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SUMMARY

The energy dissipated by viscosity at the edge of a vibrating flat plate is calculated and compared to the radiated acoustic energy. A correction to the Kirchhoff integral estimate of the noise is derived. For Helmholtz number of order unity and smaller the dissipation can be comparable to or greater than the acoustic energy.

A viscous compressible theory of the load distribution on a vibrating two-dimensional body is developed. First it is shown that load calculations based on potential theory and the Neumann uniqueness condition (continuity of potential or pressure on the surface) are not in agreement with experiment or the more correct viscous theory. For a flat plate airfoil the eigensolution of potential theory is indeterminant while viscous theory yields a unique solution that has square root singularities at the edges. It is also shown that for compact surfaces the far field acoustics depend only on the magnitude of the eigensolutions of potential theory and so will be uniquely determined by the viscous theory. It is suggested that the general viscous theory of vibrating surfaces with cross-sectional geometry will lead to results in agreement with experimentally measured load distributions.

NOMENCLATURE

a	$c \cdot (1+\tau)/4$
a_o	speed of sound
A	see Eq. (3.58)
b	$a\sqrt{(1-\tau)/(1+\tau)}$
c	chord of two-dimensional vibrating body, see Fig. 2
$C_o(t), C_1(t)$	time dependent coefficients of eigensolutions of flat plate problem, see Eq. (3.28)
ΔdB	see Eq. (2.28)
E_a	energy radiated acoustically, see Eq. (2.23)
E_v	viscous dissipation energy, see Eqs. (2.23) and (2.21)
F	force on a vibrating flat plate, see Eq. (3.29)
$F(t)$	viscous correlation function, see Eq. (2.10) and Appendix A
G_H, G_Ω	Greens functions for the viscous compressible problem, see Eqs. (3.64) and (3.65)
$G(\vec{r})$	acoustic correlation function, see Eq. (2.24) and Appendix B
h	thickness of a vibrating elliptic body, see Fig. 2
h'	perturbation enthalpy
$H_0^{(2)}(z)$	Hankel function, see Ref. 4
$\vec{i}, \vec{j}, \vec{k}$	unit vectors along the x,y,z coordinate axes, see Fig. 1
I	see Eq. (2.14)
k	ω/a
K_o	modified Bessel function, see Ref. 4
$\ell'(\vec{x})$	complex surface loading

L_{dB}	see Eq. (3.58)
M	moment on a vibrating flat plate, see Eq. (3.30)
$\vec{n}$	unit normal to S (or C with zero subscript)
p'	perturbation pressure, $(= \rho_\infty h')$
$\overline{q}$	time average of quantity q
$\|q\|$	absolute value of quantity q
$\overline{Q}$	density of viscous dissipation energy, see Eq. (2.2)
r,θ	polar coordinates
S	see Eqs. (3.36) and (3.37)
S_f	far field spherical surface for calculating the flux of acoustic energy
t	time
$\vec{t}$	unit vector tangent to C, see Fig. 2
$T_n(X)$	Chebyshev polynomial of the first kind, see Ref. 4
$U_n(X)$	Chebyshev polynomial of the second kind, see Ref. 4
$U(s),W(s)$	surface velocity components of a vibrating body, see Eqs. (3.61) and (3.62)
v_n'	perturbation normal velocity on S_f
V	domain bounded by the vibrating surface and S_f
w	$\rho e^{i\theta}$
$w_o(t)$	vertical translation velocity
$w_1(t)$	amplitude of pitch vibration
W	rate at which work is done on a fluid medium by a vibrating surface, Eq. (2.1)
$\vec{x},\vec{y}$	vector field points

X, Y	$\frac{2}{c}(x, y)$
z	$x + iy$
α	see Eq. (3.46)
γ	Euler's constant, $\cong 0.57721$
$\delta(\vec{x})$	Dirac delta function
κ	effective Helmholtz number, see Eq. (2.31)
λ	$\sqrt{\omega/2\nu}$
ν	kinematic viscosity
ρ_∞	density of fluid medium
τ	h/c
ϕ	velocity potential
Φ	complex potential
ψ	see Eq. (2.4)
ω	vibration frequency
$\vec{\omega}'$	perturbation vorticity
div	vector divergence operator
grad	vector gradient operator
Im	denotes imaginary part of a complex quantity
Re	denotes real part of a complex quantity
∇^2	Laplace operator
$(*)$	conjugate of a complex quantity

I. INTRODUCTION

The results reported in this document supplement those reported in Refs. 1 and 2. The overall objective is to incorporate viscous effects in the calculation of noise produced by a thin surface when it vibrates or interacts with an unsteady flow or an incident acoustic field. The results are of two types. First, we derive an explicit formula for the viscous dissipation at the edge of a vibrating three-dimensional surface with constant load and give an estimate for the excess noise that would be predicted with the Kirchhoff integral. The results are a direct extension of the two-dimensional results reported in Ref. 2.

Second, we investigate the surface load problem in detail and its relevance to the acoustic problem. The importance of the Neumann boundary condition and its similarity to the Kutta condition as a uniqueness criterion is illuminated. The invalidation of the Neumann condition is demonstrated with the surface loads data measured by Brooks (Ref. 3). The correct viscous origin of the load is demonstrated for a vibrating flat plate. Finally, a detailed formulation of the viscous compressible surface loads problem for a surface of finite thickness is presented.

II. VISCOUS DISSIPATION AND THE KIRCHHOFF CORRECTION FOR THREE-DIMENSIONAL SURFACES

The following derivation is the three-dimensional counterpart of the two-dimensional results derived previously in Ref. 2. The end result is a simple formula for correcting the noise estimate that would be obtained with a Kirchhoff type integral.

Consider a vibrating three-dimensional surface S in a stationary acoustic medium (see Fig. 1). From Ref. 1, Eq. (2.13), the time averaged rate at which work is done on the fluid medium by the vibrating surface is

$$W = \int_{S_f} \overline{h'v_n'}\, dz + \int_V \overline{Q}\, dV \qquad (2.1)$$

where h' is the perturbation enthalpy and v_n' is the perturbation velocity on the far-field surface S_f. The first term is the energy radiated to the far field in the

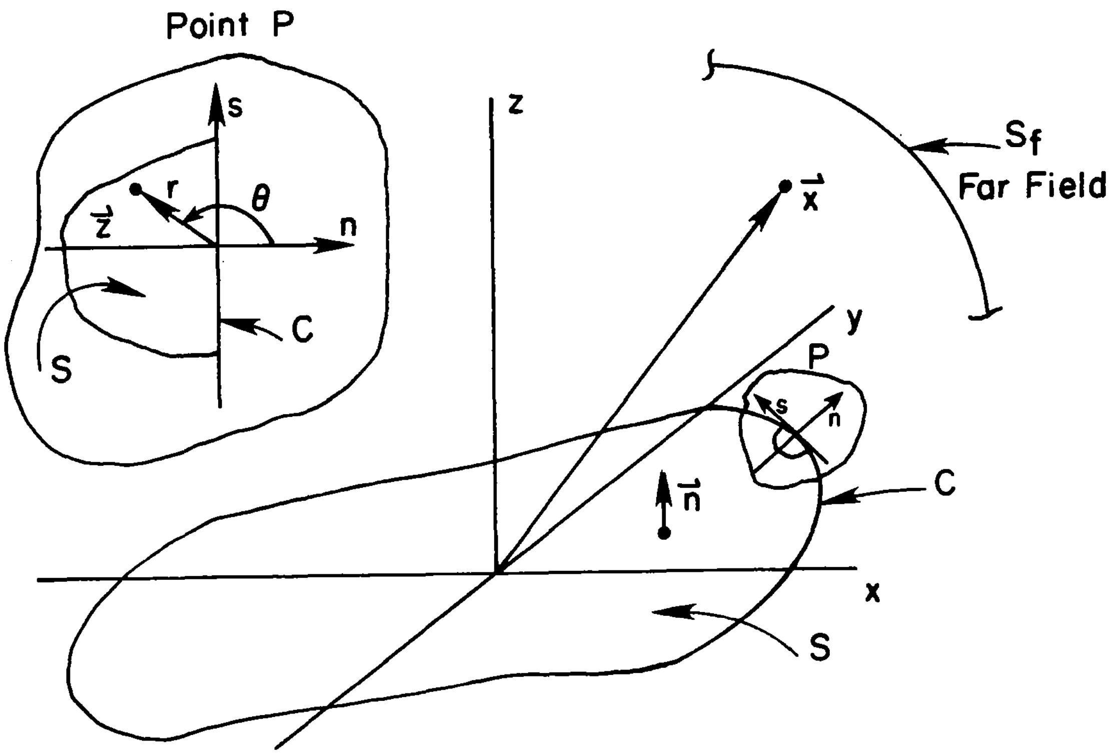


Figure 1 - Three-Dimensional Vibrating Surface

form of sound. The volume integral is the energy converted irreversibly into vorticity by viscosity. We neglect compressible viscous effects so that the time averaged dissipation can be expressed in terms of the square of the vorticity, i.e.,

$$\overline{Q} = \overline{\nu|\vec{\omega}'|^2} \tag{2.2}$$

If $\ell'(\vec{y})$ is the complex surface loading (force/unit area) then $\vec{\omega}'$ can be expressed as follows:

$$\vec{\omega}' = \vec{k} \times \text{grad Re}(\psi e^{i\omega t}) \tag{2.3}$$

where

$$\psi = -\frac{1}{4\pi\nu\rho_\infty} \int_S \ell'(\vec{y}) \frac{e^{-\alpha|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|} d\vec{y} \tag{2.4}$$

$$\alpha^2 = \frac{i\omega}{\nu} \tag{2.5}$$

and ω is the vibration frequency, ρ_∞ is the density of the fluid medium and S denotes the vibrating surface (see Fig. 1).

