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# Verification of an Acoustic Transmission Matrix Analysis of Sound Propagation in a Variable Area Duct Without Flow

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## ABSTRACT

A predicted standing wave pressure and phase angle profile for a hard wall rectangular duct with a region of converging-diverging area variation is compared to published experimental measurements in a study of sound propagation without flow.  The factor of 1/2 area variation used is of sufficient magnitude to produce large reflections.  The prediction is based on a transmission matrix approach developed for the analysis of sound propagation in a variable area duct with and without flow.  The agreement between the measured and predicted results is shown to be excellent.

## NOMENCLATURE

| | |
|---|---|
| $\mathscr{A}$ | matrix |
| A | area, $m^2$ |
| $c_o$ | isentropic speed of sound, m/s |
| f | frequency, Hz |
| H | duct height equation parameter, m |
| I | unit matrix |
| i | $(-1)^{1/2}$ |
| $k_o$ | wave number, $\omega/c_o$, $m^{-1}$ |
| L | duct length, m |
| L* | length of area variation, m |

| | |
|---|---|
| m | d ln(A)/dx, m |
| P | transformation matrix |
| p | pressure, Pa |
| R | pressure reflection factor |
| T | transmission matrix |
| t | time, s |
| u | velocity, m/s |
| x | Cartesian coordinate, m |
| $\vec{Y}$ | acoustic state vector |
| y | duct height, m |
| Z | acoustic impedance, mks Rayles |
| $\Delta x$ | region length, m |
| $\lambda$ | eigenvalue |
| $\rho_0$ | gas density, kg/m |
| $\omega$ | angular frequency, radians/s |

Superscripts and Subscripts:

| | |
|---|---|
| ( )' | scaled quantity |
| ( )$_0$ | reference state quantity |
| ( )$_1$ | perturbed quantity |

## INTRODUCTION

An acoustic transmission matrix analysis of sound propagation in a variable area duct with and without flow is discussed in Ref. 1. Reference 1 is concerned only with theoretical solutions of the sound propagation equations. This paper concentrates on the experimental verification of the analysis discussed in Ref. 1.

The analysis is compared with an experimental standing wave pressure and phase angle profile obtained in a study of sound propagation without flow in a long hard wall rectangular duct containing a region of converging-diverging area variation (Ref. 2).

This experimental data provides a good test of the theory since the gradients of the area variation in the region of the converging-diverging area variation are large enough to produce a reflection much greater than the reflection produced at the duct exit.

In the present paper, first a brief review of the theory is presented. Next, the experimental data is discussed. Then the analytical results are compared to the measurements and conclusions are drawn.

## THEORY

The following assumptions are made. The duct has an area profile $A(x)$. The acoustic equations are described by first order linearized, perturbed equations of continuity momentum, energy and state for isentropic sound propagation.

### State variable equation

The perturbed state variables are the pressure and velocity. These quantities are assumed to vary with time as $\exp(-i\omega t)$. From Ref. 1, the Fourier transform of the resulting equations is

$$\frac{d\vec{Y}}{dx} = \begin{pmatrix} dp_1/dx \\ du_1/dx \end{pmatrix} = \begin{bmatrix} 0 & \rho_0 c_0\, ik_0 \\ \dfrac{ik_0}{\rho_0 c_0} & -\dfrac{d\,\ln(A)}{dx} \end{bmatrix} \begin{pmatrix} p_1 \\ u_1 \end{pmatrix} = B\vec{Y} \qquad (1)$$

## Method of solution

The method of solution used in this study is presented in Ref. 1. For clarity, it is briefly discussed. The first step in obtaining a solution to Eq. (1) is to divide the duct into a number of regions or subsections. The region size is selected so that the area variation in each region can be approximated by an exponential area variation. Consequently, in a given region

$$A = A_0 \, e^{mx} \tag{2}$$

With this assumption the value of $d \ln(A)/dx$ is a constant in the region given by

$$m = d \ln(A)/dx \tag{3}$$

The $b$ matrix in Eq. (1) is now independent of $x$ in the region where it is to be evaluated. Thus, the solution to Eq. (1) for the jth region is

$$\vec{Y}(x_j + \Delta x) = (e^{b \Delta x})_j \, \vec{Y}(x_j) \tag{4}$$

where $x$ is the length of the jth region and $\exp(b \Delta x)$ is known as the matrix exponential. The value of $\exp(b \Delta x)$ is defined by

$$e^{b \Delta x} = 1 + (b \Delta x) + \frac{(b \Delta x)^2}{2!} + \frac{(b \Delta x)^3}{3!} + \dots \tag{5}$$

