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Scaled Runge-Kutta Algorithms for Treating the Problem of Dense Output

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SUMMARY

The problem of dense output during the numerical solution of an ordinary differential equation, using a Runge-Kutta algorithm, can severely reduce the efficiency of the integration method. A set of scaled Runge-Kutta algorithms for the third- through fifth-orders are developed to determine the solution at any point within the integration step at a relatively small increase in computing time. Each scaled algorithm is designed to be used with an existing Runge-Kutta formula, using the derivative evaluations of the defining algorithm along with an additional derivative evaluation (or two). Third-order, scaled algorithms are embedded within the existing formulas at no additional derivative expense. Such algorithms can be easily adapted to generate interpolating polynomials (or dependent variable stops) efficiently.

INTRODUCTION

The Runge-Kutta algorithm, designed to treat the initial value problem

$$\frac{dy}{dt} = f(t,y) \ , \quad y(t_o) = y_o \tag{1}$$

where y is an n-vector, is an explicit single-step algorithm. A linear combination of derivative evaluations to approximate a truncated Taylor series is used. These algorithms are frequently used in solving the ordinary differential equation (ODE) because of their high accuracy and simplicity. For Runge-Kutta methods to perform efficiently, however, the algorithms must be operated at an optimal step size that is consistent with the truncation error criteria. A significant reduction in efficiency may be encountered if frequent data output is requested. This difficulty arises because the Runge-Kutta algorithm determines the solution only at the end points of the integration steps; thus, the step size must be reduced to coincide with the requested output points. Shampine, Watts, and Davenport (ref. 1) have shown, however, that reduction of the step size by this procedure seriously impairs the efficiency of the method; i.e., the efficiency of the Runge-Kutta algorithm depends on the ability to use large step sizes whenever permissible during the solution of the ODE. (Unlike the multistep methods, a backlog of solution points for an interpolating polynomial is generally unsatisfactory. This is due to the large step sizes used by the Runge-Kutta methods, which provide too wide a spacing for an accurate interpolating polynomial.)

An alternate procedure for data determination may be developed. This procedure allows the Runge-Kutta algorithm to be operated by using a near-optimal step size while still providing the solution at an intermediate point for only a slight increase in expense. Such a procedure, first developed for Runge-Kutta-Nystroem methods (ref. 2), involves the development of a companion algorithm of equivalent order, using the function evaluations of the basic algorithm plus an additional evaluation (or two). The new scaled algorithms give the solution vector at an intermediate point $t = t_0 + h^*$ where $h^* = \sigma h$ with σ as some positive scaling parameter usually less than unity. For each set of coefficients in a Runge-Kutta formula, a family of scaled algorithms exists with varying values of σ. These values must be used in conjunction with their particular defining set of coefficients, or the efficiency features of the new algorithms are wasted.

The derivation of the scaled algorithms and a list of coefficients for varying values of σ are presented for the embedded pairs of Runge-Kutta third- through fifth-order formulas developed by Fehlberg (ref. 3) and another third-order formula, RKT(2)3, by Bettis.[1] The coefficients listed in table V have been generated in double precision on the Univac 1110 system and verified by the subroutine RKEQ,[2] which evaluates truncation error coefficients through the 12th-order for Runge-Kutta methods. These scaled Runge-Kutta coefficients have been copied directly from the generating programs into a large computer processing system for reproduction. Similarly, the scaled coefficients listed in rational form have been computed with the aid of a rational arithmetic package and verified by subroutine RKEQ before being copied into the word processing system. A discussion of the implementation of the new algorithms is included.

THE SCALED RUNGE-KUTTA ALGORITHM

The basic Runge-Kutta algorithm of order p for the solution of equation (1) assumes the form

$$y(t_0 + h) = y_1 + O(h^{p+1}) = y_0 + h \sum_{\kappa=0}^{r} C_\kappa f_\kappa + O(h^{p+1}) \qquad (2a)$$

where

$$f_0 = f(t_0, y_0) \qquad (2b)$$

and

$$f_\kappa = f\left(t_0 + \alpha_\kappa h,\ y_0 + h \sum_{\lambda=0}^{\kappa-1} \beta_{\kappa,\lambda} f_\lambda\right) \qquad (2c)$$

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[1]Private communication.

[2]M. K. Horn: A Computer Program for Determining Truncation Error Coefficients for Runge-Kutta Methods. JSC IN 80-FM-13, Feb. 1980.

for $\kappa = 1,2,\ldots,\rho$ where the α, β, and C coefficients were selected so that the algorithmic solution y_1 is equivalent to a Taylor sum of order p. Specifically, comparing the Taylor series expansion of the algorithmic solution to the Taylor series expansion of the solution vector (term by term), results in several error coefficients ($T_{i,j}$) for each order i. For an algorithm to be of order p, all error coefficients must vanish for $i = 1,2,\ldots,p$. These vanishing terms are referred to as equations of condition and are the constraints for determining the α, β, and C coefficients. (The non-vanishing error coefficients for $i = p + 1$ are also of interest because they represent how closely the algorithm approaches a method of the next higher order.) The equations of condition are developed with the additional assumptions that

$$\sum_{\lambda=0}^{\kappa-1} \beta_{\kappa,\lambda} = \alpha_\kappa \tag{3}$$

for $\kappa = 1,2,\ldots,\rho$, which define the $\beta_{\kappa,o}$ coefficients.

The development of the scaled algorithm requires knowledge of the defining algorithm because the new methods use the basic coefficients and the derivative evaluations as the core of their own system. In the scaled algorithm, it is assumed that an acceptable step of length h has been taken (i.e.; $f_1, f_2, \ldots, f_\rho$ have been evaluated and the solution has been determined at $t = t_o + h$), and now the solution at $t = t_o + h^*$ should be determined, where

$$h^* = \sigma h \tag{4}$$

with σ as some scaling factor where, generally, $\sigma \varepsilon (0,1)$. The new scaled Runge-Kutta algorithm assumes the same form as the basic Runge-Kutta algorithm with the solution and coefficients denoted by

$$y_1^* = y_o + h^* \sum_{\kappa=0}^{\rho^*} C_\kappa^* f_\kappa^* \tag{5a}$$

where

$$f_o^* = f(t_o, y_o) \tag{5b}$$

and

$$\overset{*}{f}_\kappa = f\left(t_o + \overset{*}{\alpha}_\kappa \overset{*}{h},\ y_o + \overset{*}{h} \sum_{\lambda=0}^{\kappa-1} \overset{*}{\beta}_{\kappa,\lambda}\ \overset{*}{f}_\lambda\right) \tag{5c}$$

for $\kappa = 1,2,\ldots,\rho^*$. If the α^* and β^* coefficients can be chosen such that

$$\overset{*}{f}_\kappa = f_\kappa \tag{6}$$

for $\kappa = 1,2,\ldots,\rho$, then the expense of applying the scaled algorithm will be minimal because $f_1, f_2, \ldots, f_\rho$ have already been computed. Comparing equations (2b) and (2c) to equations (5b) and (5c) show that

$\overset{*}{f}_o = f_o$ and $\overset{*}{f}_\kappa = f_\kappa$ if

$$\overset{*}{\alpha}_\kappa \overset{*}{h} = \alpha_\kappa h \tag{7a}$$

and

$$\overset{*}{\beta}_{\kappa,\lambda} \overset{*}{h} = \beta_{\kappa,\lambda}\ h \tag{7b}$$

for $\kappa = 1,2,\ldots,\rho$; $\lambda = 0,1,\ldots,\kappa-1$. Thus, only $\overset{*}{f}_{\rho+1}, \ldots, \overset{*}{f}_{\rho^*}$ need to be computed. (While a third-order algorithm exists, requiring no additional derivative evaluations, a more accurate third-order formula may be derived by using one additional evaluation. For the fourth-order algorithms, one additional evaluation is needed ($\rho^* = \rho + 1$). For the fifth-order algorithms, two extra evaluations are necessary ($\rho^* = \rho + 2$) for a fifth-order solution, although a fourth-order solution may be obtained for only one additional evaluation.) But because $h^* = \sigma h$,

$$\overset{*}{\alpha}_\kappa = \alpha_\kappa / \sigma \tag{8a}$$

and

$$\overset{*}{\beta}_{\kappa,\lambda} = \beta_{\kappa,\lambda} / \sigma \tag{8b}$$

for $\kappa = 1,2,\ldots,\rho$; $\lambda = 0,1,\ldots,\kappa-1$. Thus, many of the coefficients for each scaled algorithm are already determined by being related to their corresponding coefficients in the defining algorithm by a factor of $1/\sigma$.

As in the development of Runge-Kutta algorithms, the equations of condition for the scaled methods may be solved with one or more values of the α^* or β^* coefficients as free parameters. The main criterion for selecting these parameters is the minimization of the norm of the error coefficients for the first neglected term in the Taylor series expansion. More specifically, the accuracy of an algorithm of order p may be improved by reducing the error coefficients for order p+1 so that

$$G^*_{p+1} \equiv \left\{ \sum_{j=1}^{\lambda_{p+1}} T^{*2}_{p+1,j} \right\}^{1/2} \sigma^{p+1} \tag{9}$$

approaches a minimum, where λ_{p+1} is the number of error coefficients of order p+1. As in the Runge-Kutta algorithm development, however, secondary conditions such as moderate bounds on the coefficients, show that near-optimal values of G^*_{p+1} may produce algorithms that are more efficient than the algorithm for which G^*_{p+1} minimum is achieved. Thus, in selecting the free parameters, algorithms having a value of G^*_{p+1} close to G^*_{p+1} minimum are studied, but it is not demanded that G^*_{p+1} minimum be reached.

The stability properties of the Runge-Kutta algorithms are also of concern in the development of the coefficients. The stability analysis, applied to the single, linear differential equation

$$y' = \lambda y$$

determines the largest value of the step size h for which errors in the solution are not amplified by the algorithm itself, where λ is any eigenvalue of the Jacobian matrix $\{\frac{\delta f}{\delta y}\}_0$ of equation (1) with λ complex. For an algorithm of order p and a given λ, the step size h is restricted by the polynomial

$$P(\lambda h) = \sum_{k=0}^{p} \frac{(\lambda h)^k}{k!} + \sum_{k=p+1}^{\rho^*+1} \mu^*_k (\lambda h)^k \tag{10}$$

where μ_k^* depends upon the α^*, β^*, and C^* coefficients (ref. 4). For the algorithm to be stable, $|P(\lambda h)| < 1$. Of particular interest are the points at which the stability curves cross the negative real axis R^* and the imaginary axis I^*. In developing the scaled algorithms, care should be taken to maintain reasonable stability bounds; although, if the solution is not advanced from the point $t_0 + h^*$, the amplification during the step taken by the scaled algorithm would affect only the immediate output and not the further computations.

THIRD-ORDER SCALED ALGORITHMS FOR RK(2)3 FORMULAS

Coefficients for the RKF(2)3 algorithm (ref. 3) are found in table I while those for the RKT(2)3 algorithm having lower truncation error coefficients, are listed in table II. These paired formulas may be used with either order. The solution formed with the C coefficients is of the lower (second) order, and the solution formed with the $\hat{C}$ coefficients is of higher (third) order. (The difference between the two solutions gives an estimate of the local error that forms the basis for the step-size control of the defining algorithm. Because this step-size estimate is already available, only one scaled solution is determined, which is of the higher order.) By inserting one additional derivative evaluation, a fourth-order, scaled algorithm may be generated (although errors of the third order will be present because of the solution obtained by the defining Runge-Kutta algorithm.) An alternate solution, presented in the section entitled "Embedded Third-Order Solutions" generates a third-order, scaled algorithm at no additional derivative expense. Because of the lack of free parameters, no control may be exercised over the truncation error coefficients; hence, this scaled algorithm may not be a particularly suitable third-order method. For this reason, the scaled algorithm requiring one additional derivative evaluation is presented.

The assumptions employed in solving the equations of condition for the third-order algorithm (table III) are

$$P^*_{2,1} = \beta^*_{2,1}\,\alpha^*_1 = \alpha^{*2}_2 / 2 \tag{11a}$$

$$P^*_{3,1} = \beta^*_{3,1}\alpha^*_1 + \beta^*_{3,2}\alpha^*_2 = \alpha^{*2}_3 / 2 \tag{11b}$$

and

$$P^*_{4,1} = \beta^*_{4,1}\alpha^*_1 + \beta^*_{4,2}\alpha^*_2 + \beta^*_{4,3}\alpha^*_3 = \alpha^{*2}_4 / 2 \tag{11c}$$

along with $C_1^* = 0$. (During the development of the defining algorithm, equations (11a) and (11b) (without *) were imposed and, thus, are valid in the scaled system. Otherwise, such assumptions could not be arbitrarily imposed because the α^* and β^* coefficients involved are already determined.)

The parameters for constructing the scaled algorithms are α_4^*, $\beta_{4,1}^*$, $\beta_{4,2}^*$, and $\beta_{4,3}^*$; that is, the remaining coefficients are either scaled values of the defining coefficients or, as in the case of the C^* coefficients, are determined from these parameters. The C^* coefficients may be written as

$$C_2^* = (1/4 - (\alpha_3^* + \alpha_4^*)/3 + \alpha_3^* \alpha_4^*/2) \;/\; (\alpha_2^* (\alpha_2^* - \alpha_3^*)(\alpha_2^* - \alpha_4^*)) \tag{12}$$

with C_3^* and C_4^* determined by permuting indices, and

$$C_0^* = 1 - (C_2^* + C_3^* + C_4^*) \tag{13}$$

The $\beta_{4,K}^*$ coefficients are

$$\beta_{4,1}^* = -(C_2^* \beta_{2,1}^* + C_3^* \beta_{3,1}^*)/C_4^* \tag{14a}$$

$$\beta_{4,2}^* = \frac{P_{4,2}^* - \beta_{4,1}^* \alpha_1^{*2} - \alpha_3^*(\alpha_4^{*2}/2 - \beta_{4,1}^* \alpha_1^*)}{\alpha_2^*(\alpha_2^* - \alpha_3^*)} \tag{14b}$$

and

$$\beta_{4,3}^* = \frac{P_{4,2}^* - \beta_{4,1}^* \alpha_1^{*2} - \alpha_2^*(\alpha_4^{*2}/2 - \beta_{4,1}^* \alpha_1^*)}{\alpha_3^*(\alpha_3^* - \alpha_2^*)} \tag{14c}$$

where

$$\beta^*_{4,2} = (1/12 - c^*_2 \beta^*_{2,1}\alpha_1^{*2} - c^*_3(\beta^*_{3,1}\alpha_1^{*2} + \beta^*_{3,2}\alpha_2^{*2})) \;/\; c^*_4 \qquad (15)$$

with $\beta^*_{4,0}$ computed from equation (3). All other parameters are determined by the generating set of coefficients using equations (8a) and (8b). The solution of the equations of condition is then achieved with α^*_4 remaining as a free parameter. While the scaled algorithm for the RK(2)3 is actually of fourth order, the defining algorithm is of third-order accuracy (or second, if the lower-order solution is used), and hence, the scaled solution will only be as accurate as the computed solution at $t = t_0$, but at least additional errors due to the scaled algorithm will be of higher order. Because $G^*_4 = 0$, the value of α^*_4 is selected to reduce G^*_5 (eq. (9)), involving fifth-order error coefficients $T^*_{5,J}$, and to ensure reasonable magnitudes on all α^*, β^*, and C^* coefficients (as far as possible). These error coefficients are relatively insensitive to changes in α^*_4 for both the RKF(2)3 and the RKT(2)3 formulas. Considering the RKT(2)3 formula, a near minimal value of G^*_5 is achieved by selecting $\alpha^*_4 = 0.60$ for $\sigma \in (0.00, 0.45]$, or $\alpha^*_4 = 0.50$ for $\sigma \in (0.45, 0.70]$, $\alpha^*_4 = 0.30$ for $\sigma \in (0.70, 0.85]$, or $\alpha^*_4 = 0.10$ for $\sigma \in (0.85, 1.00)$. While G^*_5 is nearly minimal for $\alpha^*_4 = 0.60$ for $\sigma \in (0.00, 0.65]$ or $\alpha^*_4 = 0.50$ for $\sigma \in (0.65, 1.00)$ for the RKF(2)3 scaled algorithm, a scaled solution with accuracy equivalent to that of the defining algorithm may be achieved at no additional derivative expense. (See section entitled "Embedded Third-Order Solutions".) Therefore, the scaled RKF(2)3 with one additional derivative evaluation is not recommended. Values of the scaled coefficients for various values of σ are found in table II(b) for the RKT(2)3 algorithm.

