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STRESS EVALUATIONS UNDER ROLLING/SLIDING CONTACTS

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by

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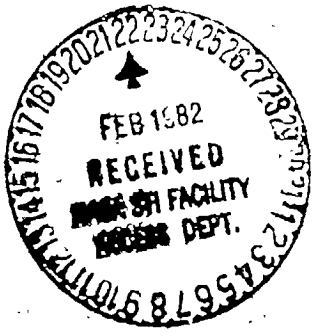

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16 Abstract

Computer models have been developed for analyzing the state-of-stress beneath traction-drive type of contacts. The analyses involve computing stresses and stress-reversals on various planes for points beneath the surface. The effect of tangential and axial friction under gross slip conditions can be evaluated with the models. Evaluations performed on an RC (rolling contact) tester configuration indicate that the classical fatigue stresses are not altered by friction-forces typical of lubricated contact. Higher values of friction ($f > 0.25$) can result in surface shear reversal that exceeds the stresses at the depth of maximum shear reversal under rolling contact.

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ABSTRACT

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INTRODUCTION

Rolling elements such as roller bearings and traction drive transmissions are subject to failure from rolling contact fatigue. Such fatigue failures cause a serious restriction on the operating life and reliability of such devices. In the case of the traction drive transmissions, rolling contact fatigue failures could conceivably restrict the use of such transmissions in motor vehicles, despite the increased efficiency such transmissions could afford. Although a number of factors contribute to fatigue failure, the effect of traction and slip on the state of stress between rolling elements is not understood. A better understanding of the role of traction on fatigue could greatly assist in the evaluation of materials for and operational limits of traction components.

The fundamental theory for rolling contact fatigue is the one published by Lundberg and Palmgren[1]* in 1947 and summarized by Coy, et al.[2]. This theory is based on the assumption that failure is in the form of shallow pitting and is related to subsurface shearing stresses within the rolling elements. In essence, any irregularity in the material beneath the surface can manifest itself as a failure initiation point. The significant stress field has been hypothesized as being bounded by the surface, the depth of the maximum and reversing shear stress, and the width of the rolling track. The magnitude and depth of the reversing shear stress have been found to be very significant with regards to fatigue failure.

The reversing stress field has been well established for pure rolling contacts[3,4]. However, when surface tractions are imposed, this stress field becomes very complicated. The classic study for line contact is by Smith and Liu[5]. Smith and Liu's research indicated that friction is a key

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*References are listed at the end of the text.

factor with regards to reversing stresses. For example, at some level of friction, the reversing stress was found to be at the surface, which would greatly reduce fatigue life. Hamilton and Goodman[6] developed a theory for a circular sliding contact. The results of their theory are in the form of useful graphs for predicting friction effects for this specific type of contact. Other researchers, such as Kuznetsov[7], have studied the influence of friction on contact stress, but not on the subsurface stress field under slip conditions.

One complication to the reversing stress field is that the stress reversals occur over the width of contact and depend on orientation as well as depth. See, for example, Figure 1. The maximum stress occurs at the center of contact and the depth and magnitude are straightforward to compute. Computing the reversing stress becomes more complicated because the reversals occur over a finite region. Consider, then, the complication when the magnitude of τ_+ or τ_- depend on the orientation out of the plane of the paper.

The goal of the project has been to develop computer models to determine the magnitude of reversing shear stresses beneath (and very near) the surface of rolling/sliding contacts. Sliding in both the tangential direction and the axial direction are considered. Considerations are given for line contacts as well as crowned rollers. In order to insure accuracy in modeling, two independent models were developed, one by Tevaarwerk (T-model) and one at Battelle (B-model)*. This approach allowed for both researchers to be heavily involved with the problem in modeling and created excellent dialogue, both for the mathematical intricacies, as well as the ramification of the predictions. A comparison between the two models is given in Table 1, which demonstrates the consistency of the two approaches. The approach for the development of the models was as follows:

- Define the stress tensor for any point beneath the surface
- Compute principal stresses and their direction cosines for each point
- Determine the direction cosines (relative to the x, y, z axes) to the octahedral plane and the plane of maximum shearing stress

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*As discussed in Appendix A, the primary difference in the two models is in the integration scheme used to determine the stress tensor.

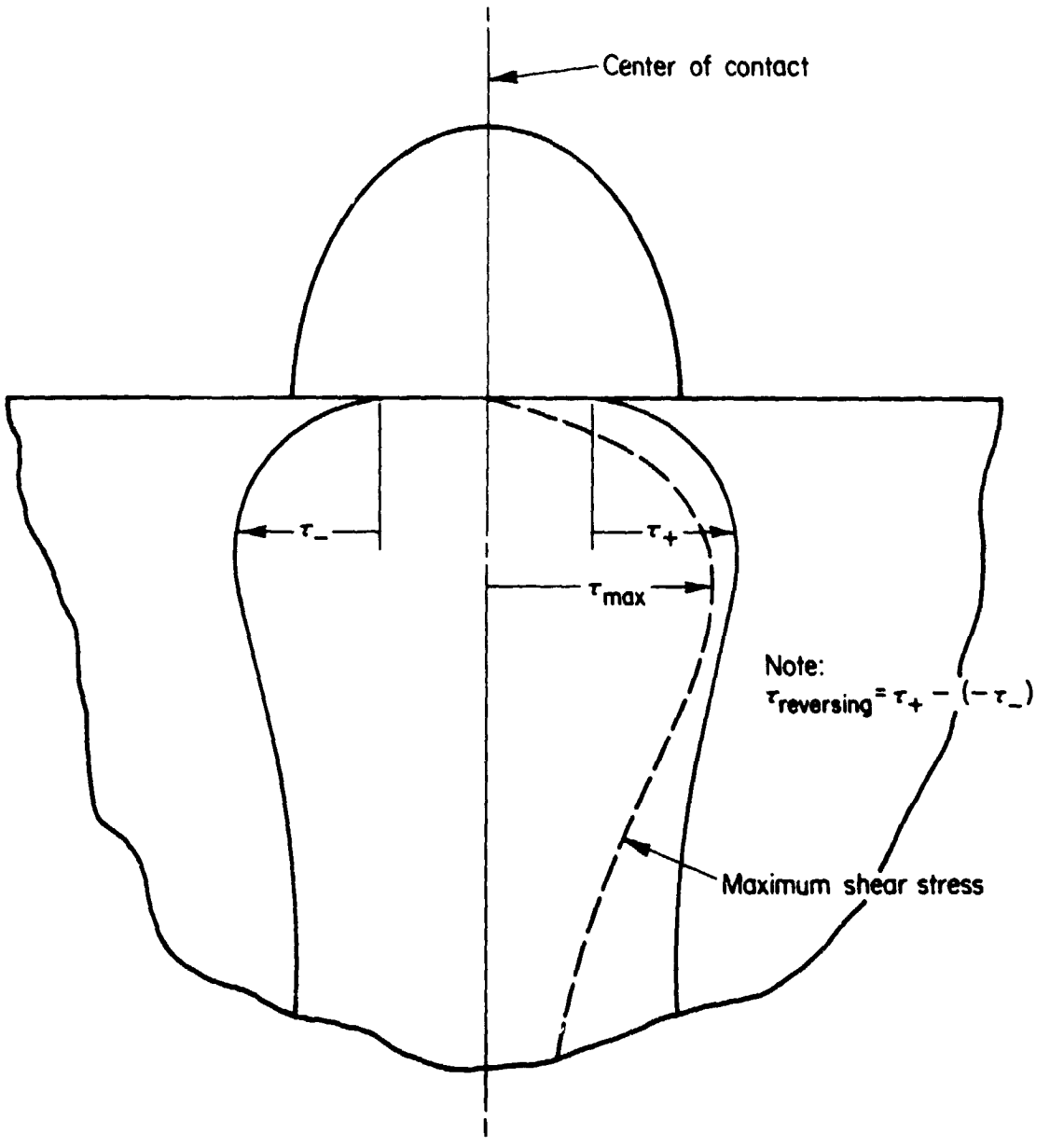

FIGURE 1. ILLUSTRATION OF SHEAR STRESS BENEATH THE SURFACE OF CONTACT

TABLE 1. COMPARISON OF TEVAARWERK MODEL WITH BATTELLE MODEL
($K = .5$, $\nu = .285$, $f_A = 0$, $z/b = .3$)

		σ_x		σ_y		σ_z		τ_{xy}		τ_{yz}		τ_{xz}	
f_T	z/b	T	B	T	B	T	B	T	B	T	B	T	B
0	-.9	-.236	-.241	-.212	-.218	-.375	-.383	0	0	0	0	.228	.231
	-.8	-.251	-.256	-.255	-.259	-.511	-.518	0	0	0	0	.222	.225
	-.5	-.329	-.333	-.361	-.369	-.798	-.807	0	0	0	0	.138	.140
	-.3	-.374	-.380	-.404	-.412	-.896	-.907	0	0	0	0	.080	.081
	0	-.401	-.408	-.428	-.436	-.948	-.958	0	0	0	0	0	0
	.3	-.374	-.380	-.404	-.412	-.896	-.907	0	0	0	0	-.080	-.081
	.5	-.329	-.333	-.361	-.369	-.798	-.807	0	0	0	0	-.138	-.141
	.8	-.251	-.256	-.255	-.259	-.511	-.518	0	0	0	0	-.222	-.225
	.9	-.236	-.240	-.212	-.218	-.375	-.383	0	0	0	0	-.228	-.230
.1	-.9	-.171	-.174	-.196	-.201	-.352	-.360	0	0	0	0	.195	.200
	-.8	-.188	-.191	-.239	-.243	-.489	-.495	0	0	0	0	.187	.190
	-.5	-.281	-.284	-.349	-.356	-.784	-.793	0	0	0	0	.095	.097
	-.3	-.344	-.349	-.396	-.414	-.888	-.898	0	0	0	0	.032	.033
	0	-.401	-.408	-.428	-.436	-.948	-.958	0	0	0	0	-.051	-.051
	.3	-.404	-.411	-.412	-.420	-.904	-.915	0	0	0	0	-.128	-.130
	.5	-.376	-.382	-.373	-.381	-.812	-.821	0	0	0	0	-.182	-.184
	.8	-.315	-.320	-.271	-.276	-.533	-.540	0	0	0	0	-.256	-.260
	.9	-.301	-.308	-.228	-.234	-.398	-.406	0	0	0	0	-.260	-.264

- Scan the planes of octahedral and maximum shearing stress in the direction of rolling and compute stress reversals.

Stress reversals can be computed by the above procedure for conditions of zero friction, axial friction, and tangential friction for line contacts and contacts corresponding to a rolling contact fatigue tester. The basic assumptions for the analyses are:

1. The contact pressure (normal stress) are predictable by Hertz theory for line or crowned contacts.
2. The contact tractions are a result of a constant traction coefficient times the pressure. This tacitly assumes a gross sliding situation.
3. The materials are all elastic and homogeneous.
4. The bodies are isothermal. That is, no thermal stresses are considered.

MODEL DEVELOPMENT

The Stress Tensor

The objective of the analyses is to evaluate subsurface stresses in rolling/sliding contacts of the type seen in traction drives. Consider, for example, the system shown in Figure 2, where two cylinders are loaded together under rolling or possibly rolling/sliding contact. The assumed Hertzian contact pressure distribution produces a state of stress throughout the cylinders. Likewise, the frictional forces associated with sliding produce an additional state of stress such that a typical point beneath the surface is undergoing a complicated and changing stress state. For crowned cylinders, this stress state varies axially (in the y direction) as a result of axially varying pressures as well as in the x direction, as shown in Figure 3.

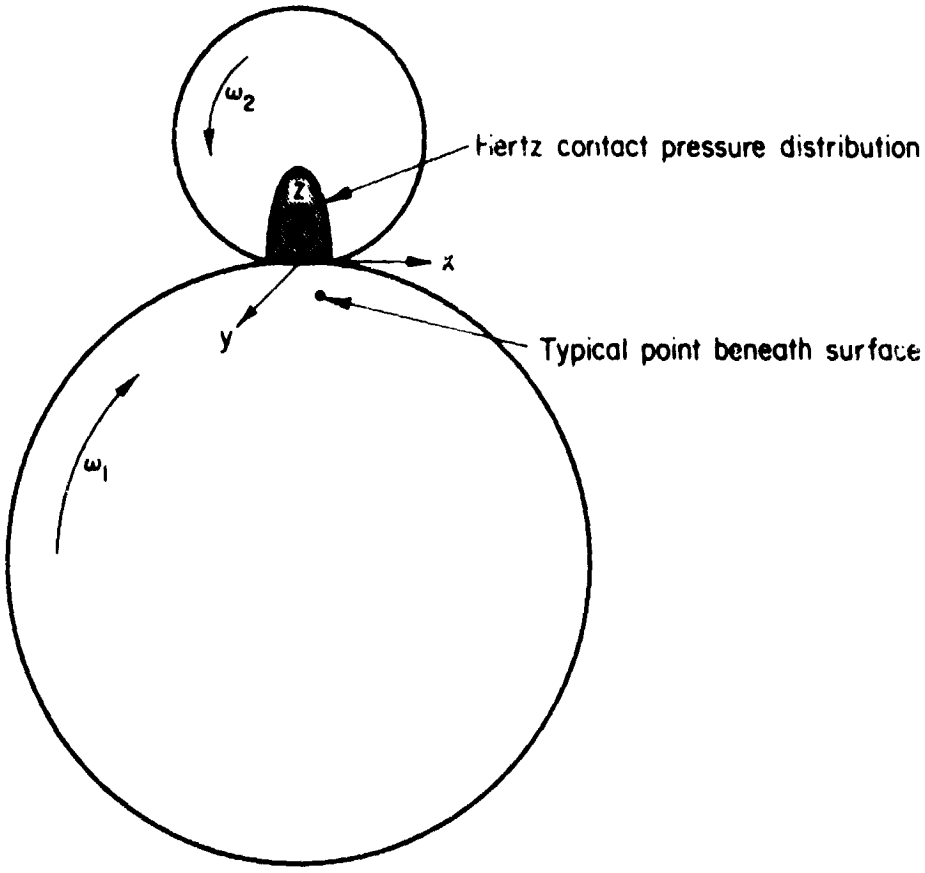

FIGURE 2. CONTACT BETWEEN ROLLING/SLIDING BODIES

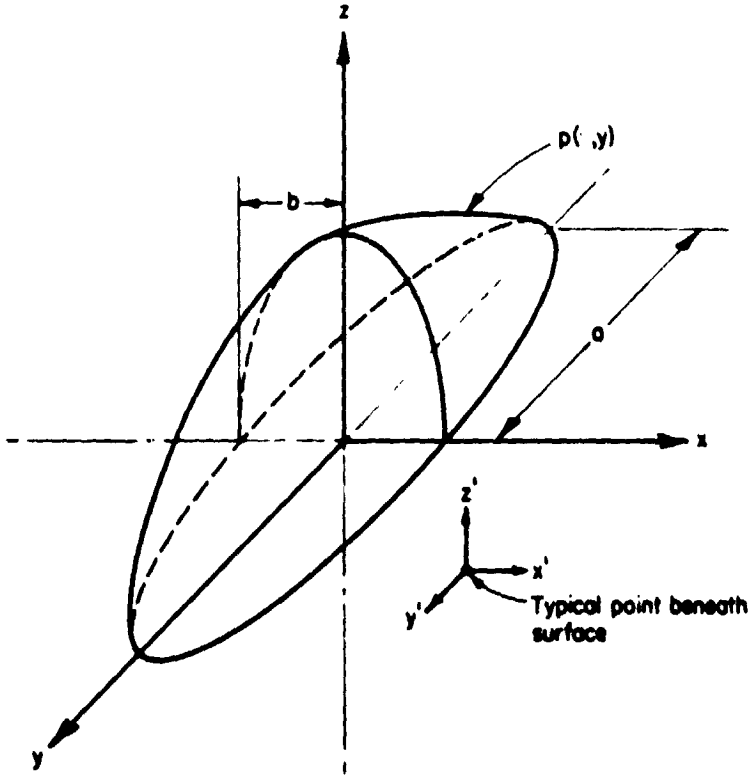

FIGURE 3. PRESSURE ELLIPSE

In tensor notation, the stress at any point can be written as

$$\sigma_{ij} \quad \text{or} \quad \bar{\sigma}_{ij} \quad , \tag{1}$$

where $$\bar{\sigma}_{ij} = \sigma_{ij}/p_H \qquad (i, j = 1, 2, 3) \quad ,$$

where p_H is the maximum Hertz contact pressure. In more conventional terminology

$$\begin{aligned} &\sigma_x = \sigma_{11} \ , \ \sigma_y = \sigma_{22} \ , \ \sigma_z = \sigma_{33} \ , \\ &\tau_{xy} = \sigma_{12} \ , \ \tau_{xz} = \sigma_{13} \text{ and } \tau_{yz} = \sigma_{23} \ . \end{aligned} \tag{2}$$

It should be noted that the stress tensor is symmetric such that $\sigma_{ij} = \sigma_{ji}$.

To determine σ_{ij} requires that expressions be developed for the effect of point loads on stresses and that the point loads be integrated over the conditions of interest. These equations are discussed in Appendix A and Equation A-27 shows the general relationship for $\bar{\sigma}_{ij}$ of interest here.

