_{ij} = t_{ij}(x_1, x_2)$. Moreover we have

$$2\varepsilon_{\alpha\beta} = u_{\alpha,\beta} + u_{\beta,\alpha}, \quad 2\varepsilon_{\alpha 3} = 2\varepsilon_{3\alpha} = u_{3,\alpha}, \quad \varepsilon_{33} = 0, \quad (50.2)$$

$$t_{i\alpha} = C_{i\alpha k\beta} u_{k,\beta} - \beta_{i\alpha} \theta, \quad (50.3)$$

$$t_{33} = C_{33 k\beta} u_{k,\beta} - \beta_{33} \theta. \quad (50.4)$$

The equilibrium equations (40.1) take the form

$$t_{i\alpha,\alpha} + f_i = 0, \quad (50.5)$$

from which it results that the generalized plane thermoelastic deformation requires that the functions f_1 will not be dependent on x_3 .

Let us assume that the following tension is prescribed on the lateral surface of the cylinder

$$t_{i\alpha} n_\alpha = p_i. \quad (50.6)$$

Thus, the functions p_i must not be dependent on x_3 . The equation (40.6) is reduced to

$$(k_{\alpha\beta} \theta_{,\beta})_{,\alpha} = -r. \quad (50.7)$$

Obviously, the state of deformation under consideration requires that the thermal loads be independent of the axial coordinate.

If the thermal flux is prescribed at the lateral surface, then in order to determine the function θ , equation (50.7) must be integrated in the domain Σ , with the boundary conditions

$$k_{\alpha\beta} \theta_{,\beta} n_\alpha = \tilde{q} \text{ on } L. \quad (50.8)$$

After determining the function θ , the system of equations (50.3), (50.5) still must be integrated in the domain Σ with the boundary conditions (50.6) on L . This system of equations may be written in the following way

$$(C_{ijkl} u_{k,\beta})_{,\alpha} - (\rho_{i\alpha} \theta)_{,\alpha} + f_i = 0 \text{ in } \Sigma. \quad (50.9)$$

The conditions (50.6) take the form

$$(C_{ijkl} u_{k,\beta} - \rho_{i\alpha} \theta) n_\alpha = p_i \text{ on } L. \quad (50.10)$$

If we know the functions u_1 and θ , we obtain from (50.4) the tension t_{33} and can thus calculate the tensions which act on the bases.

The generalized plane deformation of homogeneous elastic media was considered in various papers (see, for example, Lekhnitskii [256]) assuming that $f_3 = p_3 = 0$. This restriction is unnecessary. /202
The equilibrium conditions of the cylinder under consideration may be written in the form

$$\int_{\Sigma} f_i da + \int_L p_i ds = 0, \quad \int_{\Sigma} e_{\alpha\beta\gamma} x_\alpha f_\beta da + \int_L e_{\alpha\beta\gamma} x_\alpha f_\beta ds = 0, \quad (50.11)$$

$$\int_{\Sigma} x_\alpha f_\beta da + \int_L x_\alpha p_\beta ds = \int_{\Sigma} t_{\alpha\beta} da. \quad (50.12)$$

The relations (50.12) are similarly satisfied on the basis of relations (50.5), (50.6) so that

$$\begin{aligned} \int_{\Sigma} t_{\alpha\beta} da &= \int_{\Sigma} [t_{\alpha\beta} + x_\alpha (t_{3\beta,3} + f_\beta)] da = \\ &= \int_{\Sigma} [(x_\alpha t_{3\beta})_{,3} + x_\alpha f_\beta] da = \int_L x_\alpha p_\beta ds + \int_{\Sigma} x_\alpha f_\beta da. \end{aligned}$$

It has been shown (Iesan [203]) that in certain conditions of regularity, the relations (50.11) represent necessary and sufficient conditions for the existence of the solution of the boundary problem (50.9), (50.10).

The problem of the generalized plane thermoelastic deformation for homogeneous and anisotropic media was studied by Manolachi [276] with the help of complex variable functions. The case of the connected multiple domains was analyzed by Iesan [191], using the method of the potential.

b) Reduction of the problem to the isothermal case. Let us consider a homogeneous and anisotropic medium which is in a generalized plane thermoelastic deformation state in the absence of thermal sources and mechanical loads.

In this case the equation (50.7) becomes

$$k_{\alpha\beta} \theta_{,\alpha\beta} = 0; \quad (50.13)$$

We assume that the boundary problem for the determination of the function θ has been resolved. In the following this function will be considered as being known.

Let λ_α be the roots of the equation

$$k_{\alpha\beta} \lambda^4 - (\alpha + \beta) = 0. \quad (50.14)$$

On the basis of the relation (23.7) we can write

$$\lambda_\alpha = a - (-1)^\alpha ib, \quad i = \sqrt{-1}. \quad (50.15)$$

If we make the transformation

$$y_1 = bx_1, \quad y_2 = x_2 + ax_1, \quad (50.16)$$

then the equation (50.13) will take the form

$$\frac{\partial^2 \theta}{\partial y_1^2} + \frac{\partial^2 \theta}{\partial y_2^2} = 0. \quad (50.17)$$

Let us introduce the complex variable $\zeta = y_1 + iy_2$. Let $F(\zeta)$ be the analytical function which has as its real part the function θ

$$F(\zeta) = \theta + i\psi. \quad (50.18)$$

Obviously, we have

203

$$\frac{\partial \theta}{\partial y_1} = \frac{\partial \psi}{\partial y_2}, \quad \frac{\partial \theta}{\partial y_2} = -\frac{\partial \psi}{\partial y_1}. \quad (50.19)$$

In the variables x_α these relations are written as

$$\psi_{,\alpha} = c_{\alpha\beta} \theta_{,\beta}, \quad (50.20)$$

where

$$c_{11} = \frac{a}{b}, \quad c_{12} = -\frac{a^2 + b^2}{b}, \quad c_{21} = \frac{1}{b}, \quad c_{22} = -\frac{a}{b}. \quad (50.21)$$

Let us consider the functions u_α^* defined by

$$u_1^* = iu_2^* = \int P(\zeta) d\zeta. \quad (50.22)$$

Thus we have

$$\frac{\partial u_1^*}{\partial y_1} = \frac{\partial u_2^*}{\partial y_2} = 0, \quad \frac{\partial u_1^*}{\partial y_2} = -\frac{\partial u_2^*}{\partial y_1} = -\psi. \quad (50.23)$$

In the variables x_α these relations become

$$u_{\alpha,\beta}^* = a_{\alpha\beta} \theta_{,\beta} + b_{\alpha\beta} \psi, \quad (50.24)$$

where

$$\begin{aligned} a_{11} &= b, & a_{12} &= 0, & a_{21} &= a, & a_{22} &= 1, \\ b_{11} &= -a, & b_{12} &= -1, & b_{21} &= b, & b_{22} &= 0. \end{aligned} \quad (50.25)$$

Let us introduce the functions u'_i by the relations

$$u_i = u'_i + p_{i\beta} u_\beta^*, \quad (50.26)$$

in which $p_{i\beta}$ are constants which must be determined.

If we designate

$$l'_{ia} = C_{iak0} u'_{k,p}, \quad (50.27)$$

then we obtain from (50.3) and (50.26)

$$l_{ia} = l'_{ia} + C_{iak0} p_{kv} u'_{v,p} - \beta_{ia} \theta. \quad (50.28)$$

In view of the relations (50.24) it follows from (50.28) that

$$l_{ia} = l'_{ia} + (C_{iak0} p_{kv} a_{v,p} - \beta_{ia}) \theta + C_{iak0} p_{kv} b_{vp} \psi. \quad (50.29)$$

We require that the functions $t'_{i\alpha}$ satisfy the equations

$$l'_{ia,a} = 0. \quad (50.30)$$

We obtain from the equilibrium equations (50.5), in which $f_1 = 0$, and relations (50.28), (50.30)

/204

$$(C_{iak0} p_{kv} a_{v,p} - \beta_{ia}) \theta_{,a} + C_{iak0} p_{kv} b_{vp} \psi_{,a} = 0. \quad (50.31)$$

Keeping in mind (50.20), the relations (50.31) become

$$[C_{iak0} p_k (a_{vp} \delta_{a\mu} + b_{vp} c_{a\mu}) - \beta_{ia}] \theta_{,\mu} = 0. \quad (50.32)$$

If we have

$$C_{iak0} (a_{vp} \delta_{a\mu} + b_{vp} c_{a\mu}) p_{kv} = \beta_{ia}, \quad (50.33)$$

then the equations (50.32) are satisfied. We will determine the constants p_{kv} so that (50.33) takes place. It can be shown that the system (50.33) determines, in a unique manner, the constants $p_{i\alpha}$.

In view of (50.29) we obtain from (50.6), in which $p_1 = 0$,

$$l'_{ia} n_a = - [(C_{iak0} p_{kv} a_{v,p} - \beta_{ia}) \theta + C_{iak0} p_{kv} b_{vp} \psi] n_a \text{ on } L. \quad (50.34)$$

In this way the problem of thermoelasticity was reduced to the

resolution of a problem of elasticity ($\theta = 0$) with a certain load at the outline. The elasticity problem consists in the integration of the equations (50.27), (50.30) with the boundary conditions (50.34).

If the domain Σ is multiply connected, then, as in the case of (49.25), the function $F(\zeta)$ has the form

$$F(\zeta) = \sum_{k=1}^n B_k \log(\zeta - \zeta_k) + F_0(\zeta), \quad (50.35)$$

where ζ_k are points situated inside the outlines L_k , B_k are real constants and $F_0(\zeta)$ is a holomorphic function. It follows from (50.22) and (50.35) that

$$u_1^* + iu_2^* = \zeta \sum_{k=1}^n B_k \log(\zeta - \zeta_k) + \sum_{k=1}^n (a_k + ib_k) \log(\zeta - \zeta_k) + f_0(\zeta), \quad (50.36)$$

in which a_k, b_k are real constants and $f_0(\zeta)$ is a holomorphic function. In this case we obtain

$$[u_j']_{L_k} = 2\pi[(p_{11}a - p_{12}b)B_k x_1 + p_{11}B_k x_2 + p_{11}b_k - p_{12}a_k]. \quad (50.37)$$

Consequently, for multiply connected domains in the auxiliary problem, the displacements u_j' must have the multiform nature given by (50.37).

The results presented here were derived by Iesan [196].

51. The Problem of Saint Venant. Let B be the inside of a right cylinder with a length l , the bases of which are perpendicular on the generators. We assume that the generic transversal section Σ is limited by the closed Liapunov curve, L . We will select the reference Ox_1 so that the axis Ox_3 is parallel with the generators of the cylinder, one of the bases is in the plane $x_3 = 0$, and the other one in the plane $x_3 = l$.

We assume that B is occupied by an elastic medium which is in equilibrium under the action of a prescribed temperature field. We will consider that the mass forces and the tension on the lateral surface are zero. Adopting the principal of Saint-Venant, we assume that the tensions that act on one of the bases are statically equivalent to zero. It follows from the equilibrium conditions of the medium that the tensions which act on the other base are statically equivalent to zero. If the mechanical loads were not zero, this would imply the resolution of an auxiliary problem of elasticity.

We will study the case in which the temperature is a polynomial of degree r in the coordinate x_3 of the form

$$\theta = \sum_{k=0}^r T_k(x_1, x_2) x_3^k, \quad (51.1)$$

where the functions $T_k(x_1, x_2)$ are given. Consequently, we assume that the temperature obtained by the integration of the equation of propagation with certain boundary conditions is approximated by (51.1).

The equilibrium conditions are

$$t_{ij,j} = 0 \text{ in } B. \quad (51.2)$$

In view of the fact that the lateral surface is free of tensions, it follows that we have the conditions on this surface

$$t_{i\alpha} n_\alpha = 0. \quad (51.3)$$

On the bases situated in the plane $x_3 = 0$, we have the following conditions

$$\int_{\Sigma} t_{\alpha 3} da = 0, \quad (51.4)$$

$$\int_{\Sigma} t_{33} da = 0, \quad (51.5)$$

$$\int_{\Sigma} x_{\alpha} t_{33} da = 0, \quad (51.6)$$

$$\int_{\Sigma} \epsilon_{3\alpha\beta} x_{\alpha} t_{\beta 3} da = 0, \quad (51.7)$$

which express the fact that the tensions which act on these bases have the resultant and resulting moment zero.

Let us designate by $P^{(n)}$ the problem of the determination of the thermoelastic equilibrium of the cylinder when

$$0 \leq T_n(x_1, x_2), \quad (51.8)$$

where n is a positive integer, or zero, and $T_n(x_1, x_2)$ is a prescribed function. It is obvious that if we know the solution of the problem $P^{(n)}$ for any n , then, on the basis of the linearity of the theory, we will be able to determine the solution of the problem when the temperature has the form (51.1). In order to resolve the initial problem, we will use the method of induction. First we will resolve 206 the problem $P^{(0)}$, and then we will establish the solution of the problem $P^{(n+1)}$ when the solution of the problem $P^{(n)}$ is assumed to be known.

If $u_i^{(k)}$ are components of the displacement vector from the problem $P^{(k)}$ ($k = 0, 1, 2, \dots, r$) then the components of the displacement vector corresponding to the problem in which the temperature has the form (51.1) are given by

$$u_i = \sum_{k=0}^r u_i^{(k)}. \quad (51.9)$$

Boley and Weiner [23] studied the problem of the deformation of homogeneous and isotropic bars when the temperature is linear in x_3 . The results presented in this paragraph were derived by Iesan [190], [200], [203].

a) Homogeneous and isotropic bars. In this case the constitutive equations are

$$t_{ij} = \lambda \varepsilon_{rr} \delta_{ij} + 2\mu \varepsilon_{ij} - \beta \theta \delta_{ij}, \quad (51.10)$$

where λ , μ , β are constants and

$$2\varepsilon_{ij} = u_{i,j} + u_{j,i}. \quad (51.11)$$

Let us consider at the beginning the problem $P^{(0)}$. We will assume that

$$\theta = f(x_1, x_2), \quad (51.12)$$

where f is a given function.

Let us look for the solution of the problem in the form

$$u_1 = -\frac{1}{2} a_1 [x_3^2 + v(x_1^2 - x_2^2)] - a_2 v x_1 x_2 - a_3 v x_1 + v_1(x_1, x_2). \quad (51.13)$$

$$u_2 = -a_1 v x_1 x_2 - \frac{1}{2} a_2 [x_3^2 - v(x_1^2 - x_2^2)] - a_3 v x_2 + v_2(x_1, x_2),$$

$$u_3 = (a_1 x_1 + a_2 x_2 + a_3) x_3,$$

where $v_\alpha(x_1, x_2)$ are unknown functions, a_i are unknown constants and v represents the Poisson coefficient given by (23.20).

If we introduce the designations

$$\begin{aligned} 2\gamma_{\alpha\beta} &= v_{\alpha,\beta} + v_{\beta,\alpha}, \\ \sigma_{\alpha\beta} &= \lambda\gamma_{\alpha\beta} \delta_{\alpha\beta} + 2\mu\gamma_{\alpha\beta} - \beta f \delta_{\alpha\beta}, \end{aligned} \quad (51.14)$$

we obtain from (51.10) -- (51.13)

$$\begin{aligned} l_{\alpha\beta} &= \sigma_{\alpha\beta}, \quad l_{\alpha 3} = 0, \\ l_{33} &= E(a_1 x_1 + a_2 x_2 + a_3) + \nu \sigma_{\beta\beta} = E\alpha f, \end{aligned} \quad (51.15)$$

where E and α are constants defined in (23.19), (23.20).

The equilibrium equations (51.2) are reduced to

/207

$$\sigma_{\alpha\beta,\beta} = 0 \text{ in } \Sigma; \quad (51.16)$$

and the conditions (51.3) become

$$\sigma_{\alpha\beta} n_\beta = 0 \text{ on } L. \quad (51.17)$$

Thus, the functions $v_\alpha(x_1, x_2)$ are the components of the displacement vector from the problem of plane thermoelastic deformation (51.14), (51.16), (51.17) corresponding to the temperature $\theta = f(x_1, x_2)$. In the following we will consider this problem as being solved, and thus the functions v_α , $\sigma_{\alpha\beta}$ will be assumed to be known.

The conditions (51.4), (51.7) are satisfied on the basis of the relations (51.15). We obtain from (51.5), (51.6) and (51.15)

$$\begin{aligned} a_1 &= \frac{1}{Ed} [I_{22} M_1 - I_{12} M_2 + (x_2^0 I_{12} - x_1^0 I_{22}) P], \\ a_2 &= \frac{1}{Ed} [I_{11} M_2 - I_{12} M_1 + (x_1^0 I_{12} - x_2^0 I_{11}) P], \\ a_3 &= \frac{1}{EA} P - a_1 x_1^0 - a_2 x_2^0, \end{aligned} \quad (51.18)$$

where

$$P = \int_{\Sigma} \Lambda da, \quad M_{\alpha} = \int_{\Sigma} x_{\alpha} \Lambda da, \quad \Lambda = E \alpha f - \nu \sigma_{\alpha\beta}, \quad A = \int_{\Sigma} da, \quad (51.19)$$

$$I_{\alpha\beta} = \int_{\Sigma} (x_{\alpha} - x_{\alpha}^0)(x_{\beta} - x_{\beta}^0) da, \quad d = I_{11}I_{22} - I_{12}^2, \quad Ax_{\alpha}^0 = \int_{\Sigma} x_{\alpha} da$$

Consequently, the problem $P^{(0)}$ is reduced to the solution of a problem of plane thermoelastic deformation. The solution of the problem $P^{(0)}$ has the form of (51.13) where the constants a_1 are given by (51.18).

Let us study the process of recurrence. We will designate by $u_i^*, \epsilon_{ij}^*, t_{ij}^*$, respectively, the components of the displacement vector, the components of the deformation vector, the components of the tension vector, from the problem $P^{(n)}$ and by $u_i, \epsilon_{ij}, t_{ij}$ the similar functions from the problem $P^{(n+1)}$. Let us establish a solution of the problem $P^{(n+1)}$ assuming that the solution of the problem $P^{(n)}$ is known. In view of the fact that the solution of the problem $P^{(n)}$ is known for any $T_n(x_1, x_2)$, it is also known, therefore, in the case when the temperature has the form $\theta = T_{n+1}(x_1, x_2)x_3^n$. The problem may thus be presented in the following way: let the functions $u_i, \epsilon_{ij}, t_{ij}$ be found which satisfy the equations (51.2), (51.10), (51.11) and the conditions (51.3) -- (51.7) when the temperature has the form

$$\theta = g(x_1, x_2) x_3^{n+1}, \quad (51.20)$$

assuming that the functions $u_i^*, \epsilon_{ij}^*, t_{ij}^*$ which satisfy the equations (51.2), (51.10), (51.11) and the conditions (51.3) -- (51.7) where the temperature is

$$\theta = g(x_1, x_2) x_3^n, \quad (51.21)$$

are known.

Let us search for the functions u_1 in the form

$$u_1 = (n+1) \left\{ \int_0^{x_3} u_1^* dx_3 - \frac{1}{2} b_1 [x_3^2 + v(x_1^2 - x_2^2)] - b_2 v x_1 x_3 - \right. \\ \left. - b_3 v x_1 - \tau x_2 x_3 + w_1(x_1, x_2) \right\}, \quad (51.22)$$

$$u_2 = (n+1) \left\{ \int_0^{x_3} u_2^* dx_3 - b_1 v x_1 x_2 - \frac{1}{2} b_2 [x_3^2 - v(x_1^2 - x_2^2)] - \right. \\ \left. - b_3 v x_2 + \tau x_1 x_3 + w_2(x_1, x_2) \right\},$$

$$u_3 = (n+1) \left\{ \int_0^{x_3} u_3^* dx_3 + (b_1 x_1 + b_2 x_2 + b_3) x_3 + F(x_1, x_2) \right\},$$

where the functions $w_\alpha(x_1, x_2)$, $F(x_1, x_2)$ and the constants b_1 and τ must be determined.

We obtain from (51.22)

$$\varepsilon_{\alpha\beta} = (n+1) \left[\int_0^{x_3} \varepsilon_{\alpha\beta}^* dx_3 - v(b_1 x_1 + b_2 x_2 + b_3) \delta_{\alpha\beta} + \gamma_{\alpha\beta} \right], \quad (51.23)$$

$$\varepsilon_{\alpha 3} = (n+1) \left\{ \int_0^{x_3} \varepsilon_{\alpha 3}^* dx_3 + \frac{1}{2} [F_{,\alpha} - \tau \varepsilon_{3\alpha\beta} x_\beta + u_\alpha^*(x_1, x_2, 0)] \right\},$$

$$\varepsilon_{33} = (n+1) \left[\int_0^{x_3} \varepsilon_{33}^* dx_3 + b_1 x_1 + b_2 x_2 + b_3 + u_3^*(x_1, x_2, 0) \right],$$

where

$$2\gamma_{\alpha\beta} = w_{\alpha,\beta} + w_{\beta,\alpha}. \quad (51.24)$$

In view of (51.20), (51.21), (51.23) it follows from (51.10) that

$$\begin{aligned}
t_{\alpha\beta} &= (n+1) \left[\int_0^x t_{\alpha\beta}^* dx_3 + \pi_{\alpha\beta} + \lambda \delta_{\alpha\beta} u_3^*(x_1, x_2, 0) \right], \\
t_{\alpha 3} &= (n+1) \left\{ \int_0^x t_{\alpha 3}^* dx_3 + \mu [F_{,\alpha} - \pi e_{3\alpha\beta} x_\beta + u_3^*(x_1, x_2, 0)] \right\}, \\
t_{33} &= (n+1) \left[\int_0^x t_{33}^* dx_3 + E(b_1 x_1 + b_2 x_2 + b_3) + \nu \pi_{33} + \right. \\
&\quad \left. + (\lambda + 2\mu) u_3^*(x_1, x_2, 0) \right],
\end{aligned} \tag{51.25}$$

where

$$\pi_{\alpha\beta} = \lambda \gamma_{\alpha\beta} \delta_{\alpha\beta} + 2\mu \gamma_{\alpha\beta}. \tag{51.26}$$

The equilibrium equations are reduced to

$$\pi_{\alpha\beta,\beta} + g_\alpha = 0, \tag{51.27}$$

$$\Delta F = h \text{ in } \Sigma, \tag{51.28}$$

in which we designated

$$g_\alpha(x_1, x_2) = t_{\alpha 3}^*(x_1, x_2, 0) + \lambda u_{3,\alpha}^*(x_1, x_2, 0), \tag{51.29}$$

$$h(x_1, x_2) = -u_{3,3}^*(x_1, x_2, 0) - \frac{1}{u} t_{33}^*(x_1, x_2, 0).$$

The conditions (51.3) become

$$\pi_{\alpha\beta} n_\beta = p_\alpha, \tag{51.30}$$

$$\frac{\partial F}{\partial n} = q \text{ on } L, \tag{51.31}$$

where

$$\begin{aligned} p_\alpha &= -\lambda u_{33}^*(x_1, x_2, 0) u_\alpha, \\ q &= \tau e_{3\alpha\beta} x_\beta u_\alpha - u_\alpha^*(x_1, x_2, 0) u_\alpha. \end{aligned} \quad (51.32)$$

It follows from the above that the functions w_α satisfy the equations of the problem of elastic ($\theta = 0$) plane deformation (51.24), (51.26), (51.27), (51.30).

The conditions which are necessary and sufficient for the existence of the solutions are

$$\int_\Sigma g_\alpha da + \int_L p_\alpha ds = 0, \quad \int_\Sigma e_{3\alpha\beta} x_\beta g_\alpha da + \int_L e_{3\alpha\beta} x_\beta p_\alpha ds = 0. \quad (51.33)$$

In view of (51.29), (51.32) and the theorem of divergence, we obtain

$$\int_\Sigma g_\alpha da + \int_L p_\alpha ds = \int_\Sigma t_{\alpha 3}^*(x_1, x_2, 0) da, \quad (51.34)$$

$$\int_\Sigma e_{3\alpha\beta} x_\beta g_\alpha da + \int_L e_{3\alpha\beta} x_\beta p_\alpha ds = \int_\Sigma e_{3\alpha\beta} x_\beta t_{\beta 3}^*(x_1, x_2, 0) da,$$

so that the conditions (51.33) are satisfied because the functions t_{ij}^* satisfy the relations (51.4) -- (51.7). In the following we will assume that the functions w_α are known.

The necessary and sufficient condition for the existence of a solution of the problem (51.28), (51.31) is

$$\int_\Sigma h da - \int_L q ds. \quad (51.35)$$

If we consider (51.29), (51.32) we can write

$$\int_\Sigma h da - \int_L q ds = -\frac{1}{\mu} \int_\Sigma t_{33}^*(x_1, x_2, 0) da = 0, \quad (51.36)$$

on the basis of the fact that the functions t_{ij}^* satisfy the conditions (51.4) -- (51.7). The function F depends on the constant τ . In order to determine this constant we introduce the function of torsion $\phi(x_1, x_2)$ which satisfies the equation [382]

$$\Delta \phi = 0 \quad \text{in } \Sigma, \quad (51.37)$$

and the boundary condition

$$\frac{\partial \phi}{\partial n} = e_{3\alpha\beta} x_\alpha n_\beta \quad \text{on } L. \quad (50.38)$$

Let us consider the function ψ defined by

$$F = \tau \phi + \psi. \quad (51.39)$$

It follows from (51.28), (51.31), (51.37) -- (51.39) that the function ψ satisfies the equation

$$\Delta \psi = h \quad \text{in } \Sigma, \quad (51.40)$$

and the condition

$$\frac{\partial \psi}{\partial n} = -u_\alpha^*(x_1, x_2, 0) n_\alpha \quad \text{on } L. \quad (51.41)$$

The function ψ does not depend on τ and we will assume that it is known. We obtain from (51.25), (51.39)

$$I_{13} = (n+1) \left[\int_0^1 I_{23}^* dx_3 + \mu \tau (\phi_{,3} - e_{3\alpha\beta} x_\beta) + \right. \\ \left. + \mu \psi_{,3} + \mu u_\alpha^*(x_1, x_2, 0) \right]. \quad (51.42)$$

In view of (51.42) it follows from (51.7) that

$$\tau D = -\mu \int_\Sigma e_{3\alpha\beta} x_\alpha [\psi_{,3} + u_\beta^*(x_1, x_2, 0)] da, \quad (51.43)$$

where D is the rigidity upon torsion [382]

$$D = \mu \int_{\Sigma} (e_{3\alpha\beta} x_{\alpha} \varphi_{,\beta} + x_1^2 + x_2^2) da. \quad (51.44)$$

It is known that $D > 0$ so that the relation (51.43) determines the constant τ .

We find from (51.5), (51.6) and (51.25) that

/21

$$b_1 = \frac{1}{Ed} [I_{33}N_1 - I_{12}N_2 + (x_2^0 I_{12} - x_1^0 I_{22}) Q], \quad (51.45)$$

$$b_2 = \frac{1}{Ed} [I_{11}N_2 - I_{12}N_1 + (x_1^0 I_{12} - x_2^0 I_{11}) Q],$$

$$b_3 = \frac{1}{EA} Q - b_1 x_1^0 - b_2 x_2^0,$$

where

$$Q = \int_{\Sigma} \Omega da, \quad N_{\alpha} = \int_{\Sigma} x_{\alpha} \Omega da, \quad \Omega = -v\pi_{3\alpha} - (\lambda + 2\mu) u_3^*(x_1, x_2, 0),$$

while the other designations are given in (51.19)

The conditions (51.4) are similarly satisfied on the basis of the equilibrium equations and the boundary conditions. Thus, we have

$$\begin{aligned} \int_{\Sigma} t_{\alpha\beta} da &= \int_{\Sigma} (t_{\alpha\beta} + x_{\alpha} t_{\beta 3}) da = \int_{\Sigma} [(x_{\alpha} t_{\beta 3})_{,\beta} + x_{\alpha} t_{33,3}] da = \\ &= \int_L x_{\alpha} t_{\beta 3} n_{\beta} ds + (n+1) \int_{\Sigma} x_{\alpha} t_{33}^* da = 0, \end{aligned}$$

because the functions t_{ij}^* satisfy the equations (51.4) -- (51.7).

Consequently the problem thus posed is reduced to the integration of the equations of a problem of plane elasticity.

The study of some special problems is presented by Boley

and Weiner [23], Grindei [152], etc.

b) Bars made of different materials. Let us consider the case when the transversal section Σ is the combination of domains Σ_0 and Σ_j ($j = 1, 2, \dots, m$), Σ_0 being a multiple connected domain, limited by closed curves L_j ($j = 1, 2, \dots, m+1$) which do not have common points. We assume that the domains Σ_j are finite and simply connected, limited by the curves L_j ($j = 1, 2, \dots, m$) respectively, and L_{m+1} is the boundary of the domain Σ . We will consider the case when the bar consists of different materials, so that in each of the domains Σ_j ($j = 0, 1, 2, \dots, m$) we have a homogeneous and isotropic medium and by going from one domain to another, the thermoelastic properties of the respective media are different.

The problem of Saint-Venant for these bars in the isothermal case has been studied by Mushelisvili [296], Bors [35], et al.

We will impose that in this case the components of the displacement vectors and of tension vectors be continuous in B . These conditions, together with those on the lateral surface, may be written in the following way

$$[u_k]_i = [u_k]_0, \quad (51.46)$$

$$[t_{k\alpha}]_i n_\alpha = [t_{k\alpha}]_0 n_\alpha, \text{ on } L_i (i = 1, 2, \dots, m), \quad (51.47)$$

$$[t_{k\alpha}]_0 n_\alpha = 0, \text{ on } L_{m+1}, \forall x_3 \in [0, l], \quad (51.48)$$

where we designated by n_α the components of the vector of the unit normal which is external at the boundary of the domain Σ_0 and $[]_i$ indicates the fact that the expression in the parentheses or brackets is calculated for the medium which occupies the domain Σ_i ($i = 0, 1, 2, \dots, m$).

/212

Let us designate by $\lambda_1, \mu_1, \beta_1$ the constants which characterize the thermoelastic properties of the medium which occupies the domain Σ_1 ($i = 0, 1, 2, \dots, m$). The constants E_1, ν_1, α_1 ($i = 0, 1, 2, \dots, m$) correspond to these coefficients through the relations (23.19), (23.20).