FOURTH-ORDER SCALED ALGORITHMS FOR THE RKF(3)4 FORMULAS

The solution of the fourth-order, scaled RK algorithms is quite similar to that of the scaled RK(2)3 methods. Coefficients for a four-stage, fourth-order RK algorithm with an embedded third-order method for error estimation (denoted as RKF(3)4), are presented in table IV(a). The construction of the scaled algorithm for these coefficients requires only one additional derivative evaluation. Equations (11a), (11b), and (11c) are imposed along with

$$\overset{*}{P}_{5,1} = \sum_{i=1}^{4} \overset{*}{\beta}_{5,i} \overset{*}{\alpha}_i = \alpha_5^{*2} / 2 \tag{16}$$

and $\overset{*}{C}_1 = 0$. The free parameters for this algorithm are $\overset{*}{\alpha}_5$ and $\overset{*}{\beta}_{5,4}$. All other coefficients are determined from existing coefficients using equations (8a) or (8b), or from the equations of condition.

The C_i coefficients may be written as

$$\overset{*}{C}_{i+1} = \sum_{j=1}^{4} (a_{i,j} / ((j+1)\overset{*}{\alpha}_{i+1})) \tag{17}$$

for $i = 1, 2, 3$, and 4, where the $a_{i,j}$ coefficients are determined from the Lagrange polynomial:

$$L_i(X) = \frac{\prod_{k=1}^{n}{}' \; (X - \mu_k)}{\prod_{k=1}^{n}{}' \; (\mu_i - \mu_k)} = \sum_{j=1}^{n} a_{i,j} X^{j-1} \tag{18}$$

where $n = 4$ and where the defining set $M = \{\mu_1, \ldots, \mu_n\} = \{\overset{*}{\alpha}_2, \overset{*}{\alpha}_3, \overset{*}{\alpha}_4, \overset{*}{\alpha}_5\}$ with $'$ denoting $k \neq i$.

The unknown $\overset{*}{\beta}_{5,k}$ coefficients may be written as

$$\overset{*}{\beta}_{5,1} = -\left(\sum_{j=1}^{4} \overset{*}{C}_j \overset{*}{\beta}_{j,1} \right) / \overset{*}{C}_5 \tag{19}$$

$$\overset{*}{\beta}_{5,2} = (K_2 - K_1 \overset{*}{\alpha}_3) / \overset{*}{\alpha}_2 (\overset{*}{\alpha}_2 - \overset{*}{\alpha}_3) \tag{20}$$

and

$$\beta^*_{5,3} = (K_2 - K_1 \alpha^*_2) \,/\, \alpha^*_3 (\alpha^*_3 - \alpha^*_2) \tag{21}$$

The K_i, $i=1,2$ are defined by

$$K_i = P^*_{5,i} - \beta^*_{5,1} \alpha_1^{*i} - \beta^*_{5,4} \alpha_4^{*i}$$

with

$$P^*_{5,2} = \left(1/12 - \sum_{j=2}^{4} C^*_j P^*_{j,2}\right) \,/\, C^*_5 \tag{22}$$

$$P^*_{j,2} = \sum_{k=1}^{j-1} \beta^*_{j,k} \alpha_k^{*2} \tag{23}$$

Then $\beta^*_{5,0}$ is determined from equation (3).

While the values of α^*_5 and $\beta^*_{5,4}$ may be varied to reduce G^*_5 (eq. (9)), the system is most sensitive to the α^*_5 parameter. In fact, such a small reduction is achieved in G^*_5 by changing $\beta^*_{5,4}$, that this parameter is set equal to zero in all of the scaled RKF(3)4 algorithms. The α^*_5 parameter, however, must be selected more carefully in order to ensure a nearly minimal value of G^*_5. Values of the scaled coefficients for the RKF(3)4 algorithm, with various values of σ, are presented in table IV(b).

FOURTH- AND FIFTH-ORDER SCALED ALGORITHMS FOR THE RK(4)5 FORMULAS

The complexity of the equations of condition for the fifth-order, scaled algorithm requires that two new derivative evaluations be added. A fourth-order, scaled solution may, however, be achieved by the use of only one additional function evaluation. Quite often, for intermediate data output, the lower-order solution is sufficient and is all that is desired if the defining algorithm is being operated as a fourth-order method. (The fourth-order, scaled solution for the RK(4)5 will vary slightly from that of the RK(3)4 formula because an additional assumption has been imposed and because five derivative evaluations have already been made for the defining algorithm.) For the fourth-order, scaled algorithm, α_6^* is the only free parameter, while for the fifth-order, scaled solution α_6^*, α_7^*, $\beta_{6,2}^*$ and $\beta_{6,5}^*$ remain as free parameters. Coefficients for a six-stage, fifth-order Runge-Kutta algorithm, RKF(4)5 II, developed by Fehlberg (ref. 3), are presented in table V(a).

The simplifying assumptions, imposed during the development of the defining algorithm,

$$P_{j,1}^* = \sum_{k=1}^{j-1} \beta_{j,k}^* \alpha_k^* = \alpha_j^{*2}/2 \tag{24}$$

and

$$P_{j,2}^* = \sum_{k=1}^{j-1} \beta_{j,k}^* \alpha_k^{*2} = \alpha_j^{*3}/3 \tag{25}$$

are imposed, along with $C_1^* = 0$, for $j = 2, \ldots, \rho^*$, where $\rho^* = 6$ for the fourth-order solution and $\rho^* = 7$ for the fifth-order solution.

The C_i^* coefficients may be written as

$$C_{i+1}^* = \sum_{j=1}^{\rho^*-1} a_{i,j}/((j+1)\ \alpha_{i+1}^*) \tag{26}$$

for $i = 1, 2, \ldots, \rho^*-1$, where the $a_{i,j}$ terms are defined by equation (18) on the set $M = \{\alpha_2^*, \alpha_3^*, \ldots, \alpha_{\rho^*}^*\}$. Then

$$C_0^* = 1 - \sum_{j=2}^{\rho} C_j^* \tag{27}$$

For the fourth-order solution,

$$\beta_{6,1} = -\left(\sum_{j=2}^{5} c_j \; \beta_{j,1}\right) / c_6 \tag{28}$$

and

$$\overset{*}{\beta}_{6,2} = -\left(\sum_{j=3}^{5} \overset{*}{c}_j \; \overset{*}{\beta}_{j,2}\right) / \overset{*}{c}_6 \tag{29}$$

Then defining

$$\overset{*}{P}_{j,3} = \sum_{k=1}^{j-1} \overset{*}{\beta}_{j,k} \, \overset{*}{\alpha}_k^{3} \tag{30}$$

$$\overset{*}{P}_{6,3} = (1/20 - \left(\sum_{j=2}^{5} \overset{*}{c}_j \, \overset{*}{P}_{j,3}\right) / \overset{*}{c}_6 \tag{31}$$

and

$$\overset{*}{\beta}_{6,3} = (K_3 - K_2(\overset{*}{\alpha}_4 + \overset{*}{\alpha}_5) + K_1 \, \overset{*}{\alpha}_4 \, \overset{*}{\alpha}_5)/D_3 \tag{32}$$

where

$$K_i = \overset{*}{P}_{6,i} - \overset{*}{\beta}_{6,1} \, \overset{*}{\alpha}_1^{i} - \overset{*}{\beta}_{6,2} \, \overset{*}{\alpha}_2^{i}$$

with

$$D_3 = \overset{*}{\alpha}_3 \, (\overset{*}{\alpha}_3 - \overset{*}{\alpha}_4) \, (\overset{*}{\alpha}_3 - \overset{*}{\alpha}_5)$$

where $\overset{*}{\beta}_{6,4}$ and $\overset{*}{\beta}_{6,5}$ are determined by permuting the indices on $\overset{*}{\alpha}_3$, $\overset{*}{\alpha}_4$, and $\overset{*}{\alpha}_5$. Then $\overset{*}{\beta}_{6,0}$ can be evaluated from equation (3). (While equation (29) is not requir. for a fourth-order solution, its inclusion helps reduce higher order terms used in forming $\overset{*}{G}_5$, equation (9).)

For the fifth-order, scaled RKF(4)5 formulas, with the $\overset{*}{C}_i$ coefficients determined from equation (26),

$$\overset{*}{\beta}_{6,1} = (K_1 - K_0\,\overset{*}{\alpha}_7)/\overset{*}{C}_6(\overset{*}{\alpha}_6 - \overset{*}{\alpha}_7) \tag{33}$$

and

$$\overset{*}{\beta}_{7,1} = (K_1 - K_0\,\overset{*}{\alpha}_6)/\overset{*}{C}_7\,(\overset{*}{\alpha}_7 - \overset{*}{\alpha}_6) \tag{34}$$

where

$$K_i = -\left(\sum_{j=2}^{5} \overset{*}{C}_j\,\overset{*}{\alpha}_j^{\,i}\,\overset{*}{\beta}_{j,1}\right)$$

for $i = 1,2$ and

$$\overset{*}{\beta}_{7,2} = -\left(\sum_{j=3}^{6} \overset{*}{C}_j\,\overset{*}{\beta}_{j,2}\right)/\overset{*}{C}(7) \tag{35}$$

where $\overset{*}{\beta}_{6,2}$ is a free parameter. With $\overset{*}{\beta}_{6,5}$ as an additional free parameter, the remaining $\overset{*}{\beta}_{6,k}$ coefficients may be written as

$$\overset{*}{\beta}_{6,3} = (K_2 - K_1\,\overset{*}{\alpha}_4)/\overset{*}{\alpha}_3(\overset{*}{\alpha}_3 - \overset{*}{\alpha}_4) \tag{36}$$

and

$$\overset{*}{\beta}_{6,4} = (K_2 - K_1\,\overset{*}{\alpha}_3)/\overset{*}{\alpha}_4(\overset{*}{\alpha}_4 - \overset{*}{\alpha}_3) \tag{37}$$

where

$$K_1 = \overset{*}{P}_{6,1} - \overset{*}{\beta}_{6,1}\,\overset{*}{\alpha}_1^{\,1} - \overset{*}{\beta}_{6,2}\overset{*}{\alpha}_2^{\,1} - \overset{*}{\beta}_{6,5}\overset{*}{\alpha}_5^{\,1}$$

Using the definition for $\overset{*}{P}_{j,3}$, equation (30), for $j = 2, 3, \ldots, 6$

$$\overset{*}{P}_{7,3} = \left(1/20 - \sum_{j=2}^{6} \overset{*}{C}_j\,\overset{*}{P}_{j,3}\right)/\overset{*}{C}_7 \tag{38}$$

Using the values of $\overset{*}{P}_{7,1}$, $\overset{*}{P}_{7,2}$ and $\overset{*}{P}_{7,3}$ and defining

$$S_j = \overset{*}{C}_j \sum_{k=3}^{j-1} \overset{*}{\beta}_{j,k}\,\overset{*}{\beta}_{k,1} \,/\, \overset{*}{C}_7 \tag{39}$$

for $j = 4, 5, 6$, and 7,

$$\overset{*}{\beta}_{7,6} = \frac{-\left(\sum_{j=4}^{6} S_j\right) - \left(\sum_{j=3}^{5} A_j\,\overset{*}{\beta}_{j,1}\right)}{\sum_{j=3}^{6} B_j\,\overset{*}{\beta}_{j,1}} \tag{40}$$

where

$$A_3 = (K_3 - K_2(\overset{*}{\alpha}_4 + \overset{*}{\alpha}_5) + K_1\,\overset{*}{\alpha}_4\,\overset{*}{\alpha}_5)/\alpha_3(\alpha_3 - \overset{*}{\alpha}_4)(\alpha_3 - \overset{*}{\alpha}_5)$$

and

$$B_3 = -\overset{*}{\alpha}_6(\overset{*}{\alpha}_6^{\,2} - \overset{*}{\alpha}_6(\overset{*}{\alpha}_4 + \overset{*}{\alpha}_5) + \overset{*}{\alpha}_4\,\overset{*}{\alpha}_5)/\overset{*}{\alpha}_3(\overset{*}{\alpha}_3 - \overset{*}{\alpha}_4)(\overset{*}{\alpha}_3 - \overset{*}{\alpha}_4)$$

and

$$K_1 = \overset{*}{P}_{7,1} - \overset{*}{\beta}_{7,1}\,\overset{*}{\alpha}_1^{\,1} - \overset{*}{\beta}_{7,2}\,\overset{*}{\alpha}_2^{\,1}$$

with A_4, A_5, B_4, and B_5 determined by permuting the indices of the $\overset{*}{\alpha}$'s. ($B_6 \equiv 1.$) Then

$$\overset{*}{\beta}_{7,j} = A_j + B_j \overset{*}{\beta}_{7,6}$$

for $j = 3,4$ and 5, and $\overset{*}{\beta}_{7,0}$ may be evaluated from equation (3).

