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Multiple-Grid Acceleration of Lax-Wendroff Algorithms

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SUMMARY

A recently-introduced technique for accelerating the convergence of a one-step Lax-Wendroff method to a steady-state solution is discussed and its applicability extended to the more general class of two-step Lax-Wendroff methods. Several two-step methods which lead to quite efficient multiple-grid algorithms are discussed.

Computational results are presented using the full two-dimensional Euler equations for both subcritical and shocked, supercritical flows.

Extensions and generalizations are mentioned.

INTRODUCTION

Lax and Wendroff (1960) described a class of difference equations, of second-order accuracy in both space and time, "for approximating discontinuous time dependent solutions with prescribed initial data of hyperbolic systems of nonlinear conservation laws." Subsequently, Lax and Wendroff (1964) adapted these difference schemes to allow the computation of approximate weak solutions of multi-dimensional systems of conservation laws.

Richtmyer (1962) devised a two-step difference scheme of Lax-Wendroff type. In this scheme, each step is spatially symmetric. Two-step Lax-Wendroff schemes are inherently more efficient than one-step schemes because they do not require the evaluation of flux-vector Jacobian matrices.

MacCormack (1969) introduced a two-step Lax-Wendroff difference scheme in which the individual steps are not spatially symmetric. This method is a quite efficient member of its class. Its difference molecule contains fewer points than a spatially-symmetric scheme and it only requires information stored at grid points coincident with the spatial location of the final solution.

Lax-Wendroff algorithms have found wide acceptance in fluid mechanics and are commonly used both for the time-accurate computations of unsteady flow and for the time-asymptotic solution of steady flow problems. In the

latter case, where accurate resolution of physical transients is not required, the opportunity exists to exploit this diminished restriction in order to accelerate convergence to the steady state.

Ni (1981) introduced a multiple-grid technique for use in accelerating the convergence of a one-step Lax-Wendroff method. The present work discusses the extension of this technique to the wider class of two-step Lax-Wendroff algorithms.

Results are presented for an application of this multiple-grid technique to the solution of the full two-dimensional Euler equations. This is done mainly for ease of computation and exposition and does not imply any restrictions inherent in the technique. In particular, extensions to three dimensions and to the solution of the Navier-Stokes equations are intended.

BASIC ALGORITHMS

Consider a system of conservation laws

$$q_t = -(f_x + g_y) \tag{1}$$

such as the Euler equations describing inviscid flow, where :

$$q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix} \qquad f = \begin{bmatrix} \rho u \\ \rho u^2+p \\ \rho uv \\ (E+p)u \end{bmatrix} \qquad g = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2+p \\ (E+p)v \end{bmatrix}$$

Here ρ, u, v, p and E are respectively density, velocity components in the x- and y- directions, pressure and total energy per unit volume. Furthermore, the total energy per unit volume may be expressed as

$$E = \rho(e + \frac{1}{2}(u^2+v^2))$$

where the specific internal energy, e, is related to the pressure and density by the simple gas law

$$p = (\gamma-1)\rho e$$

with γ denoting the ratio of specific heats.

Schemes of Lax-Wendroff type may be arrived at intuitively by using Taylor's theorem to write the approximation :

$$q(t+\Delta t) = q(t) + \Delta t q_t + \frac{\Delta t^2}{2} q_{tt} \tag{2}$$

One-step Methods

Since we seek solutions to Eqn. (1), time derivatives may be expressed as space derivatives :

$$q_t = -(f_x + g_y)$$

$$q_{tt} = \left[A(f_x + g_y) \right]_x + \left[B(f_x + g_y) \right]_y$$

where A and B are the Jacobian matrices :

$$A \equiv \partial f/\partial q \qquad B \equiv \partial g/\partial q$$

Substitutions into Eqn. (2) results in :

$$q(t+\Delta t) = q(t) - \Delta t(f_x+g_y) + \frac{\Delta t^2}{2} \left\{ \left[A(f_x+g_y) \right]_x + \left[B(f_x+g_y) \right]_y \right\} \tag{3}$$

Second-order accurate spatial discretization of Eqn. (3) then yields a one-step Lax-Wendroff method.