We will consider three auxiliary problems of plane elastic deformation $A^{(k)}$ ($k = 1, 2, 3$). Let us designate by $v_a^{(k)}, \epsilon_{\alpha\beta}^{(k)}, \tau_{\alpha\beta}^{(k)}$ ($k = 1, 2, 3$) the components of the displacement vector, the components of the deformation tensor, the components of the tension tensor, respectively, from the problem $A^{(k)}$. These auxiliary problems are characterized by the equations

$$\tau_{\alpha\beta}^{(k)} = \lambda_i \epsilon_{\alpha\beta}^{(k)} + 2\mu_i \epsilon_{\alpha\beta}^{(k)}, \quad (51.49)$$

$$2\epsilon_{\alpha\beta}^{(k)} = v_{\alpha,\beta}^{(k)} + v_{\beta,\alpha}^{(k)}, \quad (51.50)$$

$$\tau_{\alpha\beta,\beta}^{(k)} = 0, \text{ in } \Sigma_i (i = 0, 1, 2, \dots, m), \quad (51.51)$$

and the conditions

$$\begin{aligned} [\tau_{\alpha\beta}^{(k)}]_i n_\beta &= [\tau_{\alpha\beta}^{(k)}]_0 n_\beta, \quad [v_a^{(k)}]_i - [v_a^{(k)}]_0 = g_{1\alpha}^{(k)}, \\ &\text{pe } I_i (i = 1, 2, \dots, m), \\ [\tau_{\alpha\beta}^{(k)}]_0 n_\beta &= 0, \text{ pe } I_{m+1}, \end{aligned} \quad (51.52)$$

where

$$g_{11}^{(1)} = \frac{1}{2} (\nu_i - \nu_0) (x_1^2 - x_2^2), \quad g_{12}^{(1)} = (\nu_i - \nu_0) x_1 x_2,$$

$$g_{11}^{(2)} = (\nu_i - \nu_0) x_1 x_2, \quad g_{12}^{(2)} = -\frac{1}{2} (\nu_i - \nu_0) (x_1^2 - x_2^2),$$

$$g_{1\alpha}^{(3)} = (\nu_i - \nu_0) x_\alpha.$$

The above auxiliary problems were considered by Mushelishvili [296] in the course of the study of the problem of Saint-Venant in the isothermal case. They were also studied by Sherman [363]. In the following we will assume that the solutions of these problems are known.

In order to solve the problem $P^{(0)}$ we are searching for the solution in the form

$$\begin{aligned} u_1 &= -\frac{1}{2}a_1[x_3^2 + v_1(x_1^2 - x_2^2)] - a_2v_1x_1x_2 - a_3v_1x_1 + \sum_{k=1}^3 a_k v_1^{(k)} + v_1(x_1, x_2), \\ u_2 &= -a_1v_1x_1x_2 - \frac{1}{2}a_2[x_3^2 - v_1(x_1^2 - x_2^2)] - a_3v_1x_2 + \sum_{k=1}^3 a_k v_2^{(k)} + \\ &\quad + v_2(x_1, x_2), \\ u_3 &= (a_1x_1 + a_2x_2 + a_3)x_3, \text{ in } \Sigma_i \times [0, l] \quad (i = 0, 1, 2, \dots, m), \end{aligned} \quad (51.53)$$

where v_α are unknown functions and a_j are unknown constants.

In view of (51.52) it follows that the functions (51.53) are /213 continuous in Σ if

$$[v_\alpha]_i = [v_\alpha]_0 \quad \text{on } l_i \quad (i = 1, 2, \dots, m). \quad (51.54)$$

We conclude from (51.53)

$$\begin{aligned} t_{13} &= \sigma_{13} + \sum_{k=1}^3 a_k \tau_{13}^{(k)}, \quad t_{23} = 0, \\ t_{33} &= E_i(a_1x_1 + a_2x_2 + a_3) + v_1 \sum_{k=1}^3 a_k \tau_{33}^{(k)} + v_1 \sigma_{33} - \\ &= E_i x_i f \text{ in } \Sigma_i \quad (i = 0, 1, 2, \dots, m), \end{aligned} \quad (51.55)$$

where

$$\begin{aligned} \sigma_{\alpha\beta} &= \lambda_i \gamma_{\alpha\beta} \delta_{\alpha\beta} + 2\mu_i \gamma_{\alpha\beta} - \beta_i f \delta_{\alpha\beta}, \\ 2\gamma_{\alpha\beta} &= v_{\alpha,\beta} + v_{\beta,\alpha} \text{ in } \Sigma_i \quad (i = 0, 1, 2, \dots, m). \end{aligned} \quad (51.56)$$

In view of (51.51) and (51.55) the equilibrium conditions (51.2) are reduced to

$$\sigma_{\alpha\beta, \beta} = 0 \text{ in } \Sigma_i (i = 0, 1, 2, \dots, m). \quad (51.57)$$

The conditions (51.47) and (51.48) become

$$\begin{aligned} [\sigma_{\alpha\beta}]_i n_\beta &= [\sigma_{\alpha\beta}]_0 n_\beta \text{ on } L_i (i = 1, 2, \dots, m), \\ [\sigma_{\alpha\beta}]_0 n_\beta &= 0 \text{ on } L_{m+1}, \end{aligned} \quad (51.58)$$

on the basis of relations (51.52).

In conclusion, the functions v_α are the components of the displacement vector from the plane thermoelastic deformation problem (51.56), (51.57), (51.58), (51.54) in which the temperature is $f(x_1, x_2)$. We will assume that this problem is solved (see Sherman [363], Iesan [193]).

The conditions (51.4) and (51.7) are satisfied on the basis of relations (51.55). We obtain from (51.5), (51.6) and (51.55)

$$\begin{aligned} a_1 &= K_{22}m_1 - K_{21}m_2, \quad a_2 = K_{11}m_2 - K_{12}m_1, \\ d_3 &= p - a_1d_1 - a_2d_2, \end{aligned} \quad (51.59)$$

where

$$\begin{aligned} K_{\alpha\beta} &= \frac{1}{d} \sum_{i=0}^m h_\alpha^{(i)} x_\beta da, \quad h_\alpha^{(i)} = E_i x_\alpha + \nu_i \tau_{\beta\beta}^{(i)} - d_\alpha [E_i + \nu_i \tau_{\beta\beta}^{(i)}], \\ d_1 &= \frac{1}{d_3} \sum_{i=0}^m \int_{\Sigma_i} [E_i x_\beta + \nu_i \tau_{\beta\beta}^{(i)}] da, \quad d_3 = \sum_{i=0}^m [E_i + \nu_i \tau_{\beta\beta}^{(i)}] da, \\ d &= K_{11}K_{22} - K_{12}^2, \quad p = \frac{1}{d_3} \sum_{i=0}^m \int_{\Sigma_i} \Lambda^{(i)} da, \quad \Lambda^{(i)} = E_i \alpha_i f - \nu_i \sigma_{\beta\beta}, \\ m_\beta &= \sum_{i=0}^m \int_{\Sigma_i} [\Lambda^{(i)} - p(E_i + \nu_i \tau_{\alpha\alpha}^{(i)})] x_\beta da. \end{aligned} \quad (51.60)$$

It can be shown (Mushelisvili [296]) that $d_3 \neq 0$, $d \neq 0$.
Let us study now the process of recurrence. We will try to find the solution of the problem of $P^{(n+1)}$ in the form

$$u_1 = (n+1) \left\{ \int_0^{x_3} u_1^* dx_3 - \frac{1}{2} b_1 [x_3^2 + v_1(x_1^2 - x_2^2)] - \right. \\ \left. - b_2 v_1 x_1 x_2 - b_3 v_1 x_1 - \tau x_2 x_3 + \sum_{k=1}^3 b_k v^{(k)} + w_1(x_1, x_2) \right\}, \quad (51.61)$$

$$u_2 = (n+1) \left\{ \int_0^{x_3} u_2^* dx_3 - b_1 v_1 x_1 x_2 - \frac{1}{2} b_2 [x_3^2 - v_1(x_1^2 - x_2^2)] - \right. \\ \left. - b_3 v_1 x_2 + \tau x_1 x_3 + \sum_{k=1}^3 b_k v^{(k)} + w_2(x_1, x_2) \right\}, \quad (51.61)$$

$$u_3 = (n+1) \left[\int_0^{x_3} u_3^* dx_3 + (b_1 x_1 + b_2 x_2 + b_3) x_3 + F(x_1, x_2) \right],$$

where the functions w_α , F and the constants b_1 , τ are unknown.

For the components of the tension tensor we obtain the expressions

$$t_{\alpha\beta} = (n+1) \left[\int_0^{x_3} t_{\alpha\beta}^* dx_3 + \pi_{\alpha\beta} + \sum_{k=1}^3 b_k \tau_{\alpha\beta}^{(k)} + \lambda_1 \delta_{\alpha\beta} u_3^*(x_1, x_2, 0) \right], \quad (51.62)$$

$$t_{\alpha 3} = (n+1) \left\{ \int_0^{x_3} t_{\alpha 3}^* dx_3 + \mu_1 [F_{,\alpha} - \tau e_{3\alpha\beta} x_\beta + u_\alpha^*(x_1, x_2, 0)] \right\},$$

$$t_{33} = (n+1) \left\{ \int_0^{x_3} t_{33}^* dx_3 + E_1 (b_1 x_1 + b_2 x_2 + b_3) + v_1 \pi_{\alpha\alpha} + v_1 \sum_{k=1}^3 b_k \tau_{\alpha\alpha}^{(k)} + \right. \\ \left. + (\lambda_1 + 2\mu_1) u_3^*(x_1, x_2, 0) \right\},$$

where

$$\pi_{\alpha\beta} = \lambda \gamma_{\alpha\beta} \delta_{\alpha\beta} + 2\mu \gamma_{\alpha\beta}, \\ 2\gamma_{\alpha\beta} = w_{\alpha,\beta} + w_{\beta,\alpha}. \quad (51.63)$$

In view of (51.52), the conditions (51.46) -- (51.48) are reduced to

$$[w_a]_i = [w_a]_0, \quad [\pi_{a\beta}]_i n_\beta = [\pi_{a\beta}]_0 n_\beta + x_a^{(i)}, \quad (51.64)$$

$$[F]_i = [F]_0, \quad \left[\mu \frac{\partial F}{\partial n} \right]_i - \left[\mu \frac{\partial F}{\partial n} \right]_0 = q_i \text{ on } L_i (i = 1, 2, \dots, m), \quad (51.65)$$

$$[\pi_{a\beta}]_0 n_\beta = p_a, \quad (51.66)$$

$$\left[\mu \frac{\partial F}{\partial n} \right]_0 n_\beta = q \text{ on } L_{m+1}, \quad (51.67)$$

in which

/215

$$\begin{aligned} x_a^{(i)} &= (\lambda_0 - \lambda_i) u_{3,a}^*(x_1, x_2, 0) n_a, \quad p_a = -\lambda_0 n_a u_{3,a}^*(x_1, x_2, 0), \\ q_i &= (\mu_0 - \mu_i) [u_{3,a}^*(x_1, x_2, 0) - \tau e_{3a\beta} x_\beta] n_a, \\ q &= \mu_0 [\tau e_{3a\beta} x_\beta - u_{3,a}^*(x_1, x_2, 0)] n_a. \end{aligned} \quad (51.68)$$

If we keep in mind (51.51) and (51.62) the equilibrium equations take the form

$$\pi_{a\beta, \beta} + g_a^{(i)} = 0, \quad (51.69)$$

$$\Delta F = h^{(i)} \text{ in } \Sigma_i (i = 0, 1, \dots, m), \quad (51.70)$$

where

$$\begin{aligned} g_a^{(i)} &= \lambda_i u_{3,a}^*(x_1, x_2, 0) + t_{33}^*(x_1, x_2, 0), \\ h^{(i)} &= -u_{3,a,a}^*(x_1, x_2, 0) - \frac{1}{\mu_i} t_{33}^*(x_1, x_2, 0). \end{aligned} \quad (51.71)$$

In the following the functions $w_\alpha(x_1, x_2)$ from (51.61) are the components of the displacement vector in the plane elastic deformation problem (51.63), (51.64), (51.66), (51.69). The necessary and sufficient conditions for the existence of this problem are (Sherman [363])

$$R_\alpha = \sum_{i=0}^m \int_{\Sigma_i} g_\alpha^{(i)} da + \int_{L_{m+1}} p_\alpha ds - \sum_{i=1}^m \int_{L_i} x_\alpha^{(i)} ds = 0,$$

$$M = \sum_{i=0}^m \int_{\Sigma_i} e_{3\alpha\beta} x_\alpha g_\beta^{(i)} da + \int_{L_{m+1}} e_{3\alpha\beta} x_\alpha p_\beta ds - \sum_{i=1}^m \int_{L_i} e_{3\alpha\beta} x_\alpha x_\beta^{(i)} ds = 0. \quad (51.72)$$

If we keep in mind (51.68), (51.71) and the meaning of the components n_α we can write

$$R_\alpha = \sum_{i=0}^m \int_{\Sigma_i} t_{33}^*(x_1, x_2, 0) da, \quad M = \sum_{i=0}^m \int_{\Sigma_i} e_{3\alpha\beta} x_\alpha t_{33}^*(x_1, x_2, 0) da,$$

so that the conditions (51.72) are satisfied on the basis of the fact that t_{ij}^* satisfies the relations (51.4) -- (51.7). We will assume that the functions w_α are determined.

For the function F we obtain the equation (51.70) and the conditions (51.65), (51.67). The necessary and sufficient condition for the existence of a solution of this problem is (Mushelisvili [296])

$$N = \sum_{i=0}^m \int_{\Sigma_i} \mu_i h^{(i)} da - \int_{L_{m+1}} q ds + \sum_{i=1}^m \int_{L_i} q ds = 0. \quad (51.73)$$

It follows from (51.68), (51.71), (51.5) that

$$N = - \sum_{i=0}^m \int_{\Sigma_i} t_{33}^*(x_1, x_2, 0) da = 0.$$

The conditions (51.4) are similarly satisfied on the basis of the equilibrium equations and the boundary conditions. This is indicated in the same way as in the case of homogeneous bars.

In order to determine the constant τ we will introduce the torsion function $\phi(x_1, x_2)$. This function satisfies the equation (Muskhelishvili [296])

$$\Delta \phi = 0 \text{ in } \Sigma_i (i = 0, 1, 2, \dots, m), \quad (51.74)$$

and the conditions

$$[\phi]_i = [\phi]_0, \quad \left[\mu \frac{\partial \phi}{\partial n} \right]_i - \left[\mu \frac{\partial \phi}{\partial n} \right]_0 = -(\mu_0 - \mu_i) e_{3\alpha\beta} x'_\beta n_\alpha, \quad (51.75)$$

$\text{on } L_i, i = 1, 2, \dots, m,$

$$\frac{\partial \phi}{\partial n} = e_{3\alpha\beta} x'_\beta n_\alpha \quad \text{on } L_{m+1}.$$

If we introduce the function ψ through (51.39), it follows that this function does not depend on τ . We obtain from (51.7) (51.39) and (51.62)

$$\tau D = - \sum_{i=0}^m \int_{\Sigma_i} e_{3\alpha\beta} x'_\alpha \mu_i [\psi_{,\beta} + n_\beta^*(x_1, x_2, 0)] da, \quad (51.76)$$

where D is the rigidity at torsion

$$D = \sum_{i=0}^m \int_{\Sigma_i} \mu_i (e_{3\alpha\beta} x'_\alpha \varphi_{,\beta} + x_1^2 + x_2^2) da. \quad (51.77)$$

Consequently the constant τ is determined by the relation (51.76). From the conditions (51.5), (51.6) and (51.62) we obtain for the constants b_1 the expressions similar to (51.59).

c) Thermal tensions in nonhomogeneous and anisotropic bars. We will study next the case in which the cylinder B is occupied by a nonhomogeneous and anisotropic medium. In this case the constitutive equations are

$$t_{ij} = C_{ijkl} u_{k,l} - \beta_{ij} \theta. \quad (51.78)$$

We will assume that

$$C_{ijkl} = C_{ijkl}(x_1, x_2), \quad \beta_{ij} = \beta_{ij}(x_1, x_2). \quad (51.79)$$

In the following we assume that Σ is a domain C^∞ -smooth (Fichera [112]) and C_{ijkl} , β_{ij} and T_k are functions of the class C^∞ . This problem may be studied also under more general conditions, but we preferred this path in order to insist on the method used.

The problem consists in the integration of the equations (51.2), (51.78) with the conditions (51.3) -- (51.7) in the case in which the temperature is given by (51.1) and the coefficients C_{ijkl} and β_{ij} have the form of (51.79).

We say that in the medium under consideration an elastic ($\theta = 0$) plane generalized deformation takes place (see Paragraph 50) parallel with the plane $x_1 O x_2$, if $u_1 = u_1(x_1, x_2)$. In this case the constitutive equations are

$$t_{ij} = C_{ijkl} u_{k,a},$$

and the equilibrium equations, in the case when the mass forces $f_1(x_1, x_2)$ are taken into account, become

$$t_{i\alpha,2} + f_i = 0 \text{ in } \Sigma.$$

/217

If the boundary conditions

$$t_{\alpha} n_{\alpha} = \tilde{t}_i \text{ on } L,$$

are considered, then the conditions necessary and sufficient for the existence of a solution of this problem are (Iesan [191], [203])

$$\int_{\Sigma} f_i da + \int_L \tilde{t}_i ds = 0, \quad \int_{\Sigma} c_{3\alpha\beta} r_{\alpha} f_{\beta} da + \int_L c_{3\alpha\beta} r_{\alpha} \tilde{t}_{\beta} ds = 0. \quad (51.80)$$

If f_i and \tilde{t}_i are of the class C^{∞} , then $u_i \in C^{\infty}(\Sigma)$ (Fichera [112]).

In the following we will use four special problems of the generalized plane elastic deformation designating them by $D^{(r)}$ ($r = 1, 2, 3, 4$). Let $v_i^{(s)}$ and $\sigma_{ij}^{(s)}$ be the components of the displacement vector and the components of the tension tensor from the problem $D^{(s)}$. These problems are defined by the equations

$$\sigma_{ij}^{(s)} = C_{ijkl} v_{k,\beta}^{(s)} \quad (s = 1, 2, 3, 4), \quad (51.81)$$

$$\sigma_{\alpha,\alpha}^{(2)} + (C_{1\alpha 33} r_{\beta})_{,\alpha} = 0, \quad (\beta = 1, 2), \quad (51.82)$$

$$\sigma_{\alpha,\alpha}^{(3)} + C_{1\alpha 33,\alpha} = 0, \quad (51.83)$$

$$\sigma_{\alpha,\alpha}^{(4)} - c_{\beta 33} (C_{1\alpha \beta 3} r_{\beta})_{,\alpha} = 0 \text{ in } \Sigma, \quad (51.84)$$

and boundary conditions

$$\sigma_{1\alpha}^{(1)} n_{\alpha} = -C_{1\alpha 33} r_{\beta} n_{\alpha}, \quad \sigma_{1\alpha}^{(3)} n_{\alpha} = -C_{1\alpha 33} n_{\alpha}, \quad (51.85)$$

$$\sigma_{1\alpha}^{(4)} n_{\alpha} = c_{\beta 33} C_{1\alpha \beta 3} r_{\beta} n_{\alpha} \text{ on } L.$$

It can easily be verified that the relations (51.80) are satisfied for each of the problems $D^{(s)}$, the fact that ensures the existence of the solutions. We will assume that the solutions of these problems are known. In the case of a homogeneous and isotropic medium, the problems $D^{(s)}$ have the following solutions

$$\begin{aligned}
v_1^{(1)} &= -\frac{\lambda}{4(\lambda + \mu)} (x_1^2 - x_2^2), \quad v_2^{(1)} = -\frac{\lambda}{2(\lambda + \mu)} x_1 x_2, \quad v_3^{(1)} = 0, \\
v_1^{(2)} &= -\frac{\lambda}{2(\lambda + \mu)} x_1 x_2, \quad v_2^{(2)} = -\frac{\lambda}{4(\lambda + \mu)} (x_1^2 - x_2^2), \quad v_3^{(2)} = 0, \\
v_4^{(3)} &= -\frac{\lambda}{2(\lambda + \mu)} x_4, \quad v_5^{(3)} = 0, \quad v_6^{(3)} = 0, \quad v_7^{(3)} = \varphi(x_1, x_2),
\end{aligned} \tag{51.86}$$

/218

where ϕ is the solution of the problem (51.37), (51.38).

Let us now study the problem $P^{(0)}$. We are looking for the solution of this problem in the form

$$\begin{aligned}
u_a &= -\frac{1}{2} a_a x_a^2 - a_4 e_{\alpha\beta} x_\beta x_3 + \sum_{i=1}^4 a_i v_i^{(0)} + v_a(x_1, x_2), \\
u_3 &= (a_1 x_1 + a_2 x_2 + a_3) x_3 + \sum_{i=1}^4 a_i v_i^{(0)} + v_3(x_1, x_2),
\end{aligned} \tag{51.87}$$

where $v_1^{(s)}$ are the solutions of the problems $D^{(s)}$ ($s = 1, 2, 3, 4$), v_1 are unknown functions and a_r ($r = 1, 2, 3, 4$) are unknown constants. We obtain from (51.87)

$$\begin{aligned}
u_{k,a} &= a_a x_3 \delta_{k3} - a_4 e_{\alpha\beta} x_3 \delta_{k\beta} + \sum_{i=1}^4 a_i v_i^{(0)} + v_{k,a}, \\
u_{k,3} &= (a_1 x_1 + a_2 x_2 + a_3) \delta_{k3} - \delta_{k\alpha} a_\alpha x_3 - \delta_{k\alpha} a_4 e_{\alpha\beta} x_\beta,
\end{aligned}$$

so that the relations (51.78) imply

$$l_{ij} = C_{ij33}(a_1 x_1 + a_2 x_2 + a_3) - C_{ij\alpha\beta} a_\alpha x_\beta + \sum_{i=1}^4 a_i \sigma_{ij}^{(0)} + \pi_{ij}, \tag{51.88}$$

where $\sigma_{ij}^{(s)}$ are given by (51.81), and

$$\pi_{ij} = C_{ijk3} v_{k,3} - \beta_{ij} f. \tag{51.89}$$

In view of (51.82) -- (51.84) and (51.88), the equilibrium equations (51.2) are reduced to

$$\pi_{\alpha, \alpha} = 0 \text{ in } \Sigma. \quad (51.90)$$

On the basis of the relations (51.85), (51.88) the conditions on the lateral surface become

$$\pi_{\alpha} n_{\alpha} = 0 \text{ on } L. \quad (51.91)$$

Consequently the functions $v_1(x_1, x_2)$ are the components of the displacement vector in the generalized plane thermoelastic deformation problem (51.89), (51.90), (51.91). We will assume that the solution of this problem is known.

From the conditions (51.5) -- (51.7) we obtain the following system for the constants a_s

$$\sum_{i=1}^4 L_{ri} a_i = H_r, \quad (r = 1, 2, 3, 4), \quad (51.92)$$

where

$$\begin{aligned} L_{\alpha\beta} &= \int_{\Sigma} x_{\alpha} [C_{3333} x_{\beta} + \sigma_{33}^{(0)}] da, \quad L_{\alpha 3} = \int_{\Sigma} x_{\alpha} [C_{3333} + \sigma_{33}^{(3)}] da, \\ L_{\alpha 4} &= \int_{\Sigma} x_{\alpha} [C_{3333} e_{\beta\alpha\beta} x_{\beta} + \sigma_{33}^{(4)}] da, \quad L_{3\alpha} = \int_{\Sigma} [C_{3333} x_{\alpha} + \sigma_{33}^{(3)}] da, \\ L_{33} &= \int_{\Sigma} [C_{3333} + \sigma_{33}^{(3)}] da, \quad L_{34} = \int_{\Sigma} [C_{3333} e_{\beta\alpha\beta} x_{\beta} + \sigma_{33}^{(4)}] da, \\ L_{4\alpha} &= \int_{\Sigma} e_{3\alpha\beta} x_{\beta} [C_{3333} x_{\alpha} + \sigma_{33}^{(3)}] da, \\ L_{43} &= \int_{\Sigma} e_{3\alpha\beta} x_{\alpha} [C_{3333} + \sigma_{33}^{(3)}] da, \\ L_{44} &= \int_{\Sigma} e_{3\alpha\beta} x_{\alpha} [C_{3333} e_{\lambda\alpha\beta} x_{\lambda} + \sigma_{33}^{(4)}] da, \\ H_{\alpha} &= - \int_{\Sigma} x_{\alpha} \pi_{33} da, \quad H_3 = - \int_{\Sigma} \pi_{33} da, \quad H_4 = - \int_{\Sigma} e_{3\alpha\beta} x_{\alpha} \pi_{\beta 3} da. \end{aligned} \quad (51.93)$$

It is known that $(L_{rs}) \neq 0$ (Iesan [203]), so that from the system (51.92) the constants a_r are determined uniquely.

The conditions (51.4) are satisfied, similarly, on the basis of the equilibrium equations and the conditions on the lateral surface. Thus, we have

$$\int_{\Sigma} t_{\alpha\beta} da = \int_{\Sigma} (t_{\alpha\beta} + x_{\alpha} t_{3\beta,3}) da = \int_{\Sigma} (x_{\alpha} t_{3\beta})_{,3} da = \int_{\Sigma} x_{\alpha} t_{3\beta} n_3 da = 0.$$

Consequently the solution of the problem $P^{(0)}$ has the form (51.87).

If we keep in mind (51.86), it can easily be seen that in the case of homogeneous and isotropic media, the constants L_{rs} are reduced to

$$L_{\alpha\beta} = E \int_{\Sigma} x_{\alpha} x_{\beta} da, \quad L_{\alpha 3} = EA x_{\alpha}^0, \quad L_{44} = 0, \quad L_{4\alpha} = D,$$

where E is Young's modulus, and A, x_{α}^0 and D are given by (51.19) and (51.44).

Next, we will study the process of recurrence. We are looking for the solution of the problem P^{n+1} when the temperature is given by (51.20) in the form

$$\begin{aligned} u_{\alpha} &= (n+1) \left[\int_0^1 u_{\alpha}^* dx_3 - \frac{1}{2} b_{\alpha} x_3^2 - b_3 c_{\alpha\beta} x_{\beta} x_3 + \sum_{i=1}^4 b_i v_i^{(n)} + w_{\alpha}(x_1, x_2) \right], \\ u_3 &= (n+1) \left[\int_0^1 u_3^* dx_3 + (b_1 x_1 + b_2 x_2 + b_3) x_3 + \sum_{i=1}^4 b_i v_i^{(n)} + w_3(x_1, x_2) \right], \end{aligned} \quad (51.94)$$

where u_i^* are components of the displacement vector from the problem $P^{(n)}$ corresponding to the temperature (51.21), $w_i(x_1, x_2)$ are unknown functions and b_r ($r = 1, 2, 3, 4$) are unknown constants.

We obtain from relations (51.78), (51.81), (51.20), (51.21), (51.94)

$$t_{ij} = (n + 1) \left[\int_0^{x_3} t_{ij}^* dx_3 + C_{ijkl}(b_1 x_1 + b_2 x_2 + b_3) - C_{ijkl} e_{\alpha\beta\gamma} b_\gamma x_\beta + \sum_{j=1}^3 b_j \sigma_{ij}^{(0)} + \sigma_{ij} + k_{ij} \right], \quad (51.95)$$

where

$$\sigma_{ij} = C_{ijkl} w_{k,\beta}, \quad (51.96)$$

$$k_{ij} = C_{ijkl} u_k^*(x_1, x_2, 0). \quad (51.97)$$

If we keep in mind (51.82) -- (51.84) and (51.95) the equilibrium equations are reduced to

$$\sigma_{i\alpha,\alpha} + G_i = 0 \text{ in } \Sigma, \quad (51.98)$$

in which we designated

$$G_i(x_1, x_2) = t_{i3}^*(x_1, x_2, 0) + k_{i\alpha,\alpha}. \quad (51.99)$$

The conditions on the lateral surface become, on the basis of relations (51.85) and (51.95)

$$\sigma_{i\alpha} n_\alpha = p_i \text{ on } J, \quad (51.100)$$

where

$$p_i(x_1, x_2) = -k_{i\alpha} n_\alpha. \quad (51.101)$$

Consequently, the functions $w_j(x_1, x_2)$ are the components of the displacement vector from the generalized plane deformation problem (51.96), (51.98), (51.100). Keeping in mind the relations (51.99) and (51.101), we can write

$$\int_{\Sigma} G_i da + \int_L p_i ds = \int_{\Sigma} t_{i3}^*(x_1, x_2, 0) da, \quad (51.102)$$

$$\int_{\Sigma} e_{3\alpha\beta} x_{\alpha} G_{\beta} da + \int_L e_{3\alpha\beta} x_{\alpha} p_{\beta} ds = \int_{\Sigma} e_{3\alpha\beta} x_{\alpha} t_{\beta 3}^*(x_1, x_2, 0) da.$$

Inasmuch as the functions t_{ij}^* satisfy the conditions (51.4) -- (51.7) it follows from (51.102) that in this case the necessary and sufficient conditions (51.80) for the existence of a solution of the problems (51.96), (51.98), (51.100) are satisfied. In the following we will assume that functions w_1 are known. From the conditions (51.5) -- (51.7) and (51.95) the following system for the determination of the constants b_r results

$$\sum_{r=1}^4 L_{rs} b_r = K_s, \quad (s = 1, 2, 3, 4), \quad (51.103)$$

where L_{rs} are given by (51.93) and

/221

$$K_1 = - \int_{\Sigma} x_3 (\sigma_{33} + k_{33}) da, \quad K_2 = - \int_{\Sigma} (\sigma_{33} + k_{33}) da,$$

$$K_4 = - \int_{\Sigma} e_{3\alpha\beta} x_{\alpha} (\sigma_{\beta 3} + k_{\beta 3}) da.$$

The conditions (51.4) are similarly satisfied on the basis of the equilibrium equations and boundary conditions. This can be shown in the same way as in the case of homogeneous and isotropic media. With this, the problem which posed is solved.

In the case of homogeneous and anisotropic media, this problem was studied by Chirita [60].

d) Bars composed of different nonhomogeneous anisotropic materials. Let $L = L_1 \cup L_2$, $L_1 \cap L_2 = \emptyset$. We assume that Γ is an arc of the curve from Σ so that $L_0 \cup \Gamma$ is the boundary of the domain Σ_0 included in Σ so that $\Sigma_1 \cap \Sigma_2 = \emptyset$. We will consider that the domains Σ_0 are occupied by anisotropic media with different thermolelastic properties, and designate by $C_{ij}^{(0)}, \beta_{ij}^{(0)}$

the corresponding coefficients. Let R_p be the three dimensional domain occupied by the medium with the coefficient $C_{ijkl}^{(\rho)}, \beta_{ij}^{(\rho)}$. We assume that

$$C_{ijkl}^{(\rho)} = C_{ijkl}^{(\rho)}(x_1, x_2), \beta_{ij}^{(\rho)} = \beta_{ij}^{(\rho)}(x_1, x_2) \text{ in } R_p. \quad (51.104)$$

Similarly, we will assume that the tensor $C_{ijkl}^{(\rho)}$ is positively defined and of the class C^∞ . The domain Σ will be considered as being occupied by an elastic medium with discontinuous coefficients at the passage through Γ .

The displacement vector and the tension vector must satisfy the conditions of continuity

$$[u_i]_1 = [u_i]_2, [t_{i\alpha}]_1 v_\alpha = [t_{i\alpha}]_2 v_\alpha \text{ on } \Gamma \times [0, l]. \quad (51.105)$$

where v_α are the components of the versor of the normal to Γ , exterior to the domain Σ_1 .

The problem of thermoelasticity consists in the integration of equations (51.2), (51.78) with the conditions (51.3), (51.105), (51.4) -- (51.7), in the case when the temperature is given by (51.1) and C_{ijkl}, β_{ij} have the form of (51.104) in R_p .