Principal Stresses

From the preceding discussions, it can be noted that the state of stress at any point has six components*. The components discussed in the preceding section describe the stress when the axis is oriented in the x, y, z system. However, the stress can look considerably different at different orientations. Since the objective of the study is to evaluate "worse case" shear stresses, it is necessary to evaluate the effect of axis rotation (see Figure 4) on stress. The first step in this phase of the study is to determine the parameters that are not a function of rotation; these are known as the invariants of the stress tensor[8]. These invariants are:

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*Actually nine components are required, but some are equal because of symmetry.

$$I_r = \sigma_{11} + \sigma_{22} + \sigma_{33} \quad , \tag{3}$$

$$II_r = \sigma_{11}\,\sigma_{22} + \sigma_{11}\,\sigma_{33} + \sigma_{22}\,\sigma_{33} - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2 \quad , \text{ and} \tag{4}$$

$$III_r = \sigma_{11}\,\sigma_{22}\,\sigma_{33} + 2\sigma_{12}\,\sigma_{23}\,\sigma_{13} - \sigma_{11}\,\sigma_{23}^2 - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2 \quad . \tag{5}$$

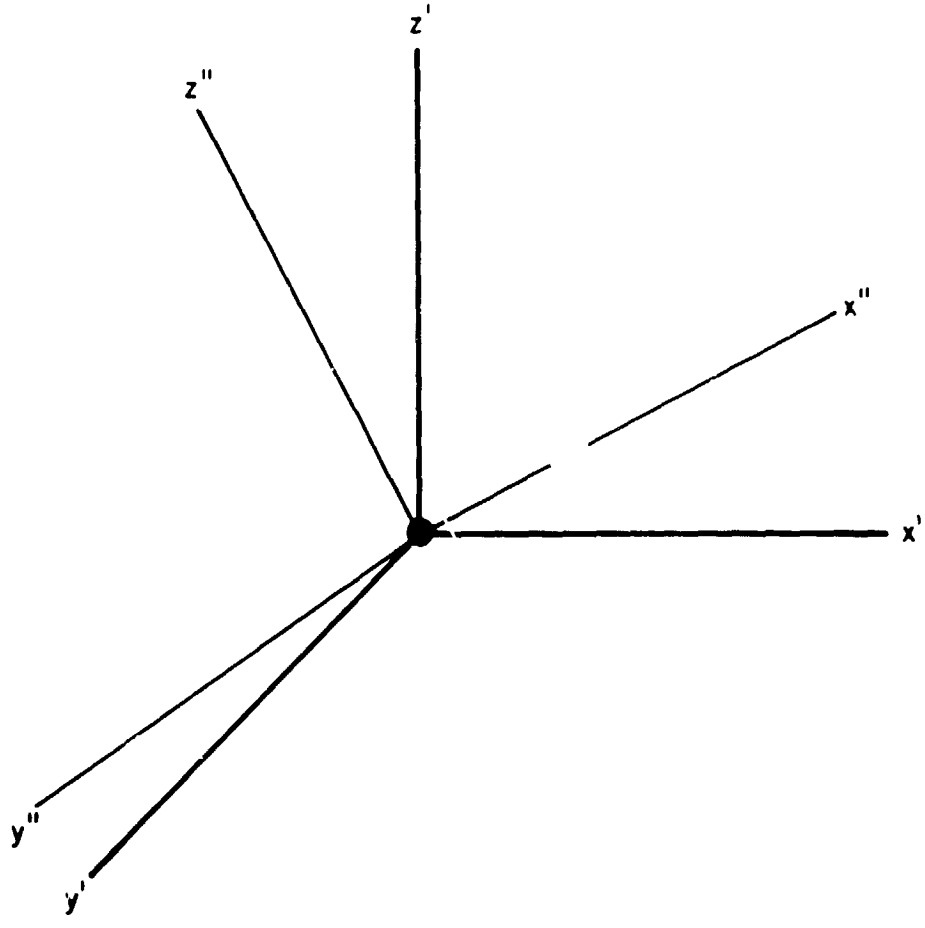

FIGURE 4. ILLUSTRATION OF AXIS ROTATION

With the knowledge of the invariants, it is possible to determine the plane of the principal stress (i.e., the rotated axis on which no shear forces exist) and the stress magnitudes. The equations for the principal stresses can be written

$$S^3 + I_r S^2 + II_r S + III_r = 0 \quad . \tag{6}$$

Equation 6 can be readily solved by cubic equation solutions and will produce three real roots for principal stresses. These roots will be known as S_1, S_2, and S_3 with S_1 being the largest and S_3 the smallest.

Angles To The Principal Stresses

In tensor computations, the angles are given in terms of direction cosines to the various axes. If we define a_{nx}, a_{ny}, and a_{nz} as the direction cosines from a given principal stress (say S_1) to the x, y, and z axis, then it can be shown that the following equations must be satisfied[8]:

$$a_{nx1}^2 + a_{ny1}^2 + a_{nx1}^2 = 1 \quad , \tag{7}$$

$$(\sigma_{11} - S_1)\, a_{nx1} + \sigma_{12}\, a_{ny1} + \sigma_{13}\, a_{nz1} = 0 \quad , \tag{8}$$

$$\sigma_{12} a_{nx1} + (\sigma_{22} - S_1)\, a_{ny1} + \sigma_{23}\, a_{nz1} = 0 \quad , \text{ and} \tag{9}$$

$$\sigma_{13}\, a_{nx1} + \sigma_{23}\, a_{ny1} + (\sigma_{33} - S_1)\, a_{nz1} = 0 \quad . \tag{10}$$

By solving Equation 7 and any two of Equations 8 through 10 simultaneously, a_{nx1}, a_{ny1} and a_{nz1} can be determined. In a like manner, the direction cosines for locating the S_2 (a_{nx2}, a_{nz2}, a_{nz2}) and the S_3 (a_{nx3}, a_{ny3}, a_{nz3}) stress can be determined.

Other Planes of Interest

The solution of equation 7 through 10 permit the determination of the location of principal stresses. However, the objective of the project is to evaluate shear stresses and by definition there are no shear stresses on these planes. Therefore, other planes must be considered.

Plane of Maximum Shear Stress

The maximum shear stress for an element lies on a plane bisecting the $S_1 - S_3$ plane. The direction cosines for the normal to this plane relative to the plane of the principal stress axes can be written

$$b_{n1} = 1/\sqrt{2} \quad , \tag{11}$$

$$b_{n2} = 0. \quad , \tag{12}$$

$$b_{n3} = 1/\sqrt{2} \quad . \tag{13}$$

To relate these cosines back to the x, y, z axis requires an angle transformation of the form

$$c_{nx} = b_{n1}\, a_{nx1} + b_{n2}\, a_{nx2} + b_{n3}\, a_{nx3} \quad , \tag{14}$$

$$c_{ny} = b_{n1}\, a_{ny1} + b_{n2}\, a_{ny2} + b_{n3}\, a_{ny3} \quad , \tag{15}$$

$$c_{nz} = b_{n1}\, a_{nz1} + b_{n2}\, a_{nz2} + b_{n3}\, a_{nz3} \quad , \tag{16}$$

where c_{n1}, c_{n2} and c_{n3} are the direction cosines from the normal (to the shear plane) to the x, y, z axes.

Plane of Octahedral Stress

The location of the normal to the octahedral stress can be written

$$b_{n1} = 1/\sqrt{3} \quad , \tag{17}$$

$$b_{n2} = 1/\sqrt{3} \quad , \tag{18}$$

$$b_{n3} = 1/\sqrt{3} \quad . \tag{19}$$

That is, the normal is situated at equal angles from the principal axis. Likewise, Equations 17 through 19 can be used in conjunction with Equations 14 through 16 to relate the normal to the x, y, z axes.

Seeking The Maximum and Minimum Shear Stresses

Using the angle transformation equations discussed in the previous section, it is possible to locate the normals to various critical shear stress planes. The final step in this process is to locate maximum and minimum shear stress on those particular planes for various x positions. Let us define the direction cosines to a given location on a shear plane as a_{sx}, a_{sy} and a_{sz}. It can be shown that(8)

$$a_{sx}\, c_{nx} + a_{sy}\, c_{ny} + a_{sz}\, c_{nz} = 0 \quad \text{and} \tag{20}$$

$$a_{sx}^2 + a_{sy}^2 + a_{sz}^2 = 1 \quad . \tag{21}$$

If, then, we choose a particular angle, say a_{sx}, we can compute a_{sy} and a_{sz} by Equations 20 and 21.

With a knowledge of the angles between the shear vector on a plane and the x, y, z axes and a knowledge of the stress tensor as oriented to the x, y, z axes, the shear stress can be computed. The equation for the stress is given by(8)

$$\begin{aligned} \tau = {} & c_{nx}\, a_{sx}\, \sigma_{11} + c_{ny}\, a_{sy}\, \sigma_{22} + c_{nz}\, a_{sz}\, \sigma \quad , \\ & + c_{nx}\, a_{sy}\, \sigma_{12} + c_{ny}\, a_{sx}\, \sigma_{12} + c_{nx}\, a_{sz}\, \sigma_{13} \quad , \\ & + c_{nz}\, a_{sx}\, \sigma_{13} + c_{ny}\, a_{sz}\, \sigma_{23} + c_{nz}\, a_{sy}\, \sigma_{23} \quad . \end{aligned} \tag{22}$$

In the computer model, values of a_{sx} were assumed in the range -1 to +1. For each value of a_{sx}, the shear stress was computed at various x positions and both the maximum and minimum values were stored. In the computation, the maximum shear stress and the maximum reversal $\tau_{max} - \tau_{min}$ were sought.

Construction Of Model

The computer model is constructed as follows:

1. Input values of κ (b/a), ν (Poissons ratio), z/b (depth) f_T (tangential friction coefficient), f_A (axial friction coefficient), and case number.
2. Compute the stress tensor for approximately eleven (11) x locations (STRSSC). For K = 0 (line contact), the Smith-Liu[5] solutions are used.
3. Compute the principal stresses at each x location (PRINC).
4. Compute the angles to the principal stresses at each location (ANGLP).
5. Compute the angle to the normals of the maximum shear plane, the octahedral shear plane, and the orthogonal plane (ANGTRAN).
6. Compute the shear stresses and shear reversals on the shear planes (RVSTRS).

All computations are in the form σ_{ij}/p_H and are applicable for any (elastic) contact pressure level. A source listing for the model is given in Appendix B and a typical printout is given in Table 2.

In this table

K = b/a (aspect ratio)

NU = (Poissons ratio)

Z = z/b (depth into surface)

F_T = f_T (tangential friction coefficient)

F_A = f_A (axial friction coefficient)

S_{XX}, S_{YY}, S_{ZZ} are the normal stresses (σ_x, σ_y, σ_z)

S_{XY}, S_{YZ}, S_{XZ} are the shear stress (τ_{xy}, τ_{yz}, τ_{xz})

S_1, S_2, S_3 are the principal stresses

X - pos = x/b is the x position being evaluated

TAU-MAX = maximum shearing stress as computed by scanning different planes

AN (1), AN (2), AN (3) are the direction cosines to the normals on the planes containing TAU-MAX and, AS (1), AS (2), AS (3) are the direction cosines to TAU-MAX on the shear plane.

TABLE 2. TYPICAL PRINTOUT FOR BATTELLE STRESS MODEL

CASE NUMBER= 100

K= .80000E+00 NU= .28500E+00 Z= .10000E+00 FT= .10000E+00 FA= 0.

IX	SXX	SYY	SZZ	SXY	SYZ	SXZ	S1	S2	S3	X-POS
1	-.59283E-01	-.17261E+00	-.13866E+00	.35355E-08	.35355E-08	.11783E+00	.25361E-01	-.17262E+00	-.22331E+00	-.10000E+01
2	-.11560E+00	-.26274E+00	-.39002E+00	.3521 -08	.35213E-08	.11627E+00	-.72964E-01	-.26274E+00	-.43265E+00	-.90000E+00
3	-.21121E+00	-.34511E+00	-.56935E+00	.35295E-08	.35295E-08	.77424E-01	-.19519E+00	-.34511E+00	-.58538E+00	-.80000E+00
4	-.45779E+00	-.51183E+00	-.85826E+00	.35244E-08	.35244E-08	-.10770E-01	-.45750E+00	-.51183E+00	-.85855E+00	-.50000E+00
5	-.56596E+00	-.57347E+00	-.95228E+00	.35446E-08	.35446E-08	-.45295E-01	-.56072E+00	-.57347E+00	-.95752E+00	-.30000E+00
6	-.65376E+00	-.61313E+00	-.10029E+01	.35196E-08	.35196E-08	-.81005E-01	-.61313E+00	-.63588E+00	-.10208E+01	0.
7	-.65600E+00	-.59489E+00	-.95857E+00	.35446E-08	.35446E-08	-.10808E+00	-.59489E+00	-.62136E+00	-.99321E+00	.30000E+00
8	-.60578E+00	-.54672E+00	-.86971E+00	.35244E-08	.35244E-08	-.12563E+00	-.54672E+00	-.55554E+00	-.91995E+00	.50000E+00
9	-.42607E+00	-.39441E+00	-.59386E+00	.35295E-08	.35295E-08	-.16825E+00	-.32196E+00	-.39441E+00	-.69797E+00	.80000E+00
10	-.33809E+00	-.31191E+00	-.41976E+00	.35213E-08	.35213E-08	-.18799E+00	-.18655E+00	-.31191E+00	-.57129E+00	.90000E+00
11	-.26966E+00	-.21612E+00	-.16702E+00	.35355E-08	.35355E-08	-.17406E+00	-.36877E-01	-.21612E+00	-.39981E+00	.10000E+01

FATIGUE SHEAR STRESS SUMMARY

PLANE OF MAX SHEAR STRESS

TAU MAX= .20372E+00 AN(1)= -.21583E+00 AN(2)= .70711E+00 AN(3)= -.67337E+00 X-REF= .30000E+00
TAU REVERSING= .32192E+00 AS(1)= -.10000E+00 AS(2)= -.70200E+00AS(3)= -.70512E+00

PLANE OF MAX OCT SHEAR STRESS

TAU MAX= .20014E+00 AN(1)= -.18715E+00 AN(2)= .57735E+00 AN(3)= .79476E+00 X-REF= .80000E+00
TAU REVERSING= .27615E+00 AS(1)= -.50000E+00 AS(2)= .64042E+00AS(3)= -.58297E+00

PLANE OF REVERSING SHEAR STRESS

TAU MAX= .18799E+00 AN(1)= 0. AN(2)= 0. AN(3)= .10000E+01 -REF=
TAU REVERSING= .30582E+00 AS(1)= -.10000E+01 AS(2)= 0. AS(3)= 0.

TAU-REVERSING is the maximum reversing shear stress on the planes specified.

ANALYSIS FOR RC FATIGUE TESTER

One objective of the project has been to evaluate the effect of magnitude and direction of slip on the stresses in a rolling contact (RC) fatigue tester. The RC tester consists of a rotating cylindrical specimen loaded between two large diameter crowned rollers. In the test, the specimen is driven by an electric motor; the crowned rollers are only driven by the specimen. The rollers are loaded to a level to produce fatigue in the specimen in a reasonable time period. Normally, the RC tester is used to evaluate rolling contact fatigue and has been used in bearing-material evaluations. The tester is also being considered as a candidate device for evaluating the effect of friction on fatigue.

Two approaches could be used to induce friction between the cylindrical specimen and the crowned rollers. One technique would be to induce a drag between the two elements by driving or braking the rollers. This approach could be quite complicated for simple fatigue testing and an alternate could be more desirable. This alternate approach could consist simply of skewing the specimen relative to the rollers. However, skewing produces an axial traction, whereas braking the rollers produces a tangential traction. The stress model can be used in determining the stress fields associated with these two types of fatigue test concepts.

Typical dimensions for an RC rig are

D_s = 9.5 mm (0.375 inch)

D_r = 190 mm (7.5 inches)

R_c = 6.4 mm (0.25 inch).

D_s is the specimen diameter, D_r is the roller diameter, and R_c is the crown radius. For these dimensions, a value of K = 0.8 has been computed. The upper limit traction coefficient should be nominally 0.1 for lubricated tractions.