For example, we may make the following finite-volume type approximations :

$$\begin{aligned}(f_x+g_y)_{i,j} = {} & \frac{1}{8\Delta x} \left[(f_{i+1,j+1}+2f_{i+1,j}+f_{i+1,j-1}) \right. \\ & \left. - (f_{i-1,j+1}+2f_{i-1,j}+f_{i-1,j-1}) \right] \\ & + \frac{1}{8\Delta y} \left[(g_{i-1,j+1}+2g_{i,j+1}+g_{i+1,j+1}) \right. \\ & \left. - (g_{i-1,j-1}+2g_{i,j-1}+g_{i+1,j-1}) \right]\end{aligned}$$

If we define the "change" in q at cell centers, such that :

$$\begin{aligned}\Delta q_{i+\frac{1}{2},j+\frac{1}{2}} \equiv {} & \frac{-\Delta t}{2\Delta x} \left[(f_{i+1,j}+f_{i+1,j+1}) - (f_{i,j}+f_{i,j+1}) \right] \\ & \frac{-\Delta t}{2\Delta y} \left[(g_{i,j+1}+g_{i+1,j+1}) - (g_{i,j}+g_{i+1,j}) \right]\end{aligned} \tag{4}$$

it then follows that

$$- \Delta t(f_x+g_y)_{i,j} = \frac{1}{4}\left[\Delta q_{i-\frac{1}{2},j-\frac{1}{2}} + \Delta q_{i-\frac{1}{2},j+\frac{1}{2}} + \Delta q_{i+\frac{1}{2},j+\frac{1}{2}} + \Delta q_{i+\frac{1}{2},j-\frac{1}{2}} \right] \tag{5}$$

Consistent with the above approximations and definition, we may write the approximation :

$$- \Delta t(f_x+g_y)_{i+\frac{1}{2},j+\frac{1}{2}} = \Delta q_{i+\frac{1}{2},j+\frac{1}{2}}$$

This motivates the definitions :

$$\Delta f_{i+\frac{1}{2},j+\frac{1}{2}} \equiv A_{i+\frac{1}{2},j+\frac{1}{2}} \Delta q_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$\Delta g_{i+\frac{1}{2},j+\frac{1}{2}} \equiv B_{i+\frac{1}{2},j+\frac{1}{2}} \Delta q_{i+\frac{1}{2},j+\frac{1}{2}}$$

We then make the approximations :

$$-\Delta t \left\{ \left[A(f_x+g_y) \right] x \right\}_{i,j} = \tag{6a}$$

$$\frac{1}{2\Delta x}\left[\Delta f_{i+\frac{1}{2},j+\frac{1}{2}} + \Delta f_{i+\frac{1}{2},j-\frac{1}{2}} - \Delta f_{i-\frac{1}{2},j-\frac{1}{2}} - \Delta f_{i-\frac{1}{2},j+\frac{1}{2}} \right]$$

$$-\Delta t \left\{ \left[B(f_x+g_y) \right] y \right\}_{i,j} = \tag{6b}$$

$$\frac{1}{2\Delta y}\left[\Delta g_{i+\frac{1}{2},j+\frac{1}{2}} + \Delta g_{i-\frac{1}{2},j+\frac{1}{2}} - \Delta g_{i-\frac{1}{2},j-\frac{1}{2}} - \Delta g_{i+\frac{1}{2},j-\frac{1}{2}} \right]$$

If we now define the "correction" to q at grid nodes :

$$\delta q_{i,j} \equiv \left[q(t+\Delta t) - q(t) \right]_{i,j}$$

we may combine Eqns. (5) and (6) to yield :

$$
\begin{aligned}
\delta q_{i,j} = & \frac{1}{4}\left[\Delta q + \frac{\Delta t}{\Delta x}\Delta f + \frac{\Delta t}{\Delta y}\Delta g\right]_{i-\frac{1}{2},j-\frac{1}{2}} \\
& + \frac{1}{4}\left[\Delta q + \frac{\Delta t}{\Delta x}\Delta f - \frac{\Delta t}{\Delta y}\Delta g\right]_{i-\frac{1}{2},j+\frac{1}{2}} \\
& + \frac{1}{4}\left[\Delta q - \frac{\Delta t}{\Delta x}\Delta f - \frac{\Delta t}{\Delta y}\Delta g\right]_{i+\frac{1}{2},j+\frac{1}{2}} \\
& + \frac{1}{4}\left[\Delta q - \frac{\Delta t}{\Delta x}\Delta f + \frac{\Delta t}{\Delta y}\Delta g\right]_{i+\frac{1}{2},j-\frac{1}{2}}
\end{aligned}
\tag{7}
$$