Let us consider the problem of the plane generalized deformation in the isothermal case for the domain Σ [203]. We will take into account the mass forces $f_i^{(\rho)} \in C^\infty(\Sigma_p)$ so that the equilibrium equations have the form

$$t_{i\alpha,\alpha} + f_i^{(\rho)} = 0 \text{ in } \Sigma_p. \quad (51.106)$$

The constitutive equations are

$$t_{i\alpha} = C_{i\alpha k\beta}^{(\rho)} u_{k,\beta} \text{ in } \Sigma_p. \quad (51.107)$$

If the components of the displacement vector and of the tension vector are continuous at the passage from one medium to another we have the conditions

$$[u_i]_1 = [u_i]_2, \quad [t_{ia}]_1 v_a = [t_{ia}]_2 v_a \quad \text{on } \Gamma. \quad (51.108)$$

Let us consider the following boundary conditions

/222

$$[t_{ia} u_a]_1 = h_i^{(p)} \quad \text{on } L_1, \quad (51.109)$$

where $h_i^{(p)}$ are functions of the class C^∞ .

If we consider the results derived by Fichera [112], it follows that under the assumption of certain regularities (see [112] p. 386) the boundary problem (51.106) -- (51.109) has a solution $u_i \in C^\infty(\Sigma_1 \cup L_1) \cap C^\infty(\Sigma_2 \cup L_2) \cap C^\infty(\Sigma)$ if and only if

$$\begin{aligned} \sum_{p=1}^2 \left[\int_{\Sigma_p} f_i^{(p)} da + \int_{L_p} h_i^{(p)} ds \right] &= 0, \\ \sum_{p=1}^2 \left[\int_{\Sigma_p} e_{3\alpha\beta} x_\alpha f_\beta^{(p)} da + \int_{L_p} e_{3\alpha\beta} x_\alpha h_\beta^{(p)} ds \right] &= 0. \end{aligned} \quad (51.110)$$

It can easily be seen that, if the conditions (51.108) are replaced by

$$[u_i]_1 = [v_i]_2, \quad [t_{ia}]_1 v_a = [t_{ia}]_2 v_a + g_i \quad \text{on } \Gamma, \quad (51.111)$$

where g_i are functions of the class C^∞ , then the conditions (51.110) are replaced by

$$\begin{aligned} \sum_{p=1}^2 \left[\int_{\Sigma_p} f_i^{(p)} da + \int_{L_p} h_i^{(p)} ds \right] + \int_{\Gamma} g_i ds &= 0, \\ \sum_{p=1}^2 \left[\int_{\Sigma_p} e_{3\alpha\beta} x_\alpha f_\beta^{(p)} da + \int_{L_p} e_{3\alpha\beta} x_\alpha h_\beta^{(p)} ds \right] + \int_{\Gamma} e_{3\alpha\beta} x_\alpha g_\beta ds &= 0. \end{aligned} \quad (51.112)$$

We will use four special generalized plane deformation problems that we will designate by $C^{(s)}$ ($s = 1, 2, 3, 4$). Let $v_1^{(s)}$ and $\sigma_{i\alpha}^{(s)}$ be the components of the displacement vector and the components of the tension tensor respectively, from the problem $C^{(s)}$. The problems $C^{(s)}$ are characterized by the equations

$$\begin{aligned}\sigma_{i\alpha}^{(1)} + (C_{i\alpha 33}^{(1)} x_\beta)_{,\alpha} &= 0, \quad (\beta = 1, 2), \\ \sigma_{i\alpha}^{(2)} + C_{i\alpha 33}^{(2)} &= 0, \\ \sigma_{i\alpha}^{(4)} - e_{\gamma\beta 3} (C_{i\alpha \gamma 3}^{(4)} x_\beta)_{,\alpha} &= 0,\end{aligned}\tag{51.113}$$

$$\sigma_{i\alpha}^{(s)} = C_{i\alpha k\beta}^{(s)} v_{k,\beta} \quad \text{in } \Sigma_s,\tag{51.114}$$

and the conditions

$$[v_i^{(s)}]_1 = [v_i^{(s)}]_2, \quad [\sigma_{i\alpha}^{(s)}]_1 v_\alpha = [\sigma_{i\alpha}^{(s)}]_2 v_\alpha + g_i^{(s)} \quad \text{on } \Gamma,\tag{51.115}$$

$$\begin{aligned}[\sigma_{i\alpha}^{(1)} n_\alpha]_0 &= -C_{i\alpha 33}^{(1)} x_\beta n_\alpha, \quad (\beta = 1, 2), \\ [\sigma_{i\alpha}^{(2)} n_\alpha]_0 &= -C_{i\alpha 33}^{(2)} n_\alpha, \\ [\sigma_{i\alpha}^{(4)} n_\alpha]_0 &= e_{\gamma\beta 3} C_{i\alpha \gamma 3}^{(4)} x_\beta n_\alpha \quad \text{on } l_s,\end{aligned}\tag{51.116}$$

where

$$\begin{aligned}g_i^{(1)} &= [C_{i\alpha 33}^{(2)} - C_{i\alpha 33}^{(1)}] x_\beta v_\alpha, \\ g_i^{(2)} &= [C_{i\alpha 33}^{(2)} - C_{i\alpha 33}^{(1)}] v_\alpha, \\ g_i^{(4)} &= e_{\gamma\beta 3} [C_{i\alpha \gamma 3}^{(1)} - C_{i\alpha \gamma 3}^{(2)}] x_\beta v_\alpha.\end{aligned}\tag{51.117}$$

It can easily be verified that for the problems $C^{(s)}$, the conditions (51.112) are satisfied. In the following we assume that the functions $v_1^{(s)}$ and $\sigma_{1j}^{(s)}$ are known.

Let us search for the solution of the problem $P^{(0)}$ in the form of (51.87) in which $v_1^{(s)}$ are the solutions of the problems $C^{(s)}$, $v_1(x_1, x_2)$ are unknown functions and a_r ($r = 1, 2, 3, 4$) are unknown constants. In this case the components of the tension tensor are expressed as

$$t_{ij} = C_{ij33}^{(0)}(a_1 x_1 + a_2 x_2 + a_3) - C_{ij\alpha\beta}^{(0)} e_{\alpha\beta\gamma} a_\gamma \tau_\beta + \sum_{s=1}^4 a_s \sigma_{ij}^{(s)} + \pi_{ij}, \quad (51.118)$$

where

$$\pi_{ij} = C_{ij33}^{(0)} v_{k,3} - \beta_{ij}^{(0)} f. \quad (51.119)$$

If we consider the equations (51.113) and the relations (51.118), it follows that the equilibrium equations are reduced to

$$\pi_{i\alpha,\alpha} = 0, \text{ in } \Sigma. \quad (51.120)$$

Keeping in mind (51.116), the conditions on the lateral surface become

$$\pi_{i\alpha} n_\alpha = 0 \text{ on } L. \quad (51.121)$$

From (51.115), (51.117), (51.87), (51.118), and (51.105) the conditions follow

$$[v_i]_1 = [v_i]_2, \quad [\pi_{i\alpha}]_1 v_\alpha = [\pi_{i\alpha}]_2 v_\alpha \text{ on } \Gamma. \quad (51.122)$$

In the following, the functions $v_1(x_1, x_2)$ are the components of the displacement vector in the problem of the plane thermoelastic deformation (51.119) -- (51.122) in which the temperature variation

is $f(x_1, x_2)$. We will assume that this problem is solved.

The conditions (51.4) are satisfied on the basis of the equilibrium equations and the conditions on the lateral surface, the same way as in Paragraph 51 (c).

From the conditions (51.5) -- (51.7) we obtain the following system for the unknowns a_r

$$\sum_{r=1}^4 M_r a_r = A_r, \quad (r = 1, 2, 3, 4), \quad (51.123)$$

where we use the designations

$$M_{\alpha\beta} = \sum_{\rho=1}^2 \int_{\Sigma_\rho} x_\alpha [C_{\beta\beta\beta\beta}^{(\rho)} x_\beta + \sigma_{\beta\beta}^{(\rho)}] da, \quad (51.124)$$

224

$$M_{\alpha\beta} = \sum_{\rho=1}^2 \int_{\Sigma_\rho} x_\alpha [C_{\beta\beta\beta\beta}^{(\rho)} + \sigma_{\beta\beta}^{(\rho)}] da,$$

$$M_{\alpha\gamma} = \sum_{\rho=1}^2 \int_{\Sigma_\rho} x_\alpha [C_{\beta\gamma\beta\beta}^{(\rho)} x_\beta + \sigma_{\beta\beta}^{(\rho)}] da,$$

$$M_{\beta\alpha} = \sum_{\rho=1}^2 \int_{\Sigma_\rho} [C_{\beta\beta\beta\beta}^{(\rho)} x_\alpha + \sigma_{\beta\beta}^{(\rho)}] da,$$

$$M_{\beta\beta} = \sum_{\rho=1}^2 \int_{\Sigma_\rho} [C_{\beta\beta\beta\beta}^{(\rho)} + \sigma_{\beta\beta}^{(\rho)}] da,$$

$$M_{\beta\gamma} = \sum_{\rho=1}^2 \int_{\Sigma_\rho} [C_{\beta\beta\gamma\beta}^{(\rho)} x_\beta + \sigma_{\beta\beta}^{(\rho)}] da,$$

$$M_{\gamma\alpha} = \sum_{\rho=1}^2 \int_{\Sigma_\rho} c_{\beta\gamma\beta} x_\alpha [C_{\beta\beta\beta\beta}^{(\rho)} x_\alpha + \sigma_{\beta\beta}^{(\rho)}] da,$$

$$M_{\gamma\beta} = \sum_{\rho=1}^2 \int_{\Sigma_\rho} c_{\beta\gamma\beta} x_\alpha [C_{\beta\beta\beta\beta}^{(\rho)} + \sigma_{\beta\beta}^{(\rho)}] da,$$

$$M_{\gamma\gamma} = \sum_{\rho=1}^2 \int_{\Sigma_\rho} c_{\beta\gamma\beta} x_\alpha [C_{\beta\beta\gamma\beta}^{(\rho)} c_{\lambda\gamma\beta} x_\lambda + \sigma_{\beta\beta}^{(\rho)}] da,$$

$$A_\alpha = - \int_{\Sigma} x_\alpha \pi_{\beta\beta} da, \quad A_\beta = - \int_{\Sigma} \pi_{\beta\beta} da, \quad A_\gamma = - \int_{\Sigma} c_{\beta\gamma\beta} x_\alpha \pi_{\beta\beta} da.$$

It can be shown (see Iesan [205]) that $\det (M_{rs}) \neq 0$, so that the system (51.123) determines the constant a_r . With this, the problem $P^{(0)}$ is solved.

Let us study now the recurrence process. Let us search for the solution of the problem $P^{(n+1)}$ in the form (51.94) where $v_i^{(s)}$ are components of the displacement vector from the problem $C_1^{(s)}$, $w_1(x_1, x_2)$ are unknown functions and b_r ($r = 1, 2, 3, 4$) are unknown constants. In view of (51.94) it follows that

$$l_{ij} = (n+1) \left[\int_0^1 t_{ij}^{(0)} dx_3 + C_{ij33}^{(0)} (b_1 x_1 + b_2 x_2 + b_3) - \right. \\ \left. - C_{ij33}^{(0)} C_{\alpha\beta 33} b_\alpha x_\beta + \sum_{k=1}^4 b_k [\sigma_{ij}^{(k)} + \sigma_{ij} + l_{ij}^{(k)}] \right], \quad (51.125)$$

where

$$\sigma_{ij} = C_{ijk\beta}^{(0)} w_{k,\beta}, \quad (51.126)$$

$$l_{ij}^{(k)} = C_{ijk3}^{(0)} v_k^*(x_1, x_2, 0). \quad (51.127)$$

The equilibrium equations are reduced to

/225

$$\sigma_{i\alpha,\alpha} + G_i^{(0)} = 0 \text{ in } \Sigma_p, \quad (51.128)$$

in which we designated

$$G_i^{(0)}(x_1, x_2) = l_{i3}^*(x_1, x_2, 0) + l_{i3,3}^{(0)}. \quad (51.129)$$

The conditions on the lateral surface become

$$[\sigma_{i\alpha} n_\alpha]_p = \mu_i^{(0)} \text{ on } L_p, \quad (51.130)$$

where

$$p_i^{(p)} = -k_{i3}^{(p)} n_3. \quad (51.131)$$

If we keep in mind the relations (51.115), then the conditions (51.105) are reduced to

$$[w_i]_1 = [w_i]_2, \quad [\sigma_{i3}]_1 v_3 = [\sigma_{i3}]_2 v_3 + q_i \text{ on } \Gamma, \quad (51.132)$$

in which

$$q_i = [k_{i3}^{(2)} - k_{i3}^{(1)}] v_3. \quad (51.133)$$

Thus, the functions w_i are components of the displacement vector from the problem of the generalized plane deformation (51.126), (51.128), (51.130), (51.132). The necessary and sufficient conditions for the existence of the solution of this problem are satisfied, which can be easily proven. In the following we will assume that the functions w_i, σ_{ij} are known.

The conditions (51.4) are similarly satisfied, which is proven similarly as in Paragraph 51 (a). From the conditions (51.5) -- (51.7) we obtain the following system for the constants b_r

$$\sum_{s=1}^4 M_{rs} b_s = B_r, \quad (r = 1, 2, 3, 4), \quad (51.134)$$

in which the constants M_{rs} are given by (51.124) and B_r takes the expressions

$$B_1 = - \sum_{p=1}^2 \int_{\Sigma_p} x_1 (\sigma_{33} + k_{33}^{(p)}) da, \quad B_3 = - \sum_{p=1}^2 \int_{\Sigma_p} [(\sigma_{33} + k_{33}^{(p)})] da,$$

$$B_4 = - \sum_{p=1}^2 \int_{\Sigma_p} e_{312} e_2 (\sigma_{23} + k_{23}^{(p)}) da.$$

The system (51.134) determines the constants b_r and consequently the problem posed is solved. Obviously, the case in which the domain Σ is occupied by n elastic media with different properties can be treated in a similar manner.

52. Successive Approximations

Let the equations of the nonlinear theory of thermoelasticity be written in the form (21.12) -- (21.15). We will consider the initial conditions (21.9) and the boundary conditions (21.10). We assume that the reference state is a natural state and that the functions $\tilde{\sigma}$ and \tilde{Q}_i are analytic. Consequently, we have

$$\tilde{\sigma} = \sum_{r=2}^{\infty} A_r(E_{ij}, T, X_i), \quad \tilde{Q}_i = \sum_{r=1}^{\infty} Q_{ir}(E_{mn}, T, T_{,j}, X_i), \quad (52.1)$$

where A_r is a homogeneous polynomial of degree r in E_{ij} and T and Q_{ir} are polynomials of degree r in the variables E_{pq} , T and $T_{,j}$, while the coefficients of these polynomials are functions of X_s . In the case of homogeneous media, the coefficients of these polynomials are constants.

We assume that the charges depend analytically on the parameter ϵ so that

$$\begin{aligned} f_i &= \sum_{n=1}^{\infty} \epsilon^n f_i^{(n)}, & \gamma &= \sum_{n=1}^{\infty} \epsilon^n \gamma^{(n)}, \\ a_i &= \sum_{n=1}^{\infty} \epsilon^n a_i^{(n)}, & b_i &= \sum_{n=1}^{\infty} \epsilon^n b_i^{(n)}, & \gamma_{i0} &= \sum_{n=1}^{\infty} \epsilon^n \gamma_{i0}^{(n)}, \\ u_i^* &= \sum_{n=1}^{\infty} \epsilon^n \tilde{u}_i^{(n)}, & P_i^* &= \sum_{n=1}^{\infty} \epsilon^n P_i^{(n)}, \\ \Theta &= T_0 + \sum_{n=1}^{\infty} \epsilon^n \mathcal{T}^{(n)}, & H^* &= \sum_{n=1}^{\infty} \epsilon^n \mathcal{H}^{(n)}. \end{aligned} \quad (52.2)$$

The specification of the parameter ϵ depends on the specific problem under consideration (see for example Green and Adkins [141]).

We assume that the solution of the problem has the form

$$u_i = \sum_{n=1}^{\infty} \epsilon^n u_i^{(n)}, \quad T = T_0 + \sum_{n=1}^{\infty} \epsilon^n T^{(n)}. \quad (52.3)$$

We will establish the equations and boundary conditions and the initial conditions for the functions $u_i^{(n)}$, $T^{(n)}$.

In view of (20.9), (21.14), (21.15), (52.1) it follows that /227
we are able to write

$$\begin{aligned} T_{ij} &= C_{ijkl} u_{k,l} - \beta_{ij} \theta + \Pi_{ij}, \\ \rho_0 \gamma &= \beta_{ij} u_{i,j} + a \theta + S, \\ Q_i &= k_{ij} \theta_{,j} + \Lambda_i, \quad \theta = T - T_0, \end{aligned} \quad (52.4)$$

where Π_{ij} and S are polynomials in $u_{r,s}$ and T which do not contain linear terms, and Λ_i are polynomials in $u_{r,s}$, T and $T_{,k}$ which do not contain linear terms. The coefficients C_{ijkl} , β_{ij} , a and k_{ij} do not depend on $u_{r,s}$ and T and have symmetry properties (23.13).

From (52.3) and (52.4) we obtain

$$T_{ij} = \sum_{k=1}^{\infty} \epsilon^k T_{ij}^{(k)}, \quad \gamma = \sum_{k=1}^{\infty} \epsilon^k \gamma^{(k)}, \quad Q_i = \sum_{k=1}^{\infty} \epsilon^k Q_i^{(k)}, \quad (52.5)$$

where

$$\begin{aligned} T_{ij}^{(1)} &= C_{ijkl} u_{k,l}^{(1)} - \beta_{ij} T^{(1)}, \\ \rho_0 \gamma^{(1)} &= \beta_{ij} u_{i,j}^{(1)} + a T^{(1)}, \\ Q_i^{(1)} &= k_{ij} T_{,j}^{(1)}, \\ T_{ij}^{(n)} &= C_{ijkl} u_{k,l}^{(n)} - \beta_{ij} T^{(n)} + \Pi_{ij}^{(n)}, \\ \rho_0 \gamma^{(n)} &= \beta_{ij} u_{i,j}^{(n)} + a T^{(n)} + S^{(n)}, \\ Q_i^{(n)} &= k_{ij} T_{,j}^{(n)} + \Lambda_i^{(n)}, \quad n \geq 2. \end{aligned} \quad (52.6)$$

In the relations (52.6), $\Pi_{ij}^{(n)}$ and $S^{(n)}$ are polynomials of a higher degree than one in the variables $u_{r,s}^{(\alpha)}$ and $T^{(\alpha)}$ ($\alpha = 1, 2, \dots, n-1$) and $\Lambda_i^{(n)}$ are polynomials of a degree larger than one in $u_{r,s}^{(\alpha)}$, $T^{(\alpha)}$ and $r_{,k}^{(\alpha)}$ ($\alpha = 1, 2, \dots, n-1$).

From (21.12), (21.13), (52.2), (52.3) and (52.5) we obtain the equations

$$\begin{aligned} T_{ij}^{(n)} + \rho_0 f_i^{(n)} &= \rho_0 \dot{u}_i^{(n)}, \\ \rho_0 T_0 \ddot{\eta}^{(n)} - Q_i^{(n)} &= \rho_0 r^{(n)} + \rho_0 \Omega^{(n-1)}, \quad n = 1, 2, \dots, \end{aligned} \quad (52.7)$$

where $\Omega^{(0)} = 0$, $\Omega^{(k)} = -(T^{(1)} \dot{\eta}^{(k)} + \dots + T^{(k)} \dot{\eta}^{(1)})$, $k = 2, 3, \dots$

If we consider the relations (52.6), we derive from (52.7)

$$\begin{aligned} (C_{ikl} u_{l,i}^{(n)})_{,j} - (\beta_{,i} T^{(n)})_{,j} + \rho_0 k_{ij}^{(n)} &= \rho_0 \dot{u}_i^{(n)}, \\ (k_{ij} T^{(n)})_{,i} - T_0 \beta_{,i} \dot{u}_{i,j}^{(n)} - a T_0 \dot{\eta}^{(n)} &= -\rho_0 W^{(n)}, \\ n &= 1, 2, \dots, \end{aligned} \quad (52.8)$$

where

$$\begin{aligned} F_i^{(n)} &= f_i^{(n)}, \quad W^{(n)} = r^{(n)}, \\ \rho_0 k_{ij}^{(n)} &= \rho_0 f_{ij}^{(n)} + H_{ij}^{(n)}, \quad W^{(n)} = r^{(n)} - \frac{1}{\rho_0} \Lambda_{i,i}^{(n)} + \\ &+ \Omega^{(n-1)} - T_0 \dot{S}^{(n)}, \quad n = 2, 3, \dots \end{aligned} \quad (52.9)$$

The initial conditions become

$$u_i^{(n)}(X, 0) = a_i^{(n)}, \quad \dot{u}_i^{(n)}(X, 0) = b_i^{(n)}, \quad r^{(n)}(X, 0) = r_0^{(n)}. \quad (52.10)$$

The boundary conditions imply

$$u_i^{(n)} = \tilde{u}_i^{(n)} \text{ on } \Sigma_1 \times [0, t_0], (C_{ijkl} u_{k,l}^{(n)} - \beta_{ij} T^{(n)}) N_j = \tau_i^{(n)} \text{ on } \Sigma_2 \times [0, t_0], \quad (52.11)$$

$$T^{(n)} = \mathcal{T}^{(n)} \text{ on } \Sigma_3 \times [0, t_0], k_{ij} T_{,j}^{(n)} N_i = \tilde{\mathcal{T}}^{(n)} \text{ on } \Sigma_4 \times [0, t_0],$$

in which

$$\tau_i^{(1)} = P_i^{(1)}, \tilde{\mathcal{T}}^{(1)} = \mathcal{K}^{(1)}, \quad (52.12)$$

$$\tau_i^{(n)} = P_i^{(n)} - \Pi_{ij}^{(n)} N_j, \tilde{\mathcal{T}}^{(n)} = \mathcal{K}^{(n)} - \Lambda_i^{(n)} N_i, \quad n = 2, 3, \dots$$

When $n = 1$ the equations are the same as those in the linear theory with loads $f_i^{(1)}, r_i^{(1)}, a_i^{(1)}, b_i^{(1)}, \eta_0^{(1)}, \tilde{u}_1^{(1)}, p_1^{(1)}, \mathcal{T}^{(1)}, \mathcal{K}^{(1)}$. Let us assume that $u_1^{(1)}, T^{(1)}, \dots, u_1^{(m-1)}, T^{(m-1)}$ are determined from the above equations and conditions. Then for $n = m$ we obtain a problem characterized by equations of the linear thermoelasticity for the same domain in which the loads depend in a known way on the functions $u_1^{(\alpha)}, T^{(\alpha)}$ ($\alpha = 1, 2, \dots, m-1$), determined in the previous pages. The solution of the problem for $n = 1$, if it exists, is unique, and consequently the loads which appear in the problem with $n = 2$ are uniquely determined. Thus, for $n = 2$, the solution is unique. In general the functions $u_1^{(n)}$ and $T^{(n)}$ are uniquely determined.

The method presented represents an extension of the method of the successive approximations of the theory of elasticity given by Signorini [365] (see also Wang and Truesdell [436]).

An interpretation of the development of (52.3) follows from the generalization (Iesan [188]) of Rivlin and Topaloglu's theorem [350] of the theory of nonlinear elasticity.

In general, if the equations (52.1) are substituted by

$$\tilde{\sigma} = \sum_{i=1}^{N+1} A_i, \quad Q_i = \sum_{j=1}^N Q_{ij},$$

we say that we are dealing with a theory of thermoelasticity of the order N .

53. The Theory of Thermoelasticity of the Second Order

/229

We will establish here the equations of the theory of thermoelasticity of the second order in the manner given by Chadwick and Seet [52].

The following considerations assume certain conditions of regularity for the functions of $\tilde{\sigma}$ and \tilde{Q}_i in a neighborhood corresponding to the reference configurations; these conditions appear in all theories of this type.

Let \mathcal{K} be set of functions $\{u_i, T\}$ corresponding to the admissible thermodynamic processes in the medium under consideration. The element $\{0, T_0\}$ from \mathcal{K} corresponds to the thermodynamic process in which the medium remains always in the reference configuration.

It is always possible to introduce (see Paragraph 35) a length l_0 and a time t_0 , which are characteristic for the material under consideration, which together with ρ_0 and T_0 permit the use of certain adimensional quantities.

Let

$$\begin{aligned} a_1 &= \sup_{\mathbf{u}} \{ |u_{i,j}| + l_0 T_0^{-1} |T_{,r}| + T_0^{-1} |\theta| \}, \\ a_2 &= \sup_{\mathbf{u}} \{ |u_{i,j}| + l_0 T_0^{-1} |T_{,r}| + T_0^{-1} |\theta| + \\ &+ l_0 |u_{i,jr}| + l_0^2 T_0^{-1} |T_{,rj}| + t_0 |\dot{u}_{i,j}| + t_0 T_0^{-1} |\dot{\theta}| \}, \end{aligned} \quad (53.1)$$

where

$$|A_0 \dots| = (A_0 \dots A_0 \dots)^{1/2}.$$

The quantities a_1 and a_2 are called the primary and secondary amplitudes, respectively, of the admissible process. Obviously $a_1 \leq a_2$. We will designate by \mathcal{X}_0 the subset of \mathcal{X} defined as the intersection of the domains of the functionals a_1 and a_2 . In order to establish the results which follow, we will limit ourselves to admissible processes for which $\{u, T\} \in \mathcal{X}_0$, $a_1^3 \ll 1$, $a_2^3 \ll 1$. As will be shown, the restriction $a_1^3 \ll 1$ is sufficient to derive the constitutive equations of the theory of the second order thermoelasticity and the condition $a_2^3 \ll 1$ is necessary to justify their substitution in the equations of motion and in the equation of energy.

We assume that $\bar{\sigma}$ is of the class C^4 with respect to F_{ij} and T in the neighborhood of the reference configuration

$$|E_{ij}| + T_0^{-1}|0| < \delta,$$

and that \bar{Q}_i are of the class C^3 with respect to F_{ij} , $T_{,r}$, T in the neighborhood

$$|E_{ij}| + l_0 T_0^{-1}|T_{,r}| + T_0^{-1}|0| < \delta,$$

while the positive constant δ was selected in such a way that

$$\{u_{i,j} | u_{i,j} \leq a_1\} \subseteq \{u_{i,j} | |E_{ij}| < \delta\}.$$

Developing in Taylor's series around the reference configuration we obtain

/230

$$\begin{aligned} \bar{\sigma} &= \frac{1}{2} C_{ijkl} E_{ij} E_{kl} - \beta_{ij} E_{ij} \theta - \frac{1}{2} a \theta^2 + \frac{1}{6} C_{ijklmn} E_{ij} E_{kl} E_{mn} + \\ &+ \frac{1}{2} C_{pqrs}^{(1)} E_{pq} E_{rs} \theta - \frac{1}{2} \beta_{pq}^{(1)} E_{pq} \theta^2 - \frac{1}{6} b \theta^3 + o(a_1^4), \\ \bar{Q}_i &= \left(k_{ij} + m_{ijr} E_{rr} + \frac{1}{2} k_{ijr} \theta_{,r} + k_{ij}^{(1)} \theta \right) \theta_{,i} + o(a_1^3), \end{aligned} \quad (53.2)$$

where

$$C_{ijkl} = \left(\frac{\partial^2 \bar{\sigma}}{\partial E_{ij} \partial E_{kl}} \right)_0, \quad C_{ijlmn} = \left(\frac{\partial^3 \bar{\sigma}}{\partial E_{ij} \partial E_{kl} \partial E_{mn}} \right)_0, \quad (53.3)$$

$$C_{ijhi}^0 = \left(\frac{\partial^2 \bar{\sigma}}{\partial E_{ij} \partial E_{hi} \partial T} \right)_0, \quad \beta_{ij} = - \left(\frac{\partial^2 \bar{\sigma}}{\partial E_{ij} \partial T} \right)_0, \quad (53.3)$$

$$\beta_{ij}^{(1)} = - \left(\frac{\partial^2 \bar{\sigma}}{\partial E_{ij} \partial T^2} \right)_0, \quad a = - \left(\frac{\partial^2 \bar{\sigma}}{\partial T^2} \right)_0, \quad b = - \left(\frac{\partial^2 \bar{\sigma}}{\partial T^3} \right)_0, \quad k_{ij} = \left(\frac{\partial \bar{Q}_i}{\partial T_{,j}} \right)_0,$$

$$k_{ijr} = \left(\frac{\partial^2 \bar{Q}_i}{\partial T_{,j} \partial T_{,r}} \right)_0, \quad k_{ij}^{(1)} = \left(\frac{\partial^2 \bar{Q}_i}{\partial T_{,j} \partial T} \right)_0, \quad m_{ijr} = \left(\frac{\partial^2 \bar{Q}_i}{\partial T_{,j} \partial E_{r,}} \right)_0.$$

The new thermoelastic coefficients have the following symmetry properties

$$C_{ijlmn} = C_{jilmn} = C_{ijlmn} = C_{mnril},$$

$$C_{ijrs}^{(1)} = C_{ijsr}^{(1)} = C_{rsij}^{(1)}, \quad (53.4)$$

$$\beta_{ij}^{(1)} = \beta_{ji}^{(1)}, \quad k_{ijr} = k_{rji}, \quad m_{ijr} = m_{ijr}.$$

Substituting (53.2) in (21.14) and setting

$$E_{ij} = \varepsilon_{ij} + \frac{1}{2} u_{r,i} u_{r,j}, \quad 2\varepsilon_{ij} = u_{i,j} + u_{j,i}, \quad (53.5)$$

we obtain

$$\begin{aligned} T_{ij} &= C_{ijrs} \varepsilon_{rs} - \beta_{ij} \theta + \frac{1}{2} C_{ijlmn} \varepsilon_{rs} \varepsilon_{mn} + C_{ijlmn} u_{m,n} u_{l,r} + \\ &+ \frac{1}{2} C_{ijlmn} u_{r,m} u_{r,n} + (C_{ijrs}^{(1)} \varepsilon_{rs} - \beta_{ij} u_{r,j}) \theta - \frac{1}{2} \beta_{ij}^{(1)} \theta^2 + o(a_1^2), \\ \rho_0 \eta &= \beta_{ij} \varepsilon_{ij} + a \theta - \frac{1}{2} C_{ijrs}^{(1)} \varepsilon_{ij} \varepsilon_{rs} + \frac{1}{2} \beta_{ij} u_{m,i} u_{m,j} + \\ &+ \beta_{ij}^{(1)} \varepsilon_{ij} \theta + \frac{1}{2} b \theta^2 + o(a_1^2). \end{aligned} \quad (53.6)$$

It follows from (53.2)₂ that

$$\bar{Q}_i = \left(k_{ij} + m_{ijrs} \varepsilon_{rs} + \frac{1}{2} k_{rjs} \theta_{,r} + k_{ij}^{(1)} \theta \right) \theta_{,j} + o(a_1^2). \quad (53.7)$$

The constitutive equations of the theory of thermoelasticity of the second order are obtained from (53.6) and (53.7) imposing the condition that $a_1^3 \ll 1$ which justifies the neglect of terms which have not been explicitly stated. When expressions (53.6), (53.7) are introduced in the equations (21.12), (21.13), terms will appear containing both $u_{1,j}$, $\theta_{,r}$, θ and also $u_{1,rs}$, $\theta_{,pq}$, $\dot{u}_{1,j}$, $\dot{\theta}$. As a general rule the terms remaining from (53.6) and (53.7) will not generate terms of the order a_1^3 . Thus, the condition $a_1^3 \ll 1$ is not sufficient to express the equations with partial derivatives of the theory with the help of the components of displacement and the temperature variation. We will replace this condition by a stronger condition $a_2^3 \ll 1$. It follows from (53.1)₂ that the terms which appear by the introduction of the constitutive equations in the equations (21.12), (21.13) are of the order a_2 . Inasmuch as a quantity which is of the order of a_1^p is of the order a_2^p , it follows that the order with respect to a_2 of the relations (53.6), (53.7) is maintained by the derivation operations implied by equations (21.12), (21.13).