Substitute Eqs. (2.2), (2.3) and (2.4) into the second term of Eq. (2.1) to obtain the dissipation energy; i.e.,

$$E_V = \int_V \bar{Q}\, dV$$

$$= -\frac{1}{32\pi^2\nu\rho_\infty^2} \int_S \ell'(\vec{y}) d\vec{y} \int_S \ell'^*(\vec{y}') d\vec{y}' \tag{2.6}$$

$$\cdot \nabla^2 \int_V \frac{e^{-\alpha|\vec{x}-\vec{y}| - \alpha^*|\vec{x}-\vec{y}'|}}{|\vec{x}-\vec{y}|\;|\vec{x}-\vec{y}'|} d\vec{x} \tag{2.7}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}, \quad \vec{y} = (y_1, y_2) \tag{2.8}$$

and the asterisk denotes complex conjugate. The volume integral is evaluated in Appendix A so that

$$E_V = -\frac{\alpha}{8\pi\rho_\infty^2\omega} \int_S \ell'(\vec{y})\,d\vec{y} \int_S \ell'^*(\vec{y}')\,d\vec{y}'$$

$$\cdot\nabla^2\cdot F(\lambda|\vec{y}-\vec{y}'|) \tag{2.9}$$

where

$$F(t) = \frac{e^{-t}\sin t}{t} \qquad t > 0 \tag{2.10}$$

and

$$\lambda = \sqrt{\frac{\omega}{2\nu}} \tag{2.11}$$

The viscous dissipation has thus been reduced to a double integral over the surface loading. The correlation between load points on the viscous length scale $\sqrt{2\nu/\omega}$ is the origin of the dissipation. Also, we will show that the principal contribution to the dissipation arises from the edge loading.

Consider the special case of a constant load surface; i.e.,

$$\ell' = \ell_o = \text{Constant} \tag{2.12}$$

Then

$$E_V = -\frac{\ell_o^2\lambda}{8\pi\rho_\infty^2\omega} \cdot I \tag{2.13}$$

where

$$I = \int_S d\vec{y}' \int_S d\vec{y}\ \nabla^2 F(\lambda|\vec{y}-\vec{y}'|) \tag{2.14}$$

The second integral can be reduced to a line integral around the contour C ; i.e.,

$$I = \int_S d\vec{y}' \cdot \oint_C ds \; F'(\lambda|\vec{y}_o-\vec{y}'|) \frac{\lambda|\vec{y}_o-\vec{y}'|\cdot\vec{n}_o}{|\vec{y}_o-\vec{y}'|} \tag{2.15}$$

where the zero subscript denotes a quantity evaluated on C. Now interchange the surface and contour integrals and let $\vec{y}' = \vec{y}_o + \vec{z}$. The result is

$$I = \oint_C ds \int_S F'(\lambda|\vec{z}|) \frac{(-\lambda\vec{n}_o\cdot\vec{z})}{|\vec{z}|} d\vec{z} \tag{2.16}$$

For $\alpha >> 1$, the support of the viscous correlation function F is of order $1/\alpha$ so that the surface integral can be approximated as follows. Assume that the contour C is smooth and introduce polar coordinates at a typical point on the contour as shown in the inset in Fig. 1; i.e.,

$$\vec{z} = (r\cos\theta,\; r\sin\theta) \tag{2.17}$$

Then

$$I = \oint_C ds \int_o^\infty r\,dr \int_{\pi/2}^{3\pi/2} d\theta \; F'(\lambda r)(-\lambda\cos\theta)$$

$$= 2 \oint_C ds \int_o^\infty \lambda r \, F'(\lambda r)\,dr$$

$$= \frac{2}{\alpha} \oint_C ds \int_o^\infty t\, F'(t)\,dt \tag{2.18}$$

But

$$\int_0^\infty t\,F'(t)\,dt = -\int_0^\infty F(t)\,dt$$

$$= -\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt = \pi/4 \qquad (2.19)$$

so that

$$I = -\frac{\pi}{2\lambda}\oint ds = -\frac{\pi C}{2\lambda} \qquad (2.20)$$

where C is the length of the contour C . Finally substitute Eq. (2.20) into Eq. (2.13) to obtain the dissipation,

$$E_V = \frac{\ell_o^2 \cdot C}{16\,\rho_\infty^2 \omega} \qquad \text{Constant Load Surface} \qquad (2.21)$$

Note that the three-dimensional dissipation is independent of the viscosity coefficient as we found previously in the two-dimensional case. (See Eq. (40) of Ref. 2.)

While Eq. (2.21) was derived rigorously for a constant load surface it is not difficult to generalize the result for a load distribution that varies slowly on the viscous length scale $\sqrt{2\nu/\omega}$. The more general result is

$$E_V = \frac{1}{16\,\rho_\infty^2 \omega}\oint_C ds\,|\ell_o|^2 \qquad \text{Slowly Varying Load} \qquad (2.22)$$

This result will now be used to derive a correction to the Kirchhoff integral estimate of the noise radiated by a vibrating surface. From Appendix B, Eq. (B.30), the acoustic energy radiated by a vibrating surface is

$$E_a = \frac{k^2}{32\pi^2\rho_\infty^2 a_o} \cdot \int_S \ell'(\vec{y})\,d\vec{y} \int_S \ell'^*(\vec{y}')\,d\vec{y}' \cdot G(\vec{y}-\vec{y}') \qquad (2.23)$$

where

$$G(\vec{r}) = \int_o^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} d\theta \sin^2\theta\cos\theta\, e^{ik\cos\theta\,\hat{x}_s\cdot\vec{r}} \qquad (2.24)$$

and

$$k = \omega/a_o$$

$$\hat{x}_s = \vec{i}\cos\theta + \vec{j}\sin\theta \qquad (2.25)$$

Note the similarity of Eqs. (2.9) and (2.23). The acoustic energy is a double integral over the surface load with a correlation function G on the length scale a_o/ω. The viscous dissipation is a double integral over the surface load with a correlation function ∇^2F on the length scale $\sqrt{2\nu/\omega}$. For compact surfaces the two integrals are comparable in magnitude. To illustrate this important point we consider an acoustically compact surface in which case (see Appendix B)

$$E_a = \frac{k^2}{24\pi\rho_\infty^2 a_o}\left|\int_S \ell_o\,d\vec{y}\right|^2 \qquad (2.26)$$

Consider the quantities

$$\frac{E_a}{W} \qquad (2.27)$$

and

$$\Delta dB = -\ 10\ \log_{10} \frac{E_a}{W} \tag{2.28}$$

which is the dB increment by which the noise will be over-estimated by a Kirchhoff formula. Then

$$\frac{E_a}{W} = \frac{E_a}{E_a + E_V} = \frac{E_a/E_V}{1 + E_a/E_V} \tag{2.29}$$

and

$$\frac{E_a}{E_V} = \frac{k^2}{24\pi\rho_\infty^2 \cdot a_o} \cdot (16\rho_\infty^2 k \cdot a_o) \frac{\left| \int_S \ell_o d\vec{y} \right|^2}{\oint_C |\ell_o|^2 ds} = \kappa^3 \tag{2.30}$$

where

$$\kappa = k \cdot \left[\frac{2}{3\pi} \frac{\left| \int_S \ell_o d\vec{y} \right|^2}{\oint_C |\ell_o|^2 ds} \right]^{1/3} \tag{2.31}$$

Finally

$$\frac{E_a}{W} = \frac{\kappa^3}{1 + \kappa^3} \tag{2.32}$$

and

$$\Delta dB = -\ 10\ \log_{10} \frac{\kappa^3}{1 + \kappa^3} \tag{2.33}$$

The Kirchhoff noise overestimate depends on the third power of the Helmholtz number based on a length scale defined by the square bracketed quantity in Eq. (2.31). For compact surfaces this scale only depends on the surface and edge loading and the geometry of the surface. It is clear from Eq. (2.31) that a fixed area surface with a very large perimeter will generate a smaller effective Helmholtz number and thus more dissipation than one with a minimum perimeter; e.g., a circular disc. Also it is evident that the Kirchhoff noise correction factor becomes small as the effective Helmholtz number exceeds unity. The correction is most important in the compact regime. The formula (2.31) can be used to correct Kirchhoff noise estimates for arbitrary compact surfaces and generalizations to moderately non-compact surfaces can be derived with the more general expression, Eq. (2.23), for the acoustic energy.

It is not our objective here to derive a collection of specific formulae with the foregoing results. Rather, it is our intention to shed some light on the deeper question of how to calculate the surface and edge loading. In reality the edge loading must go to zero on some scale. However, we have seen from the Brooks experiment, Ref. 3, that this scale must be very small. The edge load to be used in the preceding formula can be estimated from experiments like those of Brooks. However, this would be a costly endeavor if we must resort to experiment for every case and it becomes highly desirable to derive a theory that will enable us to calculate the edge load. An important conclusion based on the viscous and acoustic energy estimates in this report and Ref. 3, and the results of the Brooks experiment (Ref. 3) is that a correct load distribution theory must account for viscosity. That is the subject of the following section.

III. THE SURFACE LOAD PROBLEM

A. A Two-Dimensional Potential Problem and the Neumann Condition

Consider the typical elliptic two-dimensional body that vibrates along the vertical or y-axis as shown in Fig. 2. We consider the fluid medium to be inviscid-incompressible and calculate the surface loading from the solution of the following potential problem:

$$\nabla^2\phi = 0 \tag{3.1}$$

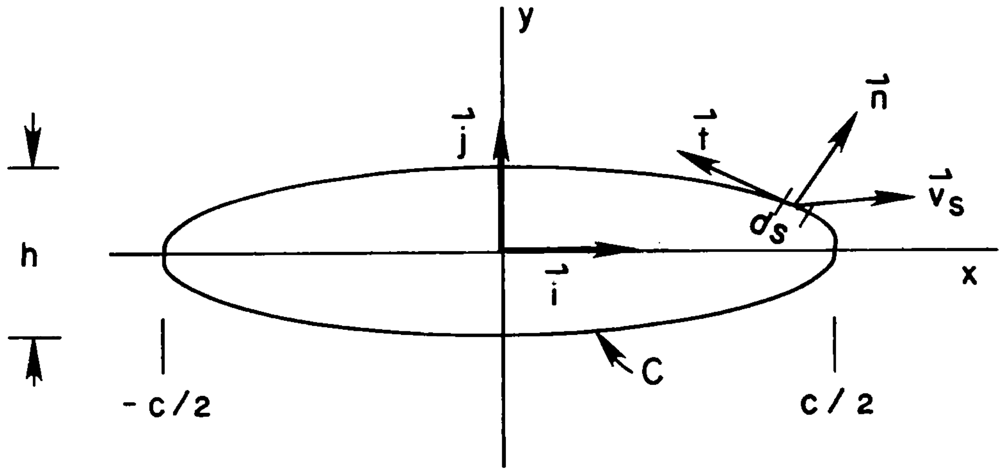


Figure 2 - Vibrating Elliptic Cross Section

$$\frac{\partial\phi}{\partial n} = \vec{v}_s \cdot \vec{n} = w_o(t)\vec{j}\cdot\vec{n} \quad \text{on} \quad C \tag{3.2}$$

$$\phi \quad \text{continuous} \quad \text{on} \quad C \quad \text{(Neumann Condition)} \tag{3.3}$$

with the perturbation pressure given by

$$p' = -\rho_\infty \frac{\partial\phi}{\partial t} \tag{3.4}$$

The purpose of this section is to investigate the consequences of the Neumann Condition (3.3) as the elliptic cross section tends to a flat plate.