Eqns. (4) and (7) constitute the one-step Lax-Wendroff method used as a basic integration scheme by Ni (1981). Notice that Ni's scheme may also be thought of as a two-step scheme with a full time-increment predictor defined by Eqn. (4) and a corrector defined by Eqn. (7). However, such an interpretation of Ni's scheme is not totally in accord with the general practice of avoiding the computation of Jacobian matrices in two-step schemes.

Two-step Methods

Following Richtmyer (1962) many two-step methods which are second-order accurate approximations of Eqn. (3) have been developed. The two-step Lax-Wendroff methods of Burstein, MacCormack and Lapidus have been chosen for use in this study of multiple-grid acceleration both because of their ubiquity and their varying degrees of similarity to the one-step method used by Ni.

Burstein (1966) used a full time-increment predictor scheme which may be written as

$$
\begin{aligned}
\Delta q_{i+\frac{1}{2},j+\frac{1}{2}} = & \frac{-\Delta t}{2\Delta x}\left(f^{n}_{i+1,j+1} + f^{n}_{i+1,j} - f^{n}_{i,j+1} - f^{n}_{i,j}\right) \\
& \frac{-\Delta t}{2\Delta y}\left(g^{n}_{i+1,j+1} + g^{n}_{i,j+1} - g^{n}_{i+1,j} - g^{n}_{i,j}\right)
\end{aligned}
\tag{8a}
$$

$$
\begin{aligned}
\delta q_{i,j} = & \frac{-\Delta t}{4\Delta x}\left(f^{n}_{i+1,j} - f^{n}_{i-1,j}\right. \\
& \left. + \tilde{f}_{i+\frac{1}{2},j+\frac{1}{2}} + \tilde{f}_{i+\frac{1}{2},j-\frac{1}{2}} - \tilde{f}_{i-\frac{1}{2},j+\frac{1}{2}} - \tilde{f}_{i-\frac{1}{2},j-\frac{1}{2}}\right) \\
& \frac{-\Delta t}{4\Delta y}\left(g^{n}_{i,j+1} - g^{n}_{i,j-1}\right. \\
& \left. + \tilde{g}_{i+\frac{1}{2},j+\frac{1}{2}} + \tilde{g}_{i-\frac{1}{2},j+\frac{1}{2}} - \tilde{g}_{i+\frac{1}{2},j-\frac{1}{2}} - \tilde{g}_{i-\frac{1}{2},j-\frac{1}{2}}\right)
\end{aligned}
\tag{8b}
$$

where :

$$\Delta q_{i+\frac{1}{2},j+\frac{1}{2}} \equiv \tilde{q}_{i+\frac{1}{2},j+\frac{1}{2}} - \frac{1}{4}\left(q^{n}_{i+1,j+1} + q^{n}_{i+1,j} + q^{n}_{i,j+1} + q^{n}_{i,j}\right)$$

$$\delta q_{i,j} \equiv \left[\, q(t+\Delta t) - q(t) \,\right]_{i,j}$$

$$\tilde{f}_{i+\frac{1}{2},j+\frac{1}{2}} \equiv f\left(\tilde{q}_{i+\frac{1}{2},j+\frac{1}{2}}\right)$$

$$\tilde{g}_{i+\frac{1}{2},j+\frac{1}{2}} \equiv g\left(\tilde{q}_{i+\frac{1}{2},j+\frac{1}{2}}\right)$$

This method centers all quantities at the point $(i,\ j,\ t + \Delta t/2)$ by means of the averaging procedure contained in Eqn. (8b).