Thus, imposing the condition $a_2^3 \ll 1$, the equations with partial derivatives of the theory of second order thermoelasticity expressed in variables u_1 , θ are obtained by replacing (53.6), (53.7) in (21.12), (21.13) and by neglecting the terms in $O(a_2^3)$.

In the case of the homogeneous and anisotropic media, these equations are

$$\begin{aligned}
 & C_{ijrs} u_{r,sj} - \beta_{ij} \theta_{,j} + C_{ijrsmn} u_{r,s} u_{m,nj} + \\
 & + C_{jrmn} (u_{m,n} u_{i,rj} + u_{m,nj} u_{i,r}) + C_{ijmn} u_{r,m} u_{r,nj} + C_{ijrs}^{(1)} (u_{r,j} \theta + \\
 & + u_{r,j} \theta_{,j}) - \beta_{jr} (u_{i,rj} \theta + u_{i,r} \theta_{,j}) - \beta_{ij}^{(1)} \theta \theta_{,j} + \rho_0 f_i = \rho_0 \ddot{u}_i, \\
 & k_{ij} \theta_{,ij} + m_{ijrs} (u_{r,s} \theta_{,j} + u_{r,s} \theta_{,ij}) + k_{irj} \theta_{,r} \theta_{,ij} + \\
 & + k_{ij}^{(1)} (\theta \theta_{,j})_{,i} - T_0 a \dot{\theta} - T_0 (\beta_{ij} \dot{u}_{i,j} - C_{ijrs}^{(1)} u_{i,j} \dot{u}_{r,s} + \\
 & + \beta_{ij} u_{m,n} \dot{u}_{m,n} + \beta_{ij}^{(1)} u_{i,j} \dot{\theta}) - (\beta_{ij} + T_0 \beta_{ij}^{(1)}) \dot{u}_{i,j} \theta - \\
 & - (a + b T_0) \theta \dot{\theta} = - \rho_0 r.
 \end{aligned} \tag{53.8}$$

If we consider the relations from Paragraph 22, we can derive the equations of second order thermoelasticity for isotropic media. Thus, in this case the constitutive equations are [52]

$$\begin{aligned}
 T_{ij} = & \lambda \epsilon_{rr} \delta_{ij} + 2 \mu \epsilon_{ij} - \beta \theta \delta_{ij} + \\
 & + \left\{ \frac{1}{2} \nu_1 \epsilon_{rr} \epsilon_{rr} + \nu_2 \epsilon_{rr} \epsilon_{rr} + \frac{1}{2} \lambda u_{m,m} u_{m,m} + \lambda^{(1)} \epsilon_{rr} \theta - \right. \\
 & - \frac{1}{2} \beta^{(1)} \theta^2 \delta_{ij} + 2(\nu_2 \epsilon_{pp} + \mu^{(1)} \theta) \epsilon_{ij} + \\
 & + (\lambda \epsilon_{pp} - \beta \theta) u_{i,i} + 4(\mu + \nu_3) \epsilon_{,i} \epsilon_{,j} - \mu u_{p,i} u_{j,p}, \\
 \rho_0 \eta = & \beta \epsilon_{rr} + \alpha \theta - \frac{1}{2} \lambda^{(1)} \epsilon_{rr} \epsilon_{rr} - \mu^{(1)} \epsilon_{ij} \epsilon_{ij} + \\
 & + \frac{1}{2} \beta u_{p,q} u_{p,q} + \beta^{(1)} \epsilon_{rr} \theta + \frac{1}{2} b \theta^2, \\
 Q_i = & k \theta_{,i} + m_1 \theta_{,i} \epsilon_{rr} + 2 m_2 \epsilon_{rr} \theta_{,i} + k^{(1)} \theta \theta_{,i}.
 \end{aligned} \tag{53.9}$$

/232

In the case of homogeneous and isotropic media, the equations (53.8) become

$$\begin{aligned}
 & \{ \lambda + \mu + (\nu_1 + \nu_2) u_{r,r} + (\lambda^{(1)} + \mu^{(1)} \theta) u_{r,r} + \\
 & + \{ \mu + (\lambda + \nu_2) u_{r,r} + (\mu^{(1)} - \beta) \theta \} u_{i,rr} + \\
 & + \{ (\lambda + \mu) u_{r,r} + 2(\nu_2 + \nu_3) u_{(r,s)} \} u_{r,ii} + 2(\mu + \nu_3) u_{(r,s)} u_{i,rs} + \\
 & + \{ (\lambda + \mu) u_{i,r} + 2(\nu_2 + \nu_3) u_{(i,r)} \} u_{j,rr} + 2(\mu + \nu_3) u_{(i,s)} u_{s,rr} - \\
 & - (\beta + \beta^{(1)} \theta - \lambda^{(1)} u_{r,r}) \theta_{,i} + (2\mu^{(1)} u_{(i,r)} - \beta u_{i,r}) \theta_{,r} + \rho_0 f_i = \rho_0 \ddot{u}_i, \\
 & (k + k^{(1)} \theta + m_1 u_{r,r}) \theta_{,ii} + 2 m_2 u_{r,s} \theta_{,rs} + \\
 & + (m_1 u_{j,rr} + 2 m_2 u_{(r,s)}) \theta_{,rr} + k^{(1)} \theta_{,rr} \theta - \\
 & - T_0 a \dot{\theta} - (a + b T_0) \theta \dot{\theta} - T_0 \beta^{(1)} u_{r,r} \dot{\theta} - \\
 & - \{ T_0 \beta + (\beta + T_0 \beta^{(1)} \theta - T_0 \lambda^{(1)} u_{r,r}) \} u_{r,i} - \\
 & - T_0 (\beta u_{r,r} - 2 \mu^{(1)} u_{(r,s)}) \dot{u}_{r,i} = - \rho_0 \dot{r}_i.
 \end{aligned} \tag{53.10}$$

where

$$u_{(i,j)} = \frac{1}{2} (u_{i,j} + u_{j,i}).$$

An incorrect version of the equations of second order thermoelasticity for homogeneous and isotropic media was given by Dillon [86]. The correct relations for this case were derived by Hermann [170].

Application. Let us consider the case of plane waves in the theory of thermoelasticity of the second order. We assume that $u_1 = u_1(x_1, t)$, $\theta = \theta(x_1, t)$. Equations (53.10) in the absence of mass forces and thermal sources become

$$\begin{aligned} & \{ \lambda + 2\mu + (3\lambda + 6\mu + \nu_1 + 6\nu_2 + 8\nu_3) u_{1,1} + \\ & + (\lambda^{(1)} + 2\mu^{(1)} - \beta) \theta \} u_{1,11} + (\lambda + 2\mu + \nu_2 + \\ & + 2\nu_3) u_{1,1} u_{1,11} - \{ \beta - (\lambda^{(1)} + 2\mu^{(1)} - \beta) u_{1,1} + \beta^{(1)} \theta \}_{,1} = \rho_0 \ddot{u}_1, \end{aligned} \quad (53.11)$$

/233

$$\begin{aligned} & \{ \mu + (\lambda + 2\mu + \nu_2 + 2\nu_3) u_{1,1} + (\mu^{(1)} - \beta) \theta \} u_{1,11} + \\ & + (\lambda + 2\mu + \nu_2 + 2\nu_3) u_{1,1} u_{1,11} + (\mu^{(1)} - \beta) u_{1,1} \theta_{,1} = \rho_0 \ddot{u}_1, \\ & \{ k + k^{(1)} \theta + (m_1 + 2m_2) u_{1,1} \} \theta_{,11} + \{ (m_1 + 2m_2) u_{1,1} + \\ & + k^{(1)} \theta_{,1} \} \theta_{,1} - T_0 a \dot{\theta} - (a + b T_0) \theta \dot{\theta} - T_0 \beta^{(1)} u_{1,1} \dot{\theta} - \\ & - \{ T_0 \beta - T_0 (\lambda^{(1)} + 2\mu^{(1)} - \beta) u_{1,1} + (\beta + T_0 \beta^{(1)}) \theta \} \dot{u}_{1,1} - \\ & - T_0 (\mu^{(1)} - \beta) u_{1,1} \dot{u}_{1,1} = 0, \end{aligned} \quad (53.11)$$

where $\Gamma, \Lambda = 2, 3$.

If we set $u_\Gamma = 0$ ($\Gamma = 2, 3$) then we obtain from (53.11) two equations in the unknowns u_1 and θ which permit the study of purely longitudinal waves. Similarly, if we set in (53.11) $u_1 = u_2 = 0$, we can study the possibility of purely transversal waves. In contrast to the linear theory, these waves will be affected by the thermal field.

Johnson [225] carried out a study of longitudinal thermoelastic waves.

54. A Stoppelli Type Theorem of Existence and Uniqueness

Let us consider a homogeneous medium which is in thermo-elastic equilibrium under the action of mass forces $\rho_0 \epsilon F_i$, of the superficial forces ϵP_i of the heat source $\rho_0 \epsilon W$ and of the thermal flux ϵH where ϵ is a parameter, and F_i, W, P_i, H are prescribed functions. If we consider the relations (21.12) -- (21.16) it follows that in this case the basic equations are

the equilibrium equations

$$T_{ji,j} + \rho_0 \epsilon F_i = 0, \quad (54.1)$$

the energy equation

$$Q_{i,i} + \rho_0 \epsilon W = 0, \quad (54.2)$$

the constitutive equations

$$\sigma_{ij} = \hat{\sigma}_{ij}(u_{k,l}, T), \quad T_{ii} = \frac{\partial \hat{\sigma}}{\partial u_{i,i}}, \quad \tau_i = - \frac{1}{\rho_0} \frac{\partial \hat{\sigma}}{\partial T}, \quad (54.3)$$

$$Q_i = \hat{Q}_i(u_{k,l}, T, T_{i,i}), \text{ in } B.$$

The boundary conditions are

$$T_{ji} N_j = \epsilon P_i, \quad Q_i N_i = \epsilon H \text{ on } \partial B. \quad (54.4)$$

The problem posed represents an extension of the problem considered by Stoppelli [397] in the nonlinear theory of elasticity to the theory of thermoelasticity. Stoppelli [397] studied the equations of the theory of nonlinear elastostatics for charges of the type which appear in (54.1), (54.4) and under certain conditions he proved a theorem of existence and uniqueness. Moreover, if $|\epsilon|$ is sufficiently small, Stoppelli showed that the solution to which the theorem of existence and uniqueness refers depends, analytically, on ϵ . Truesdell and Noll [421] and Wang and Truesdell [436] presented Stoppelli's theorem. In the following, we will present briefly Stoppelli's method with reference to the problem of thermo-elasticity (54.1) -- (54.4) (a problem studied by Nistor [309]).

It can easily be seen that in order to have the existence of a solution, the following conditions are necessary.

$$\int_{\partial B} P_i dA + \int_B \rho_0 F_i dV = 0, \quad \int_{\partial B} H dA + \int_B \rho_0 S dV = 0. \quad (54.5)$$

Following Signorini [366], we assume that we have

$$\int_{\partial B} c_{ijk} X_j P_k dA + \int_B \rho_0 c_{ijk} X_j F_k dV = 0. \quad (54.6)$$

The idea consists in the reduction of the problem of the existence and uniqueness of the solution of the problem of nonlinear thermoelasticity, when $|\epsilon|$ is sufficiently small, to the corresponding problem from the linear theory.

We assume that the reference state is the natural state. This means (see Paragraph 23) that the function $\tilde{\sigma}$ is such that we have

$$\frac{\partial \tilde{\sigma}}{\partial u_{i,j}} = 0, \quad \frac{\partial \tilde{\sigma}}{\partial T} = 0 \quad \text{for } u_{i,j} = 0, T = T_0. \quad (54.7)$$

The functions \hat{Q}_1 is satisfied by the conditions (20.9), thus we have

$$\hat{Q}_i(u_{i,j}, T, 0) = 0. \quad (54.8)$$

In certain conditions there result the existence and uniqueness of a solution $\{u_i, T\}$ with the second order derivatives which satisfy the conditions of Hölder and the condition

$$u_i(0) = 0, \quad T(0) = T_0. \quad (54.9)$$

Let the equations of nonlinear thermoelasticity be

$$T_{ji,j} + \rho_0 f_i = 0, \quad Q_{i,i} + \rho_0 r = 0, \quad (54.10)$$

with the conditions

$$T_{ji}^* N_j = \tilde{T}_i, \quad Q_i^* N_i = \tilde{H} \text{ on } \partial B, \quad (54.11)$$

and (54.9).

Let us consider the deformation $x_1(X)$ obtained by the superimposition of a rigid rotation Q , around the origin on the deformation $x_1^*(X)$. In view of the designations from Paragraph 19, we have $F = QF^*$. It can easily be seen that the relations $T_{ji,j} = Q_{fs} T_{js,j}^* Q_{r,r} = Q_{i,i}^*$ take place.

Consequently, if the functions $\{x_1^*(X), T^*(X)\}$ represent a solution of the problem (54.10), (54.11), then $\{x_1, T\}$ where $x_1 = Q_{ij} x_j^*$, $T = T^*$ represents a solution of the problem

/235

$$T_{ji,j} + \rho_0 Q_{is} f_s = 0, \quad Q_{i,i} + \rho_0 r = 0 \text{ in } B, \quad (54.12)$$

$$T_{ji} N_j = Q_{is} \tilde{T}_s, \quad Q_i N_i = \tilde{H} \text{ on } \partial B. \quad (54.13)$$

Inversely, if $\{x_j(X), T(X)\}$ is a solution of problem (54.12), (54.13) then $\{x_1^* = Q_{ji} x_j, T = T^*\}$ represents a solution of the problem (54.10), (54.11). It is appropriate to consider the problem (54.12), (54.13).

At the start we will introduce some designations. We say that the function u , defined on the domain limited by Ω , is of the class $C^{k,\lambda}$ on Ω if $u \in C^k(\Omega)$, and the derivatives of the order k satisfy the condition of Hölder with the exponent λ on Ω . Let $x^* \in \partial\Omega$. We say that (y_1, y_2, y_3) is a system of cartesian local coordinates with the origin in x^* , if $y_1 = a_{1j}(x_j - x_j^*)$, with (a_{1j}) the orthogonal matrix and the axis $x^* y_3$ is directed according to the exterior normal at the point s . The surface

$\partial\Omega$ is of the class $C^{k,\lambda}$ if there is a number $r > 0$ so that the intersection of the sphere with the center in x^* and radius r , $V(x^*, r)$ with $\partial\Omega$ may be represented by $y_3 = f(y_1, y_2)$ with $f \in C^{k,\lambda}(D)$, D being the projection of $V \cap \partial\Omega$ on $y_3 = 0$.

We designate by Z the space of the elements of the form $z = \{u_j, T\}$ where $u_i \in C^{2,\lambda}(B)$, $T \in C^{2,\lambda}(B)$. For $z \in Z$, we introduce the norm

$$\|z\| = \sum_{i=1}^4 \left\{ \max_B |z_i| + \sum_{j=1}^3 \max_B |z_{i,j}| + \sum_{j,r=1}^3 (\max |z_{i,jr}| + \beta_{ijr}) \right\}, \quad (54.14)$$

where $z_1 = u_1$, $z_4 = T$, $\beta_{1j,r}$ is the Hölder coefficient of the function $u_{1,jr}$, and $\beta_{4j,r}$ that of the function $T_{j,r}$. With this norm the space Z is a Banach space.

Let Z' be the space of the elements of the form

$$z' = \{\Phi_i, \Psi; \varphi_i, \psi\},$$

which satisfy the conditions

$$\begin{aligned} (\alpha) \quad & \Phi_i, \Psi \in C^{0,\lambda}(B), \quad \varphi_i, \psi \in C^{1,\lambda}(\partial B), \\ & \int_B \Phi_i dV + \int_{\partial B} \varphi_i dA = 0, \\ (\beta) \quad & \int_B e_{ijk} X_j \Phi_k dV + \int_{\partial B} e_{ijk} X_j \varphi_k dA = 0, \\ & \int_B \Psi dV + \int_{\partial B} \psi dA = 0. \end{aligned}$$

For $z' \in Z'$ we define the norm

$$\begin{aligned} \|z'\| = & \sum_{i=1}^3 [\max_B |\Phi_i| + \max_{\partial B} |\varphi_i| + b_i + c_i] + \\ & + \max_B |\Psi| + \max_{\partial B} |\psi| + b + c, \end{aligned} \quad (54.15)$$

where b_1, b are Hölder coefficients of the functions ϕ_1, ψ , and c_1, c are the largest of the Hölder coefficients of the derivatives of the functions ϕ_1, ψ with respect to the parameters of the representation of ∂B . It can be verified that Z' , provided with the norm (54.15) is a Banach space.

Let us designate by Z'' the subspace of the elements of Z which satisfy

$$\operatorname{rot} u = 0, u_t = 0, T = T_0 \quad \text{for } X = 0. \quad (54.16)$$

It can be shown that Z'' is also a Banach space. We will assume that

- (i) E is a compact set and ∂B is of the class $C^{2,\lambda}$
- (ii) $\tilde{\sigma}$ is of the class C^4 with respect to the variables $u_{1,j}, T$ and \hat{Q}_1 are of the class C^3 with respect to $u_{1,j}, T, r$, T in the sphere $S = \{z \mid \|z - \tilde{z}_0\| \leq R\}$ from Z , where $\tilde{z}_0 = (0, 0, 0, T_0)$;
- (iii) $z'_0 = \{\rho_0 F_0, \rho_0 W; P_0, H\} \in Z'$.

Let us consider the relations

$$\begin{aligned} T_{j,i} + \varepsilon \rho_0 Q_{i,j} F_j &= \Phi_i, & Q_{i,i} + \varepsilon \rho_0 r &= \Psi \text{ in } B, \\ -T_{j,i} N_j + \varepsilon Q_{ij} P_j &= \varphi_i & -Q_{ij} N_j + \varepsilon H &= \psi \text{ on } \partial B. \end{aligned} \quad (54.17)$$

Considering these relations as a functional transformation, we will try to determine the rigid rotation (Q_{1j}) and the functions $\{x_1(X), T(X)\}$ so that $\phi_1 = 0, \psi = 0, \phi_1 = 0, \psi_1 = 0$ and that relations (54.5), (54.6) be satisfied. Let $z = \{u_1 T\}$, $z' = \{\phi_1, \psi; \phi_1, \psi\}$, and $G(z) = z'$ be the application defined by (54.17). We will determine the rigid rotation Q_{1j} so that element z' defined by (54.17) belongs to the space Z' . Accordingly, we will have $G: Z \rightarrow Z'$. Keeping in mind the assumptions (i) -- (iii), it follows that the first condition from the definition of Z' is satisfied. It follows from

conditions (β) that Q_{ij} must satisfy the equations

$$\varepsilon e_{ijk} Q_{kr} A_{jr} = M_i, \quad (54.18)$$

where

$$A_{jr} = \int_H \rho_0 X_j F_r dV + \int_{\partial H} X_j P_r dA, \quad (54.19)$$

$$M_i = \int_H e_{ijk} T_{jk} dV.$$

Consequently Q_{ij} must satisfy (54.18) and

$$Q_{im} Q_{jm} = Q_{ri} Q_{rj} = \delta_{ij}, \det (Q_{ij}) = 1. \quad (54.20)$$

We will write the equations (54.18) in the form

$$\mathcal{F}(Q) = \frac{1}{\varepsilon} M, \quad (54.21)$$

where the application of \mathcal{F} is defined by the set of their own orthogonal matrices. It is noted that $\mathcal{F}(\delta_{ij}) = 0$. The problem of the inverting of the application of \mathcal{F} was resolved by Stoppelli and, as is known, this application is inversible if, and only if

$$\det (A_{ij} - A_{rr} \delta_{ij}) \neq 0. \quad (54.22)$$

It is known that

$$\det (A_{ij} - A_{rr} \delta_{ij}) = 0, \quad (54.23)$$

then there exists an axis so that any rotation of the body around it, while maintaining the loads, does not modify the state of equilibrium. It is said that the loads have, in this case, an equilibrium axis. Therefore, we have the following result from the theory of elasticity.

Theorem 54.1. If the loads $\{F_1, P_1\}$ do not have an equilibrium axis, then there exist two positive numbers α and β , independent of ϵ , so that if

$$|M| \leq \alpha |\epsilon|,$$

the equation (54.21) has a unique solution Q , with the properties

$$|Q_{ij} - \delta_{ij}| \leq \beta.$$

In view of assumption (ii) it must be the case that $z \in S(z_0, R)$. This implies a certain higher limitation for $|\epsilon|$. The application $G(z) = z'$ is defined on $S(z_0, R) \subset Z''$ with values in Z' .

The following stage represents the problem of inverting the application of G in the neighborhood of a point \tilde{z}_0 .

Proceeding in the same way as in the case of elasticity, it follows that this problem is reduced to the existence and to the uniqueness of the solution of the system of equations from the linear theory of thermoelasticity

$$C_{ijkl}v_{k,l} - \beta_{ij}T_{,j} = \delta\Phi_i, \quad k_{ij}T_{,j} = \delta\Psi \text{ in } B, \quad (54.24)$$

with boundary conditions

$$(C_{ijkl}v_{k,l} - \beta_{ij}T_{,j})N_j = -\delta\varphi_i, \quad -k_{ij}T_{,j}N_i = \delta\psi \text{ on } \partial B. \quad (54.25)$$

In view of the assumption (i), the definition of the space Z' and the results from the linear theory of thermoelastic equilibrium, it

follows that the boundary problem (54.24), (54.25) has a unique solution which satisfies (54.16). This solution admits second order derivatives which satisfy Hölder's condition. Therefore we have

Theorem 54.2. If conditions (i) -- (iii) occur and the system of loads (F_1, P_1) does not have an equilibrium axis, then there exist two positive numbers γ and χ so that for $|\epsilon| < \gamma$ the problem (54.1) -- (54.4) has a unique solution $z = \{u_1, T\}$ which satisfies (54.9) and $\|z - \tilde{z}_0\| \leq \chi$.

The extension of Stoppelli's method to the equations of thermoelasticity does not cause any essential difficulties.

If the functions $\tilde{\sigma}$ and \hat{Q}_1 are analytical ones, it is shown that the solution, the existence and uniqueness of which was mentioned above, does not depend analytically on ϵ .

55. Plane Deformation

/238

a) Statement of the problem. Let us consider the equations of the nonlinear theory of thermoelastostatics written in curved coordinates (see Paragraph 24). We prefer this type of writing because it facilitates the formulation of the problem in complex variables. We select a system of curved coordinates θ^1 so that $\theta^3 = x_3$. The state of plane thermoelastic deformation parallel with the plane X_1OX_2 is defined as being that state of deformation in which we have (Green and Adkins [141])

$$\begin{aligned} X_a &= X_a(\theta^1, \theta^2), \quad x_a = x_a(\theta^1, \theta^2), \quad x_3 = X_3 = \theta^3, \\ T &= T(\theta^1, \theta^2). \end{aligned} \tag{55.1}$$

In this case we can write

$$R = R^*(\theta^1, \theta^2) + \theta^3 A_3, \quad r = r^*(\theta^1, \theta^2) + \theta^3 A_3, \tag{55.2}$$

$$\begin{aligned} g_1 &= a_1, \quad G_1 = \Lambda_1, \quad g_{\alpha\beta} = a_{\alpha\beta}, \quad g_{33} = 1, \quad g_{\alpha 3} = 0, \\ G_{\alpha\beta} &= A_{\alpha\beta}, \quad G_{\alpha 3} = 0, \quad G_{33} = 1, \quad g = |a_{\alpha\beta}| = a, \quad G = |A_{\alpha\beta}| = A. \end{aligned} \quad (55.2)$$

It follows from (24.13) that

$$2\gamma_{\alpha\beta} = A_{\alpha\beta} - a_{\alpha\beta}, \quad \gamma_{33} = 0. \quad (55.3)$$

The invariants I_r defined in (24.22) take the form

$$I_1 = 1 + a^{\alpha\beta} A_{\alpha\beta}, \quad I_2 = \frac{A}{a} (a_{\alpha\beta} A^{\alpha\beta} + 1), \quad I_3 = \frac{A}{a}. \quad (55.4)$$

In view of the fact that $a_{\alpha\beta} A^{\alpha\beta} A = a^{\nu\rho} A_{\nu\rho} a$, the relation follows

$$I_3 - I_2 + I_1 - 1 = 0. \quad (55.5)$$

Let

$$I \equiv I_1, J \equiv I_3, J_1^* \equiv I_1 - 3, J_2^* \equiv I_2 - 2I_1 + 3.$$

Obviously, we can write

$$\tilde{\sigma} = (I_1, I_2, I_3, T) = \sigma_1(I, J, T) = \sigma_2(J_1^*, J_2^*, T). \quad (55.6)$$

The relations (24.71) become

$$\begin{aligned} \tau^{\alpha\beta} &= \mathcal{K} a^{\alpha\beta} + \mathcal{L} A^{\alpha\beta}, \\ \tau^{\alpha 3} &= 0, \quad \tau^{33} = \Phi + (I-1)\Psi + p, \end{aligned} \quad (55.7)$$

where

$$\begin{aligned} \mathcal{K} &= \frac{2}{\sqrt{J}} \frac{\partial \sigma_1}{\partial I} = \frac{2}{\sqrt{J}} \left(\frac{\partial \sigma_2}{\partial J_1^*} - \frac{\partial \sigma_2}{\partial J_2^*} \right), \\ \mathcal{L} &= 2\sqrt{J} \frac{\partial \sigma_1}{\partial J} = 2\sqrt{J} \frac{\partial \sigma_2}{\partial J_2^*}, \\ \Phi &= \frac{2}{\sqrt{J}} \frac{\partial \tilde{\sigma}}{\partial I_1}, \quad \Psi = \frac{2}{\sqrt{J}} \frac{\partial \tilde{\sigma}}{\partial I_2}, \quad p = 2\sqrt{J} \frac{\partial \tilde{\sigma}}{\partial I_3}. \end{aligned} \quad (55.8)$$

/239

We obtain from (24.59), (24.74) and (55.1)

$$q^a = (C_1 \delta_a^b + C_2 \gamma_a^b + C_3 \gamma_a^c \gamma_c^b) T \|_b^a, \quad q^3 = 0, \quad (55.9)$$

in which $T \|_b^a = A^{ab} T \|_a^b$, $\gamma_a^b = a^{ab} \gamma_{ab}$, and C_i are functions of the invariants $I, J, T, T \|_a^a T \|_a^a, T \|_a^a T \|_b^b \gamma_a^a, T \|_a^a T \|_b^b \gamma_a^a \gamma_b^b$.

In the absence of mass forces, the equilibrium equations are reduced to

$$\tau^{ab} \|_a = 0. \quad (55.10)$$

We will assume that there are no thermal sources, so that equation (24.68) becomes

$$q^a \|_a = 0. \quad (55.11)$$

If we represent the tensions with the help of the Airy function

$$\tau^{ab} = \epsilon^{a\eta} \epsilon^{b\rho} \varphi |_{\eta\rho}, \quad (55.12)$$

where $\epsilon^{ab} \sqrt{A} = e_{ab3}$, then equations (55.10) are satisfied.

We obtain from (55.7) and (55.12)

$$\varphi \|_{ab} = \frac{1}{a} a_{ab} \mathcal{K} + A_{ab} \mathcal{L}. \quad (55.13)$$

It is known from the theory of elasticity (Green and Zerna [139], Green and Adkins [141]) that the force which is exercised on an arc of curvature from the plane $x_1 O x_2$ measured on the unit length of the axis x_3 can be expressed such that

$$\mathbf{P} = P^a \mathbf{A}_a = \epsilon^{ab} \varphi_{,b} \mathbf{A}_a. \quad (55.14)$$

In the study of the state of plane stress from the theory of linear elasticity, it is useful to formulate the problem in complex variables. This formulation is useful also in the nonlinear theory, in particular, for the study of the second order approximation (see for example Green and Adkins [141]).