To solve the boundary value problem we consider the complex potential

$$\Phi(z) = \phi + i\,\psi \quad , \quad z = x + iy \tag{3.5}$$

and let

$$z = w + b^2/w \quad , \quad w = \rho e^{i\theta} \tag{3.6}$$

map the body into a circle

$$w = ae^{i\theta} \tag{3.7}$$

in the w-plane. The surface of the body is given by

$$z = ae^{i\theta} + \frac{b^2}{a} e^{-i\theta} = \left(a + \frac{b^2}{a}\right)\cos\theta + i\left(a - \frac{b^2}{a}\right)\sin\theta \tag{3.8}$$

so that

$$a + b^2/a = c/2$$

$$a - b^2/a = h/2 \tag{3.9}$$

or

$$a = c \cdot \frac{1 + \tau}{4}$$

$$b = a\sqrt{\frac{1 - \tau}{1 + \tau}} \tag{3.10}$$

with

$$\tau = h/c \tag{3.11}$$

In the w-plane, it is easily shown that the problem for Φ is

$$\nabla^2\Phi = 0 \tag{3.12}$$

$$\mathrm{Re}\left(\frac{\partial\Phi}{\partial\rho}\right)_{\rho=a} = \frac{2}{1 + \tau} w_o(t)\sin\theta \tag{3.13}$$

$$\Phi \text{ continuous on } C \tag{3.14}$$

and the solution is

$$\Phi = - \frac{2\ i\ a^2\ \dot{w}_o(t)}{(1+\tau)w} \tag{3.15}$$

With Eq. (3.4) the pressure on the surface is

$$p' = \frac{2\ \rho_\infty a}{(1+\tau)}\ \dot{w}_o(t)\sin\theta \tag{3.16}$$

But from Eq. (3.8)

$$x = \frac{c}{2}\cos\theta \tag{3.17}$$

and

$$\sin\theta = \sqrt{1 - (\frac{2x}{c})^2} \quad \text{for} \quad 0 < \theta < \pi \tag{3.18}$$

Thus on the upper surface

$$p' = \frac{2\rho_\infty a}{1+\tau}\ \dot{w}_o(t)\ \sqrt{1 - (\frac{2x}{c})^2}$$

or using Eq. (3.10)

$$p' = \frac{c\rho_\infty}{2}\ \dot{w}_o(t)\ \sqrt{1 - (\frac{2x}{c})^2} \tag{3.19}$$

The main points to be made about the solution are:

1) It is unique.
2) It is continuous on C.
3) Even for the flat plate the pressure tends to zero elliptically at the edges.

All of the above results are due entirely to the requirement of the Neumann condition; i.e., that the potential (or pressure in the present case) be continuous on the boundary.

In the next section we consider an alternative approach to the flat plate problem in which the Neumann condition is relaxed. We should point out here that the Neumann condition is strictly a mathematical requirement much analogous to the Kutta condition that insures uniqueness of the boundary value problem. The physical basis or in fact the invalidation of this condition will be considered in Section III-C.

B. Vibrating Flat Plate - Inviscid Incompressible Fluid Medium

Consider the flat plate illustrated in Fig. 3. We suppose that the plate can vibrate in vertical translation with velocity $w_0(t)$ or pitch about mid chord with angular velocity $w_1(t)$.

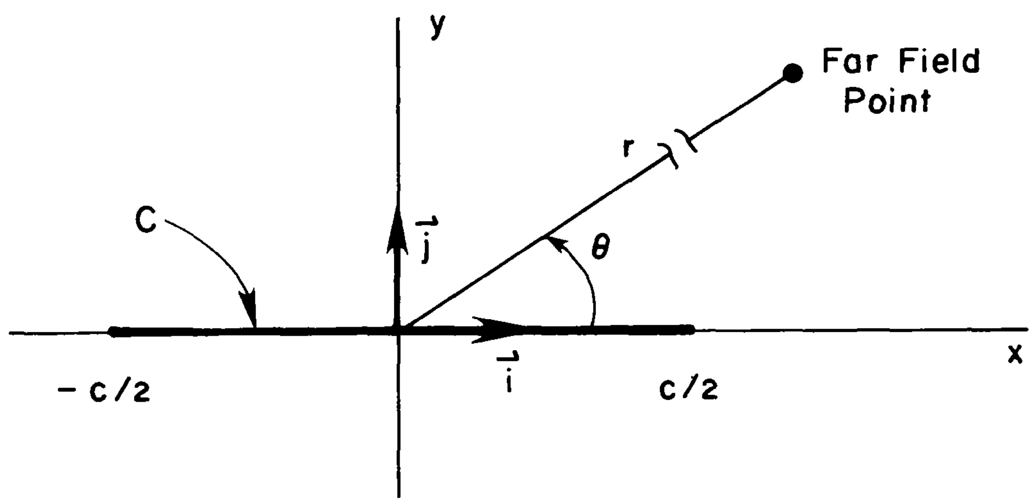


Figure 3 - Vibrating Flat Plate

The boundary value problem is

$$\nabla^2 \phi = 0 \tag{3.20}$$

$$\left.\frac{\partial \phi}{\partial y}\right|_{y=0\pm} = w_o(t) + w_1(t) \frac{2x}{c} \tag{3.21}$$

$$\text{grad } \phi \sim 0 \quad \text{at} \quad \infty \tag{3.22}$$

No requirement of continuity of ϕ on C

The solution of Eq. (3.20) that satisfies Eq. (3.22) can be expressed as a superposition of dipoles on C ; i.e.,

$$\phi = -\frac{1}{2\pi} \int_{-c/2}^{c/2} \Delta\phi \frac{\partial}{\partial y} \ell n \, R \, d\xi \tag{3.23}$$

where

$$\Delta\phi = \phi(x,0^-) - \phi(x,0^+)$$

$$R = [(x-\xi)^2 + y^2]^{1/2} \tag{3.24}$$

Now apply the boundary condition Eq. (3.21) to obtain the integral equation

$$\frac{\partial}{\partial X} \frac{1}{\pi} \int_{-1}^{1} \frac{\Delta\phi}{Y-X} \, dY = -c(w_o + w_1 \cdot X) \tag{3.25}$$

where the "slash" indicates a Cauchy principal value integral and $X = 2x/c$ is a normalized coordinate. Integrate Eq. (3.25) with respect to X to obtain

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{\Delta\phi}{Y-X}\, dY = -\frac{c}{2}\left(w_o U_1(X) + \frac{w_1}{4} U_2(X)\right) + C_1 \cdot U_o \quad (3.26)$$

where $U_n(X)$ is the Chebyshev polynomial of the second kind and $C_1(t)$ is an arbitrary constant. The general solution of Eq. (3.26) is well known (e.g., see Ref. 1, p. 28); i.e.,

$$\Delta\phi = \frac{C_o + C_1 T_1(X) - c/2(w_o T_2(X) + w_1/4\ T_3(X))}{\sqrt{1-X^2}} \quad (3.27)$$

where $C_o(t)$ is a second arbitrary constant that arises from the eigensolution of Eq. (3.26).

The pressure on the upper surface of C is given by

$$p' = -\rho_\infty \frac{\partial\phi^+}{\partial t} = \frac{\rho_\infty}{2} \frac{\partial\Delta\phi}{\partial t}$$

$$= \frac{\rho_\infty}{2}\left[\frac{\dot{C}_o + \dot{C}_1 T_1(X) - c/2(\dot{w}_o T_2(X) + \dot{w}_1/4\ T_3(X))}{\sqrt{1-X^2}}\right] \quad (3.28)$$

For arbitrary values of the coefficients C_o, C_1 the pressure has a square root singularity at both edges of the plate. Also we note that the total force and moment on the plate are proportional to $\dot{C}_o$ and $\dot{C}_1$ respectively. That is, for the vertical force

$$F = \int_{-c/2}^{c/2} \Delta p' dx = - c \int_{-1}^{1} p' \; dX$$

$$= - c \frac{\rho_\infty}{2} \dot{C}_o \int_{-1}^{1} \frac{dX}{\sqrt{1-X^2}}$$

$$= - \frac{\rho_\infty}{2} \cdot \pi c \cdot \dot{C}_o(t) \qquad (3.29)$$

and for the moment about $x = 0$,

$$M = \int_{-c/2}^{c/2} x\Delta p' \; dx = - \frac{c^2}{2} \int_{-1}^{1} Xp' \; dX$$

$$= - \frac{\rho_\infty}{2} \cdot \pi (\frac{c}{2})^2 \cdot \dot{C}_1(t) \qquad (3.30)$$

The existence of an eigensolution of the flat plate problem is a direct conseqence of the relaxation of the Neumann boundary condition; i.e., continuity of ϕ on the plate. To illustrate this point we apply the Neumann condition to Eq. (3.28). For continuity of p' at $x = \pm 1$, we must have the numerator of Eq. (3.28) vanish at both edges; i.e.,

$$\dot{C}_o + \dot{C}_1 T_1(\pm 1) - \frac{c}{2} \left(\dot{w}_o T_2(\pm 1) + \frac{\dot{w}_1}{4} T_3(\pm 1) \right)$$

$$= \left(\dot{C}_o - \frac{c}{2} \dot{w}_o \right) \pm \left(\dot{C}_1 - \frac{c}{2} \cdot \frac{\dot{w}_1}{4} \right) = 0$$

or

$$\dot{C}_o = \frac{c}{2}\,\dot{w}_o$$

$$\dot{C}_1 = \frac{c}{2}\cdot\frac{\dot{w}_1}{4} \tag{3.31}$$

For pure translation we have $\dot{w}_1 = 0$ and so

$$p' = \frac{c\rho_\infty}{2}\,\dot{w}_o\sqrt{1-X^2} \qquad \begin{pmatrix}\text{Neumann}\\ \text{Condition}\end{pmatrix} \tag{3.32}$$

which result is the same as Eq. (3.19). In the previous section we built the Neumann condition into the solution by choosing the circle plane solution as the standard. There is no a priori reason to choose the Neumann solution. In fact we will show in the next section that the solution of the viscous problem always has singularities at the edges. For mathematically sharp edges, the singularities weaken asymptotically as the viscosity tends to zero.