The fourth-order solution is not extremely sensitive to the selection of $\overset{*}{\alpha}_6$ so that acceptable error coefficients may be determined from a given $\overset{*}{\alpha}_6$ for a range of σ. For $\sigma \in (0.00, 0.30]$, $\overset{*}{\alpha}_6 = 0.60$, while for $\sigma \in (0.30, 0.55]$, $\alpha_6 = 0.40$, and for $\sigma \in (0.55, 1.00)$, $\alpha_6 = 1.00$. The parameters for the fifth-order algorithm, however, must be selected carefully in order to ensure a near-minimal value of $\overset{*}{G}_6$ (eq. (9)). The fifth-order solution is most sensitive to the $\overset{*}{\alpha}_6$ and $\overset{*}{\alpha}_7$ parameters, and (in general) $\overset{*}{\beta}_{6,2} = \overset{*}{\beta}_{6,5} = 0$ will generate a value of $\overset{*}{G}_6$ that is sufficiently close to $\overset{*}{G}_{MIN}$. Because of this, further variations in the parameters lead only to insignificant changes in $\overset{*}{G}_6$ and complicate attempts to generate coefficients in rational form. Values of the fourth-order, scaled coefficients for RKF(4)5 II are presented in table V(b), while those for the fifth-order, scaled algorithm are listed in table V(c).

EMBEDDED THIRD-ORDER SOLUTIONS AT NO ADDITIONAL DERIVATIVE EXPENSE

Embedded within each RK formula presented is a third-order, scaled algorithm that may be obtained using no additional derivative evaluations. While new C* coefficients must be generated for each value of σ, these computations are relatively inexpensive. However, because no additional derivative evaluations are imposed, no free parameters are available to reduce the nonzero truncation error coefficients. Thus, while these algorithms are of third order, the truncation error coefficients for the fourth order (and higher orders) may not be small (as they are in the case of the defining algorithms). If a low-order estimate of the solution is sufficient, this third-order algorithm may be useful because it can easily be generated for any value of σ at no additional derivative expense.

A minimum of two nonzero C_k^* coefficients, $k \geq 2$, is required to generate a third-order, scaled algorithm. The additional derivative evaluations used in the RK(3)4 and RK(4)5 formulas allow a choice of one (or two) more nonzero C_k^*'s for causing higher-order truncation error coefficients to vanish. However, the norm of the error coefficients G_4^*, G_5^*, and G_6^* (eq. (9)) is generally smaller (for the algorithms presented) when only two C_k^* coefficients are nonzero. For the RK(2)3 algorithm

$$C_i^* = \sigma(2\sigma - 3\alpha_j)/(6\ \alpha_i(\alpha_i - \alpha_j)) \tag{41}$$

For $i=2, j=3$ and $i=3, j=2$. For three nonzero C_i^*,

$$C_i^* = \frac{\sigma(3\sigma^2 - 4\sigma(\alpha_j + \alpha_k) + 6\ \alpha_j\ \alpha_k)}{12\ \alpha_i\ (\alpha_i - \alpha_j)\ (\alpha_i - \alpha_k)} \tag{42}$$

where $(i,j,k) \in \{ (2,3,4),(3,4,2),(4,2,3) \}$ for the scaled RKF(3)4 algorithm. (In general, however, equation (41) should be used for the RKF(3)4 and RKF(4)5 formulas presented, although the nonzero C_k^*'s need not be C_2^* and C_3^*.)

A case of particular interest occurs for the RKF(4)5 formulas. Using equation (42) to generate the C_k^* coefficients as a function of σ, the one nonzero fourth-order truncation error coefficient may be written as a quadratic in σ. Thus, two values of the parameter σ exist that will give a fourth-order algorithm. For the RKF(4)5 formulas, $\sigma = 0.60$ and $\sigma = 1.0$ give a fourth-order, scaled Runge-Kutta algorithm regardless of the defining set of coefficients or the selection of C_k^* coefficients. Because a fourth- and a fifth-order solution are already available at $\sigma = 1.0$, however, the only value of interest is $\sigma = 0.60$.

If the C_2^* and C_3^* coefficients are determined by equation (41) for the RKF(2)3 algorithm, the fourth-order, scaled truncation error coefficients are of the same order of magnitude as those for the defining algorithm, $G_4^* = 0.5144E-01$. Thus, in using the RKF(2)3 algorithm, no additional derivative evaluations should be used. Rather, the solution determined by equation (41) should be employed. For the RKT(2)3 formula, the fourth-order truncation error coefficients are considerably smaller: $G_4^* = 0.1447E{-}03$. While a solution obtained by using equation (41) is of third order, a scaled solution may not be as accurate as the solution obtained by the defining algorithm that has much lower fourth-order error coefficients. While this scaled solution is quite accurate near t_0 and t_0+h (for $\sigma = 0.05$, $G_4^* = 0.4158E{-}03$ and for $\sigma = 0.95$, $G_4^* = 0.4533E{-}02$), the error terms may be as large as $G_4^* = 0.1665E{-}01$ throughout the interval. However, the scaled solution presented in the section entitled "Third-Order Scaled Algorithms" is actually of fourth order. Thus, for some problems, the additional derivative evaluation may produce a more accurate scaled solution. The simplicity of the scaled solution offered by in equation (41), however, may compensate for the reduced accuracy.

A study of the RKF(3)4 algorithm shows that equation (41), defined for C_2^* and C_4^*, produces the most accurate third-order, scaled solution for $\sigma \leq 0.65$. However, for $\sigma > 0.65$, using a C_2^*, C_3^* solution with $C_4^* = 0$, will give the best results. (Eq. (42) produces a G_4^* that is approximately twice the magnitude of the best solution obtained when using eq. (41).) For $\sigma \leq 0.05$, $G_4^* \leq 0.3183E{-}03$. Throughout the interval, however, G_4^* may be as large as $0.9390E{-}02$.

The most important feature of the low-order, scaled RKF(4)5 formulas is the existence of a fourth-order solution at $\sigma = 0.60$, which requires no additional derivative evaluation. This fourth-order solution with the C_k^*'s, defined by equation (42) (using C_2^*, C_3^*, and C_5^*), gives $G_5^* = 0.1978E{-}02$. For other values of σ, a good third-order, scaled solution may be obtained using equation (41) for C_2^* and C_3^* for all values of σ, although for $\sigma \in (0.35, 0.55)$, a slightly better solution may be obtained using equation (41) to determine C_2^* and C_5^*. For $\sigma \leq 0.05$, $G_4^* \leq 0.2884E{-}03$ with $G_4^* \leq 0.5376E{-}02$ for $\sigma \leq 0.75$. For $\sigma > 0.75$, $G_4^* \leq 0.2080E{-}01$.

APPLICATION OF THE SCALED ALGORITHM

The most direct manner of applying the scaled Runge-Kutta algorithm is to demand data output at a specific value of the independent variable. With some maneuvering, however, numerical integration codes can be written that will output information by specifying values of a dependent variable or by satisfying a constraint equation. While the scaled algorithms have been developed for a wide range of σ (σ = 0.05K, K = 1, ..., 19), data will generally be requested for values of the independent variable $t = t_0 + \sigma h$, for values of σ that were not studied. The simplest manner of applying the algorithm would be to reduce the original step size slightly so that the requested output point corresponds to a $t = t_0 + \sigma h$ for a σ of a determined algorithm. For example, if the solution is required at $t = t_0 + 0.87\ h_1$ (h_1 being the predicted step size), the user would let the step size be $h = 0.87\ h_1/0.90$, which is less than a 4-percent reduction from the predicted step size. For output near t_0, the percentage of step-size reduction may be as large as 50 percent with no scaled output possible for $\sigma < 0.05$ without generating the coefficients from the scaled algorithm formulas. These problems near $t = t_0$ may be skirted in a number of ways; for example, by interpolation or by using the third-order algorithm that requires no derivative evaluations, which is valid for any choice of σ and which has low fourth-order truncation error coefficients for σ near zero.

Difficulties in applying the algorithm in such a manner may arise (1) if the value of the independent variable for data output is not known before the step is taken, (2) if storage limitations restrict the number of scaled coefficients that may be retained, or (3) if dense output is requested within the integration step. In such cases, an interpolating polynomial may be generated, using points determined by evaluating the scaled algorithm for several values of σ. Since the solution and its derivative are known at t_0 and $t_0 + h$, a fourth-order polynomial may be determined by applying a scaled algorithm at only one point. For the RKF(4)5 algorithm, a fourth-order solution is available for $\sigma = 0.60$ requiring no additional derivative evaluations. Thus, an interpolating polynomial may be generated quite inexpensively. (If output is dense in one region of the interval, a different interpolating polynomial generated by scaled solutions in the neighborhood of this region would be more accurate. However, these points can be generated inexpensively, compared to repeated Runge-Kutta solutions at small step sizes.) The disadvantage of using the interpolating polynomial is that the solution is subject to errors from the polynomial as well as from the Runge-Kutta solution; whereas, the scaled solution is equivalent to a Taylor series solution of corresponding order.

While the scaled algorithm is designed to determine information at specified values of the independent variable, the methods may be adapted to stop the integration by specifying values of a dependent variable or by satisfying a constraint equation. The usual technique for stopping the integration on a specified value of a function is to bracket the desired solution and to iterate until the value of the function is reached. In order to be operated efficiently, however, a Runge-Kutta method must take large step sizes (consistent with truncation error criteria). While using a Runge-Kutta method to trap a solution with smaller and smaller step sizes will produce an accurate solution, the expense of computing the derivative evaluations may be prohibitive. The use of scaled algorithms to insert data points allows the formulation of an interpolating polynomial. This polynomial may be analyzed efficiently to determine the value of the independent variable, which causes the constraining function to be satisfied. If the accuracy of the Runge-Kutta solution is required; i.e., if the inaccuracies introduced by the interpolating polynomial produce an unacceptable solution, an estimate may be used to determine the next step size in an attempt to satisfy the constraining function. The scaled solution in the iterative procedure can improve the convergence efficiency, particularly if the step size has been sufficiently reduced so that a lower order Runge-Kutta algorithm may be used.

CONCLUSIONS

Scaled Runge-Kutta algorithms have been developed for the third through fifth orders, which will evaluate the solution of an ODE at an intermediate point within the integration step. The expense required to output this data ranges from a few arithmetic operations to two derivative evaluations, depending on the order of the solution desired and the defining algorithm. The new scaled solutions depend on the defining algorithm, where the coefficients and determined derivative evaluations form the core of the new scaled system. By making slight adjustments to the integration step sizes, data can be determined for any point within the integration interval, using the algorithms developed for the given values of σ. If a priori knowledge of the data output point is unavailable, or if dense output is required within a given step, the scaled algorithms may be used to formulate an interpolating polynomial for data determination or may be used with an iterative procedure to reach the required output point with only minor losses in efficiency.

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2. Horn, M. K.: Developments in High-Order Runge-Kutta Nystroom Formulas. TICOM Report 77-11, Univ. of Texas at Austin, Oct. 1977.

3. Fehlberg, Erwin: Low-Order Classical Runge-Kutta Formulas with Stepsize Control and Their Application to Some Heat Transfer Problems. NASA TR R-315, July 1968.

TABLE I.- COEFFICIENTS FOR RKF(2)3 ACCORDING TO FEHLBERG (REF. 3)

$k \backslash \lambda$	α_k	$\beta_{k\lambda}$ 0	1	2	c_k	$\hat{c}_k$
0	0				$\frac{214}{891}$	$\frac{533}{2106}$
1	$\frac{1}{4}$	$\frac{1}{4}$			$\frac{1}{33}$	0
2	$\frac{27}{40}$	$-\frac{189}{800}$	$\frac{729}{800}$		$\frac{650}{891}$	$\frac{800}{1053}$
3	1	$\frac{214}{891}$	$\frac{1}{33}$	$\frac{650}{891}$	0	$-\frac{1}{78}$

EUCLIDEAN NORM OF TRUNCATION ERROR COEFFICIENTS (TEC), G_4 = 0.5144159D-01

STABILITY LIMITS R: (-0.24544D+01, 0.0)
I: (0.0, 0.17908D+01)

TABLE II.- COEFFICIENTS FOR RKT(2)3

(a) Unscaled algorithm according to Bettis[1]

k \ λ	α_k	$\beta_{k\lambda}$			c_k	$\hat{c}_k$
		0	1	2		
0					0	$\frac{1}{6}$
1	$\frac{1}{2}$	$\frac{1}{2}$			0	0
2	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$		1	$\frac{2}{3}$
3	1	0	$-\frac{143}{144}$	$\frac{287}{144}$	0	$\frac{1}{6}$

EUCLIDEAN NORM OF TEC G_4 = 0.1446759D-03

STABILITY LIMITS R: (-0.2791000D+01, 0)
I: (0, 0.2827170D+01)

[1]Private communication.

TABLE II.- Continued

(b) Scaled algorithm

SIGMA = 0.500000D-01; A(4) = 3./5.	
C0 =	419 / 2 400
C1 =	0
C2 =	37 / 56 400
C3 =	- 17 / 232 800
C4 =	45 125 / 54 708
B40 =	271 891 / 475 000
B41 =	- 74 533 / 12 996 000
B42 =	13 948 237 / 324 900 000
B43 =	- 4 559 / 475 000
SCALED NORM OF TEC	G*(5) = 0.7156214D-05
SCALED STABILITY LIMITS	R: (-0.9478000D+01, 0)
	I: (0, 0.1175900D+02)

SIGMA = 0.100000D+00; A(4) = 3./5.	
C0 =	109 / 600
C1 =	0
C2 =	17 / 6 600
C3 =	- 7 / 28 200
C4 =	3 375 / 4 136
B40 =	30 691 / 56 250
B41 =	- 1 591 / 145 800
B42 =	1 525 007 / 18 225 000
B43 =	- 517 / 28 125
SCALED NORM OF TEC	G*(5) = 0.5347556D-04
SCALED STABILITY LIMITS	R: (-0.6646000D+01, 0)
	I: (0, 0.8300200D+01)

TABLE II.- Continued

(b) Continued

SIGMA = 0.150000D+00; A(4) = 3./5.	
C0 =	451 / 2 400
C1 =	0
C2 =	93 / 16 400
C3 =	- 33 / 72 800
C4 =	36 125 / 44 772
B40 =	220 857 / 425 000
B41 =	- 53 521 / 3 468 000
B42 =	10 586 569 / 86 700 000
B43 =	- 11 193 / 425 000
SCALED NORM OF TEC	G*(5) = 0.1683013D-03
SCALED STABILITY LIMITS	R: (-0.5412667D+01, 0)
	I: (0, 0.6766133D+01)
SIGMA = 0.200000D+00; A(4) = 3./5.	
C0 =	29 / 150
C1 =	0
C2 =	14 / 1 425
C3 =	- 1 / 1 650
C4 =	500 / 627
B40 =	3 091 / 6 250
B41 =	- 2 761 / 144 000
B42 =	568 993 / 3 600 000
B43 =	- 209 / 6 250
SCALED NORM OF TEC	G*(5) = 0.3713881D-03
SCALED STABILITY LIMITS	R: (-0.4690500D+01, 0)
	I: (0, 0.5851150D+01)

TABLE II.- Continued

(b) Continued

SIGMA = 0.250000D+00; A(4) = 3./5.