The matrix $\exp(b \Delta x)$ is the transmission matrix of the jth region $(T)_j$, so that

$$(T)_j = e^{(b \Delta x)_j} = \sum_{\ell=0}^{\infty} \frac{(b \Delta x)_j^{\ell}}{\ell!} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}_j \tag{6}$$

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Consequently, the transmission matrix, T, for the variable area duct is found from

$$\vec{Y}(x)\Big|_{x=L} = \prod_{j=1}^{N} e^{(B\Delta x)_j} \vec{Y}(x)\Big|_{x=0} = T \vec{Y}(x)\Big|_{x=0} \tag{7}$$

where

$$L = \sum_{j=1}^{N} (\Delta x)_j \tag{8}$$

### Evaluation of matrix exponential

While the series given in Eq. (5) defines the matrix exponential, it was not used for computer calculations since the convergence of the series may be slow. The matrix exponential was evaluated from

$$e^{B\Delta x} = C^{-1} P e^{D\Delta x} P^{-1} C \tag{9}$$

where

$$D = P^{-1} \mathscr{A} P \tag{10}$$

$$\mathscr{A} = CBC^{-1} \tag{11}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & \rho_0 c_0 \end{bmatrix} \tag{12}$$

$$P = \begin{bmatrix} \mathscr{A}_{22} - \lambda_1 & \mathscr{A}_{22} - \lambda_2 \\ -\mathscr{A}_{21} & -\mathscr{A}_{21} \end{bmatrix} \tag{13}$$

and the $\lambda$ are the eigenvalues of the $\mathscr{A}$ matrix given by the roots of

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$$\lambda^2 + b\lambda + c = 0 \tag{14}$$

where

$$b = -(\mathscr{A}_{11} + \mathscr{A}_{22}) \tag{15}$$

and

$$c = \mathscr{A}_{11}\mathscr{A}_{22} - \mathscr{A}_{12}\mathscr{A}_{21} \tag{16}$$

A more complete discussion of the evaluation of the matrix exponential can be found in Ref. 1.

### Boundary condition

The standing wave profile is calculated as follows. First, the pressure at the duct exit (identified as location one) is assumed to be 1 Pa. A duct exit impedance is then used to calculate the velocity

$$u_1 = 1/Z_1 \tag{17}$$

where the duct exit impedance is related to the duct exit pressure reflection factor by

$$Z_1/\rho_0 c_0 = \frac{1 + R}{1 - R} \tag{18}$$

Next, the transfer matrices, $T_n$, between each of the measurement points is calculated. The state vector at the last measurement point is

$$\begin{pmatrix} p_m \\ u_m \end{pmatrix} = T_{m-1} T_{m-2} \cdots T_3 T_2 T_1 \begin{pmatrix} 1 \\ 1/Z_1 \end{pmatrix} \tag{19}$$

and the impedance at this point is

$$Z_m = p_m/u_m \tag{20}$$

Note, that this is an initial value problem where the impedance and pressure are specified at the exit which differs from the boundary value approach used by two dimensional numerical solutions.

The pressure at the point $m-1$ due to a unit pressure at the point $m$ is obtained from

$$\begin{pmatrix} p_{m-1} \\ u_{m-1} \end{pmatrix} = T_{m-1}^{-1} \begin{pmatrix} 1 \\ 1/Z_m \end{pmatrix} \tag{21}$$

This procedure is continued at each point. Finally, the pressure at the duct exit is

$$\begin{pmatrix} p_1 \\ u_1 \end{pmatrix} = T_1^{-1} T_2 T_3^{-1} \cdots T_{m-2}^{-1} \begin{pmatrix} 1 \\ 1/Z_m \end{pmatrix} \tag{22}$$