Notice that Eqn. (8a) is identical to Eqn. (4). Furthermore, if we introduce approximations of the type

$$f^{n}_{i,j} = \frac{1}{4}\left(f^{n}_{i,j+1} + 2f^{n}_{i,j} + f^{n}_{i,j-1}\right)$$

$$g^{n}_{i,j} = \frac{1}{4}\left(g^{n}_{i+1,j} + 2g^{n}_{i,j} + g^{n}_{i-1,j}\right)$$

then, for the linearized constant coefficient case, Eqn. (8b) reduces to Eqn. (7). Thus, under these restrictions, the methods of Burstein and Ni are equivalent on a step-for-step basis.

MacCormack (1969) introduced a two-step method which may be expressed as

$$\Delta q_{i,j} = \frac{-\Delta t}{\Delta x}\left(f^{n}_{i+1,j} - f^{n}_{i,j}\right) \frac{-\Delta t}{\Delta y}\left(g^{n}_{i,j+1} - g^{n}_{i,j}\right) \tag{9a}$$

$$\delta q_{i,j} = \frac{-\Delta t}{2\Delta x}\left[\left(f^{n}_{i+1,j} - f^{n}_{i,j}\right) + \left(\tilde{f}_{i,j} - \tilde{f}_{i-1,j}\right)\right] \frac{-\Delta t}{2\Delta y}\left[\left(g^{n}_{i,j+1} - g^{n}_{i,j}\right) + \left(\tilde{g}_{i,j} - \tilde{g}_{i,j-1}\right)\right] \tag{9b}$$

where :

$$\Delta q_{i,j} \equiv \tilde{q}_{i,j} - q^{n}_{i,j}$$

$$\delta q_{i,j} \equiv \left[\, q(t+\Delta t) - q(t) \,\right]_{i,j}$$

$$\tilde{f}_{i,j} \equiv f(\tilde{q}_{i,j})$$

$$\tilde{g}_{i,j} \equiv g(\tilde{q}_{i,j})$$

Here, the averaging procedure in the corrector step centers the scheme, both spatially and temporally, at the point (i, j, t + Δt/2).

Notice that, although the methods of MacCormack and Ni are similar, in that both may be interpreted as full time-increment predictor two-step Lax-Wendroff schemes, they are not equivalent on a step-for-step basis.

Lapidus (1967) used a half time-increment predictor scheme to achieve temporal centering without resort to the averaging necessary in the methods of Burstein and MacCormack. This method may be written as

$$\Delta q_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{-\Delta t}{4\Delta x}\left(f^{n}_{i+1,j+1} + f^{n}_{i+1,j} - f^{n}_{i,j+1} - f^{n}_{i,j}\right) \frac{-\Delta t}{4\Delta y}\left(g^{n}_{i,j+1} + g^{n}_{i+1,j+1} - g^{n}_{i,j} - g^{n}_{i+1,j}\right) \tag{10a}$$

$$\delta q_{i,j} = \frac{-\Delta t}{\Delta x}\left(\tilde{f}_{i+\frac{1}{2},j} - \tilde{f}_{i-\frac{1}{2},j}\right) \frac{-\Delta t}{\Delta y}\left(\tilde{g}_{i,j+\frac{1}{2}} - \tilde{g}_{i,j-\frac{1}{2}}\right) \tag{10b}$$

where :

$$\Delta q_{i+\frac{1}{2},j+\frac{1}{2}} \equiv \tilde{q}_{i+\frac{1}{2},j+\frac{1}{2}} - \frac{1}{4}\left(q^{n}_{i+1,j+1} + q^{n}_{i+1,j} + q^{n}_{i,j+1} + q^{n}_{i,j}\right)$$

$$\delta q_{i,j} \equiv \left[\, q(t+\Delta t) - q(t) \,\right]_{i,j}$$

$$\tilde{f}_{i+\frac{1}{2},j} \equiv \frac{1}{2}\left(\tilde{f}_{i+1,j} + \tilde{f}_{i,j}\right)$$

$$\tilde{g}_{i,j+\frac{1}{2}} \equiv \frac{1}{2}\left(\tilde{g}_{i,j+1} + \tilde{g}_{i,j}\right)$$

$$\tilde{f}_{i,j} \equiv f(\tilde{q}_{i,j})$$

$$\tilde{g}_{i,j} \quad g(\tilde{q}_{i,j})$$

Observe that the change as defined by Eqn. (10a) is one-half of that defined by Eqn. (4) and that, consequently, the correction as defined by Eqn. (10b) is, for the linearized constant coefficient case, twice that defined by Eqn. (7).