C0 =	19 / 96
C1 =	0
C2 =	5 / 336
C3 =	- 1 / 1 632
C4 =	375 / 476
B40 =	1 411 / 3 000
B41 =	- 7 121 / 324 000
B42 =	12 397 / 64 800
B43 =	- 119 / 3 000
SCALED NORM OF TEC	G*(5) = 0.6741268D-03
SCALED STABILITY LIMITS	R: (-0.4207200D+01, 0)
	I: (0, 0.5226880D+01)

SIGMA = 0.300000D+00; A(4) = 3./5.

C0 =	121 / 600
C1 =	0
C2 =	33 / 1 600
C3 =	- 3 / 8 200
C4 =	6 125 / 7 872
B40 =	19 557 / 43 750
B41 =	- 869 / 36 750
B42 =	101 803 / 459 375
B43 =	- 984 / 21 875
SCALED NORM OF TEC	G*(5) = 0.1080755D-02
SCALED STABILITY LIMITS	R: (-0.3858333D+01, 0)
	I: (0, 0.4766600D+01)

TABLE II.- Continued

(b) Continued

SIGMA = 0.350000D+00; A(4) = 3./5.

C0 =	491 / 2 400
C1 =	0
C2 =	931 / 34 800
C3 =	49 / 189 600
C4 =	21 125 / 27 492
B40 =	138 013 / 325 000
B41 =	- 29 099 / 1 216 800
B42 =	37 812 607 / 152 100 000
B43 =	- 16 037 / 325 000

SCALED NORM OF TEC G*(5) = 0.1589552D-02
SCALED STABILITY LIMITS R: (-0.3594286D+01, 0)
I: (0, 0.4409743D+01)

SIGMA = 0.400000D+00; A(4) = 3./5.

C0 =	31 / 150
C1 =	0
C2 =	32 / 975
C3 =	2 / 1 425
C4 =	375 / 494
B40 =	3 781 / 9 375
B41 =	- 1 817 / 81 000
B42 =	550 433 / 2 025 000
B43 =	- 494 / 9 375

SCALED NORM OF TEC G*(5) = 0.2194011D-02
SCALED STABILITY LIMITS R: (-0.3387250D+01, 0)
I: (0, 0.4123225D+01)

TABLE II.- Continued

(b) Continued

SIGMA = 0.450000D+00; A(4) = 3./5.

C0 =	499 / 2 400
C1 =	0
C2 =	351 / 9 200
C3 =	189 / 58 400
C4 =	15 125 / 20 148
B40 =	105 339 / 275 000
B41 =	- 9 061 / 484 000
B42 =	3 516 493 / 12 100 000
B43 =	- 15 111 / 275 000

SCALED NORM OF TEC	G*(5) = 0.2883997D-02
SCALED STABILITY LIMITS	R: (-0.3222000D+01, 0)
	I: (0, 0.3887333D+01)

SIGMA = 0.500000D+00; A(4) = 1./2.

C0 =	1 / 6
C1 =	0
C2 =	1 / 6
C3 =	0
C4 =	2 / 3
B40 =	5 / 16
B41 =	- 1 / 8
B42 =	3 / 8
B43 =	- 1 / 16

SCALED NORM OF TEC	G*(5) = 0.3654457D-02
SCALED STABILITY LIMITS	R: (-0.3088000D+01, 0)
	I: (0, 0.3689480D+01)

TABLE II.- Continued

(b) Continued

SIGMA = 0.550000D+00; A(4) = 1./2.	
C0 =	209 / 1 200
C1 =	0
C2 =	121 / 600
C3 =	121 / 34 800
C4 =	18 / 29
B40 =	1 421 / 4 800
B41 =	- 4 279 / 31 104
B42 =	317 251 / 777 600
B43 =	- 319 / 4 800
SCALED NORM OF TEC	G*(5) = 0.4443751D-02
SCALED STABILITY LIMITS	R: (-0.2978545D+01, 0)
	I: (0, 0.3521364D+01)

SIGMA = 0.600000D+00; A(4) = 1./2.	
C0 =	9 / 50
C1 =	0
C2 =	6 / 25
C3 =	3 / 350
C4 =	4 / 7
B40 =	7 / 25
B41 =	- 173 / 1 152
B42 =	12 677 / 28 800
B43 =	- 7 / 100
SCALED NORM OF TEC	G*(5) = 0.5264721D-02
SCALED STABILITY LIMITS	R: (-0.2889667D+01, 0)
	I: (0, 0.3377233D+01)

TABLE II.- Continued

(b) Continued

SIGMA = 0.650000D+00; A(4) = 1./2.	
C0 =	221 / 1 200
C1 =	0
C2 =	169 / 600
C3 =	169 / 10 800
C4 =	14 / 27
B40 =	423 / 1 600
B41 =	- 1 313 / 8 064
B42 =	95 069 / 201 600
B43 =	- 117 / 1 600
SCALED NORM OF TEC	G*(5) = 0.6105116D-02
SCALED STABILITY LIMITS	R: (-0.2818000D+01, 0)
	I: (0, 0.3253138D+01)

SIGMA = 0.700000D+00; A(4) = 1./2.	
C0 =	14 / 75
C1 =	0
C2 =	49 / 150
C3 =	49 / 1 950
C4 =	6 / 13
B40 =	299 / 1 200
B41 =	- 455 / 2 592
B42 =	32 543 / 64 800
B43 =	- 91 / 1 200
SCALED NORM OF TEC	G*(5) = 0.6957139D-02
SCALED STABILITY LIMITS	R: (-0.2761857D+01, 0)
	I: (0, 0.3146329D+01)

TABLE II.- Continued

(b) Continued

SIGMA = 0.750000D+00; A(4) = 3./10.

C0 =	7 / 48
C1 =	0
C2 =	51 / 88
C3 =	15 / 496
C4 =	250 / 1 023
B40 =	837 / 8 000
B41 =	- 30 079 / 48 000
B42 =	9 119 / 9 600
B43 =	- 1 023 / 8 000

SCALED NORM OF TEC	G*(5) = 0.7719994D-02
SCALED STABILITY LIMITS	R: (-0.2720267D+01, 0)
	I: (0, 0.3055080D+01)

SIGMA = 0.800000D+00; A(4) = 3./10.

C0 =	4 / 25
C1 =	0
C2 =	608 / 975
C3 =	68 / 1 425
C4 =	125 / 741
B40 =	437 / 6 250
B41 =	- 14 473 / 18 000
B42 =	536 497 / 450 000
B43 =	- 494 / 3 125

SCALED NORM OF TEC	G*(5) = 0.8401091D-02
SCALED STABILITY LIMITS	R: (-0.2693250D+01, 0)
	I: (0, 0.2978425D+01)

TABLE II.- Continued

(b) Continued

SIGMA = 0.850000D+00; A(4) = 3./10.

C0 =	203 / 1 200
C1 =	0
C2 =	19 363 / 29 400
C3 =	12 427 / 178 800
C4 =	750 / 7 301
B40 =	9 983 / 600 000
B41 =	- 283 883 / 259 200
B42 =	51 368 611 / 32 400 000
B43 =	- 124 117 / 600 000

SCALED NORM OF TEC G*(5) = 0.9969993D-02
SCALED STABILITY LIMITS R: (-0.2681647D+01, 0)
I: (0, 0.2916259D+01)

SIGMA = 0.900000D+00; A(4) = 1./10.

C0 =	7 / 50
C1 =	0
C2 =	1 431 / 2 050
C3 =	216 / 2 275
C4 =	250 / 3 731
B40 =	- 6 643 / 50 000
B41 =	- 3 991 / 3 000
B42 =	134 029 / 75 000
B43 =	- 11 193 / 50 000

SCALED NORM OF TEC G*(5) = 0.9706279D-02
SCALED STABILITY LIMITS R: (-0.2688444D+01, 0)
I: (0, 0.2869311D+01)

TABLE II.- Concluded

(b) Concluded

SIGMA = 0.9500000+00; A(4) = 1./10.		
C0 =	197 / 1 200	
C1 =	0	
C2 =	11 191 / 16 200	
C3 =	27 797 / 217 200	
C4 =	250 / 14 661	
B40 =	- 74 753 / 200 000	
B41 =	- 45 049 / 16 000	
B42 =	1 501 437 / 400 000	
B43 =	- 92 853 / 200 000	
SCALED NORM OF TEC		G*(5) = 0.1094176D-01
SCALED STABILITY LIMITS		R: (-0.2718947D+01, 0)
		I: (0, 0.2839242D+01)

TABLE III.- TRUNCATION ERROR COEFFICIENTS FOR RUNGE-KUTTA ALGORITHMS THROUGH ORDER FIVE

$$T_{1,1} = \sum_{i=0}^{\rho} c_i - 1$$

$$T_{2,1} = \left(\sum_{i=1}^{\rho} c_i \alpha_i - \frac{1}{2} \right)$$

$$T_{3,1} = \frac{1}{2} \left(\sum_{i=1}^{\rho} c_i \alpha_i^2 - \frac{1}{3} \right)$$

$$T_{3,2} = \sum_{i=2}^{\rho} c_i \sum_{j=1}^{i-1} \beta_{i,j} \alpha_j - \frac{1}{6}$$

$$T_{4,1} = \frac{1}{6} \left(\sum_{i=1}^{\rho} c_i \alpha_i^3 - \frac{1}{4} \right)$$

$$T_{4,2} = \sum_{i=2}^{\rho} c_i \alpha_i \sum_{j=1}^{i-1} \beta_{i,j} \alpha_j - \frac{1}{8}$$

$$T_{4,3} = \frac{1}{2} \left(\sum_{i=2}^{\rho} c_i \sum_{j=1}^{i-1} \beta_{i,j} \alpha_j^2 - \frac{1}{12} \right)$$

$$T_{4,4} = \sum_{i=3}^{\rho} c_i \sum_{j=2}^{i-1} \beta_{i,j} \sum_{k=1}^{j-1} \beta_{j,k} \alpha_k - \frac{1}{24}$$

$$T_{5,1} = \frac{1}{24} \left(\sum_{i=1}^{\rho} c_i \alpha_i^4 - \frac{1}{5} \right)$$

TABLE III.- Concluded

$$T_{5,2} = \frac{1}{2}\left(\sum_{i=2}^{\rho} C_i \alpha_i^2 \sum_{j=1}^{i-1} \beta_{i,j} \alpha_j - \frac{1}{10}\right)$$

$$T_{5,3} = \frac{1}{2}\left(\sum_{i=2}^{\rho} C_i \alpha_i \sum_{j=1}^{i-1} \beta_{i,j} \alpha_j^2 - \frac{1}{15}\right)$$

$$T_{5,4} = \sum_{i=3}^{\rho} C_i \alpha_i \sum_{j=2}^{i-1} \beta_{i,j} \sum_{k=1}^{j-1} \beta_{j,k} \alpha_k - \frac{1}{30}$$

$$T_{5,5} = \frac{1}{6}\left(\sum_{i=2}^{\rho} C_i \sum_{j=1}^{i-1} \beta_{i,j} \alpha_j^3 - \frac{1}{20}\right)$$

$$T_{5,6} = \sum_{i=3}^{\rho} C_i \sum_{j=2}^{i-1} \beta_{i,j} \alpha_j \sum_{k=1}^{j-1} \beta_{j,k} \alpha_k - \frac{1}{40}$$

$$T_{5,7} = \frac{1}{2}\left(\sum_{i=3}^{\rho} C_i \sum_{j=2}^{i-1} \beta_{i,j} \sum_{k=1}^{j-1} \beta_{j,k} \alpha_k^2 - \frac{1}{60}\right)$$

$$T_{5,8} = \sum_{i=4}^{\rho} C_i \sum_{j=3}^{i-1} \beta_{i,j} \sum_{k=2}^{j-1} \beta_{j,k} \sum_{l=1}^{k-1} \beta_{k,l} \alpha_l - \frac{1}{120}$$

$$T_{5,9} = \frac{1}{2}\left(\sum_{i=2}^{\rho} C_i \left(\sum_{j=1}^{i-1} \beta_{i,j} \alpha_j\right)^2 - \frac{1}{20}\right)$$

TABLE IV.- COEFFICIENTS FOR RKF(3)4

(a) Unscaled algorithm

$k \backslash \lambda$	α_k	$\beta_{k\lambda}$ 0	1	2	3	c_k	$\hat{c}_k$
0	0					$\frac{1}{6}$	$\frac{43}{288}$
1	$\frac{1}{4}$	$\frac{1}{4}$				0	0
2	$\frac{4}{9}$	$\frac{4}{81}$	$\frac{32}{81}$			$\frac{27}{52}$	$\frac{243}{416}$
3	$\frac{6}{7}$	$\frac{57}{98}$	$-\frac{432}{343}$	$\frac{1053}{686}$		$\frac{49}{156}$	$\frac{343}{1872}$
4	1	$\frac{1}{6}$	0	$\frac{27}{52}$	$\frac{49}{156}$	0	$\frac{1}{12}$

EUCLIDEAN NORM OF TEC $\quad$ G(5) = 0.7141547D-02

STABILITY LIMITS $\quad$ R: (-0.4109200D+01, 0)
I: (0, 0.3214450D+01)

TABLE IV.- COEFFICIENTS FOR SCALED RKF(3)4

(b) Scaled algorithm

SIGMA = 0.500000D-01; A(5) = 2./3.

C0 =	2 872 939	/	11 520 000
C1 =		0	
C2 =	- 12 393	/	83 200 000
C3 =	1 495 823	/	12 954 240 000
C4 =	- 3 911	/	69 600 000
C5 =	150 651	/	200 680
B50 =	29 852 683	/	47 574 000
B51 =	1 798	/	330 375
B52 =	1 063 221	/	22 906 000
B53 =	- 3 924 851	/	309 231 000
B54 =		0	

SCALED NORM OF TEC — G*(5) = 0.5603873D-05

SCALED STABILITY LIMITS — R: (-0.1011400D+02, 0)
I: (0, 0.1256900D+02)

SIGMA = 0.100000D+00; A(5) = 16./25.

C0 =	517 603	/	2 304 000
C1 =		0	
C2 =	1 152 549	/	890 240 000
C3 =	- 333 739	/	1 299 168 000
C4 =	167	/	1 755 000
C5 =	20 663 828 125	/	26 690 107 392
B50 =	4 463 312 576 104	/	7 748 935 546 875
B51 =	- 27 858 100 608	/	2 582 978 515 625
B52 =	3 330 510 639 192	/	33 578 720 703 125
B53 =	- 2 456 985 733 216	/	100 736 162 109 375
B54 =		0	

SCALED NORM OF TEC — G*(5) = 0.4156062D-04

SCALED STABILITY LIMITS — R: (-0.7095000D+01, 0)
I: (0, 0.8867800D+01)

TABLE IV.- Continued

(b) Continued

SIGMA = 0.150000D+00; A(5) = 16./25.