Figure 5 shows the effect of traction on the reversing shear stresses in the RC tester. As can be observed, the maximum stress is unaffected by the

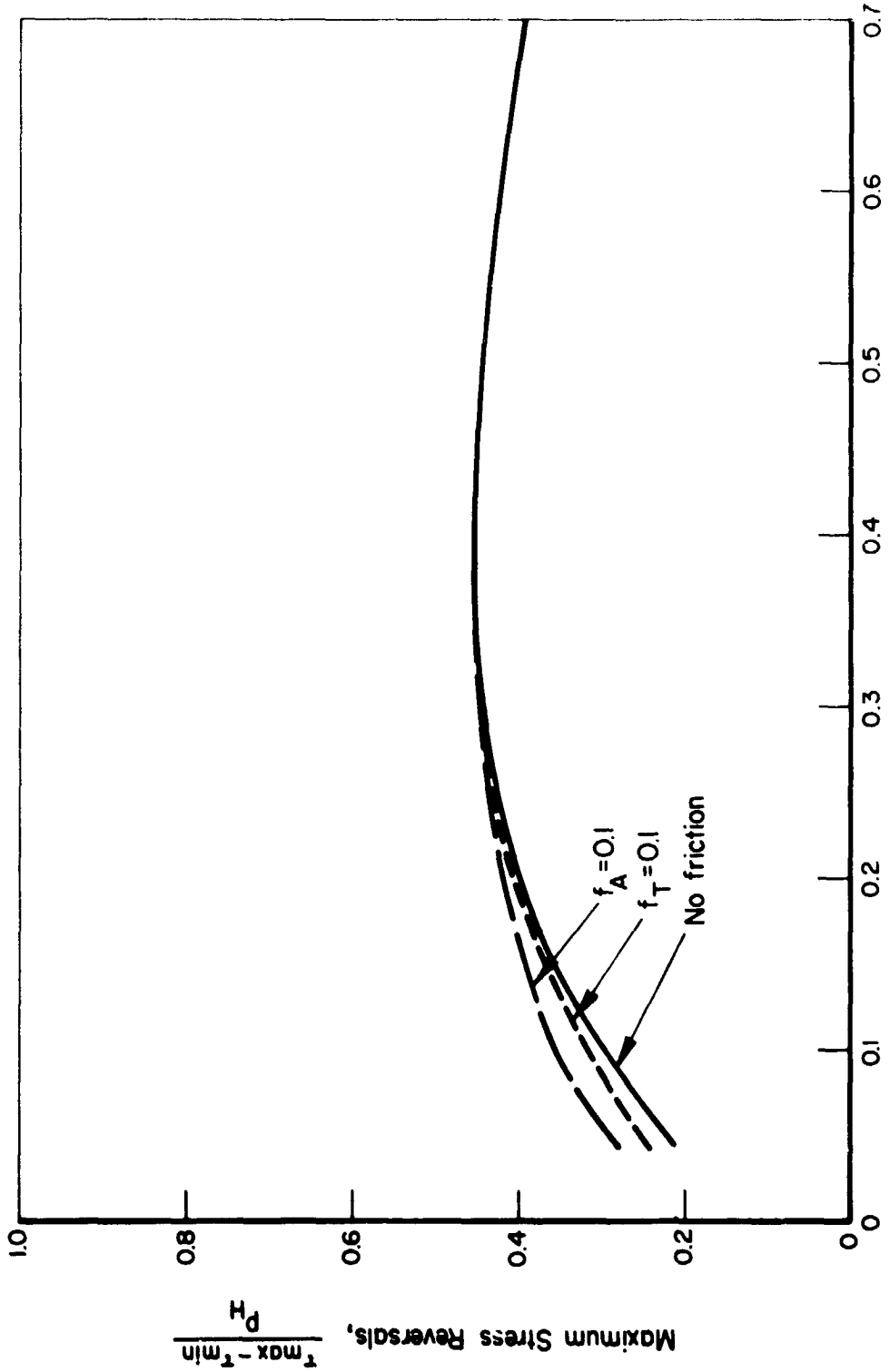

FIGURE 5. EFFECT OF FRICTION ON REVERSING SHEAR STRESS FOR RC TESTER
($K = .8$, $\nu = .285$, B-Model)

friction as is the depth to maximum stress ($z/b = 0.38$). There are some differences in the stresses near the surface, and it is impossible to be certain how these differences affect life. However, in the Lundberg-Palmgren theory, only the orthogonal shear reversal and its depth are important. If this theory is valid, it says that a traction coefficient of up to 0.1 should not alter life. There is, however, a difference in off-set between axial and tangential-friction, but this effect is not a part of fatigue theory. Fatigue tests are badly needed to clarify the effect of stress on life.

The general observation from this aspect of the analyses is that the direction of the traction is not significant with regards to fatigue type stress for low traction coefficients typical of lubricated contacts assuming Lundberg-Palmgren's theory. It should be noted that at least three stress related factors that have not been included in these stress analyses could have a profound effect on fatigue. These factors are:

(1) the effect of temperature (due to friction) on stress

(2) the effect of non-Hertzian pressure on stress

(3) the effect of asymmetric surface traction.

All of these can produce an asymmetry in stresses which can amplify the stress reversal. In fact, there will be a different stress state in the driver and driven for effects (1) and (3). The influence of the non-Hertzian pressure should be the same on both bodies provided that we do not invoke hysteresis effects in the material.

If we examine the influence of (1) and (3) on, say, the reversing orthogonal stress and assume that the traction is caused by the shearing of an elastic/plastic-like material, then the surface traction in the latter half of the contact is larger than in the front. Also the traction stresses will be negative in direction on the driver and positive on the driven. This means that the stress reversal in the driven is increased from the non-traction condition while it is decreased in the driver, possibly explaining some of the experimental findings reported elsewhere. The depth at which the maximum orthogonal acts is also reduced by an asymmetric traction stress. Traction produced by an elastic/plastic-like material may also explain why gears fail by pitting fatigue in the region directly below the pitch circle because the positive sliding conditions may produce a region of maximum surface traction

asymmetry and, hence, the largest magnitude of the reversing orthogonal shear stress. The thermal stresses produced by the sliding would also alter the stress profile for the orthogonal shear stress in a similar fashion but now because the driver (i.e., the faster moving body) would see a lower flash temperature than the driven. Thermal stresses would tend to increase the orthogonal shear stress in both bodies but more in the driven than the driver. Also besides the increases in the amplitude of the stress reversal, the depth at which this occurs is reduced, giving a further reduction in fatigue life on the driver. Further research is needed in this area to establish the influence of the effects just mentioned.

DISCUSSION OF EFFECTS OF FRICTION ON FATIGUE STRESSES

The subsurface stress model represents a useful tool for evaluating the role of friction on fatigue type stresses. The results of the evaluations of the stress analyses from the RC tester showed some interesting (although perhaps not profound) implications of the effect of lubricated traction on stresses. The purpose of this section is to further explore the effect of friction on stress.

Figures 6 through 8 show predictions of the reversing orthogonal shear stress for various conditions of friction and aspect ratio (b/a). This reversing stress is normally considered the significant stress for fatigue.[1] Note, however, in Figure 6, that friction does not alter the magnitude of the difference between maximum and minimum shear stress (i.e., stress reversals). This implies that the magnitude of the orthogonal stress reversals are not dependent on friction. Changing aspect ratio does alter the magnitude of the stress as shown in Figures 9 and 10, but this dependence of stress on aspect ratio is not affected by friction.

The effects of friction on the magnitude and depth of the maximum shear stress, are shown in Figures 11 and 12 for various values of aspect ratio. Some increase in stress accompanies an increase in friction. However, for $f_T = .1$ and $\nu = .285$ which are reasonable values for lubricated traction

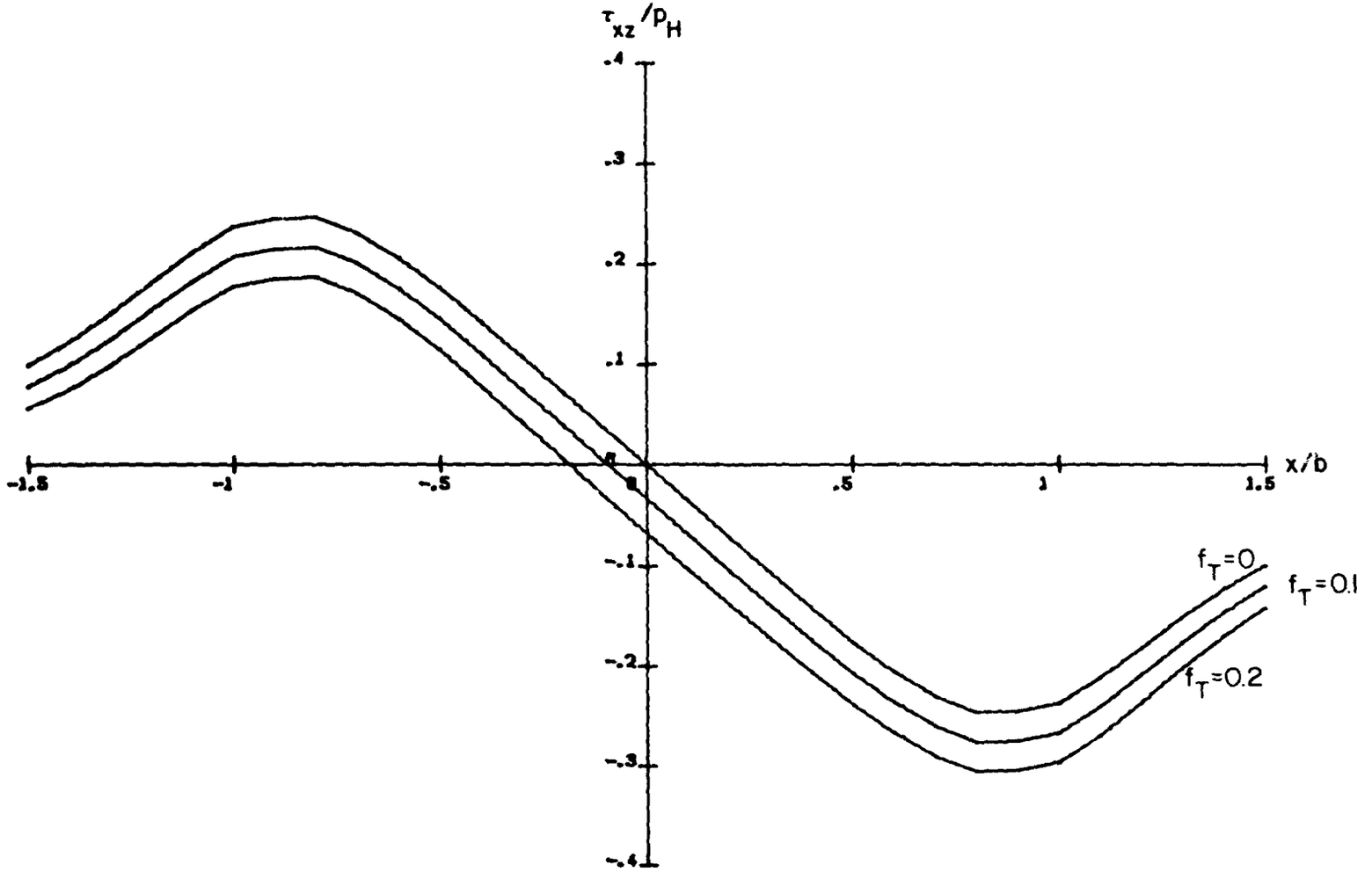

FIGURE 6. EFFECT OF FRICTION ON REVERSING ORTHOGONAL SHEAR STRESS
(K = 0, ν = .285, z/b = .5 T-Model)

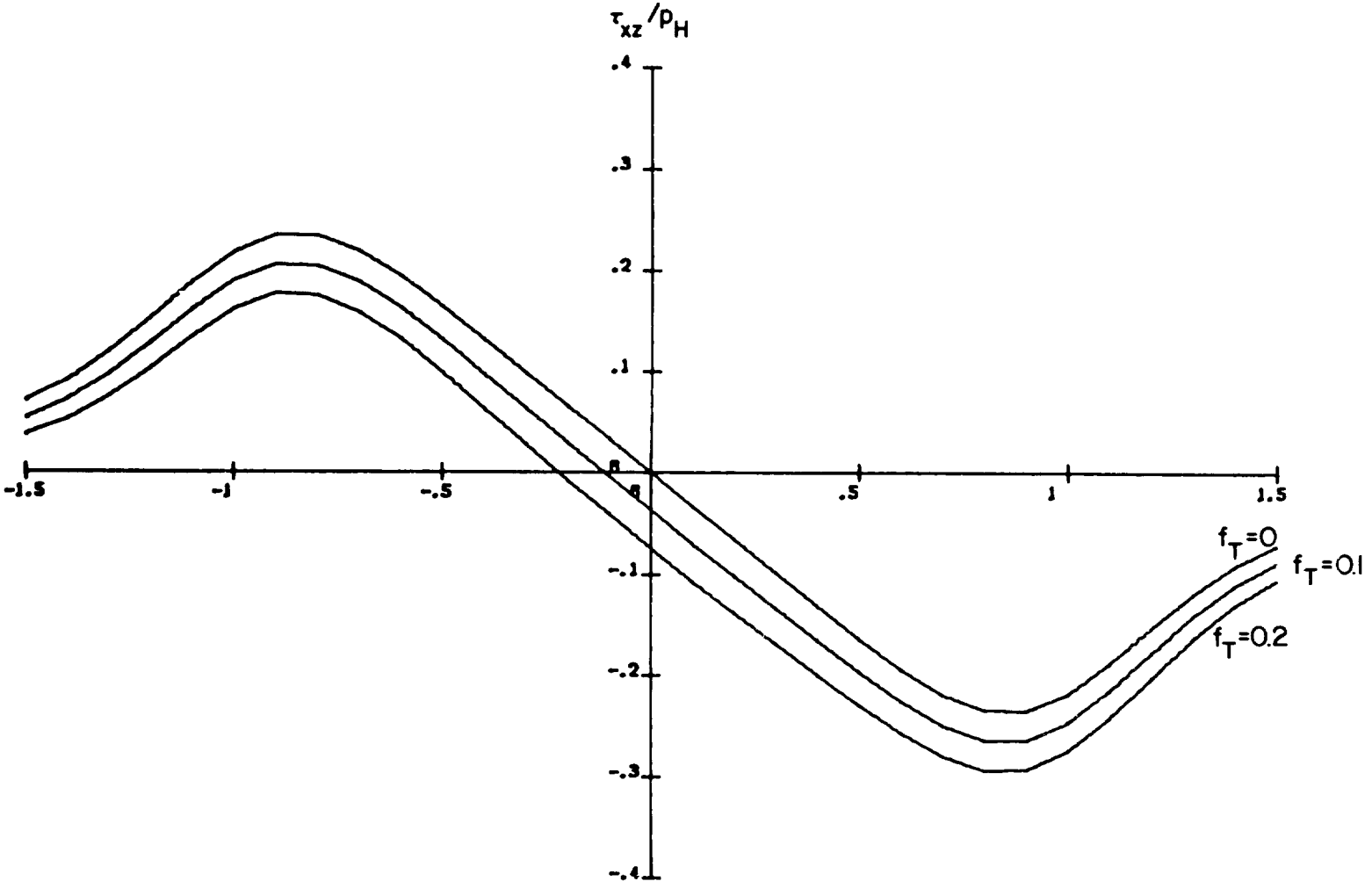

FIGURE 7. EFFECT OF FRICTION ON REVERSING ORTHOGONAL SHEAR STRESS
(K = .5, ν = .285, z/b = .435 T-Model)

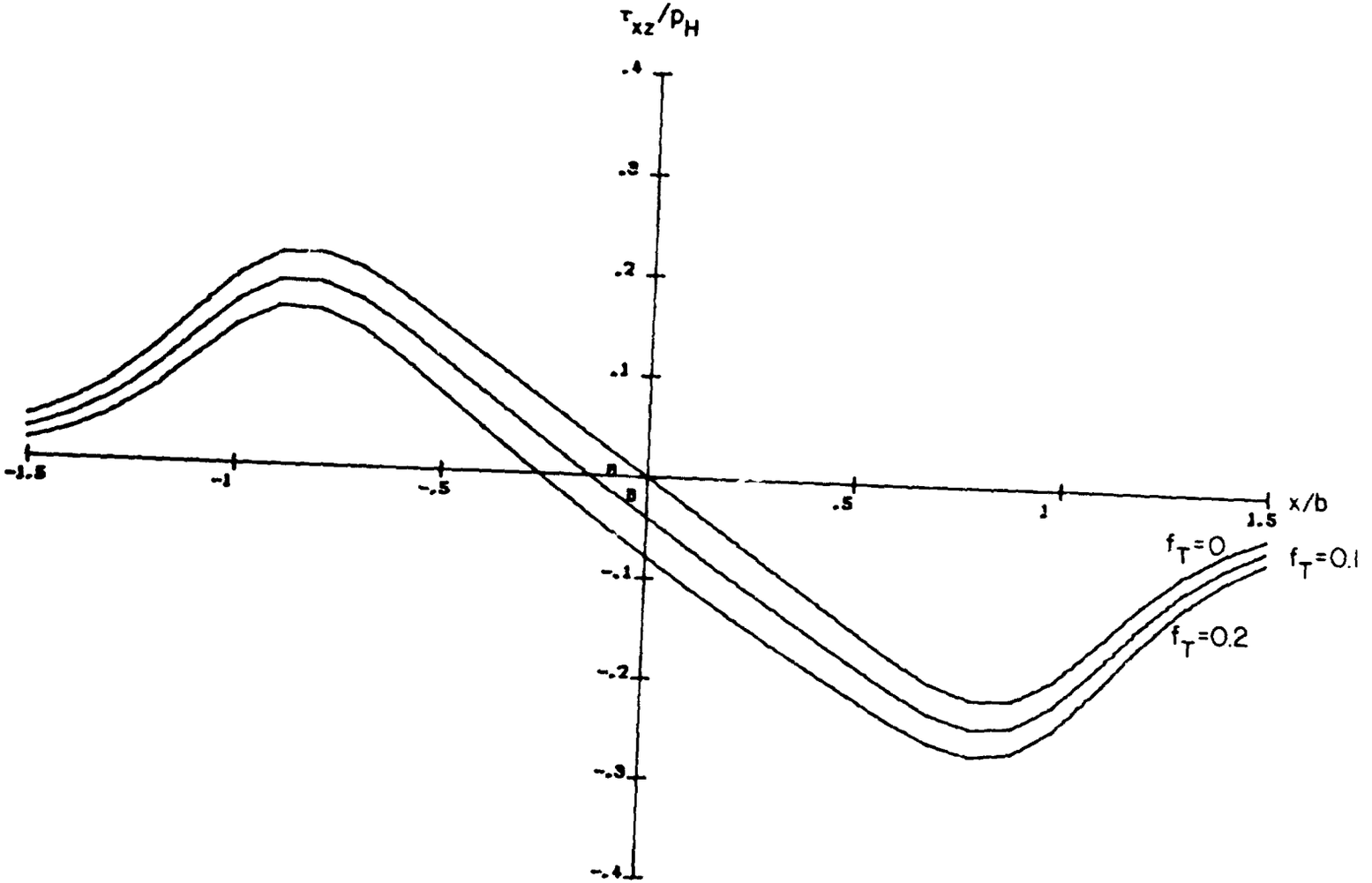

FIGURE 8. EFFECT OF FRICTION ON REVERSING ORTHOGONAL SHEAR STRESS
(K = 1, ν = .285, z/b = .35 T-Model)