Before we attack the viscous problem, we show one point about the compact acoustic problem that uses the last result. The acoustic pressure is given by

$$p_a' = \frac{-\omega\rho_\infty}{4}\int_{-c/2}^{c/2} \Delta\phi\,\frac{\partial}{\partial y}\,H_o^{(2)}(kR)\,d\xi \tag{3.33}$$

where

$$R = [(x-\xi)^2 + y^2]^{1/2} \tag{3.34}$$

$$k = \omega/a_o \tag{3.35}$$

and $H_o^{(2)}(z)$ is the Hankel function (see Ref. 4, Chapter 9). In the far field we get

$$p'_a \cong \frac{i\omega}{4} k\rho_\infty \sqrt{\frac{2}{\pi kr}}\; e^{-i(kr-\pi/4)} \cdot S \qquad (3.36)$$

where

$$S = \int_{-c/2}^{c/2} \Delta\phi \; e^{ikycos\theta} dy$$

$$= \frac{c}{2} \int_{-1}^{1} \Delta\phi \; dX + ik\left(\frac{c}{2}\right)^2 \cos\theta \int_{-1}^{1} \Delta\phi \cdot X \; dX$$

or

$$S = \frac{c}{2} \cdot \pi C_o + \left(\frac{c}{2}\right)^2 \; ik \; \frac{\pi}{2} \; C_1 \cos\theta \qquad (3.37)$$

and

$$p'_a \cong \frac{\pi c k \rho_\infty}{8} \sqrt{\frac{2}{\pi kr}}\; e^{-i(kr-\pi/4)} \sin\theta \left[(i\omega C_o) + i\,\frac{ck}{4}\,(i\omega C_1)\cos\theta\right]$$

$$(3.38)$$

for a compact plate. Thus the acoustic far field is completely determined by the magnitude of the eigensolution and therefore ultimately by the "uniqueness criterion" that we impose to solve the surface load problem. The Neumann condition is one alternative but in fact is only a mathematical condition that has no physical basis. The load distribution measured by Brooks (Ref. 3) is a prime example of the invalidation of the Neumann condition. Any viable alternative to the Neumann condition must account for viscosity in some rational way. That is the subject of the following section.

C. Vibrating Flat Plate - Viscous Compressible Fluid Medium

To further explore the importance of the eigensolution of the inviscid problem we turn now to the more general problem of a two-dimensional flat plate that vibrates in a viscous compressible fluid medium. The boundary value problem is stated as follows.

$$\frac{1}{a_o^2}\frac{\partial h'}{\partial t} + \text{div}\,\vec{v}' = 0 \tag{3.39}$$

$$\frac{\partial v'}{\partial t} + \text{grad}\,h' = -\,\nu\,\text{grad}\,\Omega' \times \vec{k} \tag{3.40}$$

$$\left.\begin{aligned} \vec{j}\cdot\vec{v}' &= w(x)\,e^{i\omega t} \\ \vec{i}\cdot\vec{v}' &= 0 \end{aligned}\right\} \quad \text{on} \quad y = 0^{\pm} \tag{3.41}$$

$$\text{Outgoing or damped waves at } \infty \tag{3.42}$$

All dependent variables are proportional to $e^{i\omega t}$ so that

$$\nabla^2 h' + k^2 h' = 0 \tag{3.43}$$

$$\nabla^2 \Omega' - \alpha^2 \Omega' = 0 \tag{3.44}$$

where

$$k = \omega / a_o \tag{3.45}$$

$$\alpha = \sqrt{\frac{\omega}{\nu}}\;e^{i\pi/4} \tag{3.46}$$

and Ω' is the perturbation vorticity. It is easily shown that the solutions of (3.43) and (3.44) that satisfy the field equations, the no-slip boundary condition and the far field boundary condition are of the following forms:

$$h' = -\frac{i}{4} \int_{-c/2}^{c/2} \Delta h' \frac{\partial}{\partial y} H_o^{(2)} (kR)\, d\xi \tag{3.47}$$

$$\Omega' = -\frac{1}{2\pi\nu} \int_{-c/2}^{c/2} \Delta h' \frac{\partial}{\partial x} K_o(\alpha R)\, d\xi \tag{3.48}$$

Now apply the boundary condition on the normal velocity component at the surface to obtain the following integral equation:

$$\frac{1}{\pi} \int_{-c/2}^{c/2} \Delta h' \left\{ \frac{\partial^2}{\partial x^2} \left[\frac{i\pi}{2} H_o^{(2)} (k|x-\xi|) + K_o(\alpha|x-\xi|) \right] + \frac{i\pi}{2} k^2 H_o^{(2)} (k|x-\xi|) \right\} d\xi$$

$$= -\ 2i\omega\ W(x) \tag{3.49}$$

We demonstrate next that Eq. (3.49) has a unique solution that is singular at both edges.

Consider the plate to be acoustically compact and of sufficiently small chord that

$$\frac{|\alpha| c}{2} << 1 \tag{3.50}$$

Then the kernel of Eq. (3.49) can be reduced asymptotically to obtain the following integral equation in terms of normalized coordinates?

$$\frac{1}{\pi} \int_{-1}^{1} \Delta h' \left(\ell n |X-Y| + \ell n \frac{\alpha c}{4} + \gamma + \frac{1}{2} \right) dY = \frac{8\nu}{c} W(X) \qquad (3.51)$$

where γ is Euler's constant ($\cong$ 0.57721) and

$$(x,y) = \frac{c}{2} (X,Y) \qquad (3.52)$$

Differentiate Eq. (3.51) with respect to X to obtain the familiar airfoil equation

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{\Delta h'}{X-Y} dY = \frac{8\nu}{c} W'(X) \qquad (3.53)$$

whose general solution is

$$\Delta h' = \frac{A_o}{\sqrt{1-X^2}} - \frac{8\nu/c}{\pi\sqrt{1-X^2}} \fint_{-1}^{1} \frac{\sqrt{1-Y^2}\, W'(Y) dY}{X-Y} \qquad (3.54)$$

where A_o is an arbitrary constant. Now multiply Eq. (3.51) by $1/\sqrt{1-X^2}$ and integrate over the plate. The result is

$$\frac{1}{\pi}\int_{-1}^{1} \Delta h' dY = \frac{8\nu/\pi c}{\left(\ln \frac{\alpha c}{8} + \gamma + \frac{1}{2}\right)} \int_{-1}^{1} \frac{W \, dY}{\sqrt{1-Y^2}} \tag{3.55}$$

which serves as a natural integral constraint to evaluate A_o ; i.e.,

$$\frac{1}{\pi}\int_{-1}^{1} \Delta h' dY = A_o \tag{3.56}$$

The solution of the viscous compressible flat plate problem is

$$\Delta h' = \frac{8\nu/\pi c}{\sqrt{1-X^2}} \left\{ \frac{\displaystyle\int_{-1}^{1} \frac{W \, dY}{\sqrt{1-Y^2}}}{\ln \frac{\alpha c}{8} + \gamma + \frac{1}{2}} - \int_{-1}^{1} \frac{\sqrt{1-Y^2}\; W'(Y)}{X-Y} dY \right\} \tag{3.57}$$

Remarks:

1) The solution of the viscous compressible vibrating flat plate problem is unique.
2) The load distribution is of the same form as the eigen-solution of the inviscid problem. Furthermore the load has square root singularities at both edges.
3) The vorticity is singular at the plate edges.

While the foregoing remarks are rigorously proved in the asymptotic regime $kc/2 \ll 1$ and $|\alpha|c/2 \ll 1$ the results remain true in the general case. The reason is that the kernel in Eq. (3.49) has a logarithmic singularity when viscosity is included. The general theory of such equations (Ref. 5, for example) proves the uniqueness of solution and the existence of edge singularities.

In Fig. 4 we illustrate the numerical solution of Eq. (3.49) for $k = 0$ and $|\alpha|c/2 = 4$. Even for this relatively large value of the viscous parameter the load distributions for a plate vibrating in translation or pitch have singularities at the edges. The results in Fig. 5 are for a plate vibrating in pitch with $|\alpha|c/2 = 10$. The singularity in the real part of the load is starting to weaken while the imaginary part still remains singular. Finally in Fig. 6 we show the results for translation when $|\alpha|c/2 = 100$. The real part of the load has the $\sqrt{1-X^2}$ dependence that is obtained with potential theory and the Neumann condition (see Section III-A). The imaginary part of the load tends to zero as the viscosity becomes smaller but appears to have a singularity at the edges.

The main conclusion of the viscous theory is that the physical effect of viscosity is absolutely essential to formulate the surface loads problem correctly; i.e., so that a unique solution can be obtained without a mathematical artiface. The second conclusion is that for the case of mathematically sharp edges, viscous effects will not eliminate the edge singularities. The effect of edge geometry and/or non-linear fluid effects must be taken into account to obtain a singularity free solution. Another implication of the viscous theory is that the Neumann condition is not the correct physical uniqueness criterion to apply to the potential theory of thin vibrating surfaces. A strong piece of experimental evidence that also supports this conclusion was obtained by Brooks (Ref. 3). The measured surface load distribution for an airfoil vibrating in pitch about the 40% chord location is illustrated in Fig. 7. Note the relatively flat nature of the loading near the edges. In fact, the load appears to increase slightly near the leading edge. The experimental results indicate the existence of a strong vorticity field near the edges and negates the validity of the Neumann condition. Recall Ref. 2 that we used this evidence to explain the discrepancy between the measured and calculated (with the Kirchhoff integral) far field noise.