Let us introduce the complex coordinates $(\zeta, \bar{\zeta})$ and (z, \bar{z}) in the nondeformed body and in the deformed body by the relations

/240

$$\begin{aligned}\zeta &= X_1 + iX_2, \quad \bar{\zeta} = X_1 - iX_2, \\ z &= x_1 + ix_2, \quad \bar{z} = x_1 - ix_2.\end{aligned}\tag{55.15}$$

Let u_α be the components of the displacement vector with respect to the system of rectangular cartesian coordinates. Let us designate

$$D = u_1 + iu_2, \quad \bar{D} = u_1 - iu_2.\tag{55.16}$$

It follows from (55.15) and (55.16) that

$$z = \zeta + D, \quad \bar{z} = \bar{\zeta} + \bar{D}.\tag{55.17}$$

If we take $\theta^1 = z, \theta^2 = \bar{z}$, we obtain

$$\begin{aligned}A_{12} &= -\frac{1}{2}, \quad A^{12} = 2, \quad A = -\frac{1}{4}, \quad A_{11} = A_{22} = A^{11} = A^{22} = 0, \\ \sqrt{a} &= \frac{i}{2} \frac{\partial(\zeta, \bar{\zeta})}{\partial(z, \bar{z})} = \frac{i}{2} \left(1 - \frac{\partial D}{\partial z} - \frac{\partial \bar{D}}{\partial \bar{z}} + \frac{\partial D}{\partial z} \frac{\partial D}{\partial \bar{z}} - \frac{\partial \bar{D}}{\partial \bar{z}} \frac{\partial \bar{D}}{\partial z} \right), \\ a_{11} &= a_{22} = \frac{\partial \bar{D}}{\partial z} \left(\frac{\partial D}{\partial \bar{z}} - 1 \right), \quad a_{12} = -\frac{1}{2\sqrt{J}} + \frac{\partial D}{\partial z} \frac{\partial \bar{D}}{\partial \bar{z}}, \\ a^{11} &= \bar{a}^{22} = \frac{1}{a} a_{22}, \quad a^{12} = -\frac{1}{a} a_{12}.\end{aligned}\tag{55.18}$$

The geometric relations take the form

$$\begin{aligned} \gamma_{11} &= \overline{\gamma_{22}} = \frac{1}{2} \frac{\partial D}{\partial z} \left(1 - \frac{\partial D}{\partial \bar{z}} \right), \\ \gamma_{12} &= \frac{1}{4} \left(\frac{\partial D}{\partial z} + \frac{\partial D}{\partial \bar{z}} - \frac{\partial D}{\partial z} \frac{\partial \bar{D}}{\partial z} - \frac{\partial \bar{D}}{\partial \bar{z}} \frac{\partial D}{\partial \bar{z}} \right). \end{aligned} \quad (55.19)$$

Similarly, we have

$$I = 1 + 2\sqrt{J} + 4J \frac{\partial D}{\partial z} \frac{\partial \bar{D}}{\partial \bar{z}}, \quad J = -\frac{1}{4a}. \quad (55.20)$$

If we designate by T^{rs} the contravariant components of the tension tensor in the coordinate system z, \bar{z}, x_3 , it follows from (55.12) that

$$T^{11} = T^{22} = -4 \frac{\partial^2 \varphi}{\partial z^2}, \quad T^{12} = 4 \frac{\partial^2 \varphi}{\partial z \partial \bar{z}}. \quad (55.21)$$

The relations (55.13) become

/241

$$\frac{\partial^2 \varphi}{\partial z^2} = J \frac{\partial D}{\partial z} \left(\frac{\partial D}{\partial z} - 1 \right), \quad \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} = \left(\frac{1}{2} \sqrt{J} + J \frac{\partial D}{\partial \bar{z}} \frac{\partial \bar{D}}{\partial z} \right) \mathcal{K} + \frac{1}{2} \mathcal{L}. \quad (55.22)$$

From (55.14) we obtain

$$P^1 = P^2 = 2i \frac{\partial \varphi}{\partial \bar{z}}. \quad (55.23)$$

b) Successive approximations. We assume, as in Paragraph 52, that the complex displacement D and the temperature T may be developed in absolutely convergent series, according to the powers of parameter ϵ . In the following we will limit ourselves to the first two terms from the developments which appear. Let

$$\begin{aligned} D(z, \bar{z}) &= \epsilon D_1(z, \bar{z}) + \epsilon^2 D_2(z, \bar{z}) + \dots, \\ T(z, \bar{z}) &= \epsilon T_1(z, \bar{z}) + \epsilon^2 T_2(z, \bar{z}) + \dots \end{aligned} \quad (55.24)$$

It follows from (55.18) and (55.20) that

$$\sqrt{J} = 1 + \varepsilon a_1 + \varepsilon^2 a_2 + \dots, \quad (55.25)$$

where

$$\begin{aligned} a_1 = \frac{\partial D_1}{\partial z} + \frac{\partial \bar{D}_1}{\partial z}, \quad a_2 = \frac{\partial D_2}{\partial z} + \frac{\partial \bar{D}_2}{\partial z} + \frac{\partial D_1}{\partial z} \frac{\partial \bar{D}_1}{\partial z} + \\ + \frac{\partial D_1}{\partial z} \frac{\partial \bar{D}_1}{\partial z} + \left(\frac{\partial D_1}{\partial z} \right)^2 + \left(\frac{\partial \bar{D}_1}{\partial z} \right)^2. \end{aligned} \quad (55.26)$$

Consequently, we have

$$J = 1 + 2a_1 \varepsilon + (a_1^2 + 2a_2) \varepsilon^2 + \dots \quad (55.27)$$

If we take into consideration (55.20), (55.25), (55.27), we obtain

$$\begin{aligned} J_1^* &= 2a_1 \varepsilon + 2 \left(a_2 + 2 \frac{\partial D_1}{\partial z} \frac{\partial \bar{D}_1}{\partial z} \right) \varepsilon^2 + \dots, \\ J_2^* &= \left(a_1^2 + 4 \frac{\partial D_1}{\partial z} \frac{\partial \bar{D}_1}{\partial z} \right) \varepsilon^2 + \dots. \end{aligned} \quad (55.28)$$

With the designations

$$\begin{aligned} A &= \frac{\partial^2 \sigma_2}{\partial J_1^{*2}}, \quad B = \frac{\partial \sigma_2}{\partial J_2^*}, \quad C = \frac{\partial^2 \sigma_2}{\partial J_1^* \partial J_2^*}, \quad G = \frac{\partial^2 \sigma_2}{\partial J_1^* \partial T}, \\ H &= \frac{\partial^3 \sigma_2}{\partial J_1^{*3}}, \quad M = \frac{1}{2} \frac{\partial^3 \sigma_2}{\partial J_1^* \partial T^2}, \quad K = \frac{\partial^2 \sigma_2}{\partial J_2^* \partial T}, \\ P &= -\frac{1}{2} \frac{\partial^3 \sigma_2}{\partial T \partial J_1^{*2}}, \quad \text{for } \varepsilon = 0, \end{aligned} \quad (55.29)$$

we have

$$\begin{aligned} \frac{\partial \sigma_2}{\partial J_1^*} &= \varepsilon (2A a_1 - G T_1) + \varepsilon^2 \left[2A a_2 - G T_2 + (2H + C) a_1^2 + \right. \\ &\quad \left. + 4(A - C) \frac{\partial D_1}{\partial z} \frac{\partial \bar{D}_1}{\partial z} - M T_1^2 - 4P a_1 T_1 \right], \end{aligned} \quad (55.30)$$

$$\frac{\partial \sigma_2}{\partial J_2^*} = B + \varepsilon (2C a_1 - K T_1) + \dots$$

It follows from (55.8), (55.25) and (55.30) that

$$\mathcal{H} = \mathcal{H}_0 + \epsilon \mathcal{H}_1 + \dots, \quad (55.31)$$

where

$$\mathcal{H}_0 = 2B, \quad \mathcal{H}_1 = 2 [(2A - 2C + B) a_1 - (G - K) T_1]. \quad (55.32)$$

Similarly, we have

$$\begin{aligned} \frac{1}{2} (\sqrt{J} \mathcal{H} + \mathcal{L}) = & \epsilon [(2A + B) a_1 - C T_1] + \\ & + \epsilon^2 \left[(2A + B) a^2 - G T_2 + 4(A - C) \frac{\partial D_1}{\partial \bar{z}} \frac{\partial \bar{D}_1}{\partial z} + \right. \\ & \left. + (2H + 3C) a_1^2 - K a_1 T_1 - M T_1^2 - 4P a_1 T_1 \right] + \dots \end{aligned} \quad (55.33)$$

If we set

$$\varphi = \mathcal{H}_0 [\epsilon \varphi_1(z, \bar{z}) + \epsilon^2 \varphi_2(z, \bar{z}) + \dots], \quad (55.34)$$

then we obtain from (55.27), (55.31), (55.22)₁

$$\begin{aligned} \frac{\partial^2 \varphi_1}{\partial z^2} = & - \frac{\partial \bar{D}_1}{\partial z}, \\ \frac{\partial^2 \varphi_2}{\partial z^2} = & - \frac{\partial \bar{D}_2}{\partial z} + \frac{2(A - C)}{B} \left(\frac{\partial D_1}{\partial z} + \frac{\partial \bar{D}_1}{\partial \bar{z}} \right) \frac{\partial \bar{D}_1}{\partial z} - \\ & - \frac{\partial \bar{D}_1}{\partial z} \frac{\partial \bar{D}_1}{\partial z} - \frac{G - K}{B} T_1 \frac{\partial D_1}{\partial z}. \end{aligned} \quad (55.35)$$

In view of (55.33), (55.27), (55.32) it follows from (55.22)₂ that

/243

$$\begin{aligned}
 \frac{\partial^2 \varphi_1}{\partial z \partial \bar{z}} = & -\frac{2A+B}{2B} \left(\frac{\partial D_1}{\partial z} + \frac{\partial \bar{D}_1}{\partial \bar{z}} \right) + \frac{G}{2B} T_1, \\
 \frac{\partial^2 \varphi_2}{\partial z \partial \bar{z}} = & -\frac{2A+B}{2B} \left(\frac{\partial D_2}{\partial z} + \frac{\partial \bar{D}_2}{\partial \bar{z}} \right) + \frac{G}{2B} T_2 - \\
 & -\frac{2A+3C+B+2H}{2B} \left(\frac{\partial D_1}{\partial z} + \frac{\partial \bar{D}_1}{\partial \bar{z}} \right)^2 - \\
 & -\frac{6A-4C-B}{2B} \frac{\partial D_1}{\partial \bar{z}} \frac{\partial \bar{D}_1}{\partial z} + \frac{2A+B}{2B} \frac{\partial D_1}{\partial z} \frac{\partial \bar{D}_1}{\partial \bar{z}} + \\
 & + \frac{K}{2B} T_1 \left(\frac{\partial D_1}{\partial z} + \frac{\partial \bar{D}_1}{\partial \bar{z}} \right) + \frac{M}{2B} T_1^2 + \frac{2P}{B} \left(\frac{\partial D_1}{\partial z} + \frac{\partial \bar{D}_1}{\partial \bar{z}} \right) T_1.
 \end{aligned} \tag{55.36}$$

We introduce the designations

$$c_1 = 1 + \frac{A}{B}, c_2 = \frac{C}{B}, c_4 = \frac{H}{B}, \tag{55.37}$$

$$\beta_1 = -\frac{G}{B}, \beta_2 = -\frac{K}{B}, \beta_3 = -\frac{M}{B}, \beta_4 = -\frac{4P}{B},$$

$$\alpha_1 = c_1 - c_2 - 1, \alpha_2 = 2c_1 + 3c_2 + 2c_4 - 1, \alpha_3 = 6c_1 - 4c_2 - 7.$$

The Poisson coefficient ν and the Young's modulus E from the linear theory are connected to A and B by the relations

$$\nu = \frac{A+B}{2A+B}, E = -\frac{4B(3A+2B)}{2A+B}. \tag{55.38}$$

With the designations thus introduced, we have from (55.35) and (55.36)

$$\begin{aligned}
 \frac{\partial^2 \varphi_1}{\partial z^2} = & -\frac{\partial \bar{D}_1}{\partial z}, \\
 \frac{\partial^2 \varphi_1}{\partial z \partial \bar{z}} = & \frac{1}{2(1-\nu)} \left(\frac{\partial D_1}{\partial z} + \frac{\partial \bar{D}_1}{\partial \bar{z}} \right) - \frac{1}{2} \beta_1 T_1,
 \end{aligned} \tag{55.39}$$

$$\frac{\partial^2 \varphi_3}{\partial z^2} = -\frac{\partial D_2}{\partial z} + 2\alpha_1 \left(\frac{\partial D_1}{\partial z} + \frac{\partial \bar{D}_1}{\partial \bar{z}} \right) \frac{\partial D_1}{\partial z} - \frac{\partial \bar{D}_1}{\partial \bar{z}} \frac{\partial \bar{D}_1}{\partial \bar{z}} + (\beta_1 - \beta_2) T_1 \frac{\partial D_1}{\partial z}, \quad (55.39)$$

$$\begin{aligned} \frac{\partial^2 \varphi_3}{\partial z \partial \bar{z}} &= \frac{1}{2(1-2\nu)} \left(\frac{\partial D_2}{\partial z} + \frac{\partial \bar{D}_2}{\partial \bar{z}} \right) - \frac{1}{2} \beta_1 T_2 - \\ &- \frac{1}{2(1-2\nu)} \frac{\partial D_1}{\partial z} \frac{\partial \bar{D}_1}{\partial \bar{z}} - \frac{1}{2} \alpha_2 \left(\frac{\partial D_1}{\partial z} + \frac{\partial \bar{D}_1}{\partial \bar{z}} \right)^2 - \\ &- \frac{1}{2} \alpha_3 \frac{\partial D_1}{\partial \bar{z}} \frac{\partial \bar{D}_1}{\partial z} - \frac{1}{2} (\beta_2 + \beta_1) T_1 \left(\frac{\partial D_1}{\partial z} + \frac{\partial \bar{D}_1}{\partial \bar{z}} \right) - \frac{1}{2} \beta_3 T_1^2, \end{aligned} \quad (55.40)$$

relations which correspond to the classical theory and to the second order approximation, respectively.

/244

The coefficients C_α from (55.9) have the form

$$C_1 = k + \varepsilon \left[2k_1 \left(\frac{\partial D_1}{\partial z} + \frac{\partial \bar{D}_1}{\partial \bar{z}} \right) + k_2 T_1 \right] + \dots, \quad (55.41)$$

$$C_2 = k_3 + \dots,$$

where

$$k = C_1, \quad k_1 = \frac{\partial C_1}{\partial J_3^*}, \quad k_2 = \frac{\partial C_1}{\partial T}, \quad k_3 = C_2, \quad \text{for } \varepsilon = 0.$$

It follows from (55.18) and (55.19) that

$$a^{11} = \bar{a}^{22} = 4\varepsilon \frac{\partial D_1}{\partial z} + \dots, \quad a^{12} = 2a + \dots, \quad (55.42)$$

$$\gamma_{11} = \bar{\gamma}_{22} = \frac{1}{2} \frac{\partial \bar{D}_1}{\partial \bar{z}} + \dots, \quad \gamma_{12} = \frac{1}{4} \varepsilon \left(\frac{\partial D_1}{\partial z} + \frac{\partial \bar{D}_1}{\partial \bar{z}} \right) + \dots$$

Since

$$T_{11}^1 = 2 \frac{\partial T}{\partial \bar{z}}, \quad T_{12}^2 = 2 \frac{\partial T}{\partial z},$$

we obtain from (55.9) and (55.41)

$$q^a = z q_{(1)}^a + z^2 q_{(2)}^a + \dots, \quad (55.43)$$

where

$$\begin{aligned} q_{(1)}^1 &= 2k \frac{\partial T_1}{\partial z}, \quad q_{(1)}^2 = 2k \frac{\partial T_1}{\partial \bar{z}}, \\ q_{(2)}^1 &= 2k \frac{\partial T_2}{\partial z} + (4k_1 + k_3) \left(\frac{\partial D_1}{\partial z} + \frac{\partial \bar{D}_1}{\partial \bar{z}} \right) \frac{\partial T_1}{\partial \bar{z}} + \\ &\quad + 2k_2 T_1 \frac{\partial T_1}{\partial \bar{z}} + 2k_3 \frac{\partial D_1}{\partial \bar{z}} \frac{\partial T_1}{\partial z}, \\ q_{(2)}^2 &= 2k \frac{\partial T_2}{\partial \bar{z}} + (4k_1 + k_3) \left(\frac{\partial D_1}{\partial z} + \frac{\partial \bar{D}_1}{\partial \bar{z}} \right) \frac{\partial T_1}{\partial z} + \\ &\quad + 2k_2 T_1 \frac{\partial T_1}{\partial z} + 2k_3 \frac{\partial D_1}{\partial z} \frac{\partial T_1}{\partial \bar{z}}. \end{aligned} \quad (55.44)$$

Equation (55.11) implies

/245

$$q_{(1)}^a \parallel_a = 0, \quad q_{(2)}^a \parallel_a = 0. \quad (55.45)$$

Let us assume that the plane domain under consideration, Σ , is simply connected and the mechanical loads are zero. For this case it was shown (Paragraph 49b) that, in the linear theory, no tensions $t_{\alpha\beta}$ appear, and the displacements are given by (49.23). We will study now what takes place in the second order approximation.

We obtain from (55.44) and (55.45)

$$\begin{aligned} \frac{\partial^2 T_1}{\partial z \partial \bar{z}} &= 0, \quad -4k \frac{\partial^2 T_2}{\partial z \partial \bar{z}} = 4k_2 \frac{\partial T_1}{\partial z} \frac{\partial T_1}{\partial \bar{z}} + (4k_1 + k_3) \left[\frac{\partial T_1}{\partial \bar{z}} \frac{\partial}{\partial z} \left(\frac{\partial D_1}{\partial z} + \frac{\partial \bar{D}_1}{\partial \bar{z}} \right) \right] + \\ &\quad + \frac{\partial T_1}{\partial z} \frac{\partial}{\partial \bar{z}} \left(\frac{\partial D_1}{\partial z} + \frac{\partial \bar{D}_1}{\partial \bar{z}} \right) + 2k_3 \left[\frac{\partial}{\partial z} \left(\frac{\partial D_1}{\partial \bar{z}} \frac{\partial T_1}{\partial z} \right) + \frac{\partial}{\partial \bar{z}} \left(\frac{\partial \bar{D}_1}{\partial z} \frac{\partial T_1}{\partial \bar{z}} \right) \right]. \end{aligned} \quad (55.46)$$

It follows from (55.46)₁ that

$$T_1(z, \bar{z}) = F'(z) + F'(\bar{z}), \quad (55.47)$$

where $F'(z) = dF/dz$.

We introduce the functions u_α^0 by

$$D_1 = D_0 + \beta_1(1-2\nu)F'(z), \quad (55.48)$$

where $D_0 = u_1^0 + iu_2^0$. We have from (55.39) and (55.48)

$$\frac{\partial^2 \varphi_1}{\partial z^2} = \frac{\partial D_0}{\partial z} \frac{\partial^2 \varphi_1}{\partial z \partial \bar{z}} - \frac{1}{2(1-2\nu)} \left(\frac{\partial D_0}{\partial z} + \frac{\partial \bar{D}_0}{\partial \bar{z}} \right). \quad (55.49)$$

We designate

$$T^{11} = T^{22} = \epsilon T_{(1)}^{11} + \epsilon^2 T_{(2)}^{11} + \dots, \quad T^{12} = \epsilon T_{(1)}^{12} + \epsilon^2 T_{(2)}^{12} + \dots \quad (55.50)$$

Obviously $T_{(1)}^{11}$, $T_{(1)}^{12}$, D_0 satisfy the equations of the linear theory of elasticity in the absence of mechanical loads. We have, therefore, $T_{(1)}^{11} = T_{(1)}^{12} = 0$ and $D_0 = 0$, up to a rigid displacement. Consequently, we obtain from (55.48)

$$D_1 = \beta_1(1-2\nu)F'(z). \quad (55.51)$$

In view of (55.47) and (55.51), equation (55.46)₂ becomes

$$\frac{\partial^2 T_2}{\partial z \partial \bar{z}} = \gamma F'''(z) F'''(\bar{z}), \quad (55.52)$$

where

$$-4k\gamma = 4k_2 + \frac{2\beta_1}{1-2\epsilon_1}(4k_1 + k_3). \quad (55.53)$$

/246

It follows from (55.52) that

$$T_2(z, \bar{z}) = \gamma F'(z) \bar{F}'(\bar{z}) + G'(z) + \bar{G}'(\bar{z}). \quad (55.54)$$

If we keep in mind (55.51) and (55.54) we obtain from (55.40)

$$\begin{aligned} \frac{\partial^2 \varphi_2}{\partial z^2} &= -\frac{\partial \bar{D}_2}{\partial z}, \\ \frac{\partial^2 \varphi_2}{\partial z \partial \bar{z}} &= \frac{1}{2(1-2\nu)} \left(\frac{\partial D_2}{\partial z} + \frac{\partial \bar{D}_2}{\partial \bar{z}} \right) - \frac{1}{2} \beta_1 [G'(z) + \bar{G}'(\bar{z})] - \\ &\quad - \frac{1}{2} [\beta_1^2(1-2\nu) + \gamma\beta_1 + 2b] F'(z) \bar{F}'(\bar{z}) - \\ &\quad - \frac{1}{2} b[(F''(z))^2 + (\bar{F}''(\bar{z}))^2], \end{aligned} \quad (55.55)$$

where

$$b = \beta_1^2(1-2\nu)^2 \alpha_2 + \beta_1(1-2\nu)(\beta_2 + \beta_4) + \beta_3. \quad (55.56)$$

We introduce the function $D_2^* (z, \bar{z})$ by

$$D_2(z, \bar{z}) = D_2^*(z, \bar{z}) + h_1 F(z) \bar{F}'(\bar{z}) + h_2 G(z) + h_3 \int (F''(z))^2 dz, \quad (55.57)$$

in which we designated

$$\begin{aligned} h_1 &= \frac{1-2\nu}{2} [\beta_1^2(1-2\nu) + \gamma\beta_1 + 2b], \\ h_2 &= \beta_1(1-2\nu), \quad h_3 = b(1-2\nu), \end{aligned} \quad (55.58)$$

It follows from (55.55) that

$$\begin{aligned}\frac{\partial^2 \varphi_2}{\partial z^2} &= -\frac{\partial D_2^*}{\partial z} - h_1 \bar{F}(\bar{z}) F''(z), \\ \frac{\partial^2 \varphi_2}{\partial z \partial \bar{z}} &= \frac{1}{2(1-2\nu)} \left(\frac{\partial D_2^*}{\partial z} + \frac{\partial \bar{D}_2^*}{\partial \bar{z}} \right).\end{aligned}\tag{55.59}$$

The first relation implies

/247

$$\frac{\partial \varphi_2}{\partial z} = -D_2^* - h_1 \bar{F}(\bar{z}) F'(z) + 4(1-\nu) \bar{\Omega}(\bar{z}).\tag{55.60}$$

We obtain from (55.60)

$$\begin{aligned}\frac{\partial D_2^*}{\partial z} + \frac{\partial \bar{D}_2^*}{\partial \bar{z}} &= -2 \frac{\partial^2 \varphi_2}{\partial z \partial \bar{z}} - 2h_1 F'(z) \bar{F}'(\bar{z}) + \\ &+ 4(1-\nu) [\Omega'(z) + \bar{\Omega}'(\bar{z})].\end{aligned}\tag{55.61}$$

In view of (55.61), a second relation from (55.59) becomes

$$\frac{\partial^2 \varphi_2}{\partial z \partial \bar{z}} = \Omega'(z) + \bar{\Omega}'(\bar{z}) - \frac{h_1}{2(1-\nu)} F'(z) \bar{F}'(\bar{z}),\tag{55.62}$$

so that we can write

$$\begin{aligned}\varphi_2(z, \bar{z}) &= z \bar{\Omega}(\bar{z}) + \bar{z} \Omega(z) + \omega(z) + \bar{\omega}(\bar{z}) - \\ &- \frac{h_1}{2(1-\nu)} F(z) \bar{F}(\bar{z}).\end{aligned}\tag{55.63}$$

We derive from (55.60) and (55.63)

$$\begin{aligned}D_2^*(z, \bar{z}) &= (3-4\nu) \Omega(z) - z \bar{\Omega}'(\bar{z}) - \bar{\omega}'(\bar{z}) - \\ &- \frac{(1-2\nu)}{2(1-\nu)} h_1 F(z) \bar{F}'(\bar{z}).\end{aligned}\tag{55.64}$$

If we designate

$$\begin{aligned}
D_2^0(z, \bar{z}) &= (3-4\nu) \Omega(z) - z\bar{\Omega}'(\bar{z}) - \bar{\omega}'(\bar{z}), \\
\varphi_2^0(z, \bar{z}) &= z\bar{\Omega}(\bar{z}) + z\Omega(z) + \omega(z) + \bar{\omega}(\bar{z}),
\end{aligned}
\tag{55.65}$$

then we have from (55.57), (55.63), (55.64)

$$\begin{aligned}
D_2(z, \bar{z}) &= D_2^0(z, \bar{z}) + \frac{h_1}{2(1-\nu)} F(z) \bar{F}'(\bar{z}) + \\
&\quad + h_2 G(z) + h_3 \int |F'(z)|^2 dz, \\
\varphi_2(z, \bar{z}) &= \varphi_2^0(z, \bar{z}) - \frac{1}{2(1-\nu)} F(z) \bar{F}(\bar{z}).
\end{aligned}
\tag{55.66}$$

These results indicate that in the case under consideration, /248
the temperature causes tensions $T_{(2)}^{11}, T_{(2)}^{12}$. Determination
of the tensions and of the displacements is reduced to the problem
(well known from the theory of elasticity, see, for example, Muskhelishvili
[296]) of the determination of the functions $\Omega(z), \omega(z)$, which
are holomorphic in the domain Σ and which, at the boundary L of
this domain, satisfy the condition

$$\Omega(\eta) + \eta\Omega'(\bar{\eta}) + \bar{\omega}'(\bar{\eta}) = \frac{h_1}{2(1-\nu)} F(\eta) \bar{F}'(\bar{\eta}), \quad \eta \in L,
\tag{55.67}$$

which expresses the fact that the tensions on L are zero.

Application. Let us consider the case when Σ is the inside of
a circle with a radius R and the temperature T_1 varies linearly with
 x_1 . We assume that the origin of the axis of coordinates is in the
center of the circle. In this case we have

$$T_1 = ax_1, \quad F'(z) = \frac{1}{2} az.
\tag{55.68}$$

We obtain from (55.67)

$$\Omega(\eta) + \eta \Omega'(\eta) + \omega'(\eta) = \frac{h_1 a^2 R^2 \eta}{16(1-\nu)}, \quad \eta \in L, \quad (55.69)$$

keeping in mind that $\eta \bar{\eta} = R^2$.

If we take

$$\Omega(z) = \frac{h_1 a^2 R^2}{32(1-\nu)} z, \quad \omega(z) = 0, \quad (55.70)$$

then the condition (55.69) is satisfied.

It follows from (55.65), (55.66), (55.70)

$$\varphi_2(z, \bar{z}) = \frac{h_1 a^2 R^2}{16(1-\nu)} \left(z\bar{z} - \frac{1}{2R^2} z^2 \bar{z}^2 \right), \quad (55.71)$$

and consequently, using (55.21), (55.33), (55.34) we derive

$$T_{(2)}^{11} = T_{(2)}^{22} = \frac{h_1 \mathcal{K}_0 a^2 z^2}{4(1-\nu)}, \quad T_{(1)}^{12} = \frac{h_1 \mathcal{K}_0 a^2 R^2}{4(1-\nu)} \left(1 - \frac{2}{R^2} z\bar{z} \right)$$

The results presented here have been derived by Iesan [192]. In the case of incompressible media, the study of plane thermoelastic deformation was made by Chaudry [53].

56. Special Problems

Various inverse problems which describe the deformation of the medium have been studied in the theory of nonlinear elasticity, determining the corresponding charges which keep the body in equilibrium ([349], [102], [141]). Moreover, the problem of finding /249 all of the deformations which satisfy the equations of equilibrium, in the absence of mass forces, regardless of the elastic medium

under consideration, has been posed. Ericksen [102] has shown that for elastic homogeneous isotropic and incompressible media, there are at least five families of deformation with this property. For elastic compressible media, the only deformations of this type are the homogeneous ones [103]. These types of solutions of the equations of nonlinear elasticity were obtained earlier by Rivlin [349]. In these problems the determination of the charges is reduced to the determination of the tensions at the boundaries; the states of deformation may be "controlled" only by boundary conditions (an important fact for experimental studies [141]). The solutions of these problems are called universal solutions [422] or controllable states [369].

In the theory of thermoelasticity, the problem of controllable states was studied by Petroski and Carlson [338], [339], Hayes, Laws and Osborn [169], Laws [252]. It was established that for compressible media, there are no essential controllable states (but only the trivial case in which the deformation is homogeneous and the temperature is constant). It was also shown that in the case of incompressible media, there might be essential controllable states and such states have been determined (Petroski and Carlson [338], [339], Laws [252]). For illustration we will present some problems of the above-mentioned type.

Let us consider the theory of thermoelastic equilibrium for homogeneous and isotropic and incompressible media, assuming that mass forces and heat sources are absent. In this case, the equations (24.40) and (24.68) become

$$\tau''_{||} = 0, \quad (56.1)$$

$$q^i_{||} = 0. \quad (56.2)$$

The constitutive equations are

$$\tau'' = p G'' + \Phi g'' + \Psi B'', \quad (56.3)$$

$$q' = (C_1 \delta_j' + C_2 \gamma_j' + C_3 \gamma_m' \gamma_n') T \|', \quad (56.4)$$

where we used the designations from Paragraph 24. In (56.3), p is an unknown function. The condition of incompressibility, $I_3 = 1$, is added to the equations under consideration. In this case, the function $\bar{\sigma}$ depends on T, I_1 and I_2 and the function C_r depends on the invariants $I_1, I_2, I_4' = T \|' T \|, I_5' = T \|' T \|, \gamma_j', I_6' = T \|' T \|, \gamma_j' \gamma_j'$ and on the temperature T .

In the first two problems, in which we will prescribe the deformation and temperature, we are considering the class of materials for which we have [339]

$$\bar{\sigma} = \bar{\sigma}(T, I_1, I_2), C_r = C_r(I_1, I_2, I_4', I_5', I_6'). \quad (56.6)$$

If the functions C_s depends on the temperature, then it can be shown [338] that in this case all controllable states have a constant temperature. In other problems we will prescribe the deformation and the heat flux [252].

(α). Let us consider the thermomechanical state defined by the relations

/250

$$\begin{aligned} x_1 &= \frac{C}{\sqrt{A}} X_1 - \frac{D}{\sqrt{B}} X_2, \\ x_2 &= \frac{D}{\sqrt{A}} X_1 + \frac{C}{\sqrt{B}} X_2, \\ x_3 &= \sqrt{AB} X_3, \\ T &= T_0 + T_1 X_3, \end{aligned} \quad \begin{aligned} (56.6) \\ (\text{sic}) \end{aligned}$$

where A, B, C, D, T_0, T_1 are constants which satisfy

$$C^2 + D^2 = 1, \quad A, B > 0. \quad (56.7)$$

If we take $\theta^1 = X_1$, then we have

$$\begin{aligned}
 G_{11} &= \frac{1}{A}, G_{22} = \frac{1}{B}, G_{33} = AB, g_{ij} = \delta_{ij}, \\
 G^{11} &= A, G^{22} = B, G^{33} = \frac{1}{AB}, G_{ij} = G^{ij} = 0, (i \neq j), G = 1, \\
 I_1 &= \frac{1}{A} + \frac{1}{B} + AB, I_2 = A + B + \frac{1}{AB}, \\
 I'_1 &= T_1^2, I'_2 = \frac{1}{2} (AB-1) T_1^2, I'_3 = \frac{1}{4} (AB-1)^2 T_1^2, \\
 B^{11} &= \frac{1}{B} + AB, B^{22} = \frac{1}{A} + AB, B^{33} = \frac{1}{A} + \frac{1}{B}, B^{ij} = 0 (i \neq j),
 \end{aligned} \tag{56.8}$$

and therefore $\tilde{\sigma}$ does not depend on X_1 and X_2 , and C_g are constant.

It follows from (56.1), (56.3), (56.8), (24.72) that the function p is independent of X_1 and X_2 . Moreover, we have

$$p = -p_0 - 2AB \left(\frac{\partial \tilde{\sigma}}{\partial I_1} + \frac{A+B}{AB} \frac{\partial \tilde{\sigma}}{\partial I_2} \right), \tag{56.9}$$

where p_0 is an arbitrary constant. In view of the relations (56.3), (56.8) and (56.9) we can easily obtain the components of the tension tensor. These are functions independent of X_1 and X_2 . It may be noted that

$$\tau^{ij} = 0, (i \neq j). \tag{56.10}$$

It follows from (56.4) and (56.8) that

/25

$$q^1 = q^2 = 0,$$

$$q^3 = \left[C_1 + \frac{1}{2} C_2 (AB - 1) + \frac{1}{4} C_3 (AB - 1)^2 \right] T_1. \quad (56.11)$$

Obviously, the equations (56.2) are similarly satisfied. Consequently, for any medium of the class under consideration, the deformation (56.6) satisfies the equations of equilibrium and the equation of energy in the absence of mass forces and heat sources. For a given body, the corresponding boundary loads can easily be determined.