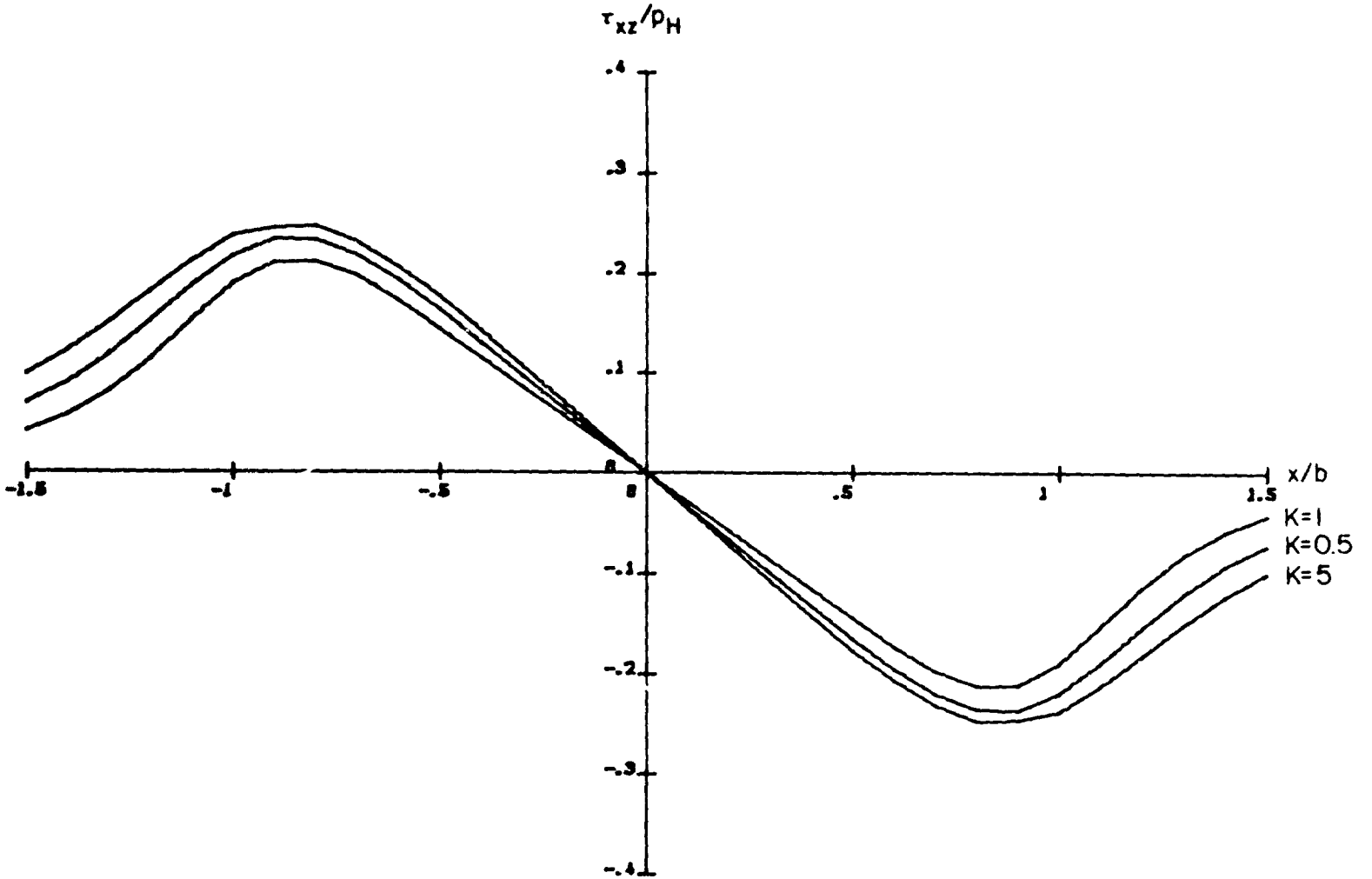

FIGURE 9. EFFECT OF ASPECT RATIO K ON REVERSING ORTHOGONAL SHEAR STRESS
($f_T = f_A = 0$, $\nu = .285$ T-Model)

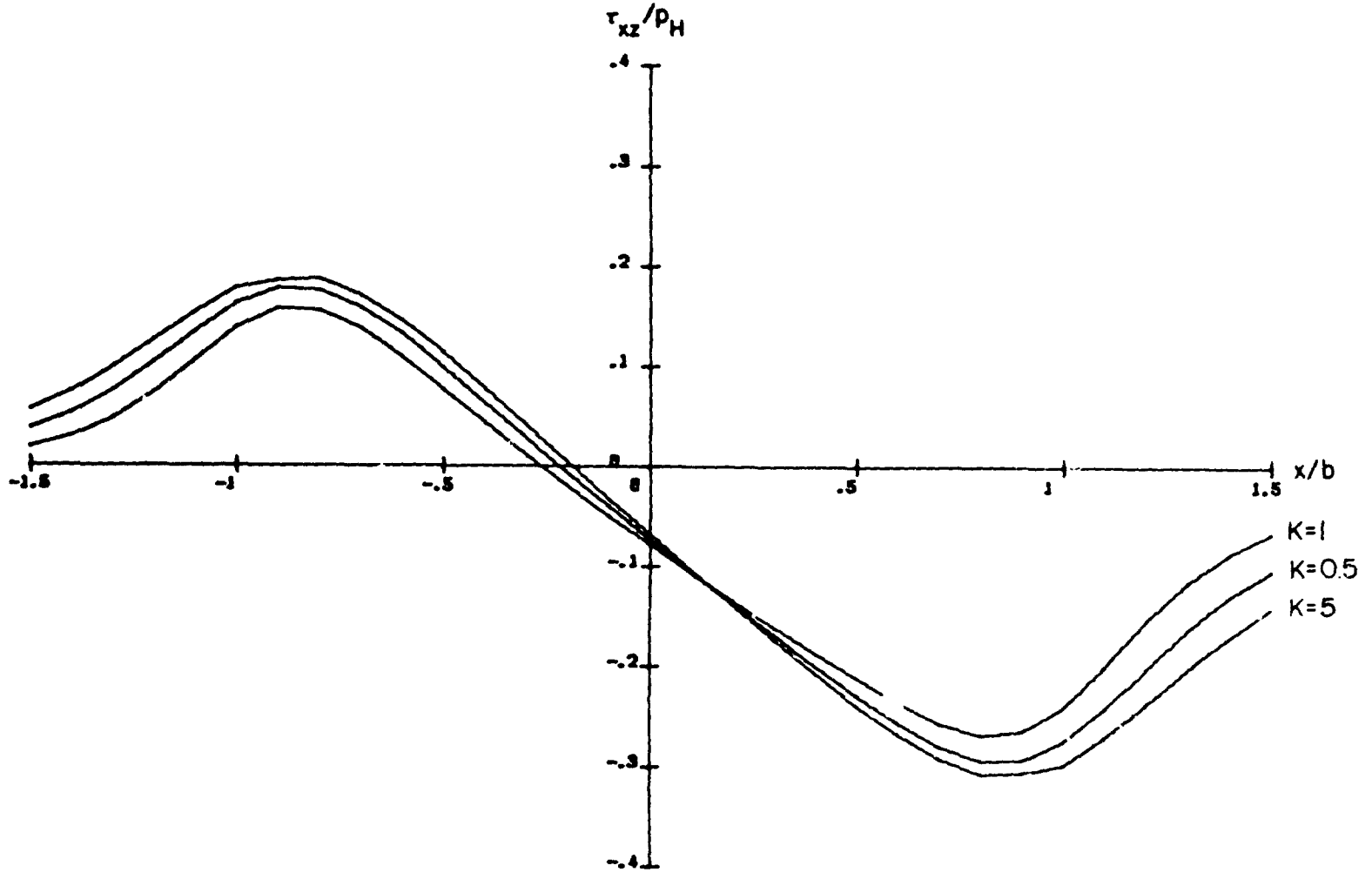

FIGURE 10. EFFECT OF ASPECT RATIO ON REVERSING ORTHOGONAL SHEAR STRESS
(f_r = .2, ν = .285 T-Model)

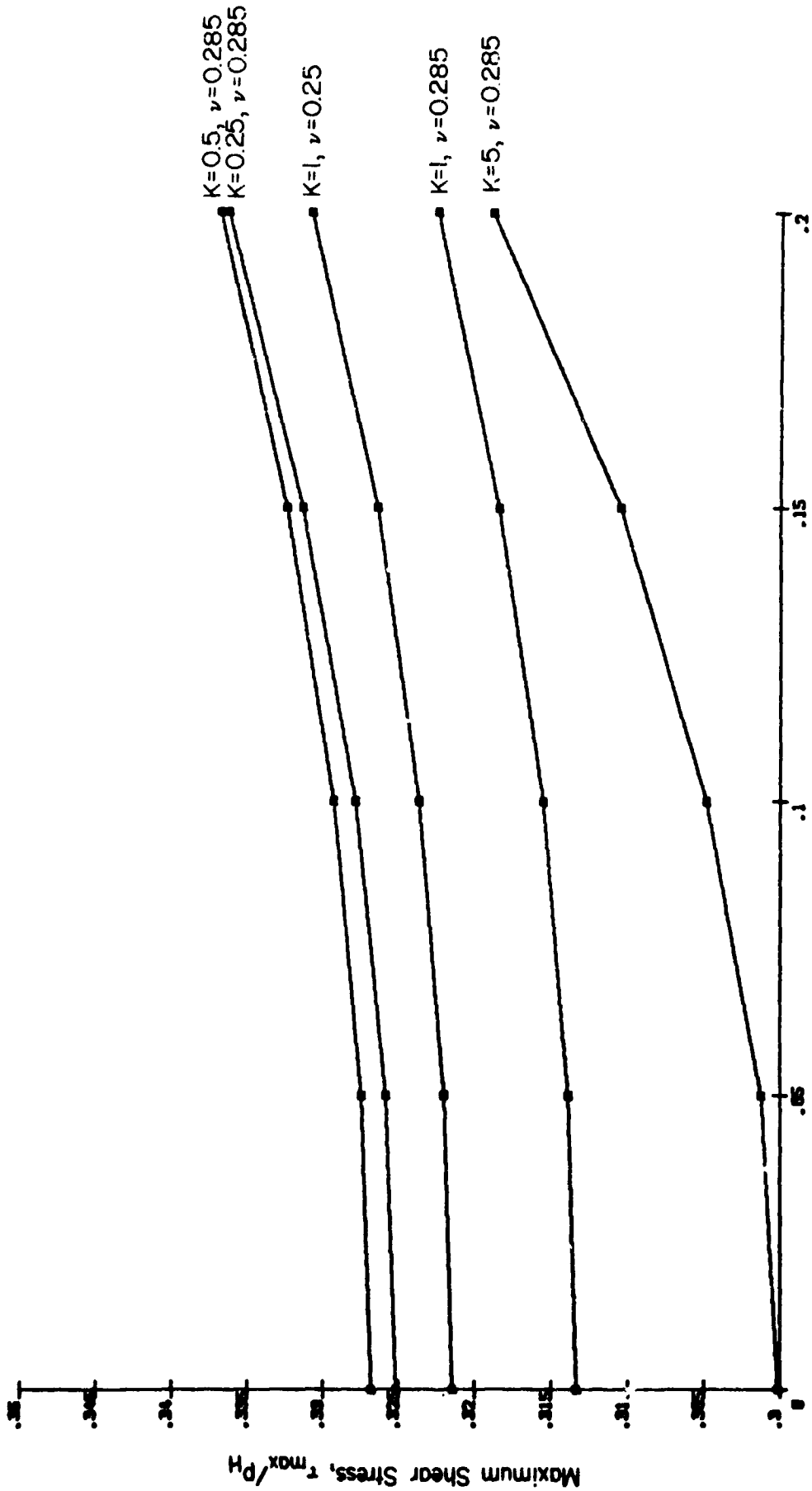

FIGURE 11. EFFECT OF FRICTION ON MAGNITUDE OF MAXIMUM SHEAR STRESS (T-Model)

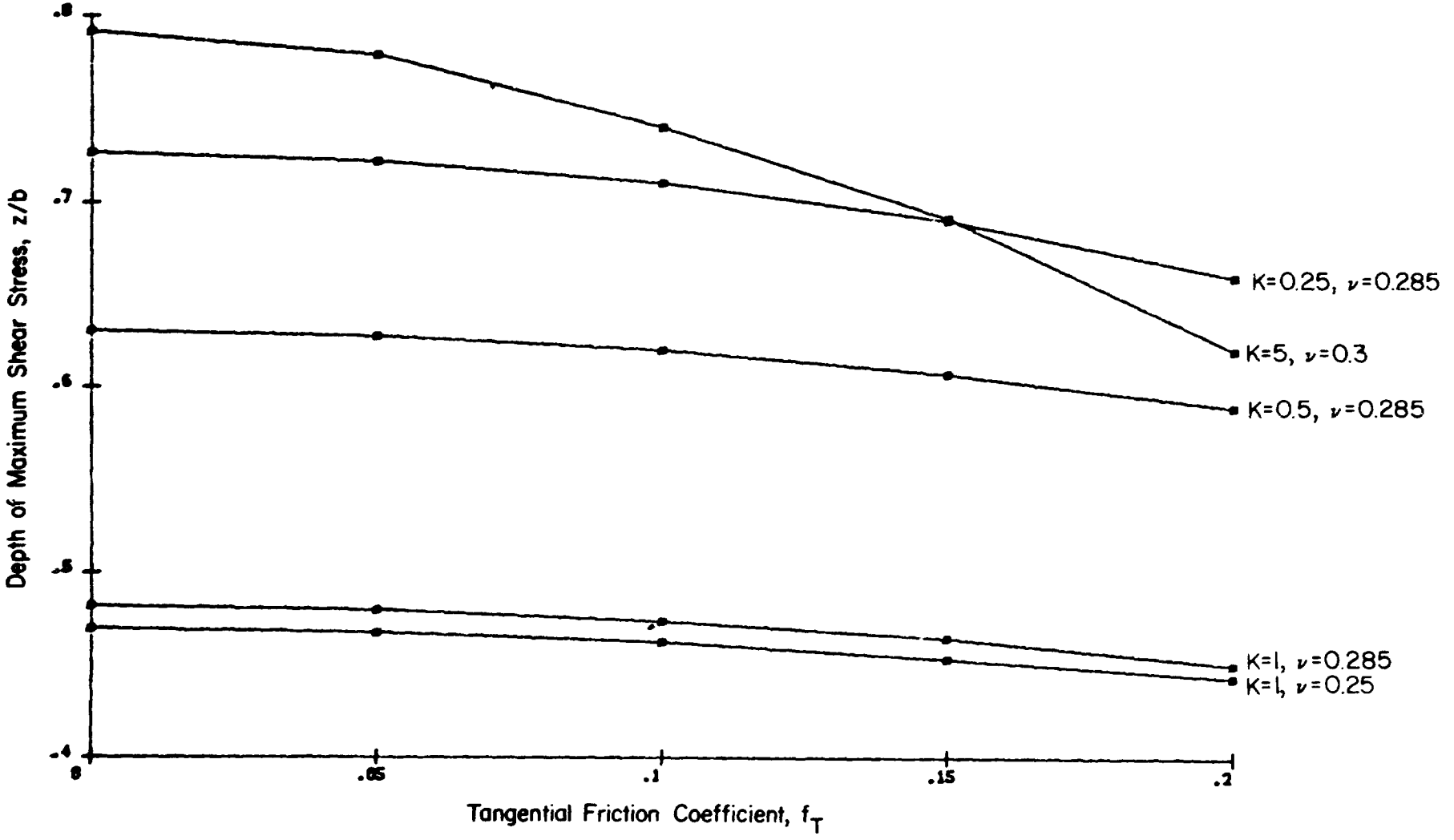

FIGURE 12. EFFECT OF FRICTION ON DEPTH OF MAXIMUM SHEAR STRESS (T-Model)

drives, only very small changes in stress depth and magnitude occur. Higher value of friction (f = .2) produces sizable increases in stress. The observations concerning shear stress also occur with regard to the octahedral shear stress as shown in Figures 13 and 14.