We conclude this section with a simple result based on the potential theory of Section III-B. Consider the normalized load distribution for a plate vibrating in pure translation (Eq. (3.28)) and expressed in dB ; i.e.,

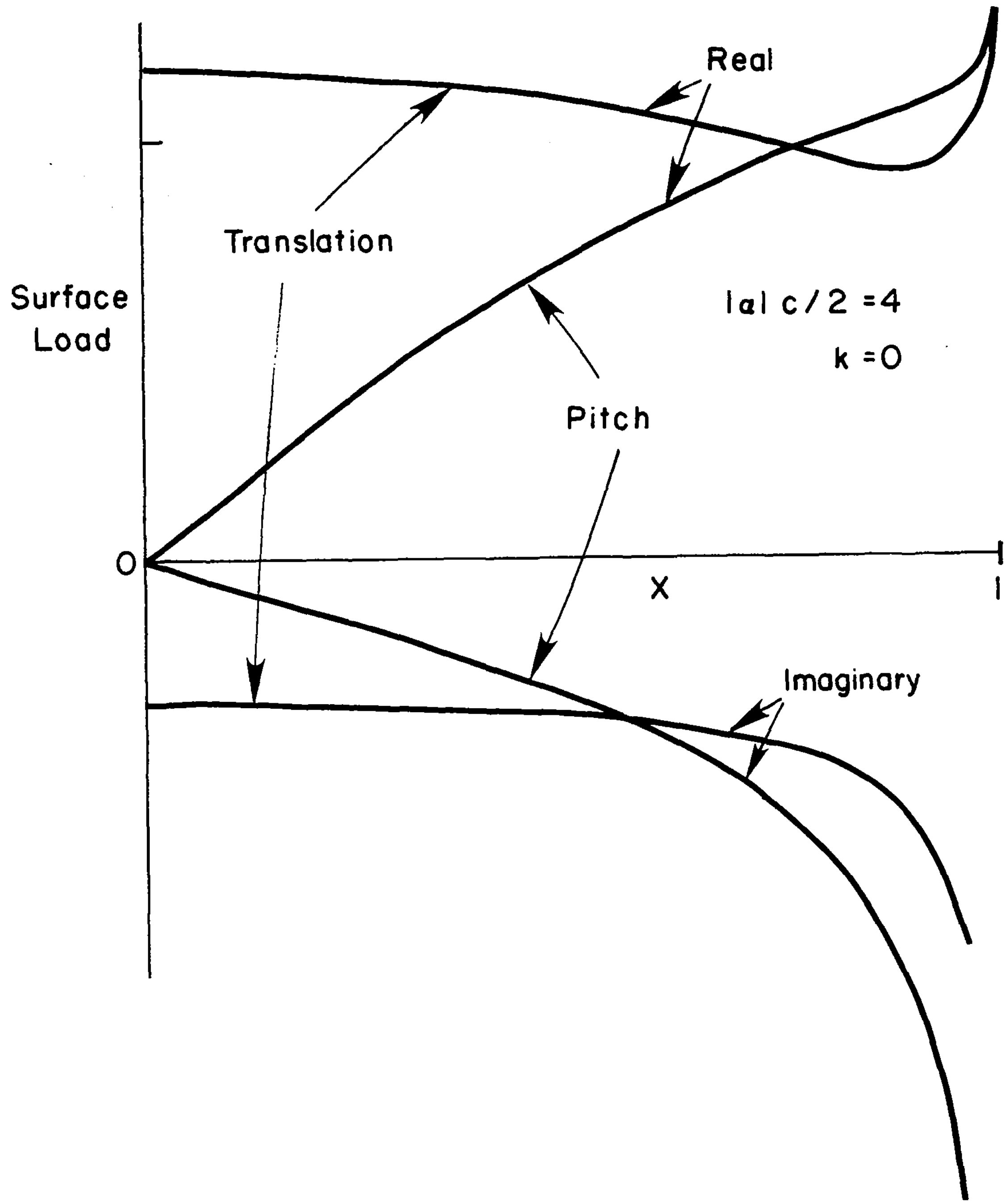


Figure 4 - Complex Surface Load on a Flat Plate Vibrating in Translation and Pitch - Viscous Theory

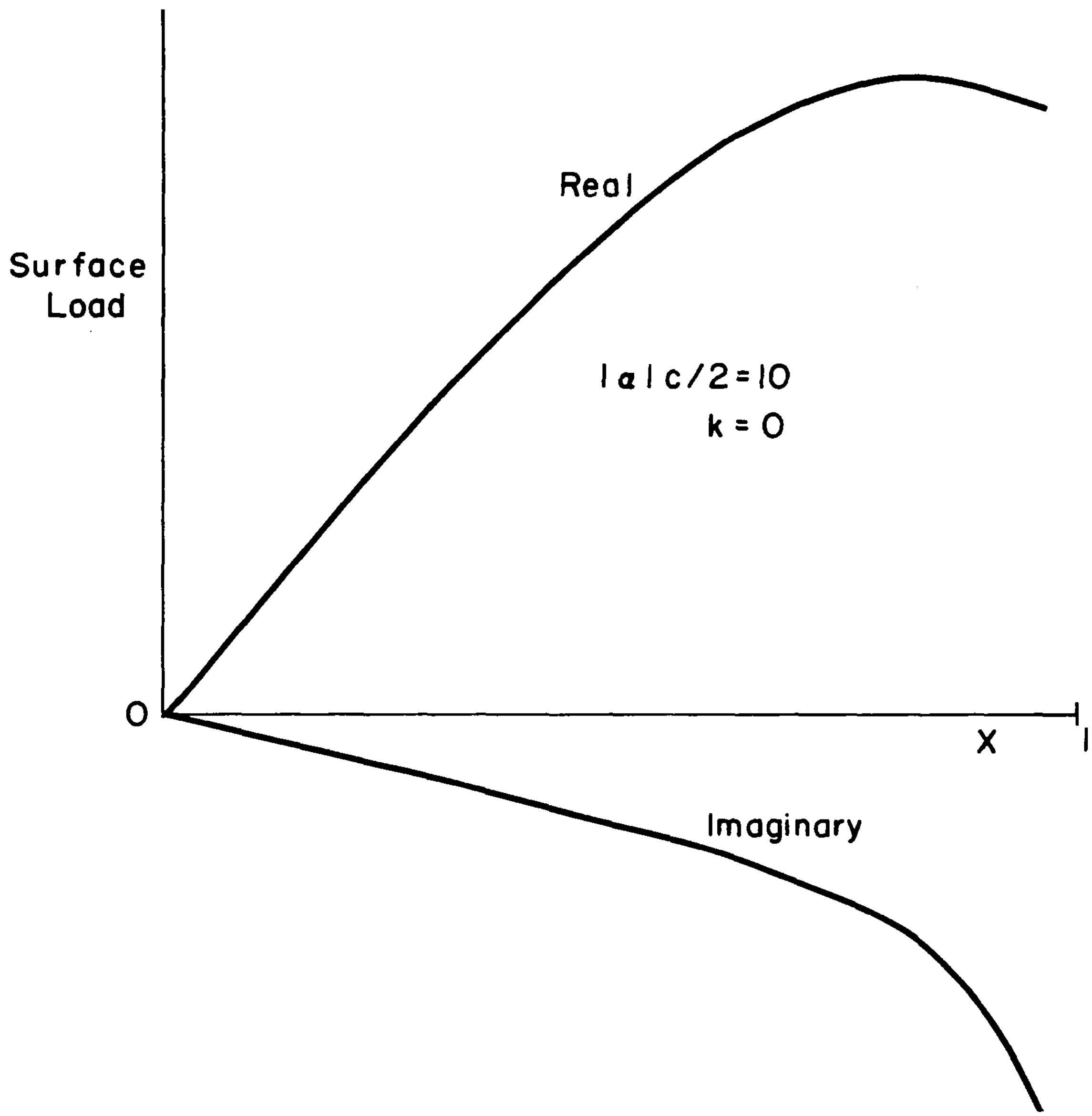


Figure 5 - Complex Surface Load on a Flat Plate Vibrating in Pitch - Viscous Theory