C0 =	43 261 / 192 000
C1 =	0
C2 =	2 184 813 / 1 630 720 000
C3 =	362 551 / 554 112 000
C4 =	- 6 687 / 18 080 000
C5 =	5 838 359 375 / 7 552 290 816
B50 =	393 604 655 104 / 729 794 921 875
B51 =	1 855 138 176 / 729 794 921 875
B52 =	1 241 843 094 576 / 9 487 333 984 375
B53 =	- 310 926 657 216 / 9 487 333 984 375
B54 =	0

SCALED NORM OF TEC G*(5) = 0.1296665D-03
SCALED STABILITY LIMITS R: (-0.5783333D+01, 0)
I: (0, 0.7225667D+01)

SIGMA = 0.200000D+00; A(5) = 1./8.

C0 =	- 262 897 / 180 000
C1 =	0
C2 =	29 671 029 / 196 300 000
C3 =	- 17 356 829 / 272 610 000
C4 =	42 103 / 1 462 500
C5 =	48 249 856 / 20 582 055
B50 =	16 014 162 551 / 72 374 784 000
B51 =	- 56 240 379 / 188 476 000
B52 =	73 783 218 033 / 313 624 064 000
B53 =	- 15 586 385 647 / 470 436 096 000
B54 =	0

SCALED NORM OF TEC G*(5) = 0.7441866D-03
SCALED STABILITY LIMITS R: (-0.5017500D+01, 0)
I: (0, 0.6246150D+01)

TABLE IV.- Continued

(b) Continued

SIGMA = 0.250000D+00; A(5) = 3./5.

C0 =	11 057 / 55 296
C1 =	0
C2 =	124 659 / 7 055 360
C3 =	2 401 / 11 860 992
C4 =	- 9 / 21 760
C5 =	628 250 / 802 791
B50 =	3 372 873 / 7 180 000
B51 =	- 431 919 / 12 565 000
B52 =	277 074 513 / 1 306 760 000
B53 =	- 8 851 689 / 186 680 000
B54 =	0

SCALED NORM OF TEC	G*(5) = 0.5096777D-03
SCALED STABILITY LIMITS	R: (-0.4507200D+01, 0)
	I: (0, 0.5578000D+01)

SIGMA = 0.300000D+00; A(5) = 3./5.

C0 =	57 851 / 288 000
C1 =	0
C2 =	4 310 577 / 247 520 000
C3 =	304 927 / 49 296 000
C4 =	- 2 637 / 820 000
C5 =	5 402 875 / 6 937 938
B50 =	9 318 525 453 / 21 611 500 000
B51 =	2 632 446 / 675 359 375
B52 =	60 133 894 317 / 280 949 500 000
B53 =	- 6 900 061 371 / 140 474 750 000
B54 =	0

SCALED NORM OF TEC	G*(5) = 0.8082074D-03
SCALED STABILITY LIMITS	R: (-0.4140333D+01, 0)
	I: (0, 0.5085633D+01)

TABLE IV.- Continued

(b) Continued

SIGMA = 0.350000D+00; A(5) = 9./16.

C0 =	19 065 713 / 103 680 000
C1 =	0
C2 =	3 252 504 213 / 59 321 600 000
C3 =	28 118 111 / 6 514 560 000
C4 =	- 536 403 / 205 600 000
C5 =	1 634 729 984 / 2 152 165 545
B50 =	5 403 228 165 441 / 13 077 839 872 000
B51 =	- 49 882 382 361 / 817 364 992 000
B52 =	45 186 311 217 159 / 170 011 918 336 000
B53 =	- 2 355 259 443 201 / 42 502 979 584 000
B54 =	0

SCALED NORM OF TEC G*(5) = 0.1172146D-02
SCALED STABILITY LIMITS R: (-0.3864000D+01, 0)
I: (0, 0.4704371D+01)

SIGMA = 0.400000D+00; A(5) = 11./20.

C0 =	446 / 2 475
C1 =	0
C2 =	24 057 / 328 250
C3 =	16 807 / 1 304 550
C4 =	- 94 / 14 625
C5 =	21 452 000 / 28 987 101
B50 =	1 226 671 061 / 3 217 800 000
B51 =	- 5 764 473 / 134 075 000
B52 =	372 592 737 / 1 394 380 000
B53 =	- 2 318 720 327 / 41 831 400 000
B54 =	0

SCALED NORM OF TEC G*(5) = 0.1594736D-02
SCALED STABILITY LIMITS R: (-0.3649500D+01, 0)
I: (0, 0.4398850D+01)

TABLE IV.- Continued

(b) Continued

SIGMA = 0.450000D+00; A(5) = 1./2.

C0 =	208 927 / 1 280 000
C1 =	0
C2 =	1 010 269 341 / 6 572 800 000
C3 =	6 043 317 / 490 880 000
C4 =	- 4 941 / 800 000
C5 =	15 776 / 23 305
B50 =	48 018 463 / 126 208 000
B51 =	- 1 170 873 / 7 888 000
B52 =	541 971 837 / 1 640 704 000
B53 =	- 1 598 723 / 25 636 000
B54 =	0

SCALED NORM OF TEC G*(5) = 0.2060460D-02
SCALED STABILITY LIMITS R: (-0.3480222D+01, 0)
I: (0, 0.4148133D+01)

SIGMA = 0.500000D+00; A(5) = 12./25.

C0 =	2 201 / 13 824
C1 =	0
C2 =	6 561 / 33 280
C3 =	45 619 / 1 617 408
C4 =	- 1 / 80
C5 =	78 125 / 124 416
B50 =	138 009 / 390 625
B51 =	- 52 704 / 390 625
B52 =	1 635 471 / 5 078 125
B53 =	- 306 936 / 5 078 125
B54 =	0

SCALED NORM OF TEC G*(5) = 0.2556559D-02
SCALED STABILITY LIMITS R: (-0.3345400D+01, 0)
I: (0, 0.3938940D+01)

TABLE IV.- Continued

(b) Continued

SIGMA = 0.550000D+00; A(5) = 2./5.		
C0 =	313 009 /	2 304 000
C1 =	0	
C2 =	563 920 137 /	1 680 640 000
C3 =	99 648 703 /	3 339 648 000
C4 =	- 226 391 /	18 720 000
C5 =	5 384 875 /	10 540 764
B50 =	6 165 113 549 /	16 154 625 000
B51 =	- 227 532 591 /	673 109 375
B52 =	7 504 919 433 /	17 500 843 750
B53 =	- 15 211 290 941 /	210 010 125 000
B54 =	0	

SCALED NORM OF TEC G*(5) = 0.3062214D-02
SCALED STABILITY LIMITS R: (-0.323[illegible]54D+01, 0)
I: (0, 0.376[illegible]82D+01)

SIGMA = 0.600000D+00; A(5) = 1./3.		
C0 =	293 /	2 500
C1 =	0	
C2 =	1 554 957 /	3 575 000
C3 =	69 629 /	1 495 000
C4 =	- 843 /	50 000
C5 =	8 463 /	20 240
B50 =	174 374 /	453 375
B51 =	- 159 072 /	352 625
B52 =	8 735 067 /	18 336 500
B53 =	- 1 804 649 /	23 575 500
B54 =	0	

SCALED NORM OF TEC G*(5) = 0.3559569D-02
SCALED STABILITY LIMITS R: (-0.3155000D+01, 0)
I: (0, 0.3613317D+01)

TABLE IV.- Continued

(b) Continued

SIGMA = 0.650000D+00; A(5) = 1./4.

C0 =	111 507 / 1 280 000
C1 =	0
C2 =	677 766 609 / 1 299 200 000
C3 =	52 094 497 / 746 880 000
C4 =	- 3 624 881 / 160 800 000
C5 =	27 300 352 / 79 361 835
B50 =	240 021 016 229 / 655 208 448 000
B51 =	- 7 218 754 803 / 13 650 176 000
B52 =	106 401 934 989 / 218 402 816 000
B53 =	- 12 231 119 663 / 163 802 112 000
B54 =	0

SCALED NORM OF TEC G*(5) = 0.4032702D-02
SCALED STABILITY LIMITS R: (-0.3092769D+01, 0)
I: (0, 0.3487415D+01)

SIGMA = 0.700000D+00; A(5) = 1./12.

C0 =	- 198 439 / 1 440 000
C1 =	0
C2 =	880 415 487 / 1 445 600 000
C3 =	554 244 439 / 6 280 560 000
C4 =	- 792 673 / 33 900 000
C5 =	24 446 592 / 52 696 985
B50 =	15 469 714 361 / 73 339 776 000
B51 =	- 406 078 043 / 1 018 608 000
B52 =	43 531 047 / 137 936 500
B53 =	- 42 452 411 309 / 953 417 088 000
B54 =	0

SCALED NORM OF TEC G*(5) = 0.4462969D-02
SCALED STABILITY LIMITS R: (-0.3051000D+01, 0)
I: (0, 0.3382686D+01)

TABLE IV.- Continued

(b) Continued

SIGMA = 0.750000D+00; A(5) = 1./25.	
C0 =	- 2 453 / 6 144
C1 =	0
C2 =	30 292 137 / 49 653 760
C3 =	1 063 643 / 7 707 648
C4 =	- 4 347 / 124 160
C5 =	14 375 000 / 20 948 799
B50 =	379 127 171 / 4 600 000 000
B51 =	- 37 545 111 / 287 500 000
B52 =	6 133 430 349 / 59 800 000 000
B54 =	0
SCALED NORM OF TEC	G*(5) = 0.4985026D-02
SCALED STABILITY LIMITS	R: (-0.3030400D+01, 0)
	I: (0, 0.3298413D+01)

TABLE IV.- Continued

(b) Continued

SIGMA = 0.800000D+00; A(5) = 1./25.	
C0 =	- 79 / 250
C1 =	0
C2 =	553 311 / 942 500
C3 =	340 942 / 1 759 875
C4 =	- 10 336 / 226 875
C5 =	8 828 125 / 15 200 988
B50 =	96 032 954 / 3 310 546 875
B51 =	28 671 192 / 1 103 515 625
B52 =	- 243 083 133 / 14 345 703 125
B53 =	84 128 884 / 43 037 109 375
B54 =	0
SCALED NORM OF TEC	G*(5) = 0.6022232D-02
SCALED STABILITY LIMITS	R: (-0.3034500D+01, 0)
	I: (0, 0.3235212D+01)

TABLE IV.- Continued

(b) Continued

SIGMA = 0.850000D+00; A(5) = 1./25.

C0	-603 317 / 2 304 000
C1 =	0
C2 =	17 122 676 913 / 30 734 080 000
C3 =	10 521 438 907 / 43 145 856 000
C4 =	- 11 352 209 / 231 840 000
C5 =	3 931 093 750 / 7 710 428 943
B50 =	- 722 270 252 129 / 23 586 562 500 000
B51 =	394 747 987 977 / 1 965 546 875 000
B52 =	- 30 750 068 734 659 / 204 416 875 000 000
B53 =	12 397 885 510 507 / 613 250 625 000 000
B54 =	0

SCALED NORM OF TEC G*(5) = 0.7423369D-02
SCALED STABILITY LIMITS R: (-0.3071176D+01, 0)
I: (0, 0.3195200D+01)

SIGMA = 0.900000D+00; A(5) = 1./25.

C0 =	- 7 279 / 32 000
C1 =	0
C2 =	1 000 703 403 / 1 911 520 000
C3 =	28 214 151 / 99 632 000
C4 =	- 202 743 / 4 820 000
C5 =	98 203 125 / 212 176 882
B50 =	- 33 318 422 659 / 392 812 500 000
B51 =	1 471 910 874 / 4 091 796 875
B52 =	- 462 500 813 457 / 1 702 187 500 000
B53 =	93 979 832 093 / 2 553 281 250 000
B54 =	0

SCALED NORM OF TEC G*(5) = 0.8875535D-02
SCALED STABILITY LIMITS R: (-0.3159222D+01, 0)
I: (0, 0.3181233D+01)

TABLE IV.- Concluded

(b) Concluded

SIGMA = 0.9500000+00; A(5) = 1./10.

C0 =	3 273 / 256 000
C1 =	0
C2 =	378 173 853 / 805 120 000
C3 =	129 147 389 / 409 728 000
C4 =	- 2 222 677 / 86 880 000
C5 =	83 048 000 / 364 430 649
B50 =	- 70 545 137 063 / 284 736 000 000
B51 =	40 552 526 367 / 41 524 000 000
B52 =	- 6 270 301 089 333 / 8 636 992 000 000
B53 =	89 886 853 133 / 925 392 000 000
B54 =	0

SCALED NORM OF TEC	G*(5) = 0.1004420D-01
SCALED STABILITY LIMITS	R: (-0.3354842D+01, 0)
	I: (0, 0.3192989D+01)

TABLE V.- COEFFICIENTS FOR RKF(4)5

(a) Unscaled six-stage, fifth-order algorithm

$k \backslash \lambda$	α_k	$\beta_{k\lambda}$ 0	1	2	3	4	c_k	$\hat{c}_k$
0							$\frac{25}{216}$	$\frac{16}{135}$
1	$\frac{1}{4}$	$\frac{1}{4}$					0	0
2	$\frac{3}{8}$	$\frac{3}{32}$	$\frac{9}{32}$				$\frac{1408}{2565}$	$\frac{6\ 656}{12\ 825}$
3	$\frac{12}{13}$	$\frac{1932}{2197}$	$-\frac{7200}{2197}$	$\frac{7296}{2197}$			$\frac{2197}{4104}$	$\frac{28\ 561}{56\ 430}$
4	1	$\frac{439}{216}$	-8	$\frac{3680}{513}$	$-\frac{845}{4104}$		$-\frac{1}{5}$	$-\frac{9}{50}$
5	$\frac{1}{2}$	$-\frac{8}{27}$	2	$-\frac{3544}{2565}$	$\frac{1859}{4104}$	$-\frac{11}{40}$		$\frac{2}{55}$

EUCLIDEAN NORM OF TEC G_6 = 0.3355745D-02

STABILITY LIMITS R: (-0.3677700D+01, 0)
I: (0.2046040D+01, 0.3606560D+01)

TABLE V.- Continued

(b) Scaled, fourth-order algorithm

SIGMA = 0.05D0; A(6) = 0.60D0

```
C(0)   =  23615711/129600000
C(1)   =  0
C(2)   =  2738144/553078125
C(3)   =  6812112671/10482436800000
C(4)   = -218411/582000000
C(5)   = -481016/155100000
C(6)   =  5362046875/6573904758
B(6,0) =  466184149149243/857927500000000
B(6,1) =  5178062961/53620468750
B(6,2) = -29286158641302/318371533203125
B(6,3) = -2241213346359503 1/17930684750000000
B(6,4) =  99128259521667/107240937500000
B(6,5) =  4989484 1462427/589825156250000
SCALED NORM OF TEC,           G*(5) = 0.1427626D-04
SCALED STABILITY LIMITS       R:  (-0.6246000+01, 0)
                              I:  (0, 0.3606560D+01)
```