(β) Let us assume that

$$(\theta^1, \theta^2, \theta^3) = (r, \varphi, z), \quad (56.12)$$

where r, φ, z are cylindric coordinates in the deformed body

$$x_1 = r \cos \varphi, \quad x_2 = r \sin \varphi, \quad x_3 = z. \quad (56.13)$$

Let ρ, θ, Z be cylindrical coordinates in the nondeformed body

$$X_1 = \rho \cos \theta, \quad X_2 = \rho \sin \theta, \quad X_3 = Z. \quad (56.14)$$

Let us consider the thermomechanical state defined by the relations

$$r = \frac{\rho}{\sqrt{\lambda}}, \quad \varphi = \theta, \quad z = \lambda Z,$$

$$T = T_0 + T_1 \theta, \quad (56.15)$$

where $\lambda > 0$, T_0, T_1 are constants.

If, in the nondeformed state, the domain occupied by the body is limited by the surfaces $X_3 = \pm h$, $\Phi = 0$, $\Phi = \alpha$, $\rho = \rho_1$, $\rho = \rho_2$ where h , α , ρ_1 and ρ_2 are constants, then the thermomechanical state (56.15) describes the deformation of the body under consideration which, in the deformed configuration, occupies the domain limited by the surfaces $z = \pm \lambda h$, $\varphi = 0$, $\varphi = \alpha$, $r_1 = \rho_1/\sqrt{\lambda}$, $r_2 = \rho_2/\sqrt{\lambda}$; the plane $\phi = 0$ is at the temperature T_0 , and the plane $\phi = \alpha$ is at the temperature $T_0 + \alpha T_1$.

The mechanical deformation described by (56.15) is a particular case of the class of deformations studied in [141].

It follows from (56.13) -- (56.15), (24.7), (24.8) that

$$\begin{aligned} G_{11} = G^{11} = G_{33} = G^{33} = 1, \quad G_{22} = r^2, \quad G^{22} = \frac{1}{r^2}, \\ G_{ij} = r^2, \quad G_{ij} = G^{ij} = 0 \quad (i \neq j), \\ g_{11} = \lambda, \quad g_{22} = \lambda r^2, \quad g_{33} = \frac{1}{\lambda^2}, \quad g = r^2, \\ g^{11} = \frac{1}{\lambda}, \quad g^{22} = \frac{1}{\lambda r^2}, \quad g^{33} = \lambda^2, \quad g_{ij} = g^{ij} = 0, \quad (i \neq j), \end{aligned} \quad (56.16)$$

and therefore, the invariants of the deformation (24.22) are

/252

$$I_1 = \frac{2}{\lambda} + \lambda^2, \quad I_2 = 2\lambda + \frac{1}{\lambda^2}, \quad I_3 = 1. \quad (56.17)$$

Keeping in mind the relations

$$T_{||1} = T_{||3} = 0, \quad T_{||2} = \frac{1}{\lambda r^2} T_1, \quad (56.18)$$

$$\gamma_1^1 = \frac{1}{2\lambda} (1 - \lambda), \quad \gamma_2^2 = \frac{1}{2\lambda} (1 - \lambda), \quad \gamma_3^3 = \frac{1}{2} (\lambda^2 - 1), \quad \gamma_i^j = 0 \quad (i \neq j),$$

we derive

$$\begin{aligned}
 I_4' &= \frac{1}{\lambda r^2} T_1^2, \quad I_5' = \frac{T_1^2}{2\lambda^2 r^2} (1 - \lambda), \\
 I_6' &= \frac{T_1^2}{4\lambda^3 r^2} (1 - \lambda)^2.
 \end{aligned}
 \tag{56.19}$$

Consequently, the free energy does not depend on r and z , and the functions C_s do not depend on ϕ and z .

It follows from (24.72), (56.16), (56.17) that

$$\begin{aligned}
 B^{11} &= \frac{1 + \lambda^3}{\lambda^2}, \quad B^{22} = \frac{1 + \lambda^3}{\lambda^2 r^2}, \\
 B^{33} &= 2\lambda, \quad B^{ij} = 0 \quad (i \neq j),
 \end{aligned}
 \tag{56.20}$$

and therefore the components of the tension tensor are expressed as

$$\begin{aligned}
 \tau^{11} &= p + \frac{1}{\lambda} \Phi + \frac{1 + \lambda^3}{\lambda^2} \Psi, \\
 \tau^{22} &= \frac{1}{r^2} p + \frac{1}{\lambda r^2} \Phi + \frac{1 + \lambda^3}{\lambda^2 r^2} \Psi, \\
 \tau^{33} &= p + \lambda^2 \Phi + 2\lambda \Psi, \\
 \tau^{23} &= \tau^{31} = \tau^{12} = 0.
 \end{aligned}
 \tag{56.21}$$

The Christoffel symbols Γ_{jr}^1 different from zero are

$$\Gamma_{22}^1 = -r, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r},$$

so that the equations of equilibrium (56.1) take the form

/253

$$\frac{\partial \tau^{11}}{\partial r} + \frac{\tau^{11} - r^2 \tau^{22}}{r} = 0, \quad \frac{\partial \tau^{22}}{\partial \phi} = 0, \quad \frac{\partial \tau^{33}}{\partial z} = 0.
 \tag{56.22}$$

It can easily be seen that these equations are satisfied if, and only if, the function p has the form

$$p = -p_0 - \frac{1}{\lambda} \Phi - \frac{1 + \lambda^3}{\lambda^2} \Psi,
 \tag{56.23}$$

where p_0 is an arbitrary constant.

It follows from (56.21) and (56.23) that

$$\tau^{11} = r^2 \tau^{22} = -p_0, \quad (56.24)$$

$$\tau^{33} = -p_0 + \left(\lambda - \frac{1}{\lambda} \right) \Phi + \frac{\lambda^3 - 1}{\lambda^2} \Psi,$$

$$\tau^{23} = \tau^{31} = \tau^{12} = 0.$$

If we keep in mind (56.18) and (56.4) it follows that

$$q^1 - q^3 = 0,$$

$$q^2 = \frac{1}{\lambda r^2} \left[C_1 + \frac{1}{2\lambda} (1-\lambda) C_2 + \frac{1}{4\lambda^2} (1-\lambda)^2 C_3 \right] T_1. \quad (56.25)$$

It ensues from (24.59), (24.60) that, in this case, we can write $Q^1 = q^1$, $Q^1|_1 = q^1|_1$. It can easily be noted that the equation (56.2) is identically satisfied.

In the case of the previously considered domain, the thermal flux on the boundary portions at the surfaces $r = r_1$, $r = r_2$, $z = \pm h$ is zero. The other charges which act at the boundary may also be calculated easily.

The equations (56.4) may be written in the form (Truesdell and Noll [421])

$$T^i = (B_1 \delta_j^i + B_2 \gamma_j^i + B_3 \gamma_m^i \gamma_j^m) q^j, \quad (56.26)$$

where the functions B_1 depend on I_1 , I_2 , $J_4 = q^1 q_1$, $J_5 = q^1 q_j \gamma_1^j$, $J_6 = q^1 q_j \gamma_r^j \gamma_1^r$ and on the temperature T . The problem of the controllable states may be posed, also, in the following way:

the functions of deformation and the components of the thermal flux for which the functions p and T exist should be determined so that equations (56.1) and (56.2) are satisfied regardless of the functions $\tilde{\sigma}$ and B_1 . The controllable states with prescribed heat flux in the case of homogeneous isotropic and incompressible media, were studied by Laws [252]. We will present two controllable states for media in which the functions B_1 depend on $I_1, I_2, J_4, J_5, J_6, T$.

(γ) Let us consider the thermoelastic deformation defined by

$$r^2 = 2AX_1 + B, \varphi = CX_2 + DX_3 + G, z = EX_2 + FX_3 + H, \\ q_r = \frac{1}{r}K, \quad q_\varphi = q_z = 0, \quad (56.27)$$

where q_r, q_φ, q_z are physical components of the vector q , and $A, B, C, D, E, F, G, H, K$ are constants which satisfy the relation

$$A(CF - DE) = 1. \quad (56.28)$$

In this case we will select curved coordinates as in (56.12). The metric tensors of the deformed body G_{ij} and G^{ij} are given by (56.16). It follows from (56.27) and (56.28) that

$$g_{11} = \frac{r^2}{A^2}, \quad g_{22} = A^2(F^2 + E^2), \quad g_{33} = A^2(D^2 + C^2), \\ g_{12} = g_{13} = 0, \quad g_{23} = -A^2(FD + EC), \quad g = r^2, \\ g^{11} = \frac{A^2}{r^2}, \quad g^{22} = D^2 + C^2, \quad g^{33} = F^2 + E^2, \\ g^{12} = g^{13} = 0, \quad g^{23} = FD + EC, \quad (56.29)$$

and therefore we have

$$I_1 = \frac{A^2}{r^2} + (D^2 + C^2)r^2 + F^2 + E^2, \quad (56.30)$$

$$I_2 = \frac{r^2}{A^2} + \frac{A^2}{r^2}(F^2 + E^2) + A^2(D^2 + C^2), I_3 = 1.$$

If we keep in mind (56.16) and (56.29), we can derive from (24.30) for the nonzero components of the tensor γ_j^1 the expressions

$$\gamma_1^1 = \frac{1}{2} \left(\frac{A^2}{r^2} - 1 \right), \gamma_2^2 = \frac{1}{2} [(D^2 + C^2)r^2 - 1], \gamma_3^3 = \frac{1}{2} (F^2 + E^2) - 1, \\ r^2 \gamma_3^2 = \gamma_2^3 = \frac{1}{2} r^2 (FD + EC). \quad (56.31)$$

In view of the fact that in this case $q^1 = q_1 = q_r$, $q^2 = q_2 = q^3 = q_3 = 0$, we obtain

$$J_4 = \frac{K^2}{r^2}, J_5 = \frac{K^2}{2r^2} \left(\frac{A^2}{r^2} - 1 \right), J_6 = \frac{K^2}{4r^2} \left(\frac{A^2}{r^2} - 1 \right)^2. \quad (56.32)$$

Similarly, in view of (24.90) and (56.27), it follows that the equation (56.2) is satisfied. We derive from (56.26), (56.27), (56.31)

$$\frac{\partial T}{\partial r} = \left[B_1 + \frac{1}{2} \left(\frac{A^2}{r^2} - 1 \right) B_2 + \frac{1}{4} \left(\frac{A^2}{r^2} - 1 \right)^2 B_3 \right] \frac{K}{r}, \quad (56.33)$$

$$\frac{\partial T}{\partial \phi} = \frac{\partial T}{\partial z} = 0.$$

Consequently the function T does not depend on the variables ϕ and z . It is noted that in this case, the functions B_s depend only on T and r . The temperature T is determined from the equation (56.33)₁. Obviously, if B_s does not depend on T , then the situation is considerably simplified. Let us now determine the tensions

/255

We obtain from (24.72), (56.16) and (56.29)

$$\begin{aligned}
 B^{11} &= I_1 \frac{A^2}{r^2} - \frac{A^4}{r^4}, \quad B^{22} = I_1(D^2 + C^2) - r^2(D^2 + C^2) - (FD + EC)^2, \\
 B^{33} &= I_1(F^2 + E^2) - (FD + EC)^2 r^2 - (F^2 + E^2)^2, \quad B^{12} = B^{13} = 0, \\
 B^{23} &= (FD + EC) [I_1 - F^2 - E^2 - r^2(D^2 + C^2)],
 \end{aligned}
 \tag{56.34}$$

and therefore, we can write

$$\begin{aligned}
 \tau^{11} &= p + \Phi \frac{A^2}{r^2} + \frac{A^2}{r^2} \left(I_1 - \frac{A^2}{r^2} \right) \Psi, \\
 \tau^{22} &= \frac{1}{r^2} p + \Phi(D^2 + C^2) + [I_1(D^2 + C^2) - \\
 &\quad - r^2(D^2 + C^2) - (FD + EC)^2] \Psi, \\
 \tau^{33} &= p + \Phi(F^2 + E^2) + [I_1(F^2 + E^2) - (FD + EC)^2 r^2 - \\
 &\quad - (F^2 + E^2)^2] \Psi, \\
 \tau^{23} &= (FD + EC) \{ \Phi + [I_1 - F^2 - E^2 - r^2(D^2 + C^2)] \Psi \}, \\
 \tau^{12} &= \tau^{13} = 0.
 \end{aligned}
 \tag{56.35}$$

If we keep the relations (56.30) and (56.33) in mind, it follows that Φ and Ψ depend only on r . In this case the equilibrium equations have the form of (56.22). Keeping in mind the relations (56.35), it follows that these equations are satisfied only if the function p does not depend on ϕ and z and satisfies the relation

$$\begin{aligned} \frac{dp}{dr} = & \frac{1}{r} \left\{ \left[r^2(D^2 + C^2) - \frac{A^2}{r^2} \right] \Phi + \right. \\ & + r^3 [I_1(D^2 + C^2) - r^2(D^2 + C^2) - (FD + EC)^2] \Psi - \\ & \left. - \frac{A^2}{r^2} \left(I_1 - \frac{A^2}{r^2} \right) \Psi \right\} - A^2 \frac{d}{dr} \left[\frac{1}{r^2} \Phi + \frac{1}{r^2} \left(I_1 - \frac{A^2}{r^2} \right) \Psi \right]. \end{aligned}$$

Consequently, the relations (56.27) characterize a thermo-mechanical state for which there are functions T and p so that the equations (56.1) and (56.2) are satisfied regardless of $\tilde{\sigma}$ and B_s .

(8) Using the designations from (56.12) -- (56.14), let us now consider the deformation defined by

/256

$$\begin{aligned} r^2 &= A\rho^2 + B, \quad \varphi = C\theta + DZ + G, \quad z = R\theta + FZ + H, \\ q_r &= \frac{1}{r} K, \quad q_\varphi = q_z = 0, \end{aligned} \quad (56.36)$$

where A, B, C, \dots, K are constants for which the relation (56.28) takes place.

The tensors of the matrix G_{ij} and G^{ij} are given by (56.16). We derive, from (56.14), (56.36) and (56.28)

$$\begin{aligned} g_{11} &= \frac{r^2}{A^2 \rho^2}, \quad g_{22} = A^2(F^2 \rho^2 + E^2), \quad g_{33} = A^2(D^2 \rho^2 + C^2), \\ g_{12} &= g_{13} = 0, \quad g_{23} = -A^2(DF \rho^2 + EC), \quad g = r^2, \\ g^{11} &= \frac{A^2 \rho^2}{r^2}, \quad g^{22} = \frac{1}{\rho^2}(D^2 \rho^2 + C^2), \quad g^{33} = \frac{1}{\rho^2}(F^2 \rho^2 + E^2), \\ g^{12} &= g^{13} = 0, \quad g^{23} = \frac{1}{\rho^2}(DF \rho^2 + EC). \end{aligned} \quad (56.37)$$

In this case we have

$$\begin{aligned}
I_1 &= \frac{A^2 \rho^2}{r^2} + \frac{1}{\rho^2} (C^2 r^2 + E^2) + D^2 r^2 + F^2, \\
I_2 &= \frac{r^2}{A^2 \rho^2} + \frac{A^2}{r^2} (F^2 \rho^2 + E^2) + A^2 (D^2 \rho^2 + C^2), \quad I_3 = 1, \\
B^{11} &= \frac{A^2}{r^2} [C^2 r^2 + E^2 + \rho^2 (D^2 r^2 + F^2)], \\
B^{22} &= \frac{1}{A^2 \rho^2} + \frac{A^2}{r^2} (D^2 \rho^2 + C^2), \\
B^{33} &= \frac{A^2}{r^2} (F^2 \rho^2 + E^2) + \frac{r^2}{A^2 \rho^2}, \\
B^{23} &= \frac{A^2}{r^2} (DF \rho^2 + EC), \quad B^{12} = B^{13} = 0.
\end{aligned} \tag{56.38}$$

The nonzero components of the tensor γ_j^1 have the expressions

$$\begin{aligned}
\gamma_1^1 &= \frac{1}{2} \left(\frac{A^2 \rho^2}{r^2} - 1 \right), \quad \gamma_2^2 = \frac{1}{2} \left(D^2 r^2 + \frac{r^2}{\rho^2} C^2 - 1 \right), \\
\gamma_3^3 &= \frac{1}{2} \left(F^2 + \frac{1}{\rho^2} E^2 - 1 \right), \quad \gamma_2^3 = \frac{1}{r^2} \gamma_3^2 = \frac{1}{2 \rho^2} (DF \rho^2 + EC).
\end{aligned} \tag{56.39}$$

The equation (56.2) is similarly satisfied. The invariants J_4, J_5 and J_6 are given by the relations

257

$$J_4 = \frac{1}{r^2} K^2, \quad J_5 = \frac{K^2}{2r^2} \left(\frac{A^2 \rho^2}{r^2} - 1 \right), \quad J_6 = \frac{K^2}{4r^2} \left(\frac{A^2 \rho^2}{r^2} - 1 \right)^2. \tag{56.40}$$

From (56.26) we derive

$$\begin{aligned}
\frac{\partial T}{\partial r} &= \left[B_1 + \frac{1}{2} \left(\frac{A^2 \rho^2}{r^2} - 1 \right) B_2 + \frac{1}{4} \left(\frac{A^2 \rho^2}{r^2} - 1 \right)^2 B_3 \right] \frac{K}{r}, \\
\frac{\partial T}{\partial \varphi} &= \frac{\partial T}{\partial z} = 0.
\end{aligned} \tag{56.41}$$

Consequently the functions T and B_s do not depend on the variables

ϕ and z . The function T is determined from (56.41). For the tensions we obtain the expressions

$$\begin{aligned}\tau^{11} &= \frac{A^2 \rho^2}{r^2} \Phi + \frac{A^2}{r^2} [C^2 r^2 + E^2 + \rho^2 (D^2 r^2 + F^2)] \Psi + p, \\ \tau^{22} &= \frac{1}{\rho^2} (D^2 \rho^2 + C^2) \Phi + \left[\frac{A^2}{r^2} (D^2 \rho^2 + C^2) + \frac{1}{A^2 \rho^2} \right] \Psi + \frac{1}{r^2} p, \\ \tau^{33} &= \frac{1}{\rho^2} (F^2 \rho^2 + E^2) \Phi + \left[\frac{A^2}{r^2} (F^2 \rho^2 + E^2) + \frac{r^2}{A^2 \rho^2} \right] \Psi + p, \\ \tau^{23} &= \frac{1}{\rho^2} (DF \rho^2 + EC) \Phi + \frac{A^2}{r^2} (DF \rho^2 + EC) \Psi, \\ \tau^{13} &= \tau^{12} = 0.\end{aligned}\tag{56.42}$$

If we eliminate the function p , it follows from (56.42) that

$$\begin{aligned}r^2 \tau^{22} &= \tau^{11} + \left[\frac{r^2}{\rho^2} (D^2 \rho^2 + C^2) - \frac{A^2 \rho^2}{r^2} \right] \Phi + \\ &\quad + \left[\frac{r^2}{\rho^2 A^2} - \frac{A^2}{r^2} (E^2 + F^2 \rho^2) \right] \Psi, \\ \tau^{33} &= \tau^{11} + \left[\frac{r^2}{\rho^2} (F^2 \rho^2 + E^2) - \frac{A^2 \rho^2}{r^2} \right] \Phi + \\ &\quad + \left[\frac{r^2}{\rho^2 A^2} - A^2 (D^2 \rho^2 + C^2) \right] \Psi, \\ \tau^{23} &= \frac{1}{\rho^2} (DF \rho^2 + EC) \left(\Phi + \frac{A^2 \rho^2}{r^2} \Psi \right), \quad \tau^{13} = \tau^{12} = 0,\end{aligned}\tag{56.43}$$

where r depends on ρ by (56.36). The functions Φ and Ψ depend only /258
on the variable r . The equilibrium equations in this case are

(56.22). It follows from these equations that p is a function only of r . In view of the fact that $r^{11} - r^2 \tau^{22}$ does not depend on p , the first equation from (56.22) determines the function τ^{11} (or p) in the form

$$\tau^{11} = \int_{r_0} (r^2 \tau^{22} - \tau^{11}) \frac{dr}{r} + \tau_0^{11},$$

where τ_0^{11} is the value of τ^{11} for $r = r_0$, and the function $r^2 \tau^{22} - \tau^{11}$ is given by (56.43)₁.

Other controllable states are presented in [169], [252], [339], [342].

REFERENCES

1. Achenbach, J.D., "Approximate transient solutions for the coupled equations of thermoelasticity," J. Acoust. Soc. Am. 36, 10-18 (1964). /259
2. Achenbach, J.D., "The propagation of stress discontinuities according to the coupled equations of thermoelasticity," Acta Mech. 3, 342-351 (1967).
3. Adkins, J.E. and R. S. Rivlin, "Propagation of electromagnetic waves in circular rods in torsion," Phil. Trans. Roy. Soc. Lond., A 255, 635-648 (1963).
4. Agmon, S., Lectures on Elliptic Boundary Value Problems, Princeton, Van Nostrand, 1965.
5. Agostinelli, C., "On the theory of thermoelasticity and some recent contributions," Meccanica 6, 179-196 (1971).
6. Ahmadi, G. and K. Firoozbakhsh, "First strain gradient theory of thermoelasticity," Int. J. Solids Structures 11, 339-345 (1975).
7. Almansi, E., "On the deformation of laterally stressed cylinders," Rend. Accad. Lincei, ser 5 10, 400-408 (1901).
8. Anderson, G.L. and D.F. Mowbray, "Free vibrations of infinite plate based on the linear coupled theory of thermoelasticity," J. Acoust. Soc. Am. 45, 646-653 (1969).
9. Antman, S.S., "The Theory of Rods," Handbuch der Physik [Handbook of Physics], vol. VIa/2 (ed. C. Truesdell), Berlin/Heidelberg/New York, Springer, 1972.
10. Arora, K.K., "Thermoelastic stresses in the flexural deformation of a block," An. Soc. Sci., Bruxelles, Ser. I 84, 359-367 (1970).
11. Baker, B.A., "Some fundamental thermoelastic problems in orthotropic slabs," J. Appl. Mech. 33, 45-51 (1966).
12. Balabukh, L.I. and L.A. Shapovalov, "Variational equations of thermoelasticity," Prikl. Math. Mekh. 24, 703-707 (1960).
13. Banerjee, D.K. and Y.H. Pao, "Thermoelastic waves in anisotropic solids," J. Acoust. Soc. Am. 56, 1444-1454 (1974).
14. Bargmann, H., "Thermal shock in rods under radiation heating," Proc. 2nd Int. Conf. Struct. Mech. Reactor Technol., Berlin, 1973.
15. Bargmann, H., "Recent developments in the field of thermally induced waves and vibrations," Nuclear Engineering and Design 27, 372-385 (1974).

16. Barrekette, E.S. "Thermoelastic stress in beams," J. Appl. Mech. 27, 465-473 (1960).
17. Belli, G. and C. Morosi, "A variational principle for the dynamic problem of linear coupled thermoelasticity," Meccanica 9, 239-243 (1974).
18. Ben-Amoz, M., "On a variational theorem in coupled thermoelasticity," J. Appl. Mech. 32, 243-245 (1965).
19. Benthien, G.W., M.E. Gurtin and T. D. Ralston, "On the semi-discrete Galerkin method for hyperbolic problems and its application to problems in elastodynamics," Arch. Rational Mech. Anal. 48, 51-63 (1972).
20. Biot, M.A., "Thermoelasticity and irreversible thermodynamics," J. Appl. Phys. 27, 240-253 (1956).
21. Biot, M.E., "Linear thermodynamics and the mechanics of solids," Proc. 3rd U.S. Nat. Cong. Appl. Mech. ASME, New York, 1958, pp. 1-18.
22. Bloom, F., "On continuous dependence for a class of dynamical problems in linear thermoelasticity," ZAMP 28, 569-579 (1975).
23. Boley, B.A. and J.H. Weiner, Theory of Thermal Stresses, John Wiley, New York, 1960.
24. Boley, B.A. and I.S. Tolins, "Transient coupled thermoelastic boundary-value problems in the half space," J. Appl. Mech. 29, 637-646 (1962).
25. Boley, B.A. and R.B. Hetnarski, "Propagation of discontinuities in coupled thermoelastic problems," J. Appl. Mech. 35, 489-494 (1968).
26. Boley, B.A. and R.B. Tiasta, "Thermal stresses in composite beams," Int. J. Solids Structures 5, 1153-1169 (1969).
27. Boley, B.A., "Survey of recent developments in the field of heat conduction in solids and thermoelasticity," Nucl. Eng. Des. 3, 377-399 (1972).
28. Borkowski, S., "Thermal stresses in solids," Survey of papers published in 1965-1967, Mech. teor. stosow. 7, 107-153 (1969). /260
29. Bors, C.I., "Axially symmetric thermal tensions in transversely isotropic bodies," An. St. Univ. Iasi. 8, 119-126 (1962).
30. Bors, C.I., "The three-dimensional problem of thermoelasticity of transversely isotropic bodies," Bull. Acad. Polon. Sci. Ser. Sci. Techn. 11, 177-181 (1963).

31. Bors, C.I., "Thermal tension of orthotropic bodies in the case of plane problems," St. cerc. st., Fil. Iasi, Acad. R.P.R. 14, 187-192 (1963).
32. Bors, C.I., "Thermal tensions of bodies which are anisotropic to the elastic plane of symmetry, for the problem of plane deformation," An. st. Univ. Iasi 9, 467-471 (1963).
33. Bors, C.I., "Elastic tensions in transversely isotropic bodies," An. st. Univ. Iasi 10, 445-452 (1964).
34. Bors, C.I., "Thermal tensions of anisotropic bodies in the case of plane deformation," An. st. Univ. Iasi, Matematica 11, 163-167 (1965).
35. Bors, C.I., Teoria elasticitatii corpurilor anizotrope [Theory of elasticity of anisotropic bodies] Editura Academiei R.S.R., Bucharest, 1970.
36. Boschi, E. and D. Iesan, "A generalized theory of linear micropolar thermoelasticity," Meccanica 8, 154-157 (1973).
37. Boschi, E. and F. Mainardi, "A representation theorem in the dynamic theory of porous media," Geophys. J.R. astr. Soc. 34, 313-320 (1973).
38. Bowen, D.E. and P.J. Chen, "Acceleration waves in anisotropic thermoelastic materials with internal state variables," Acta. Mech. 15, 95-104 (1972).
39. Boyd, D.E. and B.R. Kinshore, "Thermal stresses in axially loaded orthotropic cylinders," AIAA J. 6, 980-982 (1968).
40. Brillouin, L., Tenseurs en mécanique et en élasticité [Mechanical and elastic tensors], Paris, 1938.
41. Brun, L., "Energy methods in evolved linear systems. Uniqueness theorems," J. Mecanique 8, 167-192 (1969).
42. Burchuladze, T.B., "Thermoelasticity boundary conditions with mixed boundary conditions," Trudy Tbilis. mat. inst., AN Gruz SSR 44, 10-29 (1974).
43. Buchnell, J.C. and D.J. McCloskey, "Thermoelastic stress production in solids," J. Appl. Phys. 39, 5541-5546 (1968).
44. Bytner, S., "Variational principle for linear coupled thermo-electro-magnetoelasticity of conductors," Bull. Acad. Polon. Sci., Ser. sci. techn. 23, 207-212 (1975).
45. Caricato, G., "Menabrea's theorem for nonisothermic transformations of an elastic, connected, anisotropic and nonhomogeneous body with initial stress," I. Atti Accad. Naz. Lincei, Rend. S. VIII 44, 191-200 (1968).

46. Carlson, D.E., "Linear Thermoelasticity," Handbuch der Physik [Handbook of Physics], vol. VIa/2 (edited by C. Truesdell), Berlin/Heidelberg/New York, Springer, 1972.
47. Carslaw, H.S. and J.C. Jaeger, Conduction of heat in solids, Oxford, 1947.
48. Chadwick, P. and Sneddon, I.N., "Plane waves in an elastic solid conducting heat," J. Mech. Phys. Solids 6, 223-230 (1958).
49. Chadwick, P., "Thermoelasticity. The dynamical theory," in vol. I of Progress in Solid Mechanics (edited by J.N. Sneddon and R. Hill), Amsterdam, North-Holland, 1960.
50. Chadwick, P., "On the propagation of thermoelastic disturbances in thin plates and rods," J. Mech. Phys. Solids 10, 99-109 (1962).
51. Chadwick, P. and B. Powdrill, "Singular surfaces in linear thermoelasticity," Int. J. Engng. Sci. 3, 561-595 (1965).
52. Chadwick, P. and L.T.C. Seet, "Second-order thermoelasticity theory for isotropic and transversely isotropic materials," Trends in elasticity and thermoelasticity, Witold Nowacki anniversary volume, Wolters-Noordhoff Publ., Groningen, 1971, pp. 29-57.
53. Chaudry, H.R., "A note on second-order effects in plane strain thermoelasticity," Int. J. Eng. Sci. 9, 673-679 (1971).
54. Chen, P.J., "Thermodynamic influences on the propagation and the growth of acceleration waves in elastic materials," Arch. Rational Mech. Anal. 31, 228-254 (1968).
55. Chen, P.J., M.E. Gurtin and W.O. Williams, "On the thermodynamic of non-simple elastic materials with two temperatures," ZAMP 20, 107-112 (1969).
56. Chen, P.J., "Growth and Decay of Waves in Solids," Handbuch der Physik [Handbook of Physics], vol. VIa/3, (edited by C. Truesdell), Berlin/Heidelberg/New York, Springer, 1973.
57. Chen, W.T., "Plane thermal stress at an elliptical elastic inclusion under uniform heat flow," Q.J. Mech. Appl. Math 22, 115-123 (1969).
58. Childs, D.R., "Solution of stress-equilibrium and compatibility equations in the presence of body forces and arbitrary temperature fields," Q. Appl. Math. 26, 49-56 (1968).
59. Chirita, S., "On the existence of solutions in the linear theory of micropolar thermoelasticity," Proc. Sci. Sess. Math. Inst. Iasi, Editura Academiei R.S.R., 1976.