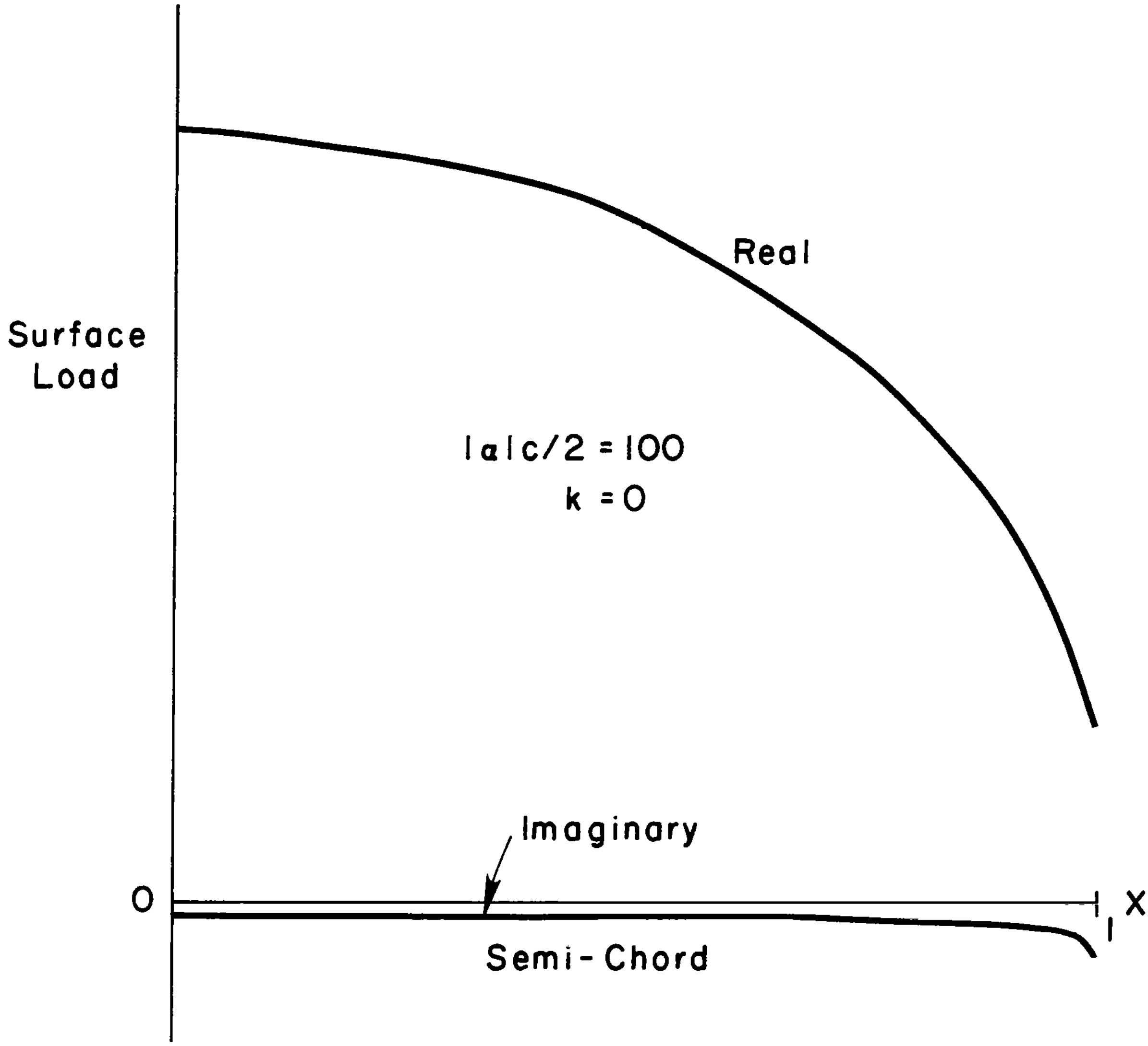


Figure 6 - Complex Surface Load on a Flat Plate Vibrating in Translation - Viscous Theory

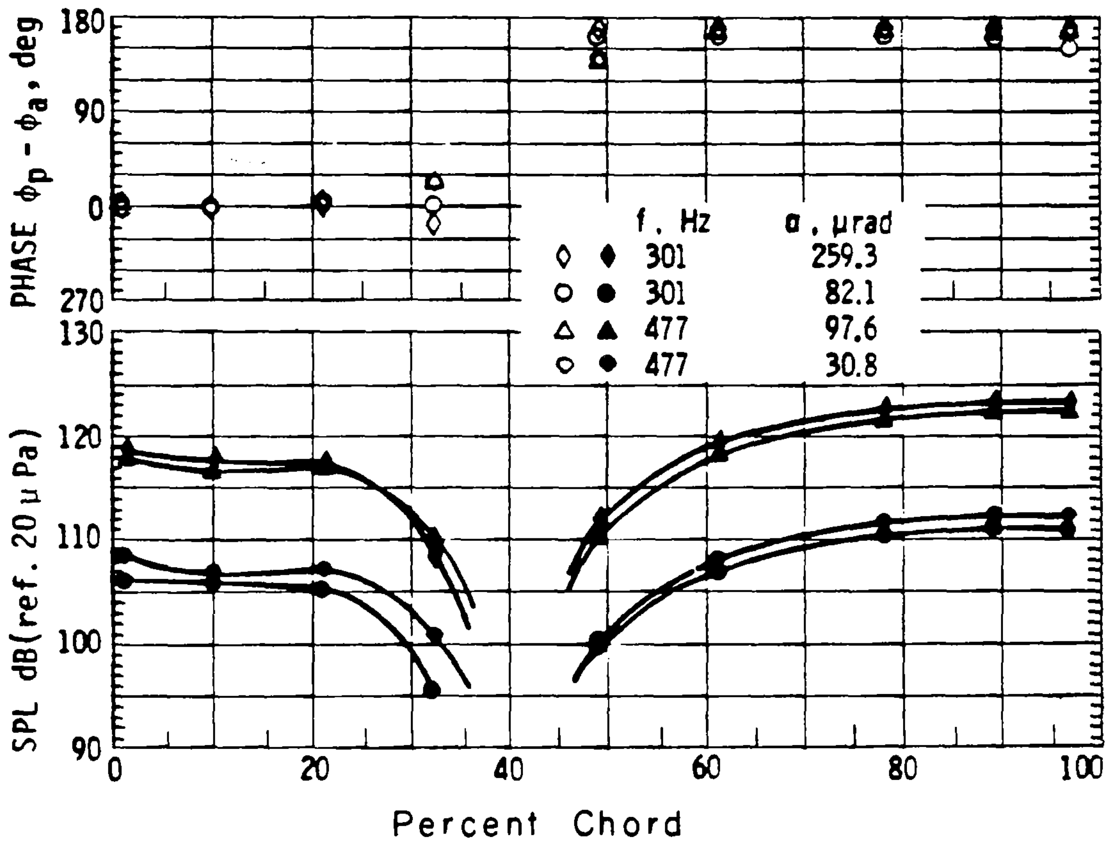


Figure 7 - Surface Load on a Vibrating NACA 0012 Airfoil
(Measured by Brooks, Ref. 3)

$$L_{dB} = 20 \log_{10} \left[\frac{(1+A) - T_2(X)}{p_r \sqrt{1-X^2}} \right] \tag{3.58}$$

where p_r is an arbitrary reference pressure. When $A = 0$, the Neumann load distribution is obtained. We compare the Neumann result with the case $A = 0.4$ in Fig. 8. The 40% increase in the magnitude of the Neumann eigensolution yields a load distribution that is very flat near the edges. By construction it does indeed have square root edge singularities. The important point is that by giving up the Neumann condition as a uniqueness criterion we can fit experimental surface loads data much more accurately. We should point out that this point is also true in regard to the Kutta condition when applied to the potential theory of lift. By giving up the Kutta condition Pinkerton (reported in Ref. 6) found that much better agreement could be obtained between measured and theoretical surface loads data. More recently, it has been shown (Ref. 7) that by including viscosity and section geometry (real and effective) in thin airfoil theory that both steady and unsteady

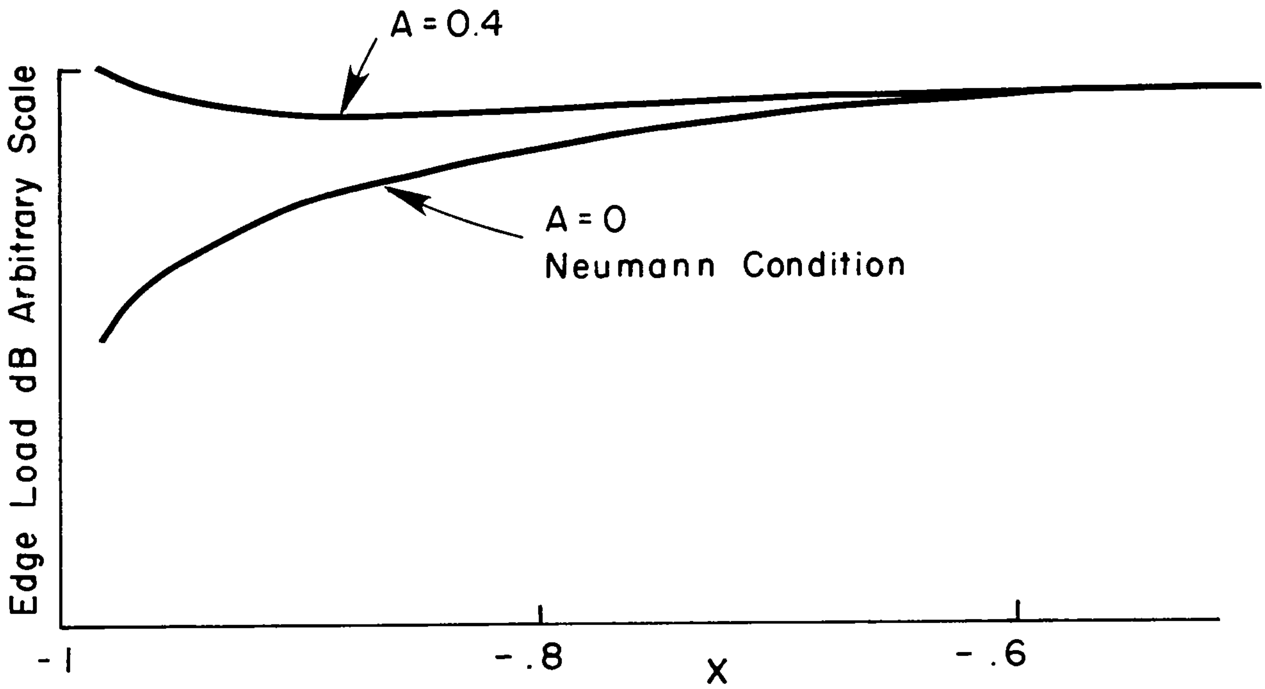


Figure 8 - Comparison of Edge Loading with and Without the Neumann Condition (See Eq. (3.58))

results can be calculated uniquely and accurately. It is the fundamental contention of the present study that viscous and geometric effects must be included simultaneously in a complete theory of the surface load required for acoustic calculations.

D. Vibrating Two-Dimensional Body - Viscous Compressible Fluid Medium

We conclude our present study with a precise formulation of the viscous compressible surface load problem for an arbitrary two-dimensional section. The boundary value problem is very similar to one posed in Section III-C, Eqs. (3.39) through (3.40). Referring to Fig. 9 we have

$$\frac{1}{a_o^2}\frac{\partial h'}{\partial t} + \mathrm{div}\,\vec{v}' = 0 \tag{3.59}$$

$$\frac{\partial \vec{v}'}{\partial t} + \mathrm{grad}\,h' = -\nu\,\mathrm{grad}\,\Omega' \times \vec{k} \tag{3.60}$$

$$\vec{n}\cdot\vec{v}' = W(s)e^{i\omega t} \quad \text{on} \quad C_o \tag{3.61}$$

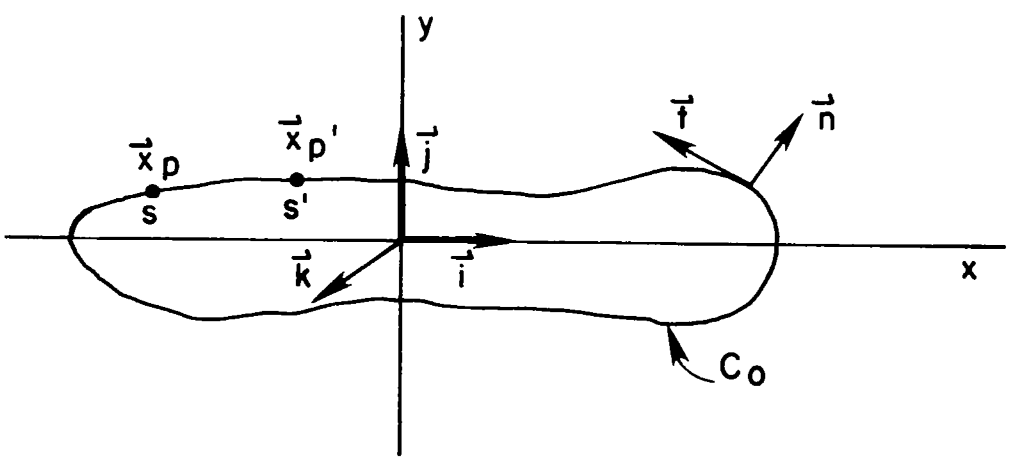


Figure 9 - Vibrating Two-Dimensional Body in a Viscous Compressible Fluid Medium

$$\vec{t}\cdot\vec{v}' = U(s)e^{i\omega t} \quad \text{on} \quad C_o \tag{3.62}$$

$$\text{Outgoing or damped waves at infinity} \tag{3.63}$$

Equations (3.43) through (3.46) are still valid. We introduce two Greens functions G_H and G_Ω as follows:

$$\nabla^2 G_H + k^2 G_H = 2\pi\delta(\vec{x}-\vec{y}) \tag{3.64}$$

$$\nabla^2 G_\Omega - \alpha^2 G_\Omega = 2\pi\delta(\vec{x}-\vec{y}) \tag{3.65}$$

The solutions are required to satisfy the boundary condition Eq. (3.63). Now use the following identities:

$$\begin{aligned} G_H\nabla^2 h' - h'\nabla^2 G_H &= \nabla(G_H\nabla h' - h'\nabla G_H) \\ &= -\ 2\pi h'\delta(\vec{x}-\vec{y}) \end{aligned} \tag{3.66}$$

and

$$\begin{aligned} G_\Omega\nabla^2\Omega' - \Omega'\nabla^2 G_\Omega &= \nabla(G_\Omega\nabla\Omega' - \Omega'\nabla G_\Omega) \\ &= -\ 2\pi\Omega'\delta(\vec{x}-\vec{y}) \end{aligned} \tag{3.67}$$

Integrate the above identities over the domain exterior to the body to obtain the following representation of the solution:

$$h'(\vec{x}) = \frac{1}{2\pi} \oint ds' \left(G_H \frac{\partial h'}{\partial n'} - h' \frac{\partial G_H}{\partial n'} \right) \qquad (3.68)$$

$$\Omega'(\vec{x}) = \frac{1}{2\pi} \oint ds' \left(G_\Omega \frac{\partial \Omega'}{\partial n'} - \Omega' \frac{\partial G_\Omega}{\partial n'} \right) \qquad (3.69)$$

Next we project the momentum equation onto the surface C_o to obtain the following pair of equations:

$$i\omega U(s) + \frac{\partial h'}{\partial s} = \nu \frac{\partial \Omega'}{\partial n} \quad \text{on} \quad C_o \qquad (3.70)$$

$$i\omega W(s) + \frac{\partial h'}{\partial n} = - \nu \frac{\partial \Omega'}{\partial s} \quad \text{on} \quad C_o \qquad (3.71)$$

These equations can be used to eliminate the normal derivatives of h' and Ω' in Eqs. (3.68) and (3.69). After integrating by parts in each term that involves a tangential derivative, we obtain

$$h'(\vec{x}) = \frac{1}{2\pi} \oint ds' \left[\nu\Omega' \frac{\partial G_H}{\partial s'} - h' \frac{\partial G_H}{\partial n'} - i\omega G_H W(s') \right] \qquad (3.72)$$

$$\nu\Omega'(\vec{x}) = - \frac{1}{2\pi} \oint ds' \left[h' \frac{\partial G_\Omega}{\partial s'} + \nu\Omega' \frac{\partial G_\Omega}{\partial n'} + i\omega G_\Omega U(s') \right] \qquad (3.73)$$

The final step is to take the limit as $\vec{x} \to \vec{x}_p(s)$ - any point on the body contour C_o . Note the following results:

$$\lim_{\vec{x}\to\vec{x}_p} \frac{1}{2\pi} \int_{s-\varepsilon}^{s+\varepsilon} ds' \frac{\partial G}{\partial n'} = - \frac{1}{2\pi} \lim_{y\to 0} \int_{-G}^{G} \frac{y}{t^2+y^2} dt = - \frac{1}{2} \tag{3.74}$$

where G can be G_H or G_Ω since both functions behave like $\ln|\vec{x}|$ near the origin. With Eq. (3.74) it is easily shown that

$$h'(s) = \frac{1}{\pi} \not\oint ds' \left(\nu\Omega' \frac{\partial G_H}{\partial s'} - h' \frac{\partial G_H}{\partial n'}\right) - \frac{i\omega}{\pi} \oint ds' G_H W(s') \tag{3.75}$$

and

$$\nu\Omega'(s) = - \frac{1}{\pi} \not\oint ds' \left(h' \frac{\partial G_\Omega}{\partial s'} + \nu\Omega' \frac{\partial G_\Omega}{\partial n'}\right) + \frac{i\omega}{\pi} \oint ds' G_\Omega U(s') \tag{3.76}$$

This is a coupled pair of inhomogeneous singular integral equations for the unknown pressure and vorticity. The combined effects of viscosity and geometric cross section are rigorously included in the formulation. It is clear from our previous discussion in Section III-C that the solution of this pair is unique. If the contour is smooth, no singularities will appear in the solution. It is firmly believed that the solution of this pair of equations has the strongest possibility of yielding results in agreement with the load distributions measured by Brooks (Ref. 3). The only physical effect that we have omitted is the possibility of nonlinear flow near the edges. Since there is nothing in the experimental results of Brooks that suggests the slightest bit of nonlinearity, we feel that such effects can be safely omitted. It is recommended that the numerical solution of the pair of integral equations derived herein be the subject of a future research effort.

IV. CONCLUSIONS

The main conclusions of this study are summarized below:

1. For Helmholtz numbers of order unity or less the energy dissipated by viscosity at the edges of a vibrating three-dimensional plate can be comparable to or greater than the acoustic energy radiated to the far field.

2. The Neumann boundary condition is not the correct uniqueness criterion to apply to the potential theory of the surface load distribution on a thin vibrating plate. The viscous theory and the experimental results of Brooks (Ref. 3) support this conclusion.

3. The correct formulation (to obtain a unique solution) of the surface loads problem must include the physical effect of viscosity and the effect of geometric shape.

4. The far field acoustic solution is determined uniquely by solution of the viscous surface loads problem.

APPENDIX A

VISCOUS CORRELATION FUNCTION

Consider the integral

$$J = \int_V T(\vec{x}-\vec{y})T^*(\vec{x}-\vec{y}\,')\,d\vec{x} \tag{A.1}$$

where

$$T(\vec{x}) = \frac{e^{-\lambda|\vec{x}|}}{|\vec{x}|}, \qquad \lambda^2 = \frac{i\omega}{\nu} \tag{A.2}$$

We introduce the three-dimensional Fourier Transform pair; i.e.,

$$\tilde{T}(\vec{\alpha}) = \int e^{-i\vec{\alpha}\cdot\vec{x}}\,T(\vec{x})\,d\vec{x}$$

$$T(\vec{x}) = \frac{1}{(2\pi)^3}\int e^{i\vec{\alpha}\cdot\vec{x}}\,\tilde{T}(\vec{\alpha})\,d\vec{\alpha} \tag{A.3}$$

Then J can be reduced to the following Parseval relation

$$J = \frac{1}{(2\pi)^3}\int e^{-i\vec{\alpha}\cdot\vec{r}}\,\tilde{T}(\vec{\alpha})\tilde{T}^*(\vec{\alpha})\,d\vec{\alpha} \tag{A.4}$$

where

$$\vec{r} = \vec{y} - \vec{y}\,' \tag{A.5}$$

Now

$$\tilde{T}(\vec{\alpha}) = \int e^{-i\vec{\alpha}\cdot\vec{x}} \cdot \frac{e^{-\lambda|\vec{x}|}}{|\vec{x}|} \, d\vec{x}$$

$$= 2\pi \int_{o}^{\pi} e^{-i\alpha r\cos\phi} \sin\phi \, d\phi \int_{o}^{\alpha} r \, e^{-\lambda r} \, dr$$

$$= \frac{4\pi}{\alpha} \int_{o}^{\infty} e^{-\lambda r} \sin\alpha r \, dr$$

$$= \frac{4\pi}{\alpha^2 + \lambda^2} = \frac{4\pi}{\alpha^2 + i\omega/\nu} \qquad \text{(A.6)}$$

Thus

$$J = \frac{2}{\pi} \int \frac{e^{-i\vec{\alpha}\cdot\vec{r}} \, d\vec{\alpha}}{\alpha^4 + \omega^2/\nu^2}$$

$$= \frac{2}{\pi} \int_{o}^{\infty} \frac{\alpha^2 d\alpha}{\alpha^4 + \omega^2/\nu^2} \cdot 2\pi \int_{o}^{\pi} e^{-i\alpha r\cos\phi} \sin\phi \, d\phi$$

$$= \frac{8}{r} \int_{o}^{\infty} \frac{\alpha \sin \alpha r \, d\alpha}{\alpha^4 + \omega^2/\nu^2}$$

$$= \frac{4\pi}{(\omega r/\nu)} \, e^{-\sqrt{\frac{\omega}{2\nu}}\, r} \sin \sqrt{\frac{\omega}{2\nu}} \, r \qquad \text{(A.7)}$$

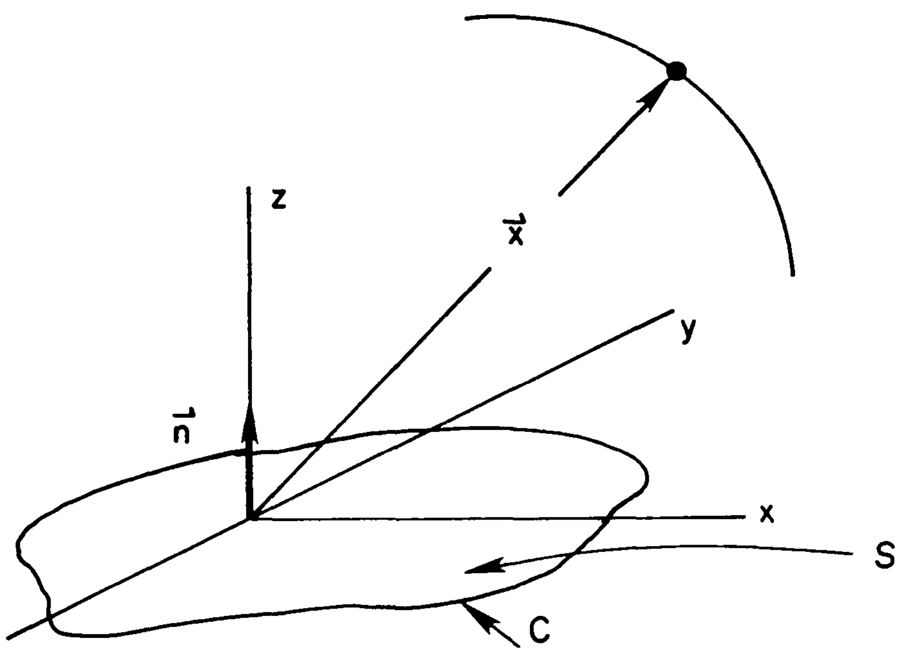
APPENDIX B

ACOUSTIC RADIATION FROM A VIBRATING SURFACE

Consider the equations

$$\frac{1}{a_o^2} \frac{\partial h'}{\partial t} + \text{div } \vec{v}' = 0 \tag{B.1}$$

$$\frac{\partial \vec{v}'}{\partial t} + \text{grad } h' = 0 \tag{B.2}$$



or

$$\nabla^2 h' - \frac{1}{a_o^2} \frac{\partial^2 h'}{\partial t^2} = 0 \tag{B.3}$$

with

$$\frac{\partial h'}{\partial z} = - \frac{\partial v_n'}{\partial t} \quad \text{on} \quad z = 0 \pm \tag{B.4}$$

Let

$$\ell(\vec{x}) = \rho_\infty(h'(\vec{x},0^-) - h'(\vec{x},0^+)) \tag{B.5}$$

and

$$h' \to h'e^{i\omega t} \tag{B.6}$$

$$k = \omega/a_o \tag{B.7}$$

Then

$$\nabla^2 h' + k^2 h' = 0 \tag{B.8}$$

and

$$h' = \frac{\partial}{\partial z} \int_S Q(\vec{y}) \frac{e^{-ik|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|} d\vec{y} \tag{B.9}$$

Also

$$\lim_{z\to 0\pm} h' = Q(\vec{x}) = \int \left(- \frac{z}{R^3} \; d\vec{y}\right) \tag{B.10}$$

$$R^3 = (z^2 + \rho^2)^{3/2}$$

$$d\vec{y} = \rho d\rho d\theta$$

$$\lim_{z\to 0\pm} h' = - 2\pi \; Q(\vec{x}) \int_o^\varepsilon \frac{\rho z \; d\rho}{(z^2+\rho^2)^{3/2}}$$

$$= 2\pi \; Q(\vec{x}) \left. \frac{z}{(z^2+\rho^2)^{1/2}} \right|_o^\varepsilon$$

$$= \mp \; 2\pi \; Q(\vec{x}) \; \text{sgn } z \tag{B.11}$$

and

$$h'^{-} - h'^{+} = 4\pi \; Q(\vec{x}) = \frac{\ell(\vec{x})}{\rho_\infty} \tag{B.12}$$

or

$$Q(\vec{x}) = \frac{\ell(\vec{x})}{4\pi\rho_\infty} \tag{B.13}$$

Finally

$$h' = \frac{1}{4\pi\rho_\infty} \frac{\partial}{\partial z} \int_S \ell(\vec{y}) \frac{e^{-ik|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|} \, d\vec{y} \tag{B.14}$$

For $|\vec{x}| \to \infty$ we get

$$h' \cong \frac{1}{4\pi\rho_\infty} \cdot \frac{\partial}{\partial z} \frac{e^{-ikr}}{r} \int_S \ell(\vec{y}) \; e^{ik\hat{x}\cdot\vec{y}} \, d\vec{y} \tag{B.15}$$

where

$$\hat{x} = \frac{\vec{x}}{|\vec{x}|} \tag{B.16}$$

$$r = |\vec{x}| \tag{B.17}$$

Or

$$h' = -\frac{ik}{4\pi\rho_\infty} \frac{z}{r^2} \; e^{-ikr} \int_S \ell(\vec{y}) e^{ik\hat{x}\cdot\vec{y}} \, d\vec{y} \tag{B.18}$$

Also in the far field on a sphere of radius r

$$\frac{\partial h'}{\partial r} = - i\omega v_n' \tag{B.19}$$

$$v_n' = \frac{i}{\omega}\frac{\partial h'}{\partial r} \tag{B.20}$$

and

$$\frac{\partial h'}{\partial r} \sim - ik\, h' \tag{B.21}$$

$$v_n' \cong \frac{k}{\omega} h' = \frac{h'}{a_o} \tag{B.22}$$

$$\overline{h'v_n'} = \frac{\overline{h'^2}}{a_o} = \frac{\overline{(\text{Re } h' \; e^{i\omega t})^2}}{a_o} \tag{B.23}$$

$$\overline{(\text{Re } h' \; e^{i\omega t})^2} = \overline{(h_R'\cos\omega t - h_I'\sin\omega t)^2}$$

$$= h_R'^2 \; \overline{\cos^2\omega t} + h_j'^2 \; \overline{\sin^2\omega t}$$

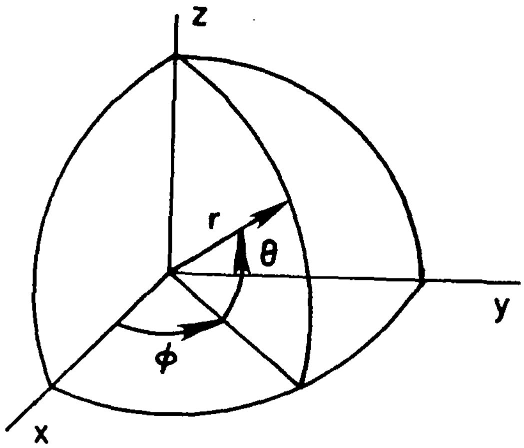
$$= \frac{1}{2} \; |h'|^2 \tag{B.24}$$

and so

$$\overline{h'v_n'} = \frac{|h'|^2}{2a_o} \tag{B.25}$$

$$\frac{|h'|^2}{2a_o} = \frac{k^2}{32\pi^2\rho_\infty^2 a_o} \cdot \frac{z^2}{r^4} \int_S \ell(\vec{y})\,d\vec{y} \int_S \ell^*(\vec{y}\,')\,d\vec{y}\,'\; e^{ik\hat{x}\cdot(\vec{y}-\vec{y}\,')} \tag{B.26}$$

$$\sin\theta = \frac{z}{r} \tag{B.27}$$



Referring to the above sketch

$$\hat{x} = \vec{i}\,\cos\theta\cos\phi + \vec{j}\,\cos\theta\sin\phi + \vec{k}\,\sin\theta$$

$$= \cos\theta(\hat{x}_s) + \vec{k}\,\sin\theta \tag{B.28}$$

$$\hat{x}_s = \vec{i}\,\cos\phi + \vec{j}\,\sin\phi$$

Thus

$$\frac{|h'|^2}{2a_o} = \frac{k^2}{32\pi^2\rho_\infty^2 a_o}\,\frac{\sin^2\theta}{r^2} \int_S \ell(\vec{y})\,d\vec{y} \int_S \ell^*(\vec{y}\,')\,d\vec{y}\,' e^{i(k\cos\theta)\hat{x}_s\cdot(\vec{y}-\vec{y}\,')} \tag{B.29}$$

Now

$$E_a = \int_{S_f} \overline{h'v'_n}\, dA = \int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} d\theta\; r^2 \cos\theta \frac{|h'|^2}{2a_o}$$

$$= \frac{k^2}{32\pi^2 \rho_\infty^2 a_o} \cdot \int_S \ell(\vec{y})\, d\vec{y} \int_S \ell^*(\vec{y}')\, d\vec{y}' \cdot G \qquad \text{(B.30)}$$

$$G = \int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} d\theta\; \sin^2\theta \cos\theta\, e^{i(k\cos\theta)\hat{x}_s \cdot (\vec{y}-\vec{y}')} \qquad \text{(B.31)}$$

For compact surfaces

$$G = 2\pi \int_{-\pi/2}^{\pi/2} \sin^2\theta \cos\theta$$

$$= 4/3\ \pi \qquad \text{(B.32)}$$

and

$$E_a = \frac{k^2}{32\pi^2 \rho_\infty^2 a_o} \cdot \frac{4}{3} \cdot \pi \cdot |L|^2$$

$$= \frac{k^2}{24\pi \rho_\infty^2 a_o} \cdot |L|^2$$

$$= \frac{k^2}{24\pi \rho_\infty^2 a_o} \cdot \left| \int_S \ell_o\, dA \right|^2 \qquad \text{(B.33)}$$

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16. Abstract

The energy dissipated by viscosity at the edge of a vibrating flat plate is calculated and compared to the radiated acoustic energy. A correction to the Kirchhoff integral estimate of the noise is derived. For Helmholtz number of order unity and smaller the dissipation can be comparable to or greater than the acoustic energy.

A viscous compressible theory of the load distribution on a vibrating two-dimensional body is developed. First it is shown that load calculations based on potential theory and the Neumann uniqueness condition (continuity of potential or pressure on the surface) are not in agreement with experiment or the more correct viscous theory. For a flat plate airfoil the eigensolution of potential theory is indeterminant while viscous theory yields a unique solution that has square root singularities at the edges. It is also shown that for compact surfaces the far field acoustics depend only on the magnitude of the eigensolutions of potential theory and so will be uniquely determined by the viscous theory. It is suggested that the general viscous theory of vibrating surfaces with cross sectional geometry will lead to results in agreement with experimentally measured load distributions.

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