MULTIPLE-GRID ACCELERATION

Consider the one-step Lax-Wendroff scheme, defined by Eqns. (4) and (7). Ni (1981) gives the following heuristic interpretation to these equations : the first calculates the change in q occurring in a control volume during the increment Δt , the second distributes the effects of the changes occurring in four nearest-neighbor control volumes to their common central nodal point where they are combined to form the correction to the vector of conservation variables, as illustrated in Fig. 1. This interpretation leads naturally to Ni's coarse grid scheme, which is illustrated in Fig. 2.

Coarse Grid Scheme

We wish to use successively coarser grids to propagate the finest-grid corrections throughout the computational domain, thus accelerating convergence to the steady state while maintaining the accuracy determined by the finest-grid discretization. Given a basic fine grid with the number of points in each direction expressible as

$$2^p + 1$$

for p a natural number, let successively coarser grids be defined by successive deletion of every other point in each coordinate direction. Ni's coarse grid solution procedure then replaces the computation of coarse grid changes by Eqn. (4) with a restriction of the latest finest-grid correction. This restricted finest-grid correction is then distributed according to a coarse-grid version of Eqn. (7) to obtain a coarse-grid correction which is then, in turn, prolonged to the finest grid to become the new finest-grid correction. One time-cycle of Ni's multiple grid scheme is composed of an application of the one-step Lax-Wendroff scheme on the finest grid followed by an application of the coarse grid solution procedure to each successively coarser grid.

In the basic integration scheme, a change at one grid point affects only its nearest neighbors while, in a k-level multiple grid scheme the same change affects all points up to 2^{k-1} mesh spacings distant. Furthermore, since the change is always determined by information from the finest grid and simply propagated by the distribution formulae for coarser grids, fine grid accuracy is maintained.

Application to Two-Step Methods

As the predictor step of two-step Lax-Wendroff schemes is analogous to the change as computed in Eqn. (4), a straightforward application of the coarse grid scheme discussed above may be used to create a simple multiple grid method.

As a basic finest-grid integration scheme, we use a two-step Lax-Wendroff method, such as one of those discussed previously. The distribution of changes on the coarser grids may then be carried out using Eqn. (7), exactly as described for Ni's coarse grid scheme. The price to be paid for the simplicity of this approach is, of course, the necessity of computing Jacobian matrices on the coarser grids. However, because of the vastly reduced density of coarser-grid points, it is advantageous to pay this price. Substantiation of this claim will be provided subsequently.

More sophisticated applications to two-step methods might attempt to eliminate the Jacobian matrix computations by adapting the coarse-grid distribution equation to model the corrector step of the two-step integration scheme. A difficulty which arises when one considers such schemes may be illustrated as follows.

$\tilde{f}$ In a one-step method, such as Ni's scheme, the implied computation of is of the form :

$$\begin{aligned} \tilde{f} &= f^n + \Delta f \\ &= f^n + A^n \Delta q \\ &= A^n \tilde{q} \qquad ; \quad \text{by Euler's theorem on homogeneous functions} \end{aligned}$$

However, in a two-step method :

$$\tilde{f} = \tilde{A}\tilde{q} \qquad ; \quad \text{by Euler's theorem on homogeneous functions}$$

Thus, the corrector step of such a method does not serve to distribute changes in the sense required for its use in a coarse grid scheme. Note that invocation of Euler's theorem is not essential to this arguement, but is used for purposes of clarity.

One possible remedy to this problem would be simply to compute $\tilde{f}$ as

$$\tilde{f} = A^n \tilde{q}$$

This would mean abandoning the original objective of eliminating all of the Jacobian matrix computations from the coarse grid scheme. However, for certain two-step methods the number of matrix operations would still be less than that required by Ni's coarse grid scheme.

We reserve the construction of alternative coarse grid distribution schemes for future study and confine ourselves here to examining the use of Ni's coarse grid scheme for multiple-grid acceleration of the two-step methods of Burstein, MacCormack and Lapidus.

RESULTS

We have computed both subcritical and shocked, supercritical flows in a straight channel with a 10% half-thick circular arc airfoil mounted on its lower wall. The physical problem, geometry and computational grid are similar to those used by Rizzi and Viviand (1981), Moretti (1980), and Ni (1981). The computational domain is illustrated in Fig. 3. The subcritical case has an isentropic inlet Mach number of 0.5 while that for the supercritical case is 0.675.