The pressure magnitude of the standing wave pressure profile is given by

$$\left| p_m \right| = (p_m p_m^{\star})^{1/2} \tag{23}$$

and the phase angle of the standing wave phase profile is given by

$$\theta = \tan^{-1} \left( \frac{I_m p_m}{R_e p_m} \right) \tag{24}$$

However, the pressure magnitude plotted is scaled with repect to the measured pressure magnitude at the duct exit $p_{exit}^{(1)}$

$$p_m' = p_m \, p_{exit}^{(1)} \Big/ p^{(1)} \tag{25}$$

The phase angle is given with respect to the angle at $x = 0$. In addition since the experimental measurements of Ref. 2 are made assuming a time factor of $\exp(+i\omega t)$ while the analysis is based on using $\exp(-i\omega t)$, the sign of the angle calculated is reversed. Consequently,

$$\theta'_m = \theta(0) - \theta_m \tag{26}$$

## THE EXPERIMENTAL DATA

The general acoustic duct system described in Ref. 3 was modified into the simple no flow apparatus shown in Fig. 1(a) and used to measure the data discussed in Ref. 2.

In the experiment discussed in Ref. 2 the duct temperature was 27 C so that the speed of sound was 348 m/s. The sound traveled in a 0.1 m by 0.0381 m hard wall rectangular test section which was 1.42 m long. It was constructed so that the 8 top plates and 8 bottom plates were detachable. The variable area test piece was inserted in place of the bottom piece at location 6.

The variable area test piece is shown in Fig. 1(b). The surface profile is prescribed by a fourth degree polynomial

$$y/H = 16(x/L^*)^2 - 32(x/L^*)^3 + 16(x/L^*)^4 \tag{27}$$

where $L^*$ is 0.0762 m and $H$ is 0.0191 m which is just half the duct height.

The sound source used to obtain the data discussed in Ref. 2 was a speaker driven by a sine wave generator at 1560 Hz. The 1560 Hz driving frequency assured that only plane waves would propagate in the straight portion of the duct far from the area variation since the first non plane mode begins propagating at a frequency of 1740 Hz. An exponential horn was

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attached to the upstream end of the test section. The horn was filled with an acoustic foam wedge to approximate a $Z = \rho_0 c_0$ impedance termination where the pressure reflection factor is $R = 0.0$ (Eq. (18)) which minimizes reflections and simulates an infinitely long duct.

Further information on the apparatus including a description of the microphone installation and the measurement procedure is given in Ref. 2.

## DISCUSSION OF RESULTS

The measured and theoretical normalized pressure magnitude profiles are compared in Fig. 2 and the measured and theoretical phase angle profiles are compared in Fig. 3. The theoretical curves calculated using as an exit boundary condition a duct exit pressure reflection factor of $R = 0.0$ which corresponds to an impedance termination of $Z = \rho_0 c_0$ are a near perfect match upstream of the area change. However, due to design and material limitations the pressure reflection factor is not exactly zero. Consequently, the pressure magnitude is not constant downstream of the area change and the experimental and theoretical curves are not in good agreement in this region.

Impedance tube theory indicates that one unique value of the exit reflection factor will generate the standing wave pattern shown in Fig. 2. In order to determine this value, small changes in the exit boundary condition were made so that the theoretical curves were in better agreement with the data downstream of the area change. The resulting boundary condition was $R = -0.030 + j\,0.070$ $(Z = \rho_0 c_0\,(0.94 + j\,0.0039))$. This boundary condition as shown in Figs. 2 and 3 yields a more perfect match not only for the downstream data points but at all data points.

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The data discussed in Ref. 2 was used to verify the "steady" state finite element theory of Astley and Eversman discussed in Ref. 4 and the transient finite difference theory of White discussed in Ref. 5 for the limiting case of plane wave sound propagation without flow. Both of these numerical methods yield a two dimensional solution. It is shown in Ref. 2 that both of these methods predict the proper trends of the data and are in excellent agreement with the measurements. In both cases, however, the calculations were made assuming that the impedance was $Z = \rho_0 c_0$. Consequently, the numerical calculations resembled the dashed curve in the downstream portion of duct (x greater than unity).