SIGMA = 0.10D0; A(6) = 0.60D0

```
C(0)   =  1555361/8100000
C(1)   =  0
C(2)   =  7115008/504984375
C(3)   =  407308421/316572300000
C(4)   = -3209/4406250
C(5)   = -68947/9075000
C(6)   =  1053709375/1315607832
B(6,0) =  664346654667/1317136718750
B(6,1) =  126594108/1053709375
B(6,2) = -7456495300464/62563994140625
B(6,3) = -1278189785 1723/50051195312500
B(6,4) =  2459753093259/1317136718 7500
B(6,5) =  1075310068 14/658568359375
SCALED NORM OF TEC,           G*(5) = 0.1018731D-03
SCALED STABILITY LIMITS       R:  (-0.4512000D+01, 0)
                              I:  (0, 0.3606560D+01)
```

TABLE V.- Continued

(b) Continued

SIGMA = 0.15D0; A(6) = 0.60D0

```
C(0)   =  315161/1600000
C(1)   =  0
C(2)   =  317696/16921875
C(3)   =  138492289/362155200000
C(4)   = -351/2000000
C(5)   = -305991/45100000
C(6)   =  1334375/1687314
B(6,0) =  103235259163/213500000000
B(6,1) =  915471/13343750
B(6,2) = -47130754/595703125
B(6,3) = -914201623309/2348500000000
B(6,4) =  75110391747/266875000000
B(6,5) =  34464232107/146781250000
SCALED NORM OF TEC,       G*(5) = 0.3053316D-03
SCALED STABILITY LIMITS   R:  (-0.3734667D+01, 0)
                          I:  (0, 0.3606560D+01)
```

SIGMA = 0.20D0; A(6) = 0.60D0

```
C(0)   =  801613/4050000
C(1)   =  0
C(2)   =  237568/24046875
C(3)   = -222404507/73641150000
C(4)   =  7487/4125000
C(5)   =  10624/1959375
C(6)   =  9284375/11782584
B(6,0) =  22313787003/46421875000
B(6,1) = -2651112/46421875
B(6,2) =  15744308256/551259765625
B(6,3) = -461266147641/882015625000
B(6,4) =  10844473266/29013671875
B(6,5) =  1723827744/5802734375
SCALED NORM OF TEC,       G*(5) = 0.6400000D-03
SCALED STABILITY LIMITS   R:  (-0.3273000D+01, 0)
                          I:  (0, 0.3606560D+01)
```

TABLE V.- Continued

(b) Continued

SIGMA = 0.25D0; A(6) = 0.60D0

```
C(0)     =  40571/207360
C(1)     =  0
C(2)     = -2656/115425
C(3)     = -27104389/2903662080
C(4)     =  889/163200
[illegible])  =  1279/36960
C(6)     =  3086875/3874878
B(6,0) =  1952712303/3951200000
B(6,1) = -1542231/6173750
B(6,2) =  292545258/1466265625
B(6,3) = -536606220891/825800800000
B(6,4) =  2267080407/4939000000
B(6,5) =  942241167/2716450000
SCALED NORM OF TEC,        G*(5) = 0.1101072D-02
SCALED STABILITY LIMITS    R:  (-0.2962400D+01, 0)
                           I:  (0, 0.3606560D+01)
```

SIGMA = 0.30D0; A(6) = 0.60D0

```
C(0)     =  9543/50000
C(1)     =  0
C(2)     = -1078528/11578125
C(3)     = -93194543/5047350000
C(4)     =  54459/5125000
C(5)     =  189243/2200000
C(6)     =  13578125/16476096
B(6,0) =  4412352476/8486328125
B(6,1) = -33320484/67890625
B(6,2) =  338943067216/806201171875
B(6,3) = -5382504160943/7094570312500
B(6,4) =  89887658679/169726562500
B(6,5) =  35427725424/93349609375
SCALED NORM OF TEC,        G*(5) = 0.1670540D-02
SCALED STABILITY LIMITS    R:  (-0.2737667D+01, 0)
                           I:  (0, 0.3606560D+01)
```

TABLE V.- Continued

(b) Continued

SIGMA = 0.35D0; A(6) = 0.40D0

```
C(0)   =  830677/7200000
C(1)   =  0
C(2)   =  14337008/41859375
C(3)   =  6313094879/765943200000
C(4)   = -143423/32250000
C(5)   = -5274409/59400000
C(6)   =  1114759375/1777574592
B(6,0) =  35745803245261/150492515625000
B(6,1) =  369196744/1114759375
B(6,2) = -962010407052128/1787098623046875
B(6,3) = -2107653010128l837/31452935765625000
B(6,4) =  17123701110416/3483623046875
B(6,5) =  42009457365554/7663970703125
SCALED NORM OF TEC,       G*(5) = 0.2290976D-02
SCALED STABILITY LIMITS   R:  (-0.2567429D+01, 0)
                          I:  (0, 0.3606560D+01)
```

SIGMA = 0.40D0; A(6) = 0.40D0

```
C(0)   =  27781/225000
C(1)   =  0
C(2)   =  4030464/12765625
C(3)   = -4826809/2915550000
C(4)   =  1114/984375
C(5)   = -40288/1753125
C(6)   =  13346875/22842288
B(6,0) =  76210981292/225228515625
B(6,1) = -11195344/66734375
B(6,2) = -3146683828928/21396708984375
B(6,3) = -4219453573891 4/47072759765625
B(6,4) =  26701738896/41708984375
B(6,5) =  58049867904/91759765625
SCALED NORM OF TEC,       G*(5) = 0.2880000D-02
SCALED STABILITY LIMITS   R:  (-0.2434750D+01, 0)
                          I:  (0, 0.3606560D+01)
```

TABLE V.- Continued

(b) Continued

SIGMA = 0.45D0; A(6) = 0.40D0		
C(0)	=	6313/50000
C(1)	=	0
C(2)	=	939984/3859375
C(3)	=	-79085409/6729800000
C(4)	=	276777/41000000
C(5)	=	1288629/17600000
C(6)	=	24690625/43936356
B(6,0)	=	7205960559/15431640625
B(6,1)	=	-97364088/123453125
B(6,2)	=	531961896544/1466005859375
B(6,3)	=	-7208130979231/6450425781250
B(6,4)	=	120503849193/154316406250
B(6,5)	=	118047108816/169748046875
SCALED NORM OF TEC,		G*(5) = 0.3502154D-02
SCALED STABILITY LIMITS		R: (-0.2328889D+01, 0)
		I: (0, 0.3606560D+01)

SIGMA = 0.50D0; A(6) = 0.40D0		
C(0)	=	721/5760
C(1)	=	0
C(2)	=	512/4275
C(3)	=	-371293/18860160
C(4)	=	107/9600
C(5)	=	98/495
C(6)	=	30625/54144
B(6,0)	=	9933211/16537500
B(6,1)	=	-43856/30625
B(6,2)	=	51378752/56109375
B(6,3)	=	-4458159667/3456337500
B(6,4)	=	677552/765625
B(6,5)	=	1213728/1684375
SCALED NORM OF TEC,		G*(5) = 0.4207160D-02
SCALED STABILITY LIMITS		R: (-0.2243600D+01, 0)
		I: (0, 0.3606560D+01)

TABLE V.- Continued

(b) Continued

SIGMA = 0.55D0; A(6) = 0.40D0		
C(0)	=	874507/7200000
C(1)	=	0
C(2)	=	-5155568/82828125
C(3)	=	-159284697/6946400000
C(4)	=	763147/58500000
C(5)	=	1459733/4200000
C(6)	=	447859375/742577472
B(6,0)	=	42737353009567/60461015625000
B(6,1)	=	-4377093688/2239296875
B(6,2)	=	100165971757878/717974560546875
B(6,3)	=	-1571898291387749/1148759296875000
B(6,4)	=	1293377129187/1399560546875
B(6,5)	=	195073554858/279912109375
SCALED NORM OF TEC,		G*(5) = 0.5955773D-02
SCALED STABILITY LIMITS		R: (-0.2174364D+01, 0)
		I: (0, 0.3606560D+01)

SIGMA = 0.60D0; A(6) = 0.10D0		
C(0)	=	23929/150000
C(1)	=	0
C(2)	=	565248/296875
C(3)	=	-6311981/19950000
C(4)	=	19899/125000
C(5)	=	-8208/3125
C(6)	=	289/168
B(6,0)	=	-1213233/1445000
B(6,1)	=	24/5
B(6,2)	=	-61850336/17159375
B(6,3)	=	265378841/302005000
B(6,4)	=	-437346/903125
B(6,5)	=	495936/1986875
SCALED NORM OF TEC,		G*(5) = 0.6109403D-02
SCALED STABILITY LIMITS		R: (-0.2117500D+01, 0)
		I: (0, 0.3606560D+01)

TABLE V.- Continued

(b) Continued

SIGMA = 0.65D0; A(6) = 0.10D0		
C(0)	=	2150839/14400000
C(1)	=	0
C(2)	=	47449792/29390625
C(3)	=	-27797593031/71227200000
C(4)	=	7951619/42000000
C(5)	=	-16920449/9900000
C(6)	=	112799/98406
B(6,0)	=	-366959163251/487291680000
B(6,1)	=	4844333/1127990
B(6,2)	=	-588204877546/180830896875
B(6,3)	=	6048781368877/9258541920000
B(6,4)	=	-725826801/22559800000
B(6,5)	=	428189853/1127990000
SCALED NORM OF TEC,		G*(5) = 0.5947044D-02
SCALED STABILITY LIMITS		R: (-0.2070923D+01, 0)
		I: (0, 0.3606560D+01)

SIGMA = 0.70D0; A(6) = 0.10D0		
C(0)	=	126439/900000
C(1)	=	0
C(2)	=	5406464/3859375
C(3)	=	-2747196907/5454900000
C(4)	=	65219/281250
C(5)	=	-1016407/825000
C(6)	=	26123/27144
B(6,0)	=	-6659987/11677550
B(6,1)	=	441028/130615
B(6,2)	=	-105258859888/4[illegible]78434375
B(6,3)	=	74987588767/184265111250
B(6,4)	=	-50534211/326537500
B(6,5)	=	81719274/179595625
SCALED NORM OF TEC,		G*(5) = 0.5457097D-02
SCALED STABILITY LIMITS		R: (-0.2032714D+01, 0)
		I: (0, 0.3606560D+01)

TABLE V.- Continued

(b) Continued

SIGMA = 0.75D0; A(6) = 0.10D0

```
C(0)   =  1019/7680
C(1)   =  0
C(2)   =  352/285
C(3)   = -371293/535040
C(4)   =  189/640
C(5)   = -1647/1760
C(6)   =  29/30
B(6,0) = -2463/7424
B(6,1) =  129/58
B(6,2) = -4206/2755
B(6,3) =  327691/1551616
B(6,4) = -279/9280
B(6,5) =  2313/5104
SCALED NORM OF TEC,        G*(5) = 0.4661104D-02
SCALED STABILITY LIMITS    R:  (-0.2001200D+01, 0)
                           I:  (0, 0.3606560D+01)
```

SIGMA = 0.80D0; A(6) = 0.10D0

```
C(0)   =  393/3125
C(1)   =  0
C(2)   =  50069504/45421875
C(3)   = -24876631/23512500
C(4)   =  18544/46875
C(5)   = -227072/309375
C(6)   =  715/612
B(6,0) = -650311/12065625
B(6,1) =  3272/3575
B(6,2) = -415033376/1146234375
B(6,3) =  18109624/193978125
B(6,4) =  76432/2234375
B(6,5) =  1834368/4915625
SCALED NORM OF TEC,        G*(5) = 0.3620387D-02
SCALED STABILITY LIMITS    R:  (-0.1974750D+01, 0)
                           I:  (0.1676275D+0, 0.2094762D+01)
```

TABLE V.- Continued

(b) Continued

```
                SIGMA = 0.85D0; A(6) = 0.10D0

C(0)     =  1721699/14400000
C(1)     =  0
C(2)     =  50457088/50765625
C(3)     = -37234375919/19060800000
C(4)     =  10236091/18000000
C(5)     = -1944103/3300000
C(6)     =  12089/6498
B(6,0) =  14522782003/52224480000
B(6,1) = -72913/120890
B(6,2) =  146868746/145715625
B(6,3) =  24076851851/574469280000
B(6,4) =  104040561/2417800000
B(6,5) =  308770881/1329790000
SCALED NORM OF TEC,        G*(5) = 0.2442667D-02
SCALED STABILITY LIMITS    R:  (-0.1952000D+01, 0)
                           I:  (0.1501106D+0, 0.2144200D+01)

                SIGMA = 0.90D0; A(6) = 0.10D0

C(0)     =  2849/25000
C(1)     =  0
C(2)     =  1879296/2078125
C(3)     = -4455516/653125
C(4)     =  114939/125000
C(5)     = -265113/550000
C(6)     =  713/112
B(6,0) =  629873/891250
B(6,1) = -8892/3565
B(6,2) =  113733424/42334375
B(6,3) =  214123/12017500
B(6,4) =  144081/8912500
B(6,5) =  328356/4901875
SCALED NORM OF TEC,        G*(5) = 0.1288702D-02
SCALED STABILITY LIMITS    R:  (-0.1931111D+01, 0)
                           I:  (0.1371556D+0, 0.2158556D+01)
```

TABLE V.- Continued

(b) Concluded

SIGMA = 0.95D0; A(6) = 0.1D0

```
C(0)      =  1567469/14400000
C(1)      =  0
C(2)      =  894368/1078125
C(3)      =  7146276371/1108800000
C(4)      =  11800007/6000000
C(5)      = -11919137/29700000
C(6)      = -69107/8694
B(6,0)    =  384031234573/298542240000
B(6,1)    = -3424921/691070
B(6,2)    =  27848646422/5830903125
B(6,3)    = -128214678219/328396464000
B(6,4)    = -201293489/13821400000
B(6,5)    = -401374449/7601770000
SCALED NORM OF TEC,      G*(5) = 0.3789098D-03
SCALED STABILITY LIMITS  R:  (-0.1910421D+01, 0)
                         I:  (0.1273168D+0, 0.2157653D+01)
```

TABLE V.- Continued

(c) Scaled, fifth-order algorithm

$\left(\beta_{k,\lambda} = \beta o_{k,\lambda}/\sigma\right.$ for $k=1,\ldots,5$; $\lambda = 1,\ldots,k-1$; where $\beta o_{k,\lambda}$ are the unscaled coefficients (table V(a))$\left.\right)$