Figure 15 shows the effect of friction coefficient, f, on surface-shear-stresses-reversals for line (K = 0) contact. As can be observed, the magnitude of reversing stresses on the surface does not exceed the reversing orthogonal shear stress for values of $f < 0.25$. This implies that for this contact situation, fatigue-related stresses are not altered by friction for $f < 0.25$. Smith and Liu[5] showed sizable effects at f = 0.33, which would be consistent with our computations. However, values of f of this magnitude should occur only in very poor lubrication situations and are not relevant to lubricated traction drive theory. As can be observed, friction does materially alter the shear stresses near the surfaces, even for low (< 0.1) friction coefficients. This implies that friction alters the overall stress field and could alter the propensity of the system to incur a surface or near surface initiated fatigue failure. Fatigue tests in conjunction with the stress model are badly needed to further extend our knowledge of the role of low levels of friction on fatigue.

Figure 16 shows a shear stress plot of τ_{xy} versus τ_{zx} for the plane of maximum shear stress at the center of contact very near the surface. The magnitude of the stress reversals envelope ($\simeq .6$) is very consistent with the Smith-Liu[5] computations and further illustrates the effect of severe friction (f = .33) on fatigue inducing stresses. However, the effect of friction on these stresses is not a factor at lower friction.

In conclusion, it can be said that based on the computer models discussed herein, low friction coefficient ($f < .1$) do not alter the fatigue stresses. At higher friction levels, changes will occur, but these cannot explain the observed effect of friction on fatigue. Other factors such as non-Hertzian pressures, thermal stress, and nonsymmetric tractions should then be included to understand the roll of friction on fatigue stress.

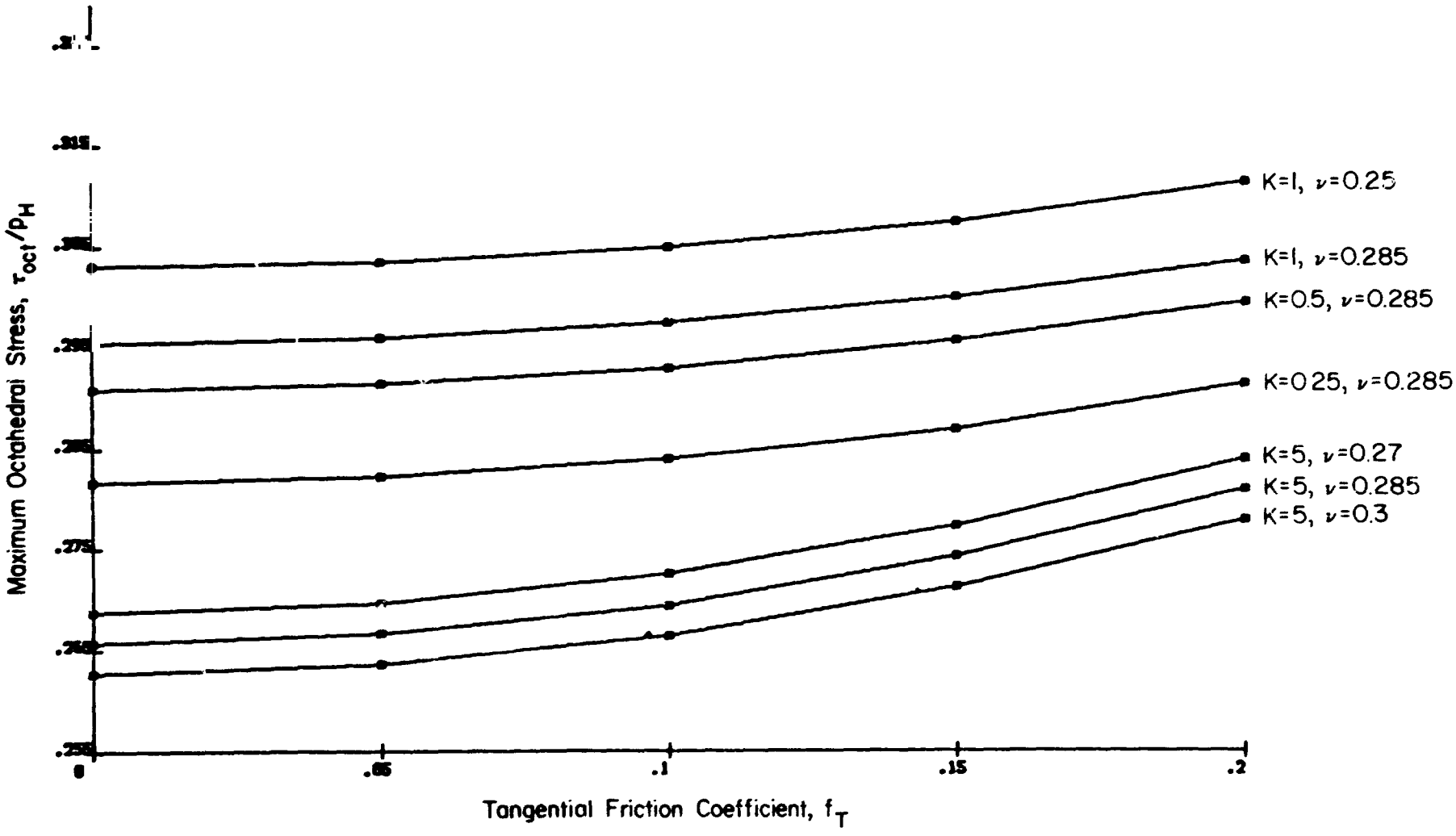

FIGURE 13. EFFECT OF FRICTION ON MAXIMUM OCTAHEDRAL SHEAR STRESS (T-Model)

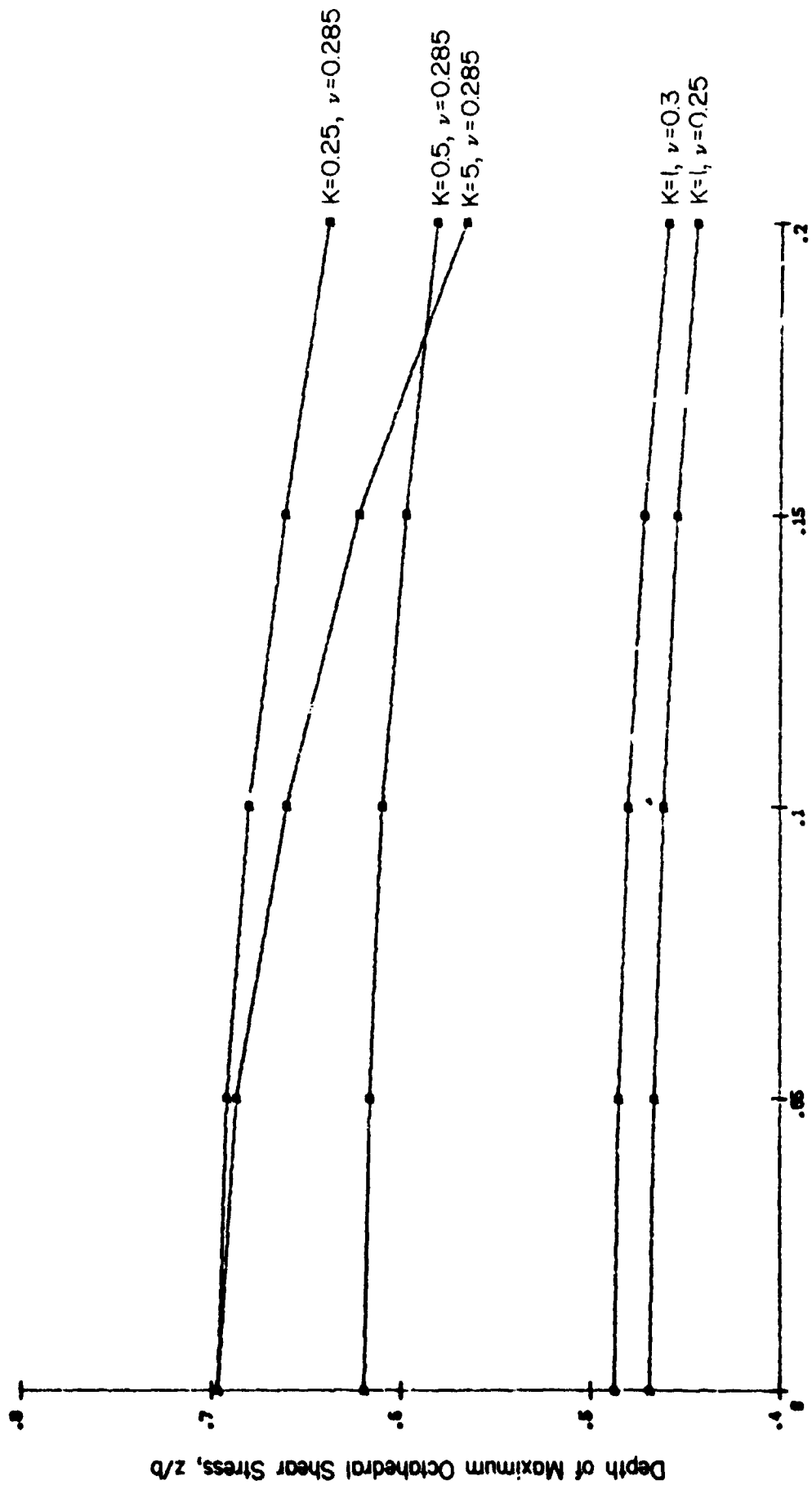

FIGURE 14. EFFECT OF FRICTION ON DEPTH OF MAXIMUM OCTAHEDRAL STRESS

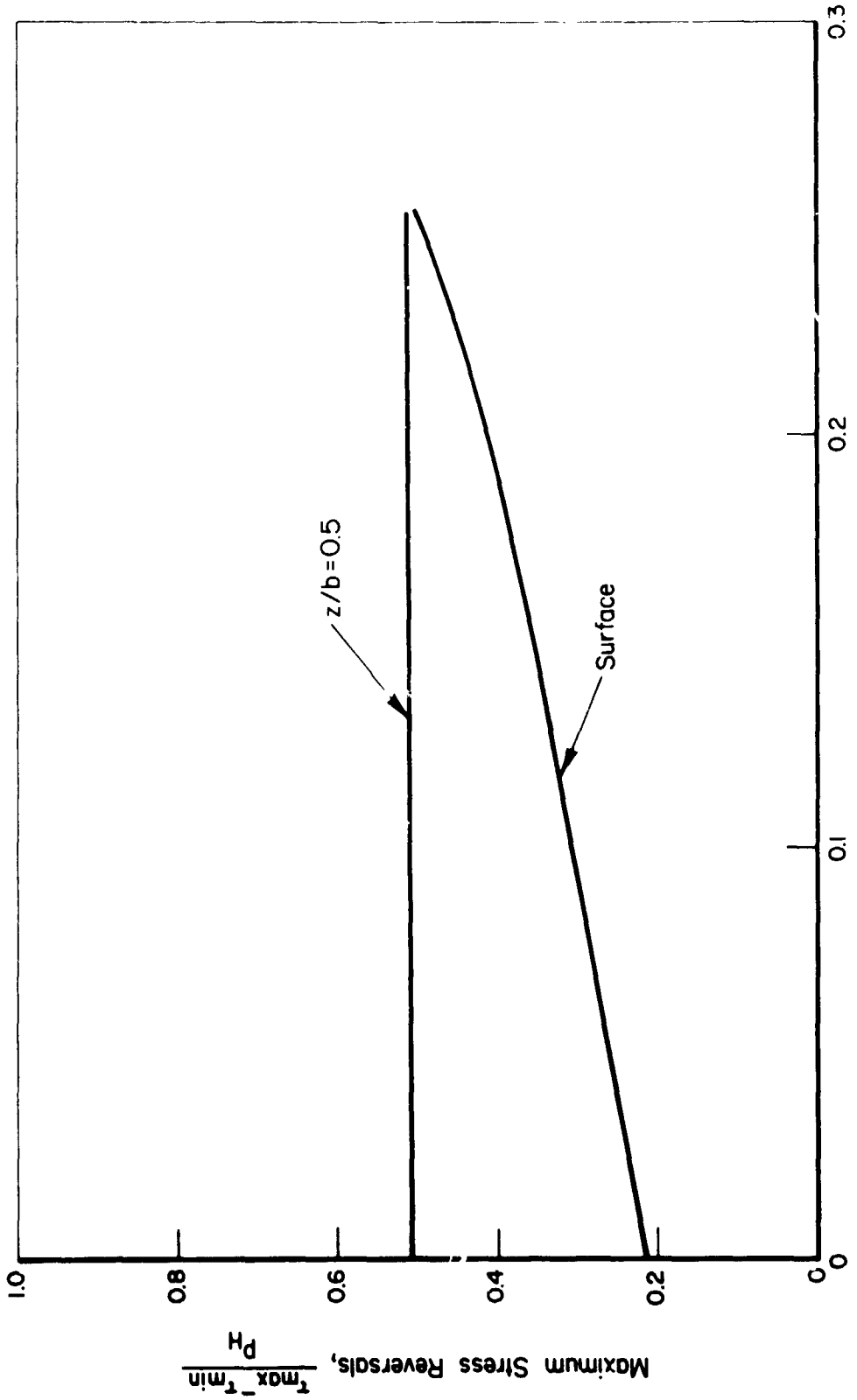

FIGURE 15. EFFECT OF FRICTION ON MAXIMUM STRESS REVERSALS ($K = 0$, $f_A = 0$, $\nu = .285$, B-Model)

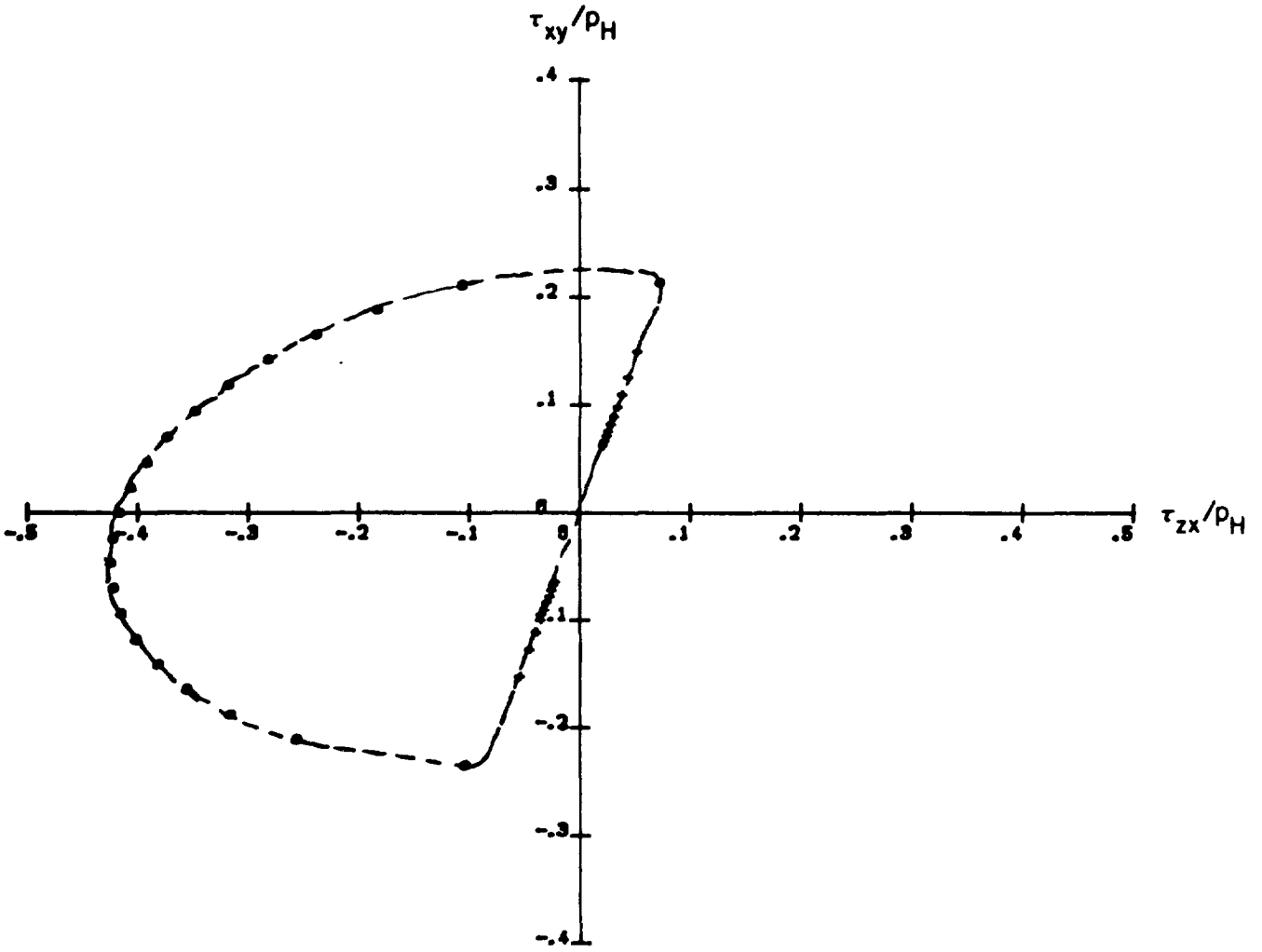

FIGURE 16. ILLUSTRATION OF VARIATION OF SHEAR STRESS ON PLANE OF MAXIMUM SHEARING STRESS
($z/b = 0$, $K = 0$, $\nu = .25$, $f_T = .333$ T-Model)

SUMMARY

The performance life of traction drives can be a very important factor in evaluating the overall effectiveness of the drive concept. Intuitively, this life should be heavily related to the state-of-stress in the elements as a result of high loads and tractions. The purpose of the project has been to develop a mathematical tool for evaluating this stress-state. Two independent, but similar, approaches (one at Battelle and one by Tevaarwerk) to developing the stress model have been pursued. Essentially, the models involve the computations of subsurface shear stresses (and stress reversals) as a result of normal and frictional loadings. Where possible the models have been checked against existing theory and against each other.