Four computer programs were written, implementing multiple-grid versions of the methods of Ni, Burstein, MacCormack and Lapidus to solve the full two-dimensional Euler equations as given in Eqn. (1). In each case the initial state, boundary conditions and (for supercritical flow) the artificial viscosity treatment were identical for all four programs. A sequence of four grids was used, as described in Table I.

The coarse grid scheme used was as described previously and is straightforward to implement for one-step methods, such as Ni's method, and for two-step methods using a full time-increment predictor, such as those of Burstein and MacCormack. Lapidus' method uses a half time-increment predictor which causes a slight complication, but no essential difficulty.

To explain this complication, we consider Lapidus' method in the one-dimensional, linearized, constant coefficient case :

$$\Delta q_{i+\frac{1}{2}} = \frac{-\Delta t}{2\Delta x}\left(f^{n}_{i+1} - f^{n}_{i}\right) \tag{11a}$$

$$\delta q_i = \left(\Delta q_{i+\frac{1}{2}} + \Delta q_{i-\frac{1}{2}}\right) - \frac{\Delta t}{\Delta x}\left(\Delta f_{i+\frac{1}{2}} - \Delta f_{i-\frac{1}{2}}\right) \tag{11b}$$

In the coarse grid scheme we wish to replace Eqn. (11b) with

$$\delta q_i = \frac{1}{2}\left(\Delta q_{i+\frac{1}{2}} + \Delta q_{i-\frac{1}{2}}\right) - \frac{\Delta t}{2\Delta x}\left(\Delta f_{i+\frac{1}{2}} - \Delta f_{i-\frac{1}{2}}\right)$$

In order for the coarse grid scheme to function properly we then choose as a restriction operator to replace Eqn. (11a), one with the property :

$$\Delta q_{coarse} = \frac{1}{2}\delta q_{fine} \;\Big|_{\text{Lapidus}}$$

With such a choice, the coarse grid scheme behaves as :

$$\Delta q_{coarse} = \delta q_{fine} \;\Big|_{\text{Ni}}$$

In the case of all the other algorithms, injection was used as a restriction operator. For all four algorithms, the prolongation operator used was linear interpolation.

All four algorithms performed as expected, in both the subcritical and supercritical test cases, producing essentially identical results. Their efficiencies were, however, not identical. With efficiency measured by the computational work required to reduce a standard error measure to a specified tolerance, one may conclude that the multiple-grid version of each method is more efficient than its single-grid counterpart. The two-step methods all have roughly comparable efficiencies and are substantially more efficient than the one-step method of Ni. This is so in spite of the fact that, on a single grid, the two-step methods require approximately half the computation time per grid point per time cycle of the one-step method. This makes application of Ni's coarse grid scheme relatively expensive.

Isomach contours for the converged solutions produced by MacCormack's method on grid sequences of lengths one through four for the subcritical and supercritical test cases are shown in Figs. 4 and 5, respectively. The corresponding convergence histories are shown in Figs. 6 and 7. We note that, for simplicity, the artificial viscosity treatment used in the supercritical computations with all methods was only applied on the finest grid. Hence, inadequate shock resolution on the coarser grids may have slowed the convergence of these cases relative to their subcritical counterparts.

The work associated with the artificial viscosity treatment on the finest grid biases computational comparisons in favor of the multiple-grid version of each method for the supercritical flow test case. Accordingly, Table II summarizes the work required by the multiple-grid algorithms to produce a converged subcritical solution on grid sequences of length one through four. Here convergence is assumed when the average absolute value of the total correction to ρu over a time-cycle does not exceed 1×10^{-5}. Notice that, under the conditions of this test, the optimum grid sequence length is three. We further note that the multiple-grid Ni algorithm requires roughly 50% more work than the multiple-grid MacCormack algorithm.

CONCLUSIONS

The coarse grid correction scheme introduced by Ni (1981) for use with a one-step Lax-Wendroff method, has been successfully applied to several representative two-step Lax-Wendroff methods.

Each resulting multiple-grid scheme is more efficient than its single-grid counterpart.