60. Chirita, S., "Thermal stresses in anisotropic cylinders," Acta. Mech. 23, 301-306 (1975).
61. Choudhuri, P., "Thermal stress around a crack in a semiinfinite solid of transversely isotropic material," Proc. nat. Inst. Sci. India, Part A, 28, 912-922 (1962).
62. Chow, P.L., "Thermoelastic wave propagation in a random medium and some related problems," Int. J. Engng. Sci. 11, 953-971 (1973).
63. Coleman, B.D., and W. Noll, "The thermodynamics of elastic materials with heat conduction and viscosity," Arch. Rational Mech. Anal. 13, 167-178 (1963).
64. Coleman, B.D. and V. Mizel, "Thermodynamics and departures from Fourier's law of heat conduction," Arch. Rational Mech. Anal. 13, 245-261 (1963).
65. Coleman, B.D. and V. Mizel, "Existence of caloric equations of state in thermodynamics," J. Chem. Phys. 40, 1116-1125 (1964).
66. Courant, R., Partial Differential Equations Vol. 2 of Methods of Mathematical Physics, R. Courant and D. Hilbert, New York, Interscience, 1962.
67. Craine, R.E., "Spherically symmetric problem in finite thermoelastostatics," Q. J. Mech. Appl. Math. 21, 279-291 (1968).
68. Cristea, M., "Concerning mixed thermoelastic equations," Rev. Univ. "C.I. Parhon" si a Polit. Bucharest, ser. st. nat. 1, 72-76 (1952).
69. Cristescu, N., Dynamic plasticity, North-Holland Publishing Company, Amsterdam, 1967.
70. Cukić, R., "The thermal shock on the shell of revolution coupled and uncoupled theory," Bull. Acad. Polon. Sci. Ser. Sci. Techn. 20, 393-400 (1972).
71. Cukić, R., "The thermal shock on the surface of the rectangular plate," Bull. Acad. Polon. Sci. Ser. Sci. Techn. 20, 365-372 (1972).
72. Czarnota-Bojarski, R.E., M. Sokolowski and H. Zorski, Trends in Elasticity and Thermoelasticity. Witold Nowacki Anniversary Volume, Wolters-Nordhoff Publ., Groningen, 1971.
73. Dafermos, C.M., "On the existence and asymptotic stability of solutions in the equations of linear thermoelasticity," Arch. Rational Mech. Anal. 29, 241-271 (1968).
74. Daimaruya, M. and H. Ishikawa, "On the propagation of thermoelastic waves according to the coupled thermoelastic theory," Bull. JSME 17, 991-999 (1974).

75. Danilowskaya, V.I., "Thermal stresses in an elastic half space arising after a sudden heating of its boundary," (in Russian), ibid 14, 316-318 (1950).
76. Danilovskaya, V.I., "Thermal waves," Prikl. Mekh. 1, 126-129 (1965).
77. Das, S.C. and K. Kishan Rao, "Axisymmetric thermal stresses in a spheroidal shell of arbitrary thickness," Acta Techn. CSAV 13, 830-843 (1968).
78. Das, B.R., "A note on thermal stresses in a long cylinder containing a penny-shaped crack," Int. J. Engng. Sci. 7, 667-676 (1969).
79. Day, W.A. and M.E. Gurtin, "On the symmetries of the conductivity tensor and other restrictions in the nonlinear theory of heat conduction," Arch. Rational Mech. Anal. 33, 26-32 (1969).
80. De Groot, S.R., Thermodynamics of Irreversible Processes, North Holland Publ. Co., Amsterdam, 1952.
81. Deresiewicz, H., "Plane waves in a thermoelastic solid," J. Acoust. Soc. Am. 29, 204-209 (1957).
82. Deresiewicz, H., "Solution of the equations of thermoelasticity," Proc. 3rd. U.S. National Congr. Appl. Mech., Brown University, 1958, pp. 287-291.
83. Derski, W., "On transient thermal stress in an infinite thin plate," Proc. Edinburg Math. Soc. (2) 12, 69-73 (1960/61).
84. De Silva, C.N. and S. J. Allen, "The non-stationary thermal problem in the linear theory of elastic shells," ZAMP 45, 225-233 (1965).
85. Dhaliwal, R. S. and K.L. Chowdhury, "Dynamical problems of thermoelasticity for cylindrical regions," Arch. Mech. Stosow. 20, 47-66 (1968).
86. Dillon, Jr., O.W., "A nonlinear thermoelasticity theory," J. Mech. Phys. Solids 10, 123-131 (1962).
87. Dillon, Jr., O.W., "Coupled thermoelasticity of bars," J. Appl. Mech. 34, 137-145 (1967).
88. Domanski, Z. and A. Piskorek, "On the initial value problem in quasi-state thermoelasticity," Bull. Acad. Polon. Ser. Sci. Techn. 19, 101-105 (1971).
89. Dragos, I., "Equations of magneto-thermoelasticity," ZAMP 17, 249-259 (1966).
90. Duhamel, J.M.C., "Second report on thermomechanical phenomena," J. Ecole Polytechn. 15, 1-57 (1937).

91. Duhamel, J.M.C., "Report on calculating molecular actions developed by changes in temperatures in solid bodies," Mémoires par Divers Savants 5, 440-498 (1938).
92. Dunn, H.S., "Transient solution of a one-dimensional thermoelastic/2.2 wave propagation problem," Q. J. Mech. Appl. Mech. 19, 157-165 (1966).
93. Duvaut, G. and J.L. Lions, "New variational inequalities encountered in thermics and thermoelasticity," C. R. Acad. Sci. Paris 269, 1198-1201 (1969).
94. Duvaut, G. and J.L. Lions, Les inéquations en mécanique et physique [Inequalities in mechanics and physics], Dunod, Paris, 1972.
95. Duvaut, G., and J.L. Lions, "Inequalities in thermoelasticity and magnetohydrodynamics," Arch. Rational Mech. Anal. 46, 241-279 (1972).
96. Dym, C.L. and H. Riesmann, "On the time-dependent heat conduction and thermoelastic problems," Q. Appl. Math. 27, 121-124 (1969).
97. Dyszlewicz, J., "Stress formulation of the "second" axially symmetric problem of micropolar theory of elasticity," Bull. Acad. Polon. Sci. Ser. Sci. Techn. 21, 45-55 (1973).
98. Eason, G., B. Noble and I.N. Sneddon, "On certain integrals of Lipschitz-Hankel involving products of Bessel functions," Phil. Trans. Roy. Ser. A 247, 529-551 (1955).
99. Eason, G. and I.N. Sneddon, "The dynamic stresses produced in elastic bodies by uneven heating," Proc. Roy. Soc. Edinburg Ser. A 65, 143-176 (1959).
100. Eason, G., "Transient thermal stresses in anisotropic bodies with spherical symmetry," Appl. Sci. Research A 13, 1-15 (1964).
101. England, A.H. and A.E. Green, "Steady-state thermoelasticity for initially stressed bodies," Phil. Trans. R. Soc. A 253, 517-542 (1961).
102. Ericksen, J.L., "Deformations possible in every isotropic incompressible, perfectly elastic body," ZAMP 5, 466-489 (1954).
103. Ericksen, J.L., "Deformations possible in every compressible, isotropic, perfectly elastic material," J. Math. Phys. 34, 126-130 (1955).
104. Ericksen, J.L., "A thermo-kinetic view of elastic stability theory," Int. J. Solids Structures 2, 573-580 (1966).

105. Ericksen, J.L., "Thermoelastic stability," Proc. 5th U.S. National Congr. of Appl. Mech. ASME, 1966, pp. 187-193.
106. Ericksen, J.L., Tensor fields, Appendix to [420].
107. Eringen, A.C., Mechanics of Continua, John Wiley and Sons, New York, 1967.
108. Eringen, A.C., Theory of Micropolar Elasticity, Fracture, vol. II (editor H. Leibowitz) Acad. Press, New York, London, 1968.
109. Eringen, A.C., "Theory of nonlocal thermoelasticity," Int. J. Solids Structures 12, 1063-1077 (1974).
110. Fabrizio, M., "A phenomenological theory for the thermodynamics of the electromagnetic field," Bollettino UMI 4, 708-719 (1971).
111. Fichera, G., Lectures on Linear Elliptic Differential Systems and Eigenvalue Problems, Berlin/Göttingen/Heidelberg/New York, Springer, 1965.
112. Fichera, G., "Existence theorems in elasticity," Handbuch der Physik [Handbook of Physics], ed. by C. Truesdell, Berlin/Heidelberg/New York, Springer, 1972.
113. Filipkowski, J., "Solution of two-dimensional problems of the theory of thermal stresses," Rozprawy Inz. 16, 139-153 (1968).
114. Fine, A.D., "Continuity conditions of wave fronts in coupled thermoelasticity," Q. Appl. Math. 25, 121-128 (1967).
115. Flavin, J.N. and A.E. Green, "Plane thermoelastic waves in an initially stressed medium," J. Mech. Phys. Solids 9, 179-190 (1961).
116. Flavin, J. H., "Thermoelastic Rayleigh waves in a prestressed medium," Proc. Cambridge Phil. Soc. 58, 532-538 (1962).
117. Florence, A.L. and J. N. Goodier, "The linear thermoelastic problem of uniform heat flow disturbed by a penny-shaped insulated crack," Int. J. Engng. Sci. 1, 533-540 (1963).
118. Fogagnolo, B., "Propagation of thermoelastic waves in an undefined isotropic ground," Rend. Ist. Lombardo (A) 106, 521-541 (1972).
119. Fong, J.T. and J.A. Simmons, "A non-equilibrium thermodynamic theory of simple materials based on a single-integral entropic functional," Arch. of Mech. 24, 363-372 (1972).
120. Fox, N., "Thermal stresses due to a moving line source on a plane boundary," Q. J. Mech. Appl. Math. 18, 25-30 (1965).

121. Fox, N. "Generalized thermoelasticity," Int. J. Engng. Sci. 7, 437-445 (1969).
122. Fox, N., "Low temperature effects and generalized thermoelasticity," J. Math. Appl. 5, 373-386 (1969).
123. Francis, P.H., "Thermo-mechanical effects in elastic wave propagation: A Survey," J. Sound Vib. 21, 181-192 (1972).
124. Frisch, H.P., "Thermally induced vibrations of long thin-walled cylinders of open section," J. Spacecraft Rockets 7, 897-905 (1970).
125. Fu Bao Lyan', "Generalized variational principles of thermoelasticity," Scientia Sinica 13, 1507-1509 (1964).
126. Fu, W.S., "Thermal stresses in an elastic solid weakened by two coplanar circular cracks," Int. J. Engng. Sci. 11, 317-330 (1973). /263
127. Galka, A., "Green's functions for the coupled problem of thermoelasticity obtained from the solutions of the theory of thermal stresses," Bull. Acad. Polon. Sci. Ser. Sci. Techn. 13, 369-376 (1965).
128. Galka, A., "Singular solutions of thermoelasticity," Bull. Acad. Polon. Sci. Ser. Sci. Techn. 13, 523-529 (1965).
129. Gascoigne, H.E. and I.K. McIvor, "Stress waves in layered thermoelastic media generated by impulsive energy deposition," AIAA J. 9, 937-943 (1971).
130. Gatewood, B.E., "Note on the thermal stresses in a long circular cylinder of $n-1$ concentric materials," Q. Appl. Math. 6, 84-86 (1948).
131. Gatewood, B.E., Thermal Stresses, McGraw-Hill, New York/Toronto/London, 1957.
132. Germain, P., Cours de Mécanique des Milieux Continus [Course on Mechanics of Continuous Media], Masson et Cie., Paris, 1973.
133. Gheorghita, St.I., Metode matematica in hidrogazodinamica subterana [Mathematical Methods in Subterranean Hydrogasdynamics], Editura Academiei, Bucharest, 1966.
134. Gonsovskiy, V.I. and Yu. Rossichin, "Plane harmonic wave propagation in an anisotropic thermoelastic medium," Prikl. Mekh. 11, 64-67 (1975).
135. Goodier, J.H., "On the integration of the thermoelastic equations," Phil. Mag. 23, 1017-1032 (1937).
136. Graffi, D., "The reciprocity theorem in the dynamics of elastic bodies," Mem. Accad. Sci. Bologna 18, 203-209 (1946-47).

137. Graffi, D., "Reciprocity in the dynamics of elastic bodies," Ing. Arch. 22, 45-52 (1954).
138. Graffi, D., "The reciprocity theorem in nonstationary phenomena," Atti. Acad. Sc. Ist. Bologna 11, 10, 33-40 (1963).
139. Green, A.E. and Zerna, W., Theoretical Elasticity, Clarendon Press, Oxford, 1954.
140. Green, A.E., J.R.M. Radok and H.S. Rivlin, "Thermoelastic similarity laws, Q. Appl. Math 15, 381-393 (1957).
141. Green A.E. and J.E. Adkins, Large Elastic Deformations, Clarendon Press, Oxford, 1960.
142. Green, A.E., "Thermoelastic stresses in initially stressed bodies," Proc. E. Soc. A, 266, 1-19 (1962).
143. Green, A.E. and R.S. Rivlin, "On Cauchy's equations of motion," ZAMP 15, 290-292 (1964).
144. Green, A.E. and R. S. Rivlin, "Multipolar continuum mechanics," Arch. Rational Mech. Anal. 17, 113-147 (1964).
145. Green, A.E., P.M. Nagudi and J.A. Trapp, "Thermodynamics of a continuum with internal constraints," Int. J. Engng. Sci. 8, 891-908 (1970).
146. Green, A.E. and K.A. Lindsay, "Thermoelasticity," J. Elasticity 2, 1-7 (1972).
147. Green, A.E., "A note on linear thermoelasticity," Mathematika 19, 69-75 (1972).
148. Grimado, P.B., "On the thermal stress induced in a semi-infinite body as a result of a suddenly applied step heat input," J. Appl. Mech. 37, 1149-1150 (1970).
149. Grindei, I., "Thermal tensions inside an elliptical homogeneous and isotropic cylinder provided with a finite number of elliptical points," St. cerc. st. mat., Acad. R.P.R., Filiala Iasi 14, 371-375 (1963).
150. Grindei, I., "Axially symmetric thermal tensions in an isotropic cylindrical body of finite length," An. st. Univ. Iasi, Matematica 12, 191-196 (1966).
151. Grindei, I., "Thermal tension in an isotropic cylindrical body placed in a discontinuous temperature field," Bul. Inst. Polit. Iasi 12, 93-98 (1966).
152. Grindei, I., Termoelasticitate [Thermoelasticity], Editura didactica si pedagogica, Bucharest, 1967.

153. Grindei, I., "Nonstationary thermal tensions in thin circular plates," An. st. Univ. Iasi Matematica 13, 103-108 (1967).
154. Grindei, I., "Thermal tensions in transverse isotropic elastic media," An. st. Univ. Iasi, Matematica 14, 169-176 (1968).
155. Grindei, I., "Stationary thermal tensions within thin plates composed of different materials," Bul. Inst. Polit. Iasi, Sect. I 15, 67-74 (1969).
156. Grioli, G., Mathematical Theory of Elastic Equilibrium (Recent Results), Springer, Berlin/Göttingen/Heidelberg, 1962.
157. Brioli, G., "On the thermodynamic potential for continua with reversible transformations - some possible types," Meccanica 1, 15-20 (1966).
158. Burtin, M.E., "Variational principles for linear elastodynamics," Arch. Rational. Mech. Anal. 16, 34-50 (1964).
159. Burtin, M.E., "On the thermodynamics of elastic materials," J. Math. Anal. Appl. 18, 38-44 (1967).
160. Gurtin, M.E. and W.O. Williams, "An axiomatic foundation for continuum thermodynamics," Arch. Rational Mech. Anal. 26, 83-117 (1967). /264
161. Gurtin, M.E., and A.C. Pipkin, "A general theory of heat conduction with finite wave speeds, Arch. Rational Mech. Anal. 31, 113-126 (1969).
162. Gurtin, M.E. and W.O. Williams, "On the first law of thermodynamics," Arch. Rational Mech. Anal. 42, 77-92 (1971).
163. Gurtin, M.E., "The Linear Theory of Elasticity," Handbuch der Physik [Handbook of Physics], vol VIa/2 (edited by C. Truesdell), Springer, Berlin/Heidelberg/New York, 1972.
164. Gurtin, M.E. and P.P. Guidugli, "The thermodynamics of constrained materials," Arch. Rational Mech. Anal. 51, 192-208 (1973).
165. Gurtin, M.E., "Thermodynamics and the potential energy of an elastic body," J. Elasticity 3, 23-26 (1973).
166. Gurtin, M.E., "Thermodynamics and the energy criterion for stability," Arch. Rational Mech. Anal. 52, 93-103 (1973).
167. Hadamard, J., Leçons sur la Propagation des Ondes et les Equations de l'Hydrodynamique [Lessons on Wave Propagation and Hydrodynamic Equations], Hermann, Paris, 1903.
168. Haimovici, M., Teori elasticitatii [Theory of Elasticity], Editura didactica si pedagogica, Bucharest, 1960.

169. Hayes, M., N. Laws and R.B. Osborn, "The paucity of universal motions in thermoelasticity," Q. Appl. Math 27, 416-420 (1969).
170. Herrmann, G., "On second-order thermoelastic effects," ZAMP 15, 253-262 (1964).
171. Hermann, G., "On variational principles in thermoelasticity and heat conduction," Q. Appl. Mech. 22, 151-155 (1964).
172. Hetnarski, R.B., "Coupled one-dimensional thermal shock problem for small times," Arch. Mech. Stosow. 13, 295-306 (1961).
173. Hetnarski, R.B., "Coupled thermoelastic problem for half-space," Bull. Acad. Polon. Sci., Ser. Sci. Techn. 12, 49-57 (1964).
174. Hetnarski, R.B., "Solution of the coupled problem of thermoelasticity in the form of series of functions," Arch. Mech. Stosow. 16, 919-941 (1964).
175. Hetnarski, R.B., "The fundamental solution of the coupled thermoelastic problem for small times," Arch. Mech. Stosow. 16, 23-31 (1964).
176. Hieke, M., "Indirect determination of Airy's plane during unstable thermal stresses," ZAMM 35, 285-294 (1955).
177. Hlavacek, I. and J. Necas, "On inequalities of Korn's type," Arch. Rational Mech. Anal. 36, 305-334 (1970).
178. Hlavacek, I., "Variational formulation of the Cauchy problem for equations with operator coefficients," Aplika matematiky 16, 46-63 (1971).
179. Hsu, T.R., "Thermal shock on a finite disk due to an instantaneous point heat source," J. Appl. Mech. 36, 113-120 (1969).
180. Hu, H., "On some variational principles in the theory of elasticity and the theory of plasticity," Sci. Sinica 4, 33-54 (1955).
181. Huang, N.C. and R.C. Shieh, "Thermomechanical coupling effect on the stability on non-conservative elastic continuous systems," Int. J. Mech. Sci. 12, 39-49 (1970).
182. Hutson, V., "A note on an integral equation occurring in thermoelasticity," Q. J. Mech. Appl. Math. 22, 125-127 (1969).
183. Iacob, C., Introduction mathématique à la mécanique des fluides [Mathematical Introduction to Fluid Mechanics], Editura Academiei R.S.R., Bucharest and Gauthier-Villars, Paris, 1969.
184. Iesan, D., "Some general theorems of the linear theory of microstructural elastic media," C. Rend. Acad. Sc. Paris 270, 960-962 (1970).

185. Iesan, D., "Thermal tension within a circular cylinder which has a longitudinal core of another material," An. St. Univ. Iasi, Matematica 11A, 347-356 (1965).
186. Iesan, D., "Variational principles within the theory of coupled thermoelasticity," Analele St. Univ. "Al. I. Cuza" Iasi, Sect. I Matematica 12, 439-456 (1966).
187. Iesan, D., "The theory of coupled micropolar thermoelasticity," C. R. Acad. Sci. Paris 265, 271-275 (1967).
188. Iesan, D., "Concerning the theory of nonlinear thermoelasticity," An. St. Univ. Iasi, Matematica 13, 161-175 (1967).
189. Iesan, D., "On the plane coupled micropolar thermoelasticity," Bull. Acad. Polon. Sci., Ser. Sci. Tech. 16, I: 653-658; II: 659-664 (1968).
190. Iesan, D., "Thermal tension in anisotropic elastic bars," An. St. Univ. Iasi, Matematica, 14, 423-441 (1968).
191. Iesan, D., "Concerning generalized plane deformation," St. cerc. mat. 21, 595-609 (1969).
192. Iesan, D., "On the thermoelastic finite plane strain," An. St. Univ. Iasi, Matematica 15, 195-207 (1969).
193. Iesan, D., "Thermoelastic plane deformation of nonhomogeneous isotropic media," An. St. Univ. Iasi, Matematica 15, 463-470 (1969).
194. Iesan, D., "On the linear coupled thermoelasticity with two temperatures," ZAMP 21, 583-591 (1970).
195. Iesan, D., "On some theorems in the linear theory of micropolar thermoelasticity," Rev. Roum. Math. Pures et Appl. 15, 1181-1195 (1970). /265
196. Iesan, D., "Thermal stresses in the generalized plane strain of anisotropic elastic solids," Bull. Acad. Polon. Sci., Ser. Sci. Tech. 18, 227-230 (1970).
197. Iesan, D., "On the plane strain of anisotropic solids in the theory of coupled thermoelasticity," An. St. Univ. Iasi, Matematica 16, 137-148 (1970).
198. Iesan, D., "On thermal stresses in plane strain of isotropic micropolar elastic solids," Acta Mech. 11, 141-144 (1971).
199. Iesan, D., "Some theorems in the linear theory of coupled micropolar thermoviscoelasticity," Bull. Acad. Polon. Sci., Ser. Sci. Techn. 19, 121-129 (1971).

200. Iesan, D., "On the thermal stresses in beams," J. Eng. Math 6, 155-163 (1972).
201. Iesan, D., "On some reciprocity theorems and variational theorems in linear dynamic theories of continuum mechanics," Memorie dell'Acad. Sci. Torino, Cl.Sci. Fis. Mat. Nat., Serie 4 17, (1974).
202. Iesan, D., "Thermal stresses in micropolar elastic cylinders," Acta Mech. 21, 261-272 (1975).
203. Iesan, D., "Saint-Venant's problem for inhomogeneous and anisotropic solids," J. Eng. Math. 9, 281-290 (1975).
204. Iesan, D., "Saint-Venant's problem for inhomogeneous and anisotropic elastic bodies," J. Elasticity 6, No. 2 (1976).
205. Iesan, D., "Saint-Venant's problem for heterogeneous anisotropic elastic solids," Ann. Mat. Pura Appl. (1976) (in print).
206. Ignaczak, J., "The Sommerfeld conditions for coupled problems of thermoelasticity. Examples of coupled stresses and temperature concentration at a cylindrical and spherical cavities," Arch. Mech. Stosow. 14, 3-14 (1962).
207. Ignaczak, J., "A completeness problem for the stress equations of motion in the linear theory of elasticity," Arch. Mech. Stosow. 15, 225-234 (1963).
208. Ignaczak, J., "On the stress equations of motion in linear thermoelasticity," Arch. Mech. Stosow. 15, 691-695 (1963).
209. Ignaczak, J., and W. Nowacki, "Singular integral equations of thermoelasticity," Rozpr. Inzyn. 13, 655-660 (1965).
210. Ignaczak, J. and W. Nowacki, "Singular integral equations of thermoelasticity," Int. J. Engng. Sci. 4, 53-68 (1966).
211. Ignaczak, J., "Tensorial equations of motion for elastic materials with microstructure," Trends in Elasticity and Thermoelasticity. Witold Nowacki Anniversary Volume, Wolters-Noordhoff Publ., Groningen, 1971, pp. 89-111.
212. Inan, E., "Coupled theory of thermoelastic plates," Acta Mech. 14, 1-29 (1972).
213. Ionescu-Cazimir, V., "On thermoelastic equilibrium equations, I. Galerkin's vector analogs," Bul. St. Acad. R.P.R., ser. mat. fiz. chim. 2, 589-595 (1950).
214. Ionescu-Cazimir, V., "On thermoelastic equilibrium equations, II, III. Relation between tension and temperature," Com. Acad. R.P.R. 1, 171-177, 385-390 (1951).

215. Ionescu-Cazimir, V., "Problem of linear coupled thermoelasticity. Theorems on reciprocity for the dynamic problem of coupled thermoelasticity. I " Bull. Acad. Polon. Sci., Ser. Sci. Techn. 12, 473-480 (1964).
216. Ionescu-Cazimir, V., "Problem of linear coupled thermoelasticity. II. Some applications of the theorems on reciprocity for the dynamic problem of coupled thermoelasticity," Bull. Acad. Polon. Sci., Ser. Sci. Techn. 12, 481-488 (1964).
217. Ionescu-Cazimir, V., "Problem of linear coupled thermoelasticity. III. Uniqueness theorem," Bull. Acad. Polon. Sci., Ser. Sci. Techn. 12, 565-573 (1964).
218. Ionescu-Cazimir, V., "Problem of linear coupled thermoelasticity. IV. Uniqueness theorem," Bull. Acad. Polon. Sci., Ser. Sci. Techn. 12, 575-579 (1964).
219. Ionescu, V., "A variational theorem for the problem of coupled thermoelasticity," An. Univ. Bucharest, Mat.-Mec. 15, 33-40 (1966).
220. Jaunzemis, W., Continuum Mechanics, MacMillan Comp., New York, 1967.
221. Jeffreys, H., "The thermodynamics of an elastic solid," Proc. Cambridge Phil. Soc. 26, 101-106 (1930).
222. Jentsch, I., "Thermoelasticity of piecewise homogeneous bodies," Arch. Rational Mech. Anal. 24, 141-172 (1967).
223. Jentsch, I., "Stationary thermoelastic variations in inhomogeneous bodies," Math. Nachr. 64, 171-231 (1974).
224. Johns, D.J., "Thermal stresses in a long circular shell with axial temperature variation," J. Aero/Space Sci. 27, 393-394 (1960).
225. Johnson, A.F., "Some second order effects in thermoelasticity theory," Report DNAM 84, National Physical Laboratory, 1970.
226. Jones, J.P., "Thermoelastic vibrations of a beam," J. Acoust. Soc. Am. 39, 542-548 (1966). /266
227. Kahnashvili, N.S., "Some estimates in the theory of thermoelasticity," Soobshch. AN GSSR 65, 547-550 (1972).
228. Kaliski, S., "Wave equations in thermoelasticity," Bull. Acad. Polon. Sci., Ser. Sci. Techn. 18, 409-416 (1965).
229. Kaliski, S., "The reciprocity theorem for the wave equations of thermo-magneto-elasticity," Proc. Vibr. Probl. 7, 85-91 (1966).

230. Kassir, M.K., "On the distribution of thermal stresses around an elliptical crack in an infinite solid," Int. J. Engng. Sci. 7, 769-784 (1969).
231. Kacs, W. and P.P. Teodorescu, Applications of the Theory of Distributions in Mechanics, Editura Academiei R.S.R. Bucharest, Abacus Press, Tunbridge Wells, Kent, 1974.
232. Kellog, O.C., Foundations of Potential Theory, Springer, Berlin, 1929.
233. Kil'chinska, G.O., "Coupled thermoelastic perturbations in half-space during thermal shock on a surface and finite heat propagation rate," Dopovidi AN URSS (A) [Reports of Ukrainian SSR Academy of Sciences (A)], 1971, pp. 368-371.
234. Kline, K.A., "Theorems of Ignaczak type for elastic materials and microstructure," Bull. Acad. Polon. Sci., Ser. sci. techn. 18, 101-107 (1970).
235. Knops, R.J. and L.E. Payne, "On uniqueness and continuous dependence in dynamical problems of linear thermoelasticity," Int. J. Solids Structures 6, 1173-1184 (1970).
236. Knops, R.J. and E.W. Wilkes, "Theory of Elastic Stability," Handbuch der Physik [Handbook of Physics], vol. VIa/3, edited by C. Truesdell, Springer, Berlin/Heidelberg/New York, 1973.
237. Koh, S.L. and A.C. Eringen, "On the foundation of non-linear thermoelasticity," Int. J. Engng. Sci. 1, 199-229 (1963).
238. Koiter, W.T., On the Thermodynamic Background of Elastic Stability Theory, Problems of Hydrodynamics and Continuum Mechanics, Sedov Anniversary Volume, SIAM, Philadelphia, 1969, pp. 243-433.
239. Koizumi, T., "Transient thermal stresses in a semi-finite body heated axi-symmetrically," ZAMM 50, 747-757 (1970).
240. Kolyano, Yu.M., "Transient phenomena and representation of the general solution of general coupled thermoelastic problems of anisotropic plates," Dopovidi AN URSS (A), 1094-1096 (1971).
241. Korenev, B.G., "Some plane thermoelasticity problems in the presence of periodic thermal effects," Dokl. AN SSSR 164, 294-297 (1965).
242. Kovalenko, A.D., Thermoelasticity. Basic Theory and Applications, Wolters-Noordhoff, Publ. Groningen, 1969.
243. Kovarik, V., "Thermoelastic stability and thermally induced vibrations of sandwich plates," Acta techn. CSAY 12, 702-729 (1967).

244. Kulshrestha, P.K. and P. Puri, "Coupled thermoelastic problem for an half-space with an embedded spherical cavity," Arch. of Mech. 23, 249-263 (1971).
245. Kupradze, V.D., Granichnye zadachi teorii kolebaniy i integral'nye uravneniya [Boundary value problems in the theory of vibrations and integral equations], Gostekhizdat, Moscow, 1950.
246. Kupradze, V.D., Metody teorii potentsiala v teorii uprugosti [Potential theory methods in elasticity theory], Fizmatgiz, Moscow, 1963.
247. Kupradze, V.D., "Dynamical Problems in Elasticity," Progress in Solid Mechanics, vol. III, 1963.
248. Kupradze, V.D., T.D. Gegelya, M.O. Bacheleishvili, T.V. Burchuladze, Trechmernyye zadachi matematicheskoi teorii uprugosti [Three dimensional problems of mathematical elasticity theory], Tbilisi Univ. Press, Tbilisi, 1968,
249. Kupradze, V.D., T.V. Burchuladze, "Boundary value problems of thermoelasticity," Diff. Uravn. 5, 3-43 (1969).
250. Kurashige, M. and A. Atsumi, "Thermal stresses in a circular cylinder with a ring of holes," J. Appl. Mech. 35, 604-606 (1968).
251. Lal, M. and A.K. Chaturvedi, "Nonaxisymmetric thermoelastic stress distribution in a solid containing a penny shaped crack," Jnanabha Sect. A 4, 27-40 (1974).
252. Laws, N., "Controllable states in thermoelasticity," Q. Appl. Math. 28, 432-436 (1970).
253. Laws, N., "The overall thermoelastic moduli of transversely isotropic composites according to the self-consistent method," Int. J. Engng. Sci. 12, 79-87 (1974).
254. Lebon, G. and J. Lambermont, "Stability of thermo-elastic bodies," Nederl. Acad. Wet., Proc. Ser. B 76, 329-340 (1975).
255. Lee, C.W., "Thermal stresses in a thick plate," Int. J. Solids Structures 6, 605-615 (1970).
256. Lekhnitskii, S.G., Theory of Elasticity of an Anisotropic Elastic Body, Holden-Day, Inc., San Francisco, 1963.
257. Lessen, M. and C.E. Duke, "On the motion of an elastic thermally conducting solid," Proc. 1st Midwest Conf. on Solid Mech., Univ. of Illinois, 1953, pp. 14-18.