One use of the stress models has been to evaluate fatigue test methodology for a rolling contact (RC) fatigue tester. The standard RC tester consists of a small driven cylinder loaded between two larger crowned cylinders. The RC tester is a candidate device to evaluate the effect of traction on fatigue, either by driving the crowned rollers (tangential slip) or by simply skewing the rollers (axial slip); the skewing technique would be easier. The stress model was used to evalute the effect of lubricated type tractions on stresses for a typical RC configuration.

The general conclusion from the RC evaluations was that the fatigue inducing stresses (shear stress) reversals and depth of shear stress are not seriously altered by tractions (axial or tangential) which are typical of lubricated contacts. There are some changes in the stress field, but these are difficult to relate to fatigue. To further explore the effect of friction on stress, additional computer cases have been evaluated.

The general conclusions from additional computer analyses is that friction coefficients of the order of 0.25 are required to materially alter the state of fatigue inducing stresses; this is much larger than would occur in lubricated contact. This implies then, that other effects not incorporated in the current stress-models should be included. These effects include non-Hertzian stresses distributions and thermal stresses. Both of these effects can impose nonsymmetrical (about the center of contact) shear stresses and thus will produce shear stress reversals which will superimpose on the normal and friction stresses.

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APPENDIX A

COMPUTATIONS OF THE STRESS TENSOR

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COMPUTATIONS OF THE STRESS TENSOR

Stress Due to Point Loads

The state of stress at any point on or beneath the surface of a system under concentrated contact is quite complex. To define this stress state at each point requires the computations of nine components of the stress tensor as related to contact pressures and tractions. The effect of pressures and tractions on stress were developed by Mindlin[9] for point loadings on the surface of a semiinfinite body using the coordinate system of Figure A-1, and letting $R^2 = x^2 + y^2 + z^2$. These equations appear as follows.

Normal Load.

$$\sigma'_{xN} = \frac{P_N}{2\pi}\left\{\frac{(1-2\nu)z}{R^3} - \frac{3x^2 z}{R^5} - \frac{(1-2\nu)}{R(R+z)}\left[1 - \frac{x^2}{R(R+z)} - \frac{x^2}{R^2}\right]\right\} \quad ,(A\text{-}1)$$

$$\sigma'_{yN} = \frac{P_N}{2\pi}\left\{\frac{(1-2\nu)z}{R^3} - \frac{3y^2 z}{R^5} - \frac{(1-2\nu)}{R(R+z)}\left[1 - \frac{y^2}{R(R+z)} - \frac{y^2}{R^2}\right]\right\} \quad ,(A\text{-}2)$$

$$\sigma'_{z_N} = -\frac{3P_N z^3}{2\pi R^5} \quad , \qquad (A\text{-}3)$$

$$\tau'_{zy_N} = \tau'_{yz_N} = -\frac{3P_N y z^3}{2\pi R^5} \quad , \qquad (A\text{-}4)$$

$$\tau'_{xz_N} = \tau'_{zx_N} = -\frac{3}{2\pi}\frac{P_N\, x\, z^2}{R^5} \quad , \tag{A-5}$$

$$\tau'_{yx_N} = \tau'_{xy_N} = \frac{P_N\, xy}{2\pi}\left[-\frac{3z}{R^5} + \frac{(1-2\nu)(2R+z)}{R^3(R+z)^2}\right] \quad . \tag{A-6}$$

Tangential Load.

$$\sigma'_{x_T} = \frac{P_T\, x}{2\pi}\left[\frac{1-2\nu}{R^3} - \frac{3x^2}{R^5} - \frac{(1-2\nu)}{R(R+z)^2}\left(3 - \frac{x^2(3R+z)}{R^2(R+z)}\right)\right] \quad , \tag{A-7}$$

$$\sigma'_{y_T} = \frac{P_T\, x}{2\pi}\left[\frac{1-2\nu}{R^3} - \frac{3y^2}{R^5} - \frac{(1-2\nu)}{R(R+z)^2}\left(1 - \frac{y^2(3R+z)}{R^2(R+z)}\right)\right] \quad , \tag{A-8}$$

$$\sigma'_{z_T} = -\frac{3P_T}{2\pi}\frac{x\, z^2}{R^5} \quad , \tag{A-9}$$

$$\tau'_{zy_T} = \tau_{yz_T} = -\frac{3P_T\, xyz}{2\pi\, R^5} \quad , \tag{A-10}$$

$$\tau'_{xz_T} = \tau'_{zx_T} = -\frac{3P_T\, x^2 z}{2\pi\, R^5} \quad , \tag{A-11}$$

$$\tau'_{yx_T} = \tau'_{xy_T} = \frac{P_T\, y}{2\pi}\left[-\frac{3x^2}{R^5} - \frac{(1-2\nu)}{R(R+z)^2}\left(1 - \frac{x^2(3R+z)}{R^2(R+z)}\right)\right] \quad . \tag{A-12}$$

Axial Load.

$$\sigma'_{x_A} = \frac{P_A\, y}{2\pi}\left[\frac{1-2\nu}{R^3} - \frac{3x^2}{R^5} - \frac{(1-2\nu)}{R(R+z)^2}\left(1 - \frac{x^2(3R+z)}{R^2(R+z)}\right)\right] \quad , \tag{A-13}$$

$$\sigma'_{y_A} = \frac{P_A\, y}{2\pi}\left\{\frac{1-2\nu}{R^3} - \frac{3y^2}{R^5} - \frac{(1-2\nu)}{R(R+z)^2}\left[3 - \frac{y^2(3R+z)}{R^2(R+z)}\right]\right\}, \qquad \text{(A-14)}$$

$$\sigma'_{z_A} = -\frac{3\,P_A\, y\, z^2}{2\pi\, R^5}, \qquad \text{(A-15)}$$

$$\tau'_{zx_A} = \tau'_{xz_A} = -\frac{3P_A\, xyz}{2\pi\, R^5}, \qquad \text{(A-16)}$$

$$\tau'_{zy_A} = \tau'_{yz_A} = -\frac{3P_A\, y^2 z}{2\pi\, R^5}, \qquad \text{(A-17)}$$

$$\tau'_{xy_A} = \tau'_{yx_A} = \frac{P_A\, x}{2\pi}\left\{-\frac{3y^2}{R^5} - \frac{(1-2\nu)}{R(R+z)^2}\left[1 - \frac{y^2(3R+z)}{R^2(R+z)}\right]\right\}. \qquad \text{(A-18)}$$

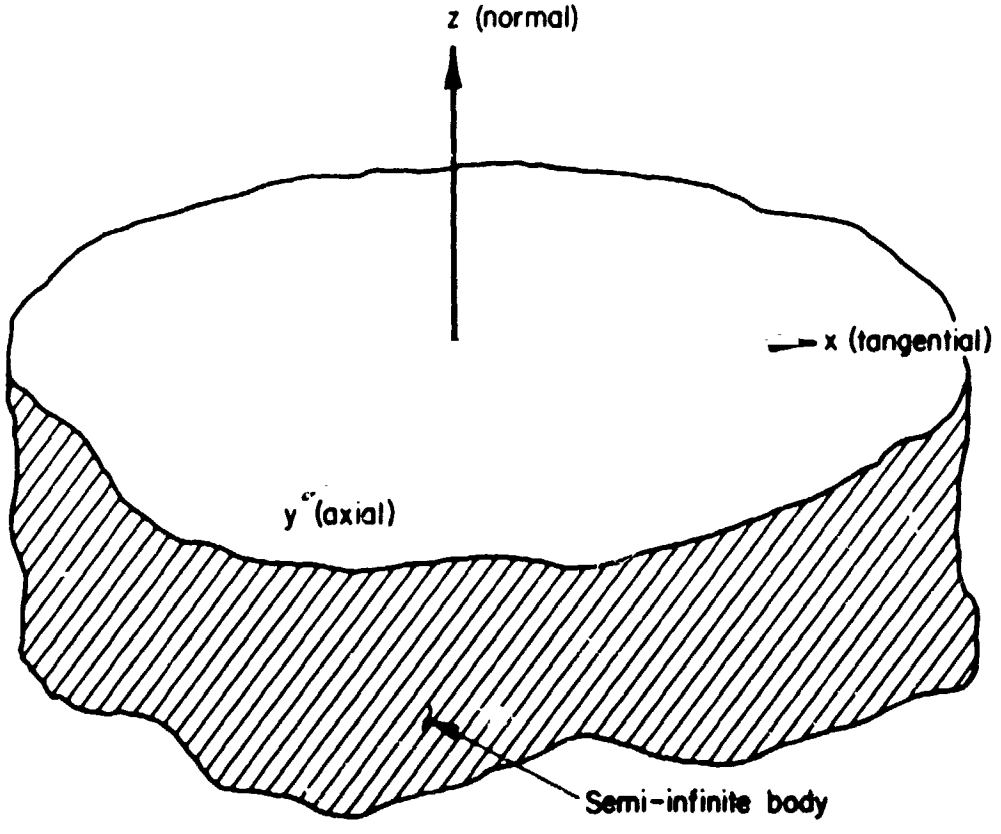

FIGURE A-1. COORDINATE SYSTEM FOR STRESS COMPUTATIONS

Hertzian Pressures and Surface Tractions

For a point on the surface, the normal load could be written

$$P_N = p dx dy \quad \text{where p is the local pressure} \tag{A-19}$$

Also, $$P_T = p f_T dx dy \quad \text{and} \tag{A-20}$$

$$P_A = p f_A dx dy \tag{A-21}$$

where f_T and f_A are the effective coefficients of friction acting tangential or axially to the surface.

If the surface were subjected to a Hertzian pressure distribution, the pressure could be expressed (see Figure A-2):

$$p = p_H \sqrt{1 - \frac{(x - x_1)^2}{b^2} - \frac{(y - y_1)^2}{a^2}} , \tag{A-22}$$

where p_H is the maximum Hertz pressure, a and b are the major and minor axes of the contact ellipse, x and y are the coordinates with origins at the center of the ellipse and x_1 and y_1 are locations of the point load relative to the center of the Hertz ellipse. The Battelle analyses were restricted to the case where $y_1 = 0$. Tevaarwerk allowed y_1 to be a variable and found $y_1 = 0$ to be the interesting region.

The general approach for determining the nine components of the stress tensor at any point can be written

$$\sigma_{ij} = p_H \int_A \int \left[\frac{\sigma'_{ij_N}}{P_N} + \frac{f_T \sigma'_{ij_T}}{P_T} + \frac{f_A \sigma'_{ij_A}}{P_A} \right] \sqrt{1 - \frac{(x - x_1)^2}{b^2} - \frac{y^2}{a^2}} \; dx dy , \tag{A-23}$$

where σ_{ij} represents nine components of the stress tensor* for Hertzian pressures and various frictional forces and σ'_{ij} are the results of Mindlin's analyses (discussed in the previous section). In further computations, we

---

* In tensor notation σ_{ij} implies σ_{11}, σ_{12}, σ_{13}, σ_{21} . . . σ_{33}, where σ_{11} is σ_x, σ_{12} is τ_{xy}, σ_{22} is σ_y, σ_{23} is τ_{yz}, etc.

shall use $\bar{\sigma}_{ij} = \sigma_{ij}/p_H$ such that all computations will be normalized on the Hertz maximum pressures.

It would serve no purpose to show all substitutions of Mindlin's equations into the general stress tensor equation. It can be noted that any of Mindlin's equations containing the term y^n (n odd) will produce a zero integral since the integration involves that function times the even Hertz pressure over the whole pressure region for $y_1 = 0$. As an example of how the substitution occurs, consider the normal stress $\bar{\sigma}_z$ for f_T and $f_A = 0$.

$$\bar{\sigma}_z \equiv \bar{\sigma}_{33} = \int_{-a}^{a} \int_{L}^{U} \frac{3}{2\pi} \frac{z^3}{R^5} \sqrt{1 - \left(\frac{x - x_1}{b}\right)^2 - \frac{y^2}{a^2}}\, dxdy \quad , \tag{A-24}$$

where the lower limit $L = x_1 - b\sqrt{1 - \frac{y^2}{a^2}}$, and

the upper limit $U = x_1 + b\sqrt{1 - \frac{y^2}{a^2}}$.

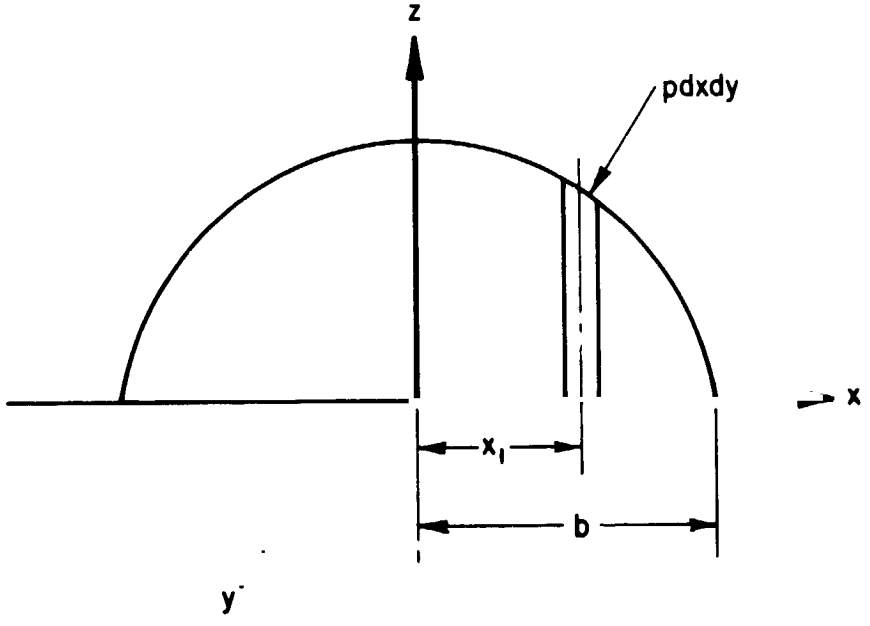

FIGURE A-2. LOADING CHARACTERIZED BY HERTZIAN ELLIPSE

Scaling the Stress Tensor Equations

The following substitutions were used to facilitate computations:

$$\bar{z} = z/b \qquad \xi = x/z \qquad \eta = y/z \qquad K = b/a \quad \text{and} \quad \bar{x}_1 = x_1/b \quad .$$

With these substitutions Equation A-24 appears

$$\bar{\sigma}_z = \frac{3}{2\pi} \int_{-1/K\bar{z}}^{1/K\bar{z}} \int_{\bar{L}}^{\bar{U}} \frac{1}{\bar{R}^5} \sqrt{1 - \left(\bar{z}\xi - \bar{x}_1\right)^2 - \left(K\bar{z}\xi\right)^2} \; d\xi d\eta \quad , \qquad \text{(A-25)}$$

where

$$\bar{L} = \bar{x}_1/\bar{z} - 1/\bar{z}\sqrt{1 - \left(\bar{z}K\eta\right)^2} \quad ,$$

$$\bar{U} = \bar{x}_1/\bar{z} + 1/\bar{z}\sqrt{1 - \left(\bar{z}K\eta\right)^2} \quad , \text{ and}$$

$$\bar{R} = \sqrt{\xi^2 + \eta^2 + 1} \quad .$$

This type of substitution permitted reasonable computations near $\bar{z} = 0$.

In the computer solution

$$\bar{\sigma}_z = \frac{3}{2\pi} \; (2) \int_0^{\infty} \int_{-\infty}^{\infty} \frac{\sqrt{A}}{R^5} \; d\xi d\eta \quad , \qquad \text{(A-26)}$$

where

$$A = \begin{cases} 1 - \left(\bar{z}\xi - \bar{x}_1\right)^2 - \left(K\bar{z}\eta\right)^2 & \text{(A positive)} \\ 0 & \text{(otherwise)} \end{cases} .$$

All other stress compon were treated in the same manner using the general expression

$$\bar{\sigma}_{ij} = 2 \int_0^\infty \int_{-\infty}^\infty \left[\frac{\bar{\sigma}'_{ijN}}{P_N} + \frac{f_T \bar{\sigma}'_{ijT}}{P_T} + \frac{f_A \bar{\sigma}'_{ijA}}{P_A} \right] \sqrt{A} \; d\xi \; d\eta \quad , \qquad \text{(A-27)}$$

where by substituting $\xi = x$, $\eta = y$ and $z = 1$, all equations for σ'_{ij} become the equations for $\bar{\sigma}'_{ij}$.