The multiple-grid one-step method requires approximately 50% more computational work than the multiple-grid two-step methods.

The possibility of constructing alternative coarse grid schemes has been discussed.

Extension of the multiple-grid acceleration technique discussed here to three dimensions and to the Navier-Stokes equations is planned.

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TABLE I. - GRID DESCRIPTIONS

GRID	1	2	3	4
NUMBER OF POINTS	65 x 17	33 x 9	17 x 5	9 x 3

TABLE II. - WORK REQUIREMENTS FOR MULTIPLE-GRID ALGORITHMS

LENGTH OF GRID SEQUENCE	1	2	3	4
MacCormack	1.76	1.16	1.00	1.03
Lapidus	1.61	1.22	1.05	1.13
Burstein	1.74	1.24	1.05	1.08
Ni	3.10	1.82	1.50	1.55

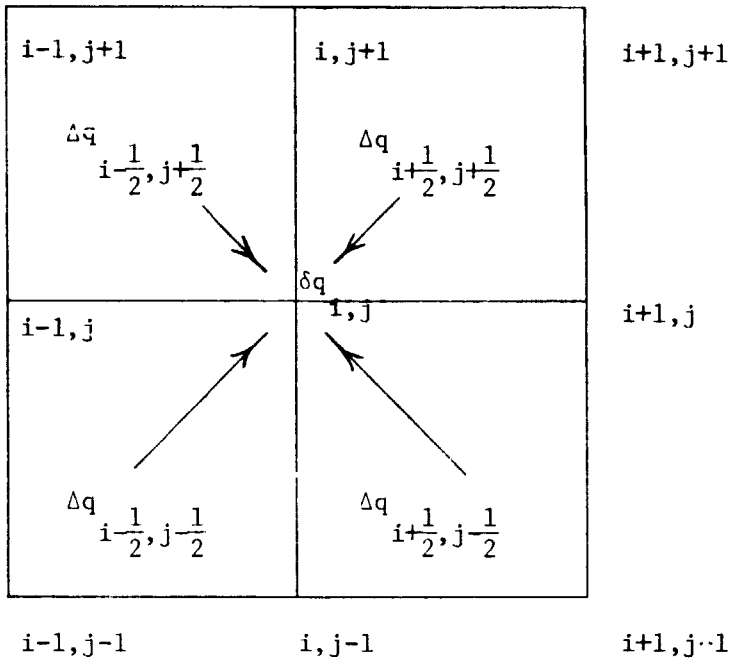


FIGURE 1. - Ni's One-Step Lax-Wendroff Scheme

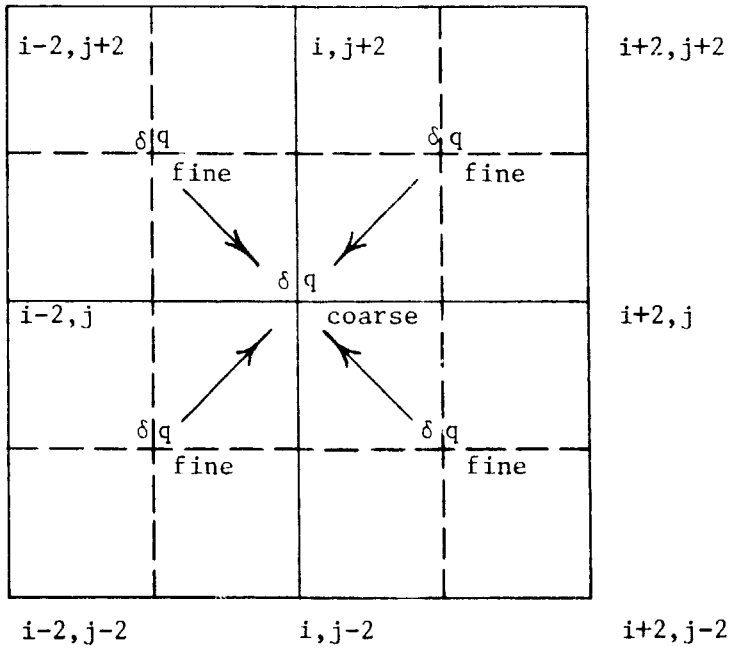