258. Lessen, M., "Note on the symmetrical property of the thermal conductivity tensor," Q. Appl. Math., 14, 208-209 (1956).
259. Lessen, M., "Thermoelasticity and thermal shock," J. Mech. Phys. Solids 5, 56-60 (1956).
260. Lessen, M., "The motion of a thermoelastic solid," Q. Appl. Math. 15, 105-108 (1957).
261. Levine, H.A., "On a theorem of Knops and Payne in dynamical linear thermoelasticity," Arch. Rational Mech. Anal. 38, 290-307 (1970).
262. Levoni, S., "Integration of equations of thermoelasticity of incompressible solids," Ann. Scuola Norm. Sup. Pisa, S III 22, 515-525 (1968).
263. Librescu, L., "Thermoelastic problem of unbounded plates and slabs, eliminating the Love-Kirchoff hypothesis," St. cerc. mec. apl. 21, 351-365 (1966).
264. Liu Hsien-Chih, "Thermoelastic stress and deformation caused by a spherical inclusion in an infinite elastic slab," Scientia Sinica 14, 1401-1414 (1965).
265. Lockett, F.J., "Effect of thermal properties of a solid on the velocity of Rayleigh waves," J. Mech. Phys. Solids 7, 71-75 (1958).
266. Lockett, F.S. and I.N. Sneddon, "Propagation of thermal stresses in an infinite medium," Proc. Edinburg Math. Soc. 11, 237-244 (1959).
267. Longcope, D.B., M.J. Forrestal and W.E. Warren, "Thermal stress in a transversely isotropic, hollow, circular cylinder," AIAA J. 7, 2174-2176 (1969).
268. Lord, H.W. and Y. Shulman, "A generalized dynamical theory of thermoelasticity," J. Mech. Phys. Solids 15, 299-309 (1967).
269. Lord, H.W. and A.A. Lopez, "Wave propagation in thermoelastic solids at very low temperature," Acta Mech. 10, 85-98 (1970).
270. Lucas, R.A., "A quasi-static thermoelastic analysis of a propagating crack," Int. J. Solids Structures 5, 175-190 (1969).
271. Lur'ye, A.I., Prostranstvennye zadachi teorii uprugosti [Three dimensional problems of elasticity theory] Gostekhizdat, Moscow, 1955.

272. Malanabis, R.K. and M. Mitra, "On a mixed problem of thermoelasticity," Proc. Nat. Inst. Sci. India, Part A 34, 143-154 (1968).
273. Manacorda, T., "Elementary waves in the thermoelasticity of incompressible solids," Atti. Accad. Sci. torino, Cl. Sci. fis. mat. natur. 101, 503-509 (1966-67).
274. Mandel, J., "Applications of thermodynamics to media whose instantaneous deformations are elastic or zero," Arch. Mech. Stosow. 20, 181-187 (1968).
275. Manfredi, B., "Connections between thermoelastic transformations of compressible and incompressible media in the linear non-stationary case," Rev. Mat. Univ. Parma II Ser. 9, 47-57 (1968).
276. Manolachi, A., "Problem of plane thermoelasticity for completely anisotropic media," An. St. Univ. Iasi, Matematica 15, 209-228 (1969).
277. Marguerre, K., "Thermoelastic slab equations," ZAMM 15, 369-372 (1935).
278. Martin, C.J., "Transient thermoelastic stresses in a slab," J. Math. Mech. 15, 465-481 (1966).
279. Mattei, G., "Cylindrical waves in the thermoelasticity of incompressible solids," Bollettino U.M.I. 4, 102-112 (1973).
280. Maysel, V.M., Temperature problems of the theory of elasticity, (Russian) Kiev, 1951.
281. Maysel, V.M., "A generalization of the Betti-Maxwell theorem to the case of thermal stresses and some of its applications," (Russian) Dokl. Acad. Sci. URSS 30, 115-118 (1941).
282. McCarthy, M.F., "Constitutive equations for thermomechanical materials with memory," Int. J. Engng. Sci. 8, 407-474 (1970).
283. McCarthy, M.F., "Wave propagation in generalized thermoelasticity," Int. J. Engng. Sci. 10, 593-602 (1972).
284. McQuillen, E.J. and M.A. Brull, "Dynamic thermoelastic response of cylindrical shells," J. Appl. Mech. 37, 661-670 (1970).
285. Meixner, J., "On the linear theory of heat conduction," Arch. Rational Mech. Anal. 39, 108-130 (1970).
286. Melan, E. and H. Parkus, Wärmespannungen infolge stationärer Temperaturfelder [Thermal Stresses Caused by Stationary Temperature Fields], Vienna, 1953.
287. Micunovic, M., "A geometrical treatment of thermoelasticity of simple inhomogeneous bodies," Bull. Acad. Polon. Sci., Ser. Sci. Techn. 22, 941-950 (1974).

288. Mikhlin, S.G., Variational Methods in Mathematical Physics, Pergamon, New York, 1964.
289. Mikhlin, S.G., The Problem of the Minimum of a Quadratic Functional, Holden-Day, Inc., San Francisco, 1965.
290. Misicu, M., Mecanica mediilor deformabile. Fundamentele elasticitatii structurale [Mechanics of deformable media. Fundamentals of structural elasticity], Editura Academiei, Bucharest, 1967.
291. Misicu, M., "Generalized theory of thermoelasticity associated with boundary problems of classical fields," Trends in Elasticity and Thermoelasticity, Witold Nowacki Anniversary Volume, 1971, pp. 161-172.
292. Moisil, Gr.C., "Preliminary theory on the system of equations with partial linear derivatives and constant coefficient," Bul. St. Acad. R.F.R. Sect. st. mat. si fiz. 4, 319-401 (1952).
293. Mountfort, M.V., "A transformation in coupled thermoelasticity problems," J. Appl. Mech. 33, 196-198 (1966).
294. Muki, R. and S. Breuer, "Coupling effects in transient thermoelastic problem," Osterr. Ing. Arch. 16, 349-368 (1962). /268
295. Mura, T. "Thermal strain and stresses in transient state," Proc. Sec. Japan Congress Appl. Mech., 1952.
296. Muskhelishvili, N.I., Some basic problems of the mathematical theory of elasticity, Groningen, 1953.
297. Müller, H. and K. Stork, "Propagation of thermoelastic waves," Ann. der Physik 18, 214-218 (1966).
298. Müller, I. "On the entropy inequality," Arch. Rational Mech. Anal. 26, 118-141 (1967).
299. Müller, I., "The coldness, a universal function in thermoelastic bodies," Arch. Rational Mech. Anal. 41, 319-332 (1971).
300. Naghdi, P.M., "The Theory of Shells and Plates," Handbuch der Physik [Handbook of Physics], vol. VIa/2 (ed. C. Truesdell), Springer, Berlin/Heidelberg/New York, 1972.
301. Naghdi, P.M. and J.A. Trapp, "On the general theory of stability for elastic bodies," Arch. Rational Mech. Anal. 51, 165-191 (1973).
302. Nariboli, G.A., "Spherically symmetric shock in a medium with thermal and elastic deformations coupled," Quart. J. Mech. Appl. Math. 14, 1 (1961).

303. Nariboli, G.A. and Nayayadhish, "One dimensional thermoelastic waves," Q. J. Mech. Appl. Math. 16, 473-482 (1973).
304. Nemat-Nassar, S., "On thermomechanics of elastic stability," ZAMP 21, 538-552 (1970).
305. Neumann, F., Vorlesungen über die Theorie der Elastizität [Lectures on elastic theory], Teubner, Leipzig, 1885.
306. Newman, M. and M. Forray, "Bending stresses due to temperature in hollow circular plates," J. Aerospace Sci. 27, 792-793 (1960).
307. Nickell, R.E. and J.I. Sackman, "Variational principles for linear coupled thermoelasticity," Q. Appl. Math. 26, 11-26 (1968).
308. Nickell, R.E. and J.I. Sackman, "Approximate solutions in linear coupled thermoelasticity," J. Appl. Mech. 35, 255-266 (1968).
309. Nistor, I., "A theorem of existence, uniqueness and analyticity of the solution of the equations of nonlinear thermoelasticity," An. St. Univ. Iasi Matematica 19, 465-476 (1973).
310. Noll, W., "A mathematical theory of the mechanical behavior of continuous media," Arch. Rational. Mech. Anal. 2, 197-226 (1958).
311. Noll, W., The Foundations of Mechanics and Thermodynamics, Selected Papers, Springer, Berlin/Heidelberg/New York, 1974.
312. Nordgren, R.P., "On the method of Green's function in the thermoelastic theory of shallow shells," Int. J. Engng. Sci. 1, 279-308 (1963).
313. Norwood, F.R. and W.E. Warren, "Wave propagation in the generalized dynamical theory of thermoelasticity," Q. J. Mech. Appl. Math. 22, 283-290 (1969).
314. Nowacki, W., "Thermal stresses in anisotropic bodies, I" (In Polish), Arch. Mech. Stosow. 6, 481-492 (1954).
315. Nowacki, W., Thermoelasticity, Pergamon Press, Oxford/London/New York/Paris, 1962.
316. Nowacki, W., "Green functions for the thermoelastic medium II," Bull. Acad. Polon. Sci., Ser. Sci. Techn. 14, 505-513 (1966).
317. Nowacki, W., Dynamiczne zagadnienia termospreszystości [Dynamic Problems of Thermoelasticity] Warsaw, 1966.
318. Nowacki, W., "Couple-stresses in the theory of thermoelasticity III," Bull. Acad. Polon. Sci., Ser. Sci. Techn. 14, 505-513 (1966).

319. Nowacki, W., Dinamicheskiye zadachi termouprugosti [Dynamic problems of thermoelasticity], Moscow, 1970.
320. Nowacki, W., Theory of Micropolar Elasticity, CISM, Udine, Springer, 1970.
321. Nowinski, J., "Biharmonic solutions to the steady-state thermoelastic problems in three dimensions," ZAMP 12, 132-149 (1961).
322. Nowinski, J.L., "A mean value theorem for an arbitrary steady state thermoelastic problem for a solid sphere," J. Soc. Industr. Appl. Math. 11, 623-631 (1963).
323. Nowinski, J.L., "Thermal waves in an elastic highly stretched cylindrical bar," Acta Mech. 7, 45-57 (1969).
324. Oden, J.T. and J. Poe, "On the numerical solution of a class of problems in dynamic coupled thermoelasticity," In Proc. 5th SECTAM 1970 (D. Frederick, ed.), Pergamon Press, Oxford, 1970.
325. Oden, J.T., Finite Elements of Non-Linear Continua, McGraw-Hill, New York, 1972.
326. Olesiak, Z. and I.N. Sneddon, "The distribution of thermal stress in an infinite elastic solid containing a penny-shaped crack," Arch. Rational Mech. Anal. 3, 238-254 (1960).
327. Olszak, W., and P. Perzyna, "Thermal effects in viscoplasticity," In IUTAM Symp. (B.A. Boley, ed.) East Kilbride, 1968, Springer-Verlag, Vienna (1970).
328. Owen, D.R., "Thermodynamics of materials with elastic range," Arch. Rational Mech. Anal. 31, 91-112 (1968).
329. Padovan, J., "Thermoelasticity of an anisotropic half space," J. Appl. Mech. 41, 935-940 (1974).
330. Pal, R., "Thermal stresses and displacements in an infinite elastic slab due to a moving line source of heat on its plane boundary," Bull. Acad. Sci. Polon., Ser. Sci. Techn. 18, 709-718 (1970). /269
331. Paolini, G., "Thermodynamic analysis of isotropic elastic solids stressed to fatigue," Rend. Ist. Lombardo (A) 101, 117-158 (1967).
332. Paria, G., "Coupling of elastic and thermal deformations," I. Appl. Sci. Res. Sect. A 7, 463-475 (1958).
333. Paria, G., "Coupling of elastic and thermal deformations," Appl. Sci. Research. Series A 7, 463-475 (1959).
334. Parkus, H., Instationäre Wärmespannungen [Nonstationary Thermal Stresses], Springer-Verlag, Vienna, 1959.

335. Parkus, H., "Thermal stresses during random surface temperature," ZAMM 42, 499-507 (1962).
336. Parkus, H., Thermoelasticity, Blaisdell Publishing Co., Waltham, Mass., 1968.
337. Perzyna, P. and A. Sawczuk, "Problems of thermoplasticity," Nucl. Eng.-Des. 24, 1-55 (1973).
338. Petroski, H.J. and D.E. Carlson, "Controllable states of elastic heat conductors," Arch. Rational Mech. Anal. 31, 127-150 (1968).
339. Petroski, H.J. and D.E. Carlson, "Some exact solutions to the equations of nonlinear thermoelasticity," J. Appl. Mech. 37, 1151-1154 (1970).
340. Petroski, H.J., "On spherizing thermoelastic annuli," Arch. of Mech. 24, 547-554 (1972).
341. Petroski, H.J., "On the use of incomplete constitutive information in thermoelasticity," Arch. Rational Mech. Anal. 53, 229-245 (1974).
342. Pipkin, A.C. and R. S. Rivlin, "The formulation of constitutive equations in continuum physics," Div. Appl. Math., Brown Univ. Report, September, 1958.
343. Possi, G., "Thermomechanical transformations of two-dimensional continua," Ann. Scuola Norm. Sup. Pisa, s. III 23, 205-222 (1969).
344. Podstrigach, Y.S. and R.N. Shvets, "Quasistatic problem of coupled thermoelasticity," Prikl. mekh. 5, 43-45 (1969).
345. Popov, E.B., "Dynamic coupled thermoelastic problem for the half-space taking into account finiteness of heat propagation rate," Prikl. Mat. Mekh. 31, 328-334 (1967).
346. Predeleanu, M., "On thermal stresses in viscoelastic bodies," Bull. Math., Soc. Sci. Math., Phys. de la R.P.R. 3 (51), 223-228 (1959).
347. Puri, P., "Plane waves in generalized thermoelasticity," Int. J. Engng. Sci. 11, 735-744 (1973).
348. Rafalski, P., "A variational principle for the coupled thermoelastic problem," Int. J. Engng. Sci. 6, 465-471 (1968).
349. Rivlin, R.S., "Large elastic deformations of isotropic materials IV. Further developments of the general theory," Phil. Trans. Royal Soc. London 241 A, 377-427 (1948).

350. Rivlin, R.S. and C.Topakoglu, "A theorem in the theory of finite elastic deformations," J. Rat. Mech. Anal. 3, 581-589 (1954).
351. Rivlin, R.S., "Red herrings and sundry unidentified fish in non-linear continuum mechanics" (In preprint).
352. Rogge, T., "A coupled thermoelastic problem," SIAM Review 11, 277-282 (1969).
353. Rosen, B.W., "Thermoelastic energy functions and minimum energy principles for composite materials," Int. J. Engng. Sci. 8, 5-18 (1970).
354. Roznowski, t., "Radiation conditions of Sommerfeld type for coupled thermoelasticity," Bull. Acad. Polon. Sci., Ser. Sci. Techn. 19, 511-544 (1971).
355. Roy Choudhuri, S.K., "Thermoelastic stress in a rod due to disturbed time-dependent heat sources," AIAA J. 10, 531-533 (1972).
356. Rubinfeld, L., "Non-axisymmetric, thermoelastic stress distribution in a solid containing an external crack," Int. J. Engng. Sci. 8, 499-509 (1970).
357. Ruhadze, Z.A., "Fundamental boundary-contact problems of thermoelastic vibration theory," Soobshch. AN GSSR 56, 285-288 (1969).
358. Rudiger, D., "Note on integration of fundamental thermoelastic equations," Osterr. Ing. Arch. 18, 121-122 (1964).
359. Shail, R., "On some steady state thermoelastic stress distributions in a slab," Proc. Edinburg Math. Soc. S II 14, 303-310 (1965).
360. Shail, R., "Some steady state thermoelastic stresses distributions in the vicinity of an external crack in an infinite solid," Int. J. Engng. Sci. 6, 685-694 (1968).
361. Shanker, M.U. and R.S. Dhaliwal, "Dynamic coupled thermoelastic problems in micropolar theory," I Int. J. Engng. Sci. 13, 121-148 (1975).
362. Shaw, R.P., "Discontinuities in one-dimensional thermoelastic wave propagation in an inhomogeneous medium," AIAA J. 6, 1626-1627 (1968).
363. Sherman, D.I., On the problem of plane strain in nonhomogeneous media. Non-homogeneity in Elasticity and Plasticity, Pergamon Press, London, 1959.
364. Shoya, S., "On the thermal stresses of a semi-infinite plate with a circular inclusion under uniform heat flow," Ing. Arch. 38, 343-357 (1969).

365. Signorini, A., "Finite thermoelastic deformations," Proc. 3rd Int. Congr. Appl. Mech. 2, 80-89 (1930). /270
366. Signorini, A., "Finite thermoelastic transformations. Report 3a, incompressible solids," Ann. Mat. Pura Appl. 39, 147-201 (1955).
367. Silli, C., "Extraordinary discontinuity waves in a thermoelastic membrane," Bolettino U.M.I. 8, 518-536 (1973).
368. Singh, M. and A.C. Pipkin, "Note on Ericksen's problem," ZAMP 16, 706-709 (1965).
369. Singh, H. and A. Singh, "Thermal deformations of a sphere by radiation from a point source," Proc. Indian Acad. Sci.(A) 76, 247-262 (1972).
370. Sinitsyn, A.P., "Vibrations and stability of shells due to loads and temperature," Int. J. Solids Structures 5, 1333-1337 (1969).
371. Sneddon, I.N., Fourier Transforms, McGraw-Hill Book Co., New York, 1951.
372. Sneddon, I.N. and S.D. Berry, "The classical theory of elasticity," Handbuch der Physik [Handbook of Physics], Vol. VI (S. Flügge, ed.), Springer, Berlin/Göttingen, Heidelberg, 1958.
373. Sneddon, I.N. "The propagation of thermal stresses in thin metallic rods," Proc. Roy. Soc. Edinburg, Sec. A 65 (1959).
374. Sneddon, I.N., "Boundary value problems in thermoelasticity," In Boundary Problems in Differential Equations. Proc. of the 2nd Symp. on Diff. Eqs., University of Wisconsin Press, Madison, Wis., 1960, pp. 231-240.
375. Sneddon, I.N. and F.J. Lockett, "On the steady-state thermoelastic problem for the half-space and the thick plate," Q. Appl. Math. 18, 145-153 (1960).
376. Sneddon, I.N. and F.J. Lockett, "The steady-state thermoelastic problem for an elastic layer resting on a rigid foundation," Ann. Mat. Pura Appl. s IV 50, 309-317 (1960).
377. Sneddon, I.N., "Solutions of the equations of thermoelastic equilibrium," Arch. Mech. Stosow. 14, 113-120 (1962).
378. Sneddon, I.N. and M. Lowengrub, Crack Problems in the Classical Theory of Elasticity, Wiley and Sons, New York, 1969.
379. Sneddon, I.N., "Lectures on the linear theory of thermoelasticity," International Centre for Mechanical Sciences, Udine, Italy, June, 1972.

380. Sorczyk, K., "Thermoelastic field in semi-space bounded by a rough surface," Proc. Vibr. Probl. 9, 169-181 (1968).
381. Sobolev, S.L., Some Applications of Functional Analysis in Mathematical Physics, Leningrad, 1950; Translated by F. Browder, Providence: A.M.S., 1963.
382. Sokolnikoff, I.S., Mathematical Theory of Elasticity, McGraw-Hill, New York, 1956.
383. Soler, A.I. and M.A. Brull, "On the solution to transient coupled thermoelastic problems by perturbation techniques," J. Appl. Mech. 32, 389-399 (1965).
384. Solomon, L., Elasticité lineaire [Linear Elasticity], Masson, Paris, 1968.
385. Soós, E., "Galerkin's representation and Beltrami-Michell's conditions in the coupled theory of linear thermoelasticity," An. Univ. Bucharest, Ser. St. Mat., Mat.-Mec. 14, 49-57 (1965).
386. Soós, E., "The Green's function (for short time) in the linear theory of the coupled thermoelasticity," Arch. Mech. Stosow. 18, 101-109 (1966).
387. Soós, E., "Thermoelastic waves produced by a point force in an unbounded material," Rev. Roum. Math. Pures et Appl. 13, 261-272 (1968).
388. Soós, E., "Reciprocity theorems for elastic and thermo-elastic materials with microstructure," Bull. Acad. Polon. Sci., Ser. Sci. Techn. 16, 895-900 (1968).
389. Srivastava, K.N. and R.M. Palaiya, "A symmetric distribution of thermal stress in a semi-infinite elastic solid containing a penny-shaped crack," ZAMP 50, 721-735 (1970).
390. Srivastav, R.P. and K.S. Parihar, "Distribution of thermal stresses in wedge-shaped solids," Acta Mechn. 10, 145-160 (1970).
391. Stavsky, Y. and I. Smolash, "Thermoelasticity of heterogeneous orthotropic cylindrical shells," Int. J. Solids Structures 6, 1211-1231 (1970).
392. Stefaniak, J., "A generalization of Galerkin's functions for asymmetric thermoelasticity," Bull. Acad. Polon. Sci., Ser. Sci. Techn. 16, 391-396 (1968).
393. Sternberg, E. and E.L. McDowell, "On the steady-state thermoelastic problem for the half-space," Q. Appl. Math. 14, 381-398 (1957).

394. Stickler, D.C. and R.E. Nickell, "Thermoelastic damping of stress waves induced by volumetric energy deposition," J. Acoust. Soc. Am. 50, 224-229 (1971).
395. Stojanovitch, R., "On the relation in non-linear thermoelasticity," Int. J. Non-linear Mech. 4, 217-233 (1969).
396. Stojanovich, R., Nonlinear thermoelasticity. CISM. Courses and Lectures No. 120, Springer, Vienna/New York, 1972.
397. Stoppelli, F., "A theorem of existence and uniqueness relative to the isothermic elastostatic equations for finite deformations," Ricerche Mat. 3, 247-267 (1954).
398. Stoppelli, F., "Possibility of developing in series of powers a parameter for solution of isothermic elastostatic equations," Ricerche Mat. 4, 58-73 (1955). /271
399. Stoppelli, F., "Existence of solutions for isothermic elastostatic equations in the case of stresses from the axis of equilibrium," Ricerche Mat. 6, 241-287 (1957); 7, 71-101; 138-152 (1958).
400. Sundapa Raja Jyengar, K.T. and K. Chandrashekhara, "Thermal stresses in a finite solid cylinder due to steady temperature variation along the curved and end surfaces," Int. J. Engng. Sci. 5, 393-413 (1967).
401. Sve, C., "Thermoelastic waves in a periodically laminated medium," Int. J. Solids Structures 7, 1363-1373 (1971).
402. Sve, C. and J. Miklowitz, "Thermally induced stress waves in an elastic layer," J. Appl. Mech. 40, 161-167 (1973).
403. Svoboda, M., "Plane thermal stresses around holes under distribution of temperature," Acta Mech. 6, 232-238 (1968).
404. Suhubi, E.S., "A two dimensional thermoelastic problem for an infinite medium with cavity," Bull. Techn. Univ. Istanbul 27, 54-73 (1972).
405. Suhubi, E.S., "Propagation of a plane wave in an initially stressed thermoelastic medium," Bull. Techn. Univ. Istanbul 27, 98-107 (1974).
406. Takeuti, Y. and N. Noda, "The effects of dynamical treatment of thermal stresses in a circular region due to continuous point heat source," Int. J. Engng. Sci. 12, 261-271 (1974).
407. Tao, L.N., "Integration of the dynamic thermoelastic equations," Int. J. Engng. Sci. 9, 489-505 (1971).
408. Teodorescu, P.P., "The plane thermoelasticity problem," Rev. de Mec. Appl. 3, 333-340 (1958).

409. Teodorescu, P.P., Probleme plane in teoria elasticitatii [Plane Problems in the Theory of Elasticity], vol. I, II, Editura Academiei, Bucharest, 1961; 1966.
410. Teodorescu, P.P., Probleme spatiale in teoria elasticitatii [Spatial Problems in the Theory of Elasticity], Editura Academiei, Bucharest, 1970.
411. Teodorescu, P.P., Dynamics of linear elastic bodies, Editura Academiei RSR, Bucharest, Romania and Abacus Press, Tunbridge Wells, Kent, England, 1975.
412. Teodosiu, C., "Non-linear theory of the materials of grade two with initial stresses and hyperstress. II. Constitutive equations," Bull. Acad. Polon. Sci., Ser. Sci. Techn. 15, 95-102 (1967).
413. Thomas, T.Y., Plastic Flow and Fracture in Solids, Acad. Press, New York, 1961.
414. Thomas, T.Y., Concepts from Tensor Analysis and Differential Geometry, Acad. Press, New York, 1961.
415. Thrun, Z., "The method of subregions in coupled thermoelasticity," Ing. Arch. 37, 369-373 (1969).
416. Tokuoka, T., "Thermo-acoustical waves in linear thermo-elastic materials," J. Eng. Math. 7, 115-122 (1973).
417. Tong, P. and T.H.H. Pian, "The convergence of finite element method in solving linear elastic problems," Int. J. Solids Structures 3, 705-711 (1967).
418. Tonti, E., "On the variational formulation for linear initial value problems," Ann. Mat. Pura Appl. s IV 95, 331-359 (1973).
419. Trapp, J.A., "Reinforced materials with thermomechanical constraints," Int. J. Engng. Sci. 9, 891-908 (1971).
420. Truesdell, C. and R. Toupin, "The classical field theories," vol III/1 of Handbuch der Physik [Handbook of Physics], edited by S. Flügge, Springer, Berlin/Göttingen/Heidelberg, 1960.
421. Truesdell, C. and W. Noll, "The non-linear field theories of mechanics," Handbuch der Physik [Handbook of Physics], vol. III/3, edited by S. Flügge, Springer, Berlin/Heidelberg/New York, 1965.
422. Truesdell, C., The Elements of Continuum Mechanics, Springer, Berlin/Heidelberg/New York, 1966.
423. Truesdell, C., Rational Thermodynamics, McGraw-Hill, New York, 1969.

424. Truesdell, C., A First Course in Rational Continuum Mechanics, (in Russian), Moscow, 1975.
425. Tweed, J., "A note on the thermoelastic problem for a penny-shaped crack," Glasgow Math. J. 10, 169-172 (1969).
426. Van der Pol, B. and H. Bremmer, Operational calculus based on the two-sided Laplace integral, Cambridge, 1950.
427. Valcovici, V., "Relationships among tensions," Com. Acad. R.P. Romane 1, 337-339 (1951).
428. Verruijt, A., "The completeness of Biot's solution of the coupled thermoelastic problem," Q. Appl. Math 26, 485-490 (1969).
429. Viehmann, G., "Investigation of the behavior of nonstationary elastic thermal stresses in the wall of plates, hollow cylinders and hollow spheres," Mitt. Verein, Grosskesselbesitzer, Dtsch. 98, 321-328 (1965).
430. Villaggio, P., "Conditions of stability for thermoelastic continua," Meccanica 7, 19-22 (1972). /272
431. Vishik, M.I. and O.A. Ladythenskaya, "Boundary value problems for partial differential equations and some classes of operator equations," Uspchi Mat. Nauk 11, 6(72), 41-97 (1956).
432. Voigt, W., Kompndium der theoretischen Physik [Compendium of Theoretical Physics], Vol. 1, Leipzig, Von Veit & Comp., 1895.
433. Volterra, V., "Equilibrium of multiply connected elastic bodies," Ann. Ecole Norm. 24, 401-517 (1907).
434. Wah, T., "Thermal stresses in thin beams," Int. J. Solids Structures 2, 293-305 (1966).
435. Wang, C.C., "On the symmetry of the heat conduction tensor," Appendix 1 to C. Truesdell: Rational Thermodynamics, McGraw-Hill, New York, 1969.
436. Wang, C.C. and C. Truesdell, Introduction to Rational Elasticity, Noordhoff, Leyden, 1973.
437. Warren, W.E., "A note on the transient axisymmetric thermoelastic problem for the solid sphere," J. Appl. Mech. 31, 348-350 (1964).
438. Warren, W.E. and P.J. Chen, "Wave propagation in the two temperature theory of thermoelasticity," Acta Mech. 16, 21-33 (1973).

439. Washizu, K., Variational Methods in Elasticity and Plasticity, Pergamon, Oxford, 1968,
440. Weiner, J.H., "A uniqueness theorem for the coupled thermo-elastic problem," Q. Appl. Math. 15, 102-105 (1957).
441. Wheeler, L.T. and E. Sternberg, "Some theorems in classical elastodynamics," Arch. Rational Mech. Anal. 31, 51-90 (1968).
442. Wheeler, L.T., "Some results in the linear dynamical theory of anisotropic elastic solids," Q. Appl. Math. 28, 91-101 (1970).
443. Wilde, P., "The thermal buckling of a thin plate in the form of a minimal surface," Arch. Mech. Stosow. 10, 81-92 (1964).
444. Williams, H.E., "Axisymmetric thermal stresses in sandwich shells of revolution," AIAA J. 5, 981-990 (1967).
445. Williams, W.E., "A solution of the steady state thermoelastic equations," ZAMP 12, 452-455 (1961).
446. Wilms, E.V., "On coupling effects in transient thermoelasticity," J. Appl. Mech. 31, 719-722 (1964).
447. Wilms, E.V., "Temperature induced in a medium due to a suddenly applied pressure inside a spherical cavity," J. Appl. Mech. 33, 941-943 (1966).
448. Woelke, S., "Dynamic stresses in a thin circular disc induced by the action of an unsteady source of heat," Bull. Acad. Polon. Sci. Ser. Sci. Techn. 18, 107-115 (1969).
449. Wozniak, C., "Hyperstress in linear thermoelasticity," Bull. Acad. Polon. Sci., Ser. Sci. Techn. 14, 1079-1084 (1966).
450. Wozniak, C., "Theory of thermoelasticity of non-simple materials," Arch. Mech. Stosow. 19, 485-492 (1967).
451. Wyrwinski, J., "A generalization of Weber's theorem for problems of the coupled thermoelasticity," Bull. Acad. Polon. Sci., Ser. Sci. Techn. 14, 637-645 (1966).
452. Yen, A.C. and P.G. Kirmser, "On the thermal stresses in a finite circular cylinder," J. Eng. Math. 5, 19-32 (1971).
453. Zagorskiy, T.Ja. and G.S. Ivanenko, "The Cauchy problem for the system of equations of thermoelasticity theory," Lvov. Polt. Inst. 31, 23-28 (1969).

454. Zaker, T.A., "Dynamic thermal shock in hollow spheres," Q. Appl. Math. 26, 503-520 (1969).
455. Ziegler, F., "Plane wave propagation in half-space during random excitations and coupling stress and temperature field," Acta Mech. 2, 307-327 (1966).
456. Zienkiewicz, O.C. and Y.K. Cheung, The Finite Element Method in Structural and Continuum Mechanics, McGraw-Hill, New York, 1967.
457. Zorski, H., "Singular solutions for thermoelastic media," Bull. Acad. Polon. Sci. Ser. Sci. Techn. 6, 331-339 (1958).
458. Zorski, H. and W.C. Lyons, "Dynamics of thermoelastic plates," Arch. Mech. Stosow. 17, 497-515 (1965).