SIGMA = 0.5000000D-01;

```
A(1)   =  .5000000000000000D+01
A(2)   =  .7500000000000000D+01
A(3)   =  .1846153846153846D+02
A(4)   =  .2000000000000000D+02
A(5)   =  .1000000000000000D+02
A(6)   =  .3500000000000000D+00
A(7)   =  .8500000000000000D+00
C(0)   =  .1049397167478102D+00
C(1)   =  .0000000000000000D+00
C(2)   = -.3224643108141323D-04
C(3)   = -.1766214053432440D-05
C(4)   =  .9424013548044395D-06
C(5)   =  .1535439210883020D-04
C(6)   =  .5214284718493537D+00
C(7)   =  .3736495272545072D+00
BO(6)  =  .3361810372988544D+00
B(6,1) =  .1367307147892028D-01
B(6,2) =  .0000000000000000D+00
B(6,3) =  .6521568195420585D-02
B(6,4) = -.6375676973195225D-02
B(6,5) =  .0000000000000000D+00
BO(7)  = -.3658580877472001D+00
B(7,1) = -.2014534506081222D-01
B(7,2) =  .1087644986101981D-02
B(7,3) = -.1281692109667515D-01
B(7,4) =  .1138192692070809D-01
B(7,5) =  .3116907521207185D-02
B(7,6) =  .1233233874476670D+01
  SCALED NORM OF TEC,          G*(6) = .2546697D-06
  SCALED STABILITY LIMITS      R:  (-.1114D+02, 0.0)
                               I:  (0.0, 0.0)
```

TABLE V.- Continued

(a) Continued

SIGMA = 0.1000000D+00		
A(1)	=	.2500000000000000D+01
A(2)	=	.3750000000000000D+01
A(3)	=	.9230769230769231D+01
A(4)	=	.1000000000000000D+02
A(5)	=	.5000000000000000D+01
A(6)	=	.2500000000000000D+00
A(7)	=	.8000000000000000D+00
C(0)	=	.4451274470899470D-01
C(1)	=	.0000000000000000D+00
C(2)	=	-.3809536323995350D-03
C(3)	=	-.2611573247839896D-04
C(4)	=	.1403588681849550D-04
C(5)	=	.2046345300480637D-03
C(6)	=	.4800625645242541D+00
C(7)	=	.4756130897147626D+00
B0(6)	=	.2209411548434050D+00
B(6,1)	=	.4133458669904354D-01
B(6,2)	=	.0000000000000000D+00
B(6,3)	=	-.6587223327993963D-01
B(6,4)	=	.5359649173749108D-01
B(6,5)	=	.0000000000000000D+00
B0(7)	=	-.4295673021976385D+00
B(7,1)	=	-.4751223112567721D-01
B(7,2)	=	.5651231138285231D-02
B(7,3)	=	.4675086696685042D-01
B(7,4)	=	-.4084578133162770D-01
B(7,5)	=	.1644610650843368D-01
B(7,6)	=	.1249077110041374D+01

SCALED NORM OF TEC, G*(6) = .4513416D-05
SCALED STABILITY LIMITS R: (-.7030D+01, 0.0)
I: (0.0, .3604D+01)

SIGMA = 0.1500000D+00		
A(1)	=	.1666666666666667D+01
A(2)	=	.2500000000000000D+01
A(3)	=	.6153846153846154D+01
A(4)	=	.6666666666666667D+01
A(5)	=	.3333333333333333D+01
A(6)	=	.4500000000000000D+00
A(7)	=	.9000000000000000D+00

TABLE V.- Continued

(a) Continued

SIGMA = 0.1500000D+00 - Concluded		
C(0)	=	.1481595058054086D+00
C(1)	=	.0000000000000000D+00
C(2)	=	.4986595268659456D-02
C(3)	=	.2380882100418878D-03
C(4)	=	-.1262747900722378D-03
C(5)	=	-.2228476440022090D-02
C(6)	=	.5994112564550854D+00
C(7)	=	.2495593054908990D+00
BO(6)	=	.5120271325556084D+00
B(6,1)	=	-.1699243281016840D+00
B(6,2)	=	.0000000000000000D+00
B(6,3)	=	.6529719757685093D+00
B(6,4)	=	-.5450747802224337D+00
B(6,5)	=	.0000000000000000D+00
BO(7)	=	-.7252120584774573D+00
B(7,1)	=	.4835916557695700D+00
B(7,2)	=	-.7917604295877293D-01
B(7,3)	=	-.1524709383928742D+01
B(7,4)	=	.1279935176012341D+01
B(7,5)	=	-.4379297707736362D-02
B(7,6)	=	.1469949951290798D+01

SCALED NORM OF TEC G*(6) = .1162557D-03

SCALED STABILITY LIMITS R: (-.4333D+01, 0.0)

I: (0.0, 0.0)

SIGMA = 0.2000000D+00		
A(1)	=	.1250000000000000D+01
A(2)	=	.1875000000000000D+01
A(3)	=	.4615384615384615D+01
A(4)	=	.5000000000000000D+01
A(5)	=	.2500000000000000D+01
A(6)	=	.4166666666666667D+00
A(7)	=	.1000000000000000D+01
C(0)	=	.1249895238095238D+00
C(1)	=	.0000000000000000D+00
C(2)	=	-.1741280111844235D-01
C(3)	=	-.1726641291560977D-03
C(4)	=	.8289177489177480D-04
C(5)	=	.4130724386724384D-02
C(6)	=	.6268760256948224D+00

TABLE V.- Continued

(a) Continued

SIGMA = 0.2000000D+00 - Concluded

```
C(7)   =  .2615062995816361D+00
BO(6)  =  .3254591429778346D+00
B(6,1) =  .1025525465501136D+00
B(6,2) =  .0000000000000000D+00
B(6,3) = -.3988396535323532D-01
B(6,4) =  .2853894249195378D-01
B(6,5) =  .0000000000000000D+00
BO(7)  = -.2425211742757191D+00
B(7,1) = -.3082978320350140D+00
B(7,2) =  .1087182249969155D+00
B(7,3) =  .5726610466691963D-02
B(7,4) = -.9043353051320651D-02
B(7,5) =  .4706619454105489D-01
B(7,6) =  .1398351329357391D+01
  SCALED NORM OF TEC,       G*(6) = .3767397D-04
  SCALED STABILITY LIMITS   R:  (-.5910D+01, 0.0)
                            I:  (0.0, 0.0)
```

SIGMA = 0.2500000D+00

```
A(1)   =  .1000000000000000D+01
A(2)   =  .1500000000000000D+01
A(3)   =  .3692307692307692D+01
A(4)   =  .4000000000000000D+01
A(5)   =  .2000000000000000D+01
A(6)   =  .2500000000000000D+00
A(7)   =  .1000000000000000D+01
C(0)   =  .1655712632275131D-01
C(1)   =  .0000000000000000D+00
C(2)   = -.2093054859370649D+00
C(3)   = -.3710393274856326D-02
C(4)   =  .1899140211640211D-02
C(5)   =  .5601808905380333D-01
C(6)   =  .5740214933894104D+00
C(7)   =  .5645200302343159D+00
BO(6)  = -.1203859697625723D+00
B(6,1) =  .6481601383594973D+00
B(6,2) =  .0000000000000000D+00
B(6,3) = -.1606106242091659D+01
B(6,4) =  .1328332073494734D+01
B(6,5) =  .0000000000000000D+00
```

TABLE V.- Continued

(c) Continued

SIGMA = 0.2500000D+00 - Concluded

```
BO(7)  =  .421355663904306 1D+00
B(7,1) = -.1014313258174223D+01
B(7,2) =  .5391994767100782D+00
B(7,3) =  .1293016808117887D+01
B(7,4) = -.1127796866749287D+01
B(7,5) =  .1259148909495025D+00
B(7,6) =  .7626232852417356D+00
  SCALED NORM OF TEC,          G*(6) = .1642604D-02
  SCALED STABILITY LIMITS      R:  (-.2500D+01, 0.0)
                               I:  (0.0, .2516D+01)
```

SIGMA = 0.3000000D+00

```
A(1)   =  .8333333333333333D+00
A(2)   =  .1250000000000000D+01
A(3)   =  .3076923076923077D+01
A(4)   =  .3333333333333333D+01
A(5)   =  .1666666666666667D+01
A(6)   =  .3750000000000000D+00
A(7)   =  .1000000000000000D+01
C(0)   =  .1109941269841270D+00
C(1)   =  .0000000000000000D+00
C(2)   = -.1613703745076978D+00
C(3)   = -.2768584710515157D-03
C(4)   =  .1128738143144574D-03
C(5)   =  .1775009635525763D-01
C(6)   =  .5762960567119235D+00
C(7)   =  .4564940791131267D+00
BO(6)  =  .2715404261444098D+00
B(6,1) =  .1119600064787936D+00
B(6,2) =  .0000000000000000D+00
B(6,3) = -.2085435304556424D-01
B(6,4) =  .1235392042236091D-01
B(6,5) =  .0000000000000000D+00
BO(7)  =  .5751495035304226D-01
B(7,1) = -.6919211292481326D-01
B(7,2) =  .1798826360429507D+00
B(7,3) = -.1171397638691662D+00
B(7,4) =  .7850184104819526D-01
B(7,5) =  .8140789914156719D-01
```

TABLE V.- Continued

(a) Continued

SIGMA = 0.3000000D+00 - Concluded

```
B(7,6) =  .7890245502082241D+00
  SCALED NORM OF TEC,        G*(6) = .2347786D-03
  SCALED STABILITY LIMITS    R:  (-.4040D+01, 0.0)
                             I:  (.3213D+01, .4014D+01)
```

SIGMA = 0.3500000D+00

```
A(1)    =  .7142857142857143D+00
A(2)    =  .1071428571428571D+01
A(3)    =  .2637362637362637D+01
A(4)    =  .2857142857142857D+01
A(5)    =  .1428571428571429D+01
A(6)    =  .3500000000000000D+00
A(7)    =  .1000000000000000D+01
C(0)    =  .1032286673280424D+00
C(1)    =  .0000000000000000D+00
C(2)    = -.8356817496767217D+00
C(3)    = -.3189722973458971D-03
C(4)    =  .1163503908247497D-03
C(5)    =  .3443927419894305D-01
C(6)    =  .5408204216019118D+00
C(7)    =  .1157396008454346D+01
B0(6)   =  .2507234172155856D+00
B(6,1)  =  .1033357672502307D+00
B(6,2)  =  .0000000000000000D+00
B(6,3)  =  .4384345507637338D-02
B(6,4)  = -.8443529973453679D-02
B(6,5)  =  .0000000000000000D+00
B0(7)   =  .1955200897459099D+00
B(7,1)  =  .3616055310818360D+00
B(7,2)  =  .1180200252746648D+00
B(7,3)  = -.9034508634562335D-01
B(7,4)  =  .6158814766864584D-01
B(7,5)  =  .4988355248092283D-01
B(7,6)  =  .3037277400936440D+00
 SCALED NORM OF TEC         G*(6) = .4977249D-03
 SCALED STABILITY LIMITS    R:  (-.3514D+01, 0.0)
                            I:  (.2314D+01, .3492D+01)
```

TABLE V.- Continued

(a) Continued

SIGMA = 0.4000000D+00

A(1)	=	.6250000000000000D+00
A(2)	=	.9375000000000000D+00
A(3)	=	.2307692307692308D+01
A(4)	=	.2500000000000000D+01
A(5)	=	.1250000000000000D+01
A(6)	=	.3125000000000000D+00
A(7)	=	.1000000000000000D+01
C(0)	=	.9098835978835973D-01
C(1)	=	.0000000000000000D+00
C(2)	=	.1511689766638819D+01
C(3)	=	-.1233204211195182D-02
C(4)	=	.5517641723355987D-03
C(5)	=	.9218955266955256D-01
C(6)	=	.4898813273626355D+00
C(7)	=	-.1184067566420507D+01
B0(6)	=	.1997487995087127D+00
B(6,1)	=	.1349930681999413D+00
B(6,2)	=	.0000000000000000D+00
B(6,3)	=	-.1043230585626922D+00
B(6,4)	=	.8208119085403821D-01
B(6,5)	=	.0000000000000000D+00
B0(7)	=	.2400248761202671D+00
B(7,1)	=	.1342029463421628D+01
B(7,2)	=	-.2692274450121969D+00
B(7,3)	=	.1212229776127820D+00
B(7,4)	=	-.7216879014020694D-01
B(7,5)	=	-.7744341804365747D-01
B(7,6)	=	-.2844376639586156D+00

SCALED NORM OF TEC, G*(6) = .8575443D-03
SCALED STABILITY LIMITS R: (-.3425D+01, 0.0)
I: (.2586D+01, .3581D+01)

SIGMA = 0.4500000D+00

A(1)	=	.5555555555555556D+00
A(2)	=	.8333333333333333D+00
A(3)	=	.2051282051282051D+01
A(4)	=	.2222222222222222
A(5)	=	.1111111111111111D+01
A(6)	=	.2727272727272727D+00
A(7)	=	.1000000000000000D+01

TABLE V.- Continued

(c) Continued

SIGMA = 0.4500000D+00 - Concluded

```
C(0)     =   .7756103531746098D-01
C(1)     =   .0000000000000000D+00
C(2)     =   .8028613631375733D+00
C(3)     =  -.2922890071788901D-02
C(4)     =   .1358100363367199D-02
C(5)     =   .2768295086840867D+00
C(6)     =   .4372140992518053D+00
C(7)     =  -.5929020166825045D+00
BO(6)    =   .1603284068985282D+00
B(6,1)   =   .1444873883601111D+00
B(6,2)   =   .0000000000000000D+00
B(6,3)   =  -.1651287642084786D+00
B(6,4)   =   .1330402416771120D+00
B(6,5)   =   .0000000000000000D+00
BO(7)    =  -.5813416834062475D-01
B(7,1)   =   .3023191015738038D+01
B(7,2)   =  -.1433450595013702D+01
B(7,3)   =   .5620883331165248D+00
B(7,4)   =  -.3351779271591276D+00
B(7,5)   =  -.2222157495600493D+00
B(7,6)   =  -.5363009087810589D+00
  SCALED NORM OF TEC,          G*(6) = .1500875D-02
  SCALED STABILITY LIMITS      R:  (-.3202D+01, 0.0)
                               I:  (.2360D+01, .3407D+01)
```

SIGMA = 0.5000000D+00

```
A(1)     =   .5000000000000000D+00
A(2)     =   .7500000000000000D+00
A(3)     =   .1846153846153846D+01
A(4)     =   .2000000000000000D+01
A(5)     =   .1000000000000000D+01
A(6)     =   .5200000000000000D+00
A(7)     =   .4700000000000000D+00
C(0)     =   .1098854553641793D+00
C(1)     =   .0000000000000000D+00
C(2)     =   .1044158498385370D+01
C(3)     =   .7965416675785862D-02
C(4)     =  -.4048207871737286D-02
C(5)     =  -.6578827357129257D-01
C(6)     =  -.3612366225695946D+01
```