Review

This is the most involved section in the report and may be difficult for the reader to readily grasp. Basically, all it involves is to take Mindlin's equations (A-1 - A-18), replace x with ξ, y with η, and z with 1 and integrate equation A-27. For example, suppose we want to calculate $\bar{\sigma}_{13} = \tau_{xz}$ from the normal load equation

$$\frac{\bar{\sigma}'_{13\ N}}{P_N} = -\frac{3}{2\pi} \frac{\xi}{R^5} \quad \text{(Equation A-5)} \quad . \qquad \text{(A-28)}$$

From the tangential load equation

$$\frac{\bar{\sigma}'_{13\ T}}{P_T} = -\frac{3}{2\pi} \frac{\xi^2}{R^5} \quad \text{(Equation A-11)} \quad . \qquad \text{(A-29)}$$

From the axial load equation

$$\frac{\bar{\sigma}'_{13\ A}}{P_A} = -\frac{3}{2\pi} \frac{\xi\eta}{R^5} \quad \text{(Equation A-16)} \quad , \qquad \text{(A-30)}$$

where $\bar{R} = \sqrt{1 + \xi^2 + \eta^2}$.

Equation A-27 then becomes

$$\bar{\sigma}_{13} = -2 \int_0^{\infty} \int_{-\infty}^{\infty} \frac{3\,\bar{R}^5}{2\pi} \left[\xi + \xi^2 + \xi\eta\right] \sqrt{A}\; d\xi\; d\eta \quad . \tag{A-31}$$

This equation can be solved by numerical computation. In the Battelle computations, the integral on ξ was performed by a Gaussian quadrature approach and the y integration was performed by a Trapezoidal rule.

Check of Numerical Solution

Hamilton and Goodman[6] developed an exact solution for circular contacts (K = 1). The numerical solution is compared with the exact solution in Table A-1; the exact solution was generated as a part of the work of Tevaarwerk. The agreement tends to verify the numerical accuracy.

Smith & Liu[5] developed an exact solution for line contact (K = 0). The numerical solution is compared with the exact solution in Table A-2.

TABLE A-1. COMPARISON OF NUMERICAL SOLUTION WITH EXACT SOLUTION FOR K = 1[6]
($\nu = 0.3$, $f_T = f_A = 0$)

		σ_x		σ_y		σ_z		τ_{xy}		τ_{yz}		τ_{xz}	
x/b	z/b	Num	Exact	Num	Exact	Num	Exact	Num	Exact	Num	Exact	Num	Exact
.1	-.8	-.305	-.300	-.361	-.356	-.579	-.573	0	0	0	0	.121	.119
	-.5	-.507	-.501	-.532	-.524	-.862	-.853	0	0	0	0	.057	.056
	-.3	-.581	-.575	-.591	-.583	-.953	-.943	0	0	0	0	.031	.031
	0	-.620	-.614	-.623	-.614	-1.00	-.99	0	0	0	0	0	0
	.3	-.581	-.575	-.591	-.583	-.953	-.943	0	0	0	0	-.031	-.031
	.5	-.507	-.501	-.532	-.524	-.861	-.853	0	0	0	0	-.057	-.056
	.8	-.305	-.300	-.360	-.356	-.579	-.573	0	0	0	0	-.121	-.119
.4	-.8	-.184	-.184	-.128	-.129	-.450	-.445	0	0	0	0	.213	.212
	-.5	-.212	-.212	-.198	-.199	-.717	-.709	0	0	0	0	.150	.149
	-.3	-.235	-.235	-.230	-.231	-.818	-.809	0	0	0	0	.090	.090
	0	-.252	-.250	-.249	-.250	-.871	-.862	0	0	0	0	0	0
	.3	-.235	-.235	-.230	-.231	-.817	-.809	0	0	0	0	-.090	-.090
	.5	-.212	-.212	-.198	-.199	-.717	-.709	0	0	0	0	-.150	-.149
	.8	-.185	-.184	-.128	-.129	-.450	-.445	0	0	0	0	-.213	-.212

TABLE A-2. COMPARISON OF NUMERICAL SOLUTION WITH EXACT SOLUTION FOR K = 0[5]
(ν = .285, f_A = 0)

			σ_x		σ_y		σ_z		τ_{xy}		τ_{yz}		τ_{yz}	
f_T	z/b	x/b	Num	Exact	Num	Exact	Num	Exact	Num	Exact	Num	Exact	Num	Exact
0	.5	-.8	-.298	-.299	-.240	-.227	-.506	-.500	0	0	0	0	.249	.247
		-.5	-.311	-.313	-.314	-.301	-.751	-.745	0	0	0	0	.180	.176
		-.3	-.328	-.329	-.346	-.334	-.850	-.842	0	0	0	0	.109	.107
		0	-.342	-.342	-.367	-.352	-.905	-.894	0	0	0	0	0	0
		.3	-.328	-.329	-.346	-.334	-.850	-.842	0	0	0	0	-.109	-.107
		.5	-.311	-.313	-.314	-.301	-.751	-.745	0	0	0	0	-.180	-.176
		.8	-.298	-.299	-.240	-.227	-.506	-.500	0	0	0	0	-.249	-.247
.15	.5	-.8	-.228	-.229	-.209	-.200	-.468	-.462	0	0	0	0	.203	.202
		-.5	-.259	-.263	-.290	-.280	-.724	-.718	0	0	0	0	.133	.129
		-.3	-.295	-.297	-.331	-.320	-.834	-.826	0	0	0	0	.059	.057
		0	-.342	-.342	-.367	-.353	-.905	-.894	0	0	0	0	-.052	-.051
		.3	-.361	-.362	-.360	-.348	-.867	-.858	0	0	0	0	-.159	-.157
		.5	-.363	-.364	-.337	-.324	-.779	-.771	0	0	0	0	-.227	-.222
		.8	-.369	-.369	-.272	-.260	-.543	-.536	0	0	0	0	-.295	-.292