## CONCLUDING REMARKS

The variable area transmission matrix theory (Ref. 1) produced good agreement with the axial data both in terms of the pressure magnitude and phase. Similar results can be obtained by numerical methods (Refs. 4 and 5). However, these latter methods are more complex than necessary if only plane wave propagation is of interest. The variable area transmission matrix approach is simple to use by comparison with two dimensional numerical methods. Consequently, the acoustic transmission matrix analysis of sound propagation in a variable area duct is an ideal procedure to use for plane wave propagation of sound in a hard wall duct with large area variation.

## REFERENCES

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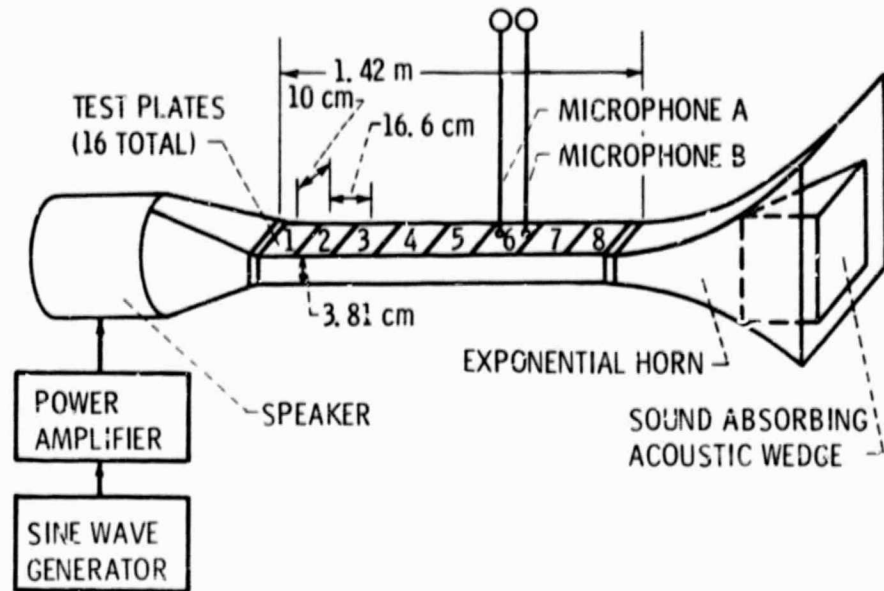
(a) No flow acoustic duct test section and instrumentation.
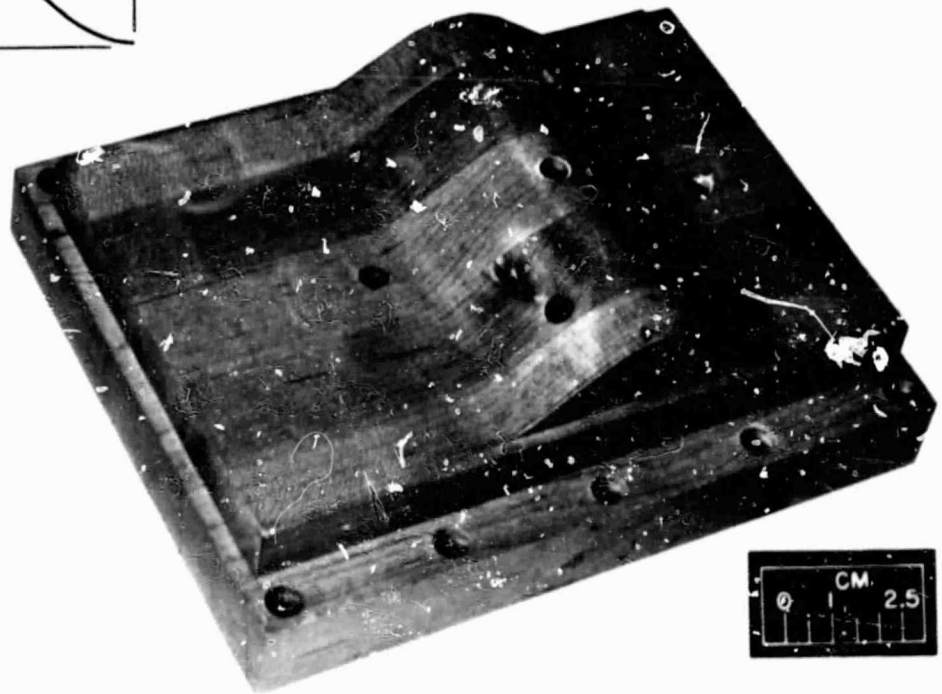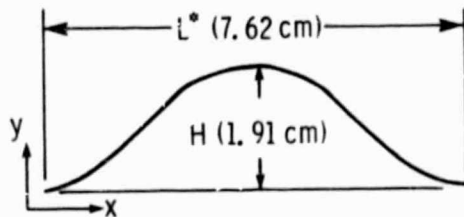


(b) Variable area test piece with 5 microphone locations (microphone plugs not shown).
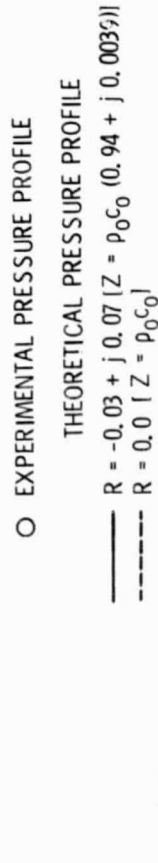
Figure 1. - Apparatus.

O   EXPERIMENTAL PHASE ANGLE PROFILE

      THEORETICAL PHASE ANGLE
              PROFILE

   —— R = -0.03 + j 0.07
             $[Z = \rho_0 c_0 (0.94 + j\ 0.0039)]$
   - - - R = 0.0 $[Z = \rho_0 c_0]$

PHASE ANGLE, deg

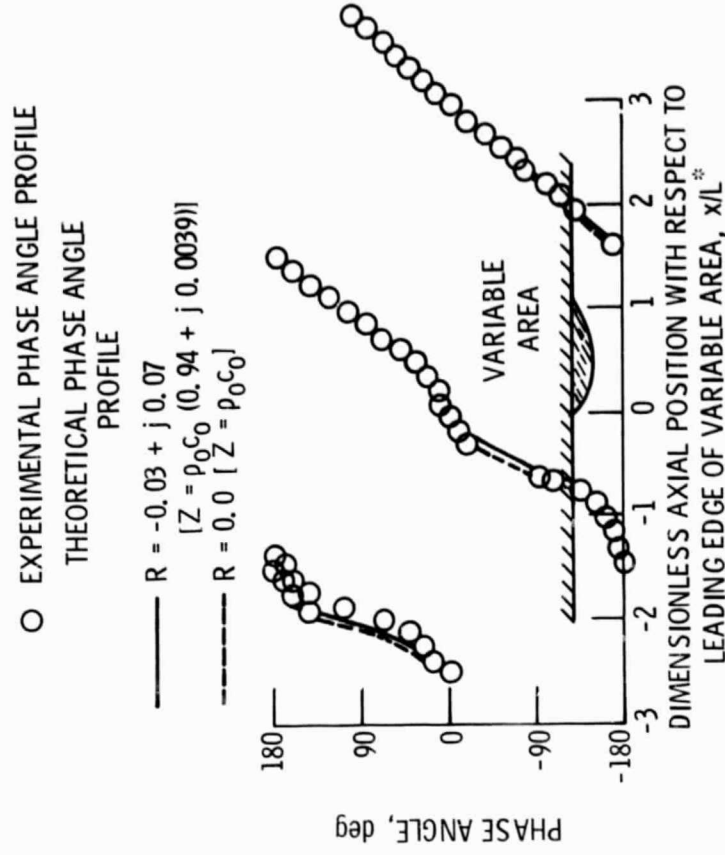DIMENSIONLESS AXIAL POSITION WITH RESPECT TO
LEADING EDGE OF VARIABLE AREA, x/L*

VARIABLE
AREA

Figure 3. - Phase angle profile.

O   EXPERIMENTAL PRESSURE PROFILE

      THEORETICAL PRESSURE PROFILE

   —— R = -0.03 + j 0.07 $[Z = \rho_0 c_0 (0.94 + j\ 0.0039)]$
   - - - R = 0.0 $[Z = \rho_0 c_0]$

MAGNITUDE, |p'|

VARIABLE AREA

DIMENSIONLESS AXIAL POSITION WITH RESPECT TO LEADING
EDGE OF VARIABLE AREA, x/L*

Figure 2. - Pressure profile.