TABLE V.- Continued

(c) Continued

SIGMA = 0.5000000D+00 - Concluded

```
C(7)    =  .3520193336713641D+01
BO(6)   =  .2336973664450301D+00
B(6,1)  =  .2892124854846433D+00
B(6,2)  =  .0000000000000000D+00
B(6,3)  =  .2331250273933592D-01
B(6,4)  = -.2622235466900938D-01
B(6,5)  =  .0000000000000000D+00
BO(7)   =  .2185627558257097D+00
B(7,1)  =  .2011231844500803D+00
B(7,2)  = -.5017373261354327D-01
B(7,3)  =  .5057904966913352D-01
B(7,4)  = -.4336585528124070D-01
B(7,5)  = -.1589391712042916D-01
B(7,6)  =  .1091685150702896D+00
  SCALED NORM OF TEC,        G*(6) = .3295725D-03
  SCALED STABILITY LIMITS    R:  (-.3402D+01, 0.0)
                             I:  (0.0, .1181D+01)
```

SIGMA = 0.5500000D+00

```
A(1)    =  .4545454545454545D+00
A(2)    =  .6818181818181818D+00
A(3)    =  .1678321678321678D+01
A(4)    =  .1818181818181818D+01
A(5)    =  .9090909090909091D+00
A(6)    =  .2600000000000000D+00
A(7)    =  .8300000000000000D+00
C(0)    =  .7785029189100709D-01
C(1)    =  .0000000000000000D+00
C(2)    =  .3979374013823073D+00
C(3)    =  .2019117455361709D-02
C(4)    = -.1028655979078399D-02
C(5)    =  .2213041356159321D+00
C(6)    =  .3940643461671919D+00
C(7)    = -.9214663653272163D-01
BO(6)   =  .1304931589926324D+00
B(6,1)  =  .1695041979055356D+00
B(6,2)  =  .0000000000000000D+00
B(6,3)  = -.2107469964831939D+00
B(6,4)  =  .1707496395850258D+00
B(6,5)  =  .0000000000000000D+00
```

TABLE V.- Continued

(a) Continued

SIGMA = 0.5500000D+00 - Concluded

```
BO(7)  = -.1218612445012908D+01
B(7,1) =  .1169830944584696D+02
B(7,2) = -.6046577641630584D+01
B(7,3) =  .3063253945594024D+01
B(7,4) = -.1846156898922458D+01
B(7,5) = -.2128387752352545D+01
B(7,6) = -.2691828653522487D+01
  SCALED NORM OF TEC,        G*(6) = .3221258D-02
  SCALED STABILITY LIMITS    R:  (-.2764D+01, 0.0)
                             I:  (.1988D+01, .2991D+01)
```

SIGMA = 0.6000000D+00

```
A(1)   =  .4166666666666667D+00
A(2)   =  .6250000000000000D+00
A(3)   =  .1538461538461538D+01
A(4)   =  .1666666666666667D+01
A(5)   =  .8333333333333333D+00
A(6)   =  .3300000000000000D+00
A(7)   =  .7200000000000000D+00
C(0)   =  .9356034471701149D-01
C(1)   =  .0000000000000000D+00
C(2)   = -.6489274121253184D+00
C(3)   =  .1482179832556871D-01
C(4)   = -.7113612427683299D-02
C(5)   = -.2477401180821677D-02
C(6)   =  .5625659895314581D+00
C(7)   =  .9875702931597852D+00
BO(6)  =  .1382626113915072D+00
B(6,1) =  .2346911060269583D+00
B(6,2) =  .0000000000000000D+00
B(6,3) = -.2203622318524367D+00
B(6,4) =  .1774085144339713D+00
B(6,5) =  .0000000000000000D+00
BO(7)  =  .1691491924658049D+00
B(7,1) =  .1686177657726157D+00
B(7,2) = -.2725776674174607D-02
B(7,3) = -.5296532249363373D-01
B(7,4) =  .2281937886921284D-01
B(7,5) =  .1929426157938227D+00
```

TABLE V.- Continued

(a) Continued

SIGMA = 0.6000000D+00 - Concluded

```
B(7,6) =  .2221621462663522D+00
  SCALED NORM OF TEC,          G*(6) = .3812914D-02
  SCALED STABILITY LIMITS      R:  (-.2700D+01, 0.0)
                               I:  (.1870D+01, .2978D+01)
```

SIGMA = 0.6500000D+00

```
A(1)    =  .3846153846153846D+00
A(2)    =  .5769230769230769D+00
A(3)    =  .1420118343195266D+01
A(4)    =  .1538461538461538D+01
A(5)    =  .7692307692307692D+00
A(6)    =  .4500000000000000D+00
A(7)    =  .6000000000000000D+00
C(0)    =  .1019736151528440D+00
C(1)    =  .0000000000000000D+00
C(2)    = -.1664646463782353D+02
C(3)    =  .3534560752026778D-01
C(4)    = -.1655062970211546D-01
C(5)    = -.1697810990360506D+00
C(6)    =  .2717921870222296D+01
C(7)    =  .1497755527366629D+02
BO(6)   =  .1620346888601602D+00
B(6,1)  =  .3012732832271356D+00
B(6,2)  =  .0000000000000000D+00
B(6,3)  = -.4942796664665503D-01
B(6,4)  =  .3611999455935920D-01
B(6,5)  =  .0000000000000000D+00
BO(7)   =  .1359503855491384D+00
B(7,1)  =  .4594121042544327D+00
B(7,2)  = -.2395746072222467D-01
B(7,3)  =  .1325363243626399D-01
B(7,4)  = -.8761545135178338D-02
B(7,5)  =  .2931812700755386D-02
B(7,6)  =  .2117107091681247D-01
  SCALED NORM OF TEC,          G*(6) = .1775157D-02
  SCALED STABILITY LIMITS      R:  (-.3209D+01, 0.0)
                               I:  (.2108D+01, .3636D+01)
```

TABLE V.- Continued

(a) Continued

SIGMA = 0.7000000D+00		
A(1)	=	.3571428571428571D+00
A(2)	=	.5357142857142857D+00
A(3)	=	.1318681318681319D+01
A(4)	=	.1428571428571429D+01
A(5)	=	.7142857142857143D+00
A(6)	=	.4600000000000000D+00
A(7)	=	.5900000000000000D+00
C(0)	=	.9938619271813810D-01
C(1)	=	.0000000000000000D+00
C(2)	=	-.1164829345465175D+02
C(3)	=	.5606947796913903D-01
C(4)	=	-.2523500352943741D-01
C(5)	=	-.3070367817657410D+00
C(6)	=	.4963332454942884D+01
C(7)	=	.7861777114316768D+01
B0(6)	=	.1571494738728330D+00
B(6,1)	=	.3031440894261848D+00
B(6,2)	=	.0000000000000000D+00
B(6,3)	=	.1862196774786961D-01
B(6,4)	=	-.1891553104688739D-01
B(6,5)	=	.0000000000000000D+00
B0(7)	=	.1009320434329384D+00
B(7,1)	=	.5122075870474418D+00
B(7,2)	=	-.7802727655440578D-01
B(7,3)	=	.1537488043439003D-01
B(7,4)	=	-.4453448339869780D-02
B(7,5)	=	-.4789538986188459D-02
B(7,6)	=	.4875575296569375D-01

SCALED NORM OF TEC, G*(6) = .1430145D-02
SCALED STABILITY LIMITS R: (-.3907D+01, 0.0)
I: (0.0, 0.0)

SIGMA = 0.7500000D+00		
A(1)	=	.3333333333333333D+00
A(2)	=	.5000000000000000D+00
A(3)	=	.1230769230769231D+01
A(4)	=	.1333333333333333D+01
A(5)	=	.6666666666666667D+00
A(6)	=	.4500000000000000D+00
A(7)	=	.5900000000000000D+00

TABLE V.- Continued

(a) Continued

SIGMA = 0.7500000D+00 - Concluded

```
C(0)    =  .9587638580619913D-01
C(1)    =  .0000000000000000D+00
C(2)    = -.9934257866889451D+01
C(3)    =  .8389383603325926D-01
C(4)    = -.3592910703709527D-01
C(5)    = -.5843105264300945D+00
C(6)    =  .6498344899417760D+01
C(7)    =  .4876382379099422D+01
BO(6)   =  .1505270475603677D+00
B(6,1) =  .2921683987467562D+00
B(6,2) =  .0000000000000000D+00
B(6,3) =  .5731899393434687D-01
B(6,4) = -.5001444024147078D-01
B(6,5) =  .0000000000000000D+00
BO(7)   =  .3404453725296851D-01
B(7,1) =  .6907242066214459D+00
B(7,2) = -.2264502174892697D+00
B(7,3) =  .4487947377765493D-02
B(7,4) =  .1726173816542408D-01
B(7,5) = -.1373019934089543D-01
B(7,6) =  .8366198741256116D-01
  SCALED NORM OF TEC,       G*(6) = .3651727D-02
  SCALED STABILITY LIMITS   R:  (-.5201D+01, 0.0)
                            I:  (0.0, 0.0)
```

SIGMA = 0.8000000D+00

```
A(1)    =  .3125000000000000D+00
A(2)    =  .4687500000000000D+00
A(3)    =  .1153846153846154D+01
A(4)    =  .1250000000000000D+01
A(5)    =  .6250000000000000D+00
A(6)    =  .4300000000000000D+00
A(7)    =  .5800000000000000D+00
C(0)    =  .9177701437947632D-01
C(1)    =  .0000000000000000D+00
C(2)    = -.8899911158150710D+01
C(3)    =  .1190999330473690D+00
C(4)    = -.4779650070206462D-01
C(5)    = -.7674883851950577D+00
C(6)    =  .6791125691350474D+01
```

TABLE V.- Continued

(c) Continued

SIGMA = 0.8000000D+00 - Concluded

```
C(7)   =  .3713193405270513D+01
BO(6)  =  .1321587392059206D+00
B(6,1) =  .2964715493885388D+00
B(6,2) =  .0000000000000000D+00
B(6,3) =  .1985878378477856D-01
B(6,4) = -.18489072379237
B(6,5) =  .0000000000000000D+00
BO(7)  = -.2799871066013902D-02
B(7,1) =  .8198193282626641D+00
B(7,2) = -.3747013763688526D+00
B(7,3) =  .7113607481103113D-01
B(7,4) = -.3002685456127201D-01
B(7,5) =  .8076674027364075D-02
B(7,6) =  .8849602489507914D-01
  SCALED NORM OF TEC,        G*(6) = .3770641D-02
  SCALED STABILITY LIMITS    R:  (-.2700D+01, 0.0)
                             I:  (.1606D+01, .3120D+01)
```

SIGMA = 0.8500000D+00

```
A(1)   =  .2941176470588235D+00
A(2)   =  .4411764705882353D+00
A(3)   =  .1085972850678733D+01
A(4)   =  .1176470588235294D+01
A(5)   =  .5882352941176471D+00
A(6)   =  .4200000000000000D+00
A(7)   =  .5500000000000000D+00
C(0)   =  .8791034006448265D-01
C(1)   =  .0000000000000000D+00
C(2)   = -.1432598557850752D+02
C(3)   =  .1637724502289815D+00
C(4)   = -.6025067094271176D-01
C(5)   = -.6051223123962511D+00
C(6)   =  .1221191469286895D+02
C(7)   =  .3527761078684070D+01
BO(6)  =  .1184034763470802D+00
B(6,1) =  .3012366421081962D+00
B(6,2) =  .0000000000000000D+00
B(6,3) =  .9087546933044339D-02
B(6,4) = -.8727665388320754D-02
B(6,5) =  .0000000000000000D+00
```

TABLE V.- Continued

(c) Continued

SIGMA = 0.8500000D+00 - Concluded

```
BO(7)  =  .8853309076199897D-02
B(7,1) =  .7227588300733380D+00
B(7,2) = -.3160632625050489D+00
B(7,3) =  .5339983556124819D-01
B(7,4) = -.1974686788002205D-01
B(7,5) =  .6057946312381548D-02
B(7,6) =  .9474020936190329D-01
  SCALED NORM OF TEC,        G*(6) = .3611939D-02
  SCALED STABILITY LIMITS    R:  (-.2788D+01, 0.0)
                             I:  (.1605D+01, .3232D+01)
```

SIGMA = 0.9000000D+00

```
A(1)   =  .2777777777777778D+00
A(2)   =  .4166666666666667D+00
A(3)   =  .1025641025641026D+01
A(4)   =  .1111111111111111D+01
A(5)   =  .5555555555555556D+00
A(6)   =  .4100000000000000D+00
A(7)   =  .5000000000000000D+00
C(0)   =  .8376275842044271D-01
C(1)   =  .0000000000000000D+00
C(2)   = -.4632917846616554D+02
C(3)   =  .2117284305017549D+00
C(4)   = -.6815849169530956D-01
C(5)   =  .1372517577079254D+00
C(6)   =  .4329054809328235D+02
C(7)   =  .3674045917948378D+01
BO(6)  =  .1057551781670995D+00
B(6,1) =  .3050248490658752D+00
B(6,2) =  .0000000000000000D+00
B(6,3) = -.2194594564576725D-02
B(6,4) =  .1414567331602021D-02
B(6,5) =  .0000000000000000D+00
BO(7)  =  .9727583942245832D-01
B(7,1) =  .3084586106037708D+00
B(7,2) = -.7426131069934527D-02
B(7,3) =  .3707095614794195D-01
B(7,4) = -.2403224686884023D-01
B(7,5) = -.3610608663030326D-01
```

TABLE V.- Concluded

(a) Concluded

SIGMA = 0.9000000D+00 - Concluded

```
B(7,6) =  .1247590583949670D+00
  SCALED NORM OF TEC,          G*(6) = .3609105D-02
  SCALED STABILITY LIMITS      R:  (-.2269D+01, 0.0)
                               I:  (0.0, .1142D+01)
```

SIGMA = 0.9500000D+00

```
A(1)    =  .2631578947368421D+00
A(2)    =  .3947368421052632D+00
A(3)    =  .9716599190283401D+00
A(4)    =  .1052631578947368D+01
A(5)    =  .5263157894736842D+00
A(6)    =  .3750000000000000D+00
A(7)    =  .4500000000000000D+00
C(0)    =  .7875613278958027D-01
C(1)    =  .0000000000000000D+00
C(2)    = -.1391893465558086D+02
C(3)    =  .2491545555366749D+00
C(4)    = -.6196645457142458D-01
C(5)    =  .7677737840819843D+00
C(6)    =  .1113317718958223D+02
C(7)    =  .2752039448161809D+01
BO(6)   =  .1107858516039181D+00
B(6,1)  =  .1600964963528354D+00
B(6,2)  =  .0000000000000000D+00
B(6,3)  =  .3048371470891898D-01
B(6,4)  = -.2636606266567253D-01
B(6,5)  =  .0000000000000000D+00
BO(7)   =  .1570928567435419D+00
B(7,1)  = -.1949399739289811D-01
B(7,2)  =  .2592967819784316D+00
B(7,3)  = -.3563381480848514D+00
B(7,4)  =  .2603695796445432D+00
B(7,5)  =  .1340898607825139D+00
B(7,6)  =  .1498306632871896D-01
  SCALED NORM OF TEC,          G*(6) = .1910657D-01
  SCALED STABILITY LIMITS      R:  (-.1946D+01, 0.0)
                               I:  (.1260D+01, .2208D+01)
```