APPENDIX B

SOURCE LISTING FOR BATTELLE STRESS MODEL

```
      PROGRAM SUBSTS(INPUT,OUTPUT,TAPE 1)                                000100
      COMMON/F/ ISTRSS,X1(22,1),GNU,Z,Y,XK,FT,FA,MX                      000110
      COMMON/C/ STRSS(3,3,22)                                            000120
      DIMENSION AN(3,3),ASM(3,22),AS(3,22),BN1(3),BN2(3)                 000130
     1,CN1(3,22),CN2(3,22),CN3(3,22)                                     000140
      REWIND 1                                                           000150
      BN1(1)=.70711                                                      000160
      BN1(2)=0.                                                          000170
      BN1(3)=-.70711                                                     000180
      BN2(1)=.57735                                                      000190
      BN2(2)=.57735                                                      000200
      BN2(3)=.57735                                                      000210
      CN3(1,1)=0.                                                        000220
      CN3(2,1)=0.                                                        000230
      CN3(3,1)=1.                                                        000240
      SIGN=+1.                                                           000250
      SIGNA=+1.                                                          000260
8     PRINT 6                                                            000270
6     FORMAT( * INPUT,ASPECT RATIO,NU,Z,FT,FA,X-POSS,CASE-NO*)           000280
      READ*,XK,GNU,Z,FT,FA,MX,NCASE                                      000290
      IF(XK.LT.0.) GO TO 7                                               000300
      WRITE(1,13) NCASE                                                  000310
13    FORMAT(///,50X,* CASE NUMBER=*,I8,///)                             000320
      WRITE(1,9) XK,GNU,Z,FT,FA                                          000330
      PRINT 9,XK,GNU,Z,FT,FA                                             000340
9     FORMAT( * K=*,E12.5, * NU=*,E12.5,* Z=*,E12.5,* FT=*,E12.5,* FA=*,000350
     1E12.5,//)                                                          000360
      TMXXM=TMXXO=TAURVM=TAURVO=0.                                       000370
      IF(XK.LT.1.E-6) CALL STRSSC(1)                                     000380
      IF(XK.GT.1.E-6) CALL STRSSC(6)                                     000390
      WRITE(1,12)                                                        000400
12    FORMAT(*  IX        SXX          SYY          SZZ          SXY     000410
     1SYZ          SXZ          S1          S2          S3     X-POS    000420
     1*//)                                                               000430
      DO 1 IX=1,MX                                                       000440
      CALL PRINC(STRSS,IX,S1,S2,S3)                                      000450
      CALL ANGLP(STRSS,S1,S2,S3,AN,IX,SIGNA)                             000460
      CALL ANGTRAN(AN,BN1,CN1,IX)                                        000470
      CALL ANGTRAN(AN,BN2,CN2,IX)                                        000480
      WRITE(1,11) IX,STRSS(1,1,IX),STRSS(2,2,IX),STRSS(3,3,IX),          000490
     1STRSS(1,2,IX),STRSS(2,3,IX),STRSS(1,3,IX),S1,S2,S3,X1(IX,1)        000500
11    FORMAT(I5,10E12.5)                                                 000510
1     CONTINUE                                                           000520
      WRITE(1,28)                                                        000530
28    FORMAT(///,45X,* FATIGUE SHEAR STRESS SUMMARY * //)                000540
      DO 21 IXRF=1,MX                                                    000550
      XRF=X1(IXRF,1)                                                     000560
      CALL RVSTRS(STRSS,CN1,ASM,TRVM,TMAX,IXRF,SIGN)                     000570
      PRINT 99,TMAX                                                      000580
99    FORMAT( * TMAX=*,E12.5)                                            000590
      IF(ABS(TRVM).GT.ABS(TAURVM)) TAURVM=TRVM                           000600
      IF(ABS(TMAX).GT.ABS(TMXXM)) 22,23                                  000610
22    TMXXM=TMAX                                                         000620
      IXRFM=IXRF                                                         000630
      SIGN=-1.                                                           000640
23    CALL RVSTRS(STRSS,CN2,AS,TRVMO,TMAXO,IXRF,SIGN)                    000650
      IF(ABS(TRVMO).GT.ABS(TAURVO)) TAURVO=TRVMO                         000660
      IF(ABS(TMAXO).GT.ABS(TMXXO)) 24,25                                 000670
24    TMXXO=TMAXO                                                        000680
      IXRFO=IXRF                                                         000690
25    CONTINUE                                                           000700
      PRINT 29,IXRF,TRVM,TMAX,TRVMO,TMXXO                                000710
29    FORMAT(I5,4E14.5)                                                  000720
```

```
21    CONTINUE                                                          000730
      WRITE(1,15)                                                       000740
15    FORMAT(//,20X,* PLANE OF MAX SHEAR STRESS*//)                     000750
      WRITE(1,18) TMXXM,CN1(1,IXRFM),CN1(2,IXRFM),CN1(3,IXRFM),X1(IXRFM,000760
     1,E13.5,* X-REF=*,E13.5)                                           000790
      WRITE(1,19) TAURVM,ASM(1,IXRFM),ASM(2,IXRFM),ASM(3,IXRFM)         000800
19    FORMAT( * TAU REVERSING=*,E13.5,* AS(1)=*,E13.5,* AS(2)=*,E13.5,  000810
     1*AS(3)=*,E13.5)                                                   000820
      WRITE(1,16)                                                       000830
16    FORMAT(//,20X,* PLANE OF MAX OCT SHEAR STRESS *//)                000840
      WRITE(1,18) TMXXO,CN2(1,IXRFO),CN2(2,IXRFO),CN2(3,IXRFO),X1(IXRFO,000850
     11)                                                                000860
      WRITE(1,19) TAURVO,AS(1,IXRFO),AS(2,IXRFO),AS(3,IXRFO)            000870
      CALL RVSTRS(STRSS,CN3,AS,TRVM,TMAX,1,SIGN)                        000880
      WRITE(1,17)                                                       000890
17    FORMAT(//,20X,* PLANE OF REVERSING SHEAR STRESS *//)              000900
      WRITE(1,18) TMAX,CN3(1,1),CN3(2,1),CN3(3,1)                       000910
      WRITE(1,19) TRVM,AS(1,1),AS(2,1),AS(3,1)                          000920
      WRITE(1,20)                                                       000930
20    FORMAT(1H1)                                                       000940
      GO TO 8                                                           000950
7     CONTINUE                                                          000960
      END                                                               000970
        SUBROUTINE STRSSC(IFLAG)                                        000980
      COMMON/F/ IS,X1(22,1),GNU,Z,Y,XK,FT,FA,MX1                        000990
      COMMON/C/ STRSS(3,3,22)                                           001000
      DIMENSION FI(22,17),GINT(22),ICALC(22),TEMP1(22),FX(22),FIY(17)   001010
     1,Y1(17)                                                           001020
      PI=3.14159                                                        001030
      MY=17                                                             001040
      MX=MX1                                                            001050
      CALL XPOS                                                         001060
      IF(IFLAG.EQ.1) GO TO 10                                           001070
      DO 9 ISTRSS=1,6                                                   001080
      IS=ISTRSS                                                         001090
      DO 4 JY=1,MY                                                      001100
      IF(JY.EQ.1) Y=0.                                                  001110
      IF(JY.EQ.2) Y=.2                                                  001120
      IF(JY.EQ.3) Y=.4                                                  001130
      IF(JY.EQ.4) Y=.6                                                  001140
      IF(JY.EQ.5) Y=.8                                                  001150
      IF(JY.EQ.6) Y=1.2                                                 001160
      IF(JY.EQ.7) Y=1.6                                                 001170
      IF(JY.EQ.8) Y=2.                                                  001180
      IF(JY.EQ.9) Y=2.5                                                 001190
      IF(JY.EQ.10) Y=3.                                                 001200
      IF(JY.EQ.11) Y=4.                                                 001210
      IF(JY.EQ.12) Y=6.                                                 001220
      IF(JY.EQ.13) Y=8.                                                 001230
      IF(JY.EQ.14) Y=10.                                                001240
      IF(JY.EQ.15) Y=20.                                                001250
      IF(JY.EQ.16) Y=30.                                                001260
      IF(JY.EQ.17) Y=50.                                                001270
      Y1(JY)=Y                                                          001280
      BL=-2./Z                                                          001290
      IF(BL.LT.-20.) BL=-20.                                            001300
      BU=-BL                                                            001310
      IELE=1                                                            001320
      CALL GAUSS(MX,1,20,1,1.E-3,1.E-3,BL,BU,GINT,ICALC,TEMP1,FX)       001330
      DO 5 IX=1,MX                                                      001340
      FI(IX,JY)=GINT(IX)                                                001350
5     CONTINUE                                                          001360
4     CONTINUE                                                          001370
```

```
      DO 6 IX=1,MX                                                      001380
      DO 8 IY=1,MY                                                      001390
    8 FIY(IY)=FI(IX,IY)                                                 001400
      CALL YINT(FIY,Y1,SIG1,MY)                                         001410
      IF(IS.EQ.1) STRSS(1,1,IX)=SIG1                                    001420
      IF(IS.EQ.2) STRSS(2,2,IX)=SIG1                                    001430
      IF(IS.EQ.3) STRSS(3,3,IX)=SIG1                                    001440
      IF(IS.EQ.4) STRSS(1,2,IX)=STRSS(2,1,IX)=SIG1                      001450
      IF(IS.EQ.5) STRSS(2,3,IX)=STRSS(3,2,IX)=SIG1                      001460
      IF(IS.EQ.6) STRSS(1,3,IX)=STRSS(3,1,IX)=SIG1                      001470
    6 CONTINUE                                                          001480
    9 CONTINUE                                                          001490
      RETURN                                                            001500
   10 CONTINUE                                                          001510
      DO 12 ISTRSS=1,6                                                  001520
      IS=ISTRSS                                                         001530
      DO 11 IX=1,MX                                                     001540
      X=X1(IX,1)                                                        001550
      XK1=(1.+X)**2+Z**2                                                001560
      XK2=(1.-X)**2+Z**2                                                001570
      C1=SQRT(XK2/XK1)                                                  001580
      C2=SQRT(2.*C1+(XK1+XK2-4.)/XK1)                                   001590
      SI=PI/XK1*(1.-C1)/C1/C2                                           001600
      SIB=PI/XK1*(1.+C1)/C1/C2                                          001610
      SXN=-1./PI*Z*((1.+2.*X**2+2.*Z**2)*SIB-2.*PI-3.*X*SI)             001620
      SZN=-Z/PI*(SIB-X*SI)                                              001630
      TXZN=-1./PI*Z**2*SI                                               001640
      CTZ=-1./PI*((1.+2.*X**2+2.*Z**2)*Z*SIB-2.*PI*Z-3.*X*Z*SI)         001650
      TXZT=FT*CTZ                                                       001660
      TYZT=FA*CTZ                                                       001670
      CXT=-1./PI*((2.*X**2-2.-3.*Z**2)*SI+2.*PI*X+2.*(1.-X**2-Z**2)*X   001680
     1*SIB)                                                             001690
      SXT=FT*CXT                                                        001700
      SYT=FA*CXT                                                        001710
      SZT=-(FT+FA)/PI*Z**2*SI                                           001720
      IF(IS.EQ.1) STRSS(1,1,IX)=SXN+SXT                                 001730
      IF(IS.EQ.2) STRSS(2,2,IX)=GNU*(SXN+SXT+SZN+SZT)+SYT               001740
      IF(IS.EQ.3) STRSS(3,3,IX)=SZN+SZT                                 001750
      IF(IS.EQ.4) STRSS(1,2,IX)=STRSS(2,1,IX)=0.                        001760
      IF(IS.EQ.5) STRSS(2,3,IX)=STRSS(3,2,IX)=TYZT                      001770
      IF(IS.EQ.6) STRSS(1,3,IX)=STRSS(3,1,IX)=TXZN+TXZT                 001780
   11 CONTINUE                                                          001790
   12 CONTINUE                                                          001800
      RETURN                                                            001810
      END                                                               001820
        SUBROUTINE YINT(FI,Y1,SIG1,MY)                                  001830
      DIMENSION FI(17),Y1(17)                                           001840
      C1=FI(1)*(Y1(2)-Y1(1))                                            001850
      C2=FI(MY)*(Y1(MY)-Y1(MY-1))                                       001860
      SUM=0.                                                            001870
      MY1=MY-1                                                          001880
      DO 1 I=2,MY1                                                      001890
    1 SUM=SUM+FI(I)*(Y1(I+1)-Y1(I-1))                                   001900
      SIG1=C1+C2+SUM                                                    001910
      RETURN                                                            001920
      END                                                               001930
        SUBROUTINE FEVAL(X,FX,JF,ICALC)                                 001940
      COMMON/F/ ISTRSS,X1(22,1),GNU,ZB,Y,XK,FT,FA,MX                    001950
      DIMENSION FX(22),ICALC(22)                                        001960
      PI=3.14159                                                        001970
      X2=X**2                                                           001980
      Y2=Y**2                                                           001990
      GNU1=1.-2.*GNU                                                    002000
```

```
      R=SQRT(X**2+Y**2+1.)                                                  002010
      R2=R**2                                                               002020
      R3=R**3                                                               002030
      R5=R**5                                                               002040
      RZ=R+1.                                                               002050
      IF(ISTRSS.EQ.1) 1,2                                                   002060
    1 C1=(GNU1/R3-3.*X2/R5-GNU1/R/RZ*(1.-X2/R/RZ-X2/R2))/2./PI              002070
      C2=(GNU1/R3-3.*X2/R5-GNU1/R/RZ**2*(3.-X2*(3.*R+1.)/R2/RZ))/2./PI*X    002080
      C3=0.                                                                 002090
      GO TO 10                                                              002100
    2 IF(ISTRSS.EQ.2) 3,4                                                   002110
    3 C1=(GNU1/R3-3.*Y2/R5-GNU1/R/RZ*(1.-Y2/R/RZ-Y2/R2))/2./PI              002120
      C2=(GNU1/R3-3.*Y2/R5-GNU1/R/RZ**2*(1.-Y2*(3.*R+1.)/R2/RZ))*X/2./PI    002130
      C3=0.                                                                 002140
      GO TO 10                                                              002150
    4 IF(ISTRSS.EQ.3) 5,6                                                   002160
    5 C1=-3./R5/2./PI                                                       002170
      C2=-3./2./PI*X/R5                                                     002180
      C3=0.                                                                 002190
      GO TO 10                                                              002200
    6 IF(ISTRSS.EQ.4) 7,8                                                   002210
    7 C1=0.                                                                 002220
      C2=0.                                                                 002230
      C3=(-3.*X2/R5-GNU1/R/RZ**2*(1.-X2*(3.*R+1.)/R2/RZ))/2./PI*X           002240
      GO TO 10                                                              002250
    8 IF(ISTRSS.EQ.5) 9,11                                                  002260
    9 C1=0.                                                                 002270
      C2=0.                                                                 002280
      C3=-3./2./PI*Y2/R5                                                    002290
      GO TO 10                                                              002300
   11 C1=-3./2./PI*X/R5                                                     002310
      C2=-3./2./PI*X2/R5                                                    002320
      C3=0.                                                                 002330
   10 ARG=C1*FT*C2*FA*C3                                                    002340
      DO 14 I=1,JF                                                          002350
      IF(ICALC(I))14,13,13                                                  002360
   13 CONTINUE                                                              002370
      ARGU=1.-(ZB*X-X1(I,1))**2-(ZB*Y*XK)**2                                002380
      IF(ARGU.LT.0.) 15,16                                                  002390
   15 FY(I)=1.E-12                                                          002400
      GO TO 17                                                              002410
   16 CONTINUE                                                              002420
      FX(I)=SQRT(ARGU)*ARG                                                  002430
   17 CONTINUE                                                              002440
   14 CONTINUE                                                              002450
      RETURN                                                                002460
      END                                                                   002470
        SUBROUTINE PRINC(STRSS,IX,SM,SIM,SMN)                               002480
      DIMENSION STRSS(3,3,22)                                               002490
      C1=STRSS(1,1,IX)+STRSS(2,2,IX)+STRSS(3,3,IX)                          002500
      C1=-C1                                                                002510
      C2=STRSS(1,1,IX)*STRSS(2,2,IX)+STRSS(1,1,IX)*STRSS(3,3,IX)+           002520
     1STRSS(2,2,IX)*STRSS(3,3,IX)-STRSS(1,2,IX)**2-                         002530
     2STRSS(1,3,IX)**2-STRSS(2,3,IX)**2                                     002540
      C3=STRSS(1,1,IX)*STRSS(2,2,IX)*STRSS(3,3,IX)+                         002550
     12.*STRSS(1,2,IX)*STRSS(2,3,IX)*STRSS(1,3,IX)-                         002560
     2STRSS(1,1,IX)*STRSS(2,3,IX)**2-                                       002570
     3STRSS(2,2,IX)*STRSS(1,3,IX)**2-                                       002580
     4STRSS(3,3,IX)*STRSS(1,2,IX)**2                                        002590
      C3=-C3                                                                002600
      CALL CUBIC(C1,C2,C3,S1,S2,S3)                                         002610
      SM=S1                                                                 002620
      IF(S2.GT.SM) SM=S2                                                    002630
```

```
      IF(S3.GT.SM) SM=S3                                                002640
      SMN=S1                                                            002650
      IF(S2.LT.SMN) SMN=S2                                              002660
      IF(S3.LT.SMN) SMN=S3                                              002670
      SIM=S1                                                            002680
      IF(S2.GT.SMN.AND.S2.LT.SM) SIM=S2                                 002690
      IF(S3.GT.SMN.AND.S3.LT.SM) SIM=S3                                 002700
      RETURN                                                            002710
      END                                                               002720
        SUBROUTINE CUBIC(C1,C2,C3,S1,S2,S3)                             002730
      PI=3.14159                                                        002740
      A=1./3.*(3.*C2-C1**2)                                             002750
      B=1./27.*(2.*C1**3-9.*C1*C2+27.*C3)                               002760
      IF(A.LT.0.) GO TO 3                                               002770
      S1=S2=S3=0.                                                       002780
      RETURN                                                            002790
    3 CONTINUE                                                          002800
      PHI=ACOS(-B/2./SQRT(-A**3/27.))                                   002810
      S1=2.*SQRT(-A/3.)*COS(PHI/3.)-C1/3.                               002820
      S2=2.*SQRT(-A/3.)*COS(PHI/3.+2.*PI/3.)-C1/3.                      002830
      S3=2.*SQRT(-A/3.)*COS(PHI/3.+4.*PI/3.)-C1/3.                      002840
      RETURN                                                            002850
      END                                                               002860
        SUBROUTINE ANGLP(STRSS,S1,S2,S3,AN,IX,SIGNA)                    002870
      DIMENSION STRSS(3,3,22),AN(3,3)                                   002880
      S11=STRSS(1,1,IX)                                                 002890
      S12=STRSS(1,2,IX)                                                 002900
      S13=STRSS(1,3,IX)                                                 002910
      S22=STRSS(2,2,IX)                                                 002920
      S23=STRSS(2,3,IX)                                                 002930
      S33=STRSS(3,3,IX)                                                 002940
      ER=1.E-3                                                          002950
      DO 2 I=1,3                                                        002960
      IF(I.EQ.1) S=S1                                                   002970
      IF(I.EQ.2) S=S2                                                   002980
      IF(I.EQ.3) S=S3                                                   002990
      IF(ABS(S11-S).LT.ER) 3,4                                          003000
    3 AN(I,1)=1.                                                        003010
      AN(I,2)=AN(I,3)=0.                                                003020
      GO TO 14                                                          003030
    4 IF(ABS(S22-S).LT.ER) 5,6                                          003040
    5 AN(I,2)=1.                                                        003050
      AN(I,1)=AN(I,3)=0.                                                003060
      GO TO 14                                                          003070
    6 IF(ABS(S33-S).LT.ER) 7,8                                          003080
    7 AN(I,3)=1.                                                        003090
      AN(I,1)=AN(I,2)=0.                                                003100
      GO TO 14                                                          003110
    8 IF(ABS(S12).LT.ER.AND.ABS(S13).LT.ER) 9,10                        003120
   10  A=-(S12*S13-(S11-S)*S23)/(S12**2-(S11-S)*(S22-S))                 003130
      C11=((-S12*A-S13)/(S11-S))**2+1.+A**2                             003140
      AN(I,3)=SIGNA*SQRT(1./C11)                                        003150
      AN(I,2)=A*AN(I,3)                                                 003160
      AN(I,1)=(-S12*AN(I,2)-S13*AN(I,3))/(S11-S)                        003170
      GO TO 14                                                          003180
    9 IF(ABS(S12).LT.ER.AND.ABS(S23).LT.ER) 11,12                       003190
   11 A=-(S12*S23-(S22-S)*S13)/(S23**2-(S22-S)*(S33-S))                 003200
      C11=((-S12-S23*A)/(S22-S))**2+1.+A**2                             003210
      AN(I,1)=SIGNA*SQRT(1./C11)                                        003220
      AN(I,3)=A*AN(I,1)                                                 003230
      AN(I,2)=(-S23*AN(I,3)-S12*AN(I,1))/(S22-S)                        003240
      GO TO 14                                                          003250
   12 CONTINUE                                                          003260
```

```
      A=-(S13*S23-S12*(S33-S))/(S13**2-(S11-S)*(S33-S))                 003270
      C11=((-S23-S13*A)/(S33-S))**2+1.+A**2                             003280
      AN(I,2)=SIGNA*SQRT(1./C11)                                        003290
      AN(I,1)=A*AN(I,2)                                                 003300
      AN(I,3)=(-S13*AN(I,1)-S23*AN(I,2))/(S33-S)                        003310
   14 CONTINUE                                                          003320
    2 CONTINUE                                                          003330
      RETURN                                                            003340
      END                                                               003350
        SUBROUTINE RVSTRS(STRSS,AN,AS,TRVM,TMXX,I,SIGN)                 003360
      COMMON/F/ ISTRSS,X1(22,1),GNU,ZR,Y,XK,FT,FA,MX                    003370
      DIMENSION STRSS(3,3,22),AN(3,22),AS(3,22)                         003380
      TRVM=0.                                                           003390
      TMXX=0.                                                           003400
      ER=1.E-3                                                          003410
      DO 1 IAS=1,21                                                     003420
      AS1=-1.+.1*FLOAT(IAS-1)                                           003430
      IF(ABS(AN(2,I)).LT.ER.AND.ABS(AN(1,I)).LT.ER) 8,9                 003440
    8 AS3=0.                                                            003450
      AS2=SIGN*SQRT(1.-AS1**2)                                          003460
      GO TO 13                                                          003470
    9 IF(ABS(AN(1,I)).LT.ER.AND.ABS(AN(3,I)).LT.ER) 10,11               003480
   10 AS2=0.                                                            003490
      AS3=SIGN*SQRT(1.-AS1**2)                                          003500
      GO TO 13                                                          003510
   11 IF(ABS(AN(2,I)).LT.ER.AND.ABS(AN(3,I)).LT.ER) GO TO 12            003520
      IF(ABS(AN(2,I)).LT.ER) 16,17                                      003530
   16 AS3=-AN(1,I)*AS1/AN(3,I)                                          003540
      CC=1.-AS1**2-AS3**2                                               003550
      IF(CC.LT.0.) GO TO 22                                             003560
      AS2=SIGN*SQRT(CC)                                                 003570
      GO TO 13                                                          003580
   17 IF(ABS(AN(3,I)).LT.ER) 18,19                                      003590
   18 AS2=-AS1*AN(1,I)/AN(2,I)                                          003600
      CC=1.-AS1**2-AS2**2                                               003610
      IF(CC.LT.0.) GO TO 22                                             003620
      AS3=SIGN*SQRT(CC)                                                 003630
      GO TO 13                                                          003640
   19 IF(ABS(AN(1,I)).LT.ER) 20,21                                      003650
   20 AS2=AN(3,I)*SIGN*SQRT(1.-AS1**2)                                  003660
      AS3=-AN(2,I)/AN(3,I)*AS2                                          003670
      GO TO 13                                                          003680
   21 A=AN(2,I)**2+AN(3,I)**2                                           003690
      B=2.*AN(1,I)*AN(3,I)*AS1                                          003700
      C=AS1**2*AN(1,I)**2+AN(2,I)**2*AS1**2-AN(2,I)**2                  003710
      CC=B**2-4.*A*C                                                    003720
      IF(CC.LT.0.) GO TO 22                                             003730
      AS3=(-B+SIGN*SQRT(CC))/2./A                                       003740
      AS2=-(AS3*AN(3,I)+AS1*AN(1,I))/AN(2,I)                            003750
   13 CONTINUE                                                          003760
      TMX=-1.E+20                                                       003770
      TMN=1.E+20                                                        003780
      DO 2 IX=1,MX                                                      003790
      TAU=AN(1,I)*AS1*STRSS(1,1,IX)+AN(2,I)*AS2*STRSS(2,2,IX)           003800
     1+AN(3,I)*AS3*STRSS(3,3,IX)+AN(1,I)*AS2*STRSS(1,2,IX)              003810
     2+AN(2,I)*AS1*STRSS(1,2,IX)+AN(1,I)*AS3*STRSS(1,3,IX)              003820
     3+AN(3,I)*AS1*STRSS(1,3,IX)+AN(2,I)*AS3*STRSS(2,3,IX)              003830
     4+AN(3,I)*AS2*STRSS(2,3,IX)                                        003840
      IF(TAU.GT.TMX) 3,4                                                003850
    3 TMX=TAU                                                           003860
      AST1=AS1                                                          003870
      AST2=AS2                                                          003880
      AST3=AS3                                                          003890
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      4     IF(TAU.LT.TMN) 5,6                                          003900
      5     TMN=TAU                                                     003910
      6     CONTINUE                                                    003920
      2     CONTINUE                                                    003930
            TRV=TMX-TMN                                                 003940
            IF(ABS(TRV).GT.ABS(TRVM)) TRVM=TRV                          003950
            IF(ABS(TMN).GT.ABS(TMX)) TMX=ABS(TMN)                       003960
            IF(TMX.GT.TMXX) 23,22                                       003970
      23    TMXX=TMX                                                    003980
            AS(1,I)=AST1                                                003990
            AS(2,I)=AST2                                                004000
            AS(3,I)=AST3                                                004010
      22    CONTINUE                                                    004020
      1     CONTINUE                                                    004030
            RETURN                                                      004040
      12    WRITE(1,15)                                                 004050
      15    FORMAT( * SOLUTION NOT APPROPRIATE *)                       004060
            RETURN                                                      004070
            END                                                         004080
              SUBROUTINE ANGTRAN(AN,BN,CN,IX)                           004090
            DIMENSION AN(3,3),BN(3),CN(3,22)                            004100
            CN(1,IX)=BN(1)*AN(1,1)+BN(2)*AN(2,1)+BN(3)*AN(3,1)          004110
            CN(2,IX)=BN(1)*AN(1,2)+BN(2)*AN(2,2)+BN(3)*AN(3,2)          004120
            CN(3,IX)=BN(1)*AN(1,3)+BN(2)*AN(2,3)+BN(3)*AN(3,3)          004130
            RETURN                                                      004140
            END                                                         004150
              SUBROUTINE XPOS                                           004160
            COMMON/F/ IS,X1(22,1),GNU,Z,Y,XK,FT,FA,MX                   004170
            IF(MX.EQ.7) 1,2                                             004180
      1     X1(1,1)=-.8                                                 004190
            X1(2,1)=-.5                                                 004200
            X1(3,1)=-.3                                                 004210
            X1(4,1)=0.                                                  004220
            X1(5,1)=.3                                                  004230
            X1(6,1)=.5                                                  004240
            X1(7,1)=.8                                                  004250
            RETURN                                                      004260
      2     IF(MX.EQ.11) 3,4                                            004270
      3     X1(1,1)=-1.                                                 004280
            X1(2,1)=-.9                                                 004290
            X1(3,1)=-.8                                                 004300
            X1(4,1)=-.5                                                 004310
            X1(5,1)=-.3                                                 004320
            X1(6,1)=0.                                                  004330
            X1(7,1)=.3                                                  004340
            X1(8,1)=.5                                                  004350
            X1(9,1)=.6                                                  004360
            X1(10,1)=.9                                                 004370
            X1(11,1)=1.                                                 004380
            RETURN                                                      004390
      4     IF(MX.EQ.15) 5,6                                            004400
      5     X1(1,1)=-1.)                                                004410
            X1(2,1)=-1.                                                 004420
            X1(3,1)=-.9                                                 004430
            X1(4,1)=-.8                                                 004440
            X1(5,1)=-.7                                                 004450
            X1(6,1)=-.5                                                 004460
            X1(7,1)=-.3                                                 004470
            X1(8,1)=0.                                                  004480
            X1(9,1)=.3                                                  004490
            X1(10,1)=.5                                                 004500
            X1(11,1)=.7                                                 004510
            X1(12,1)=.8                                                 004520
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      X1(13,1)=.9                                                    004530
      X1(14,1)=1.                                                    004540
      X1(15,1)=1.1                                                   004550
      RETURN                                                         004560
    6 PRINT 10                                                       004570
   10 FORMAT( * MATRIX NOT AVAILABLE * )                             004580
      RETURN                                                         004590
      END                                                            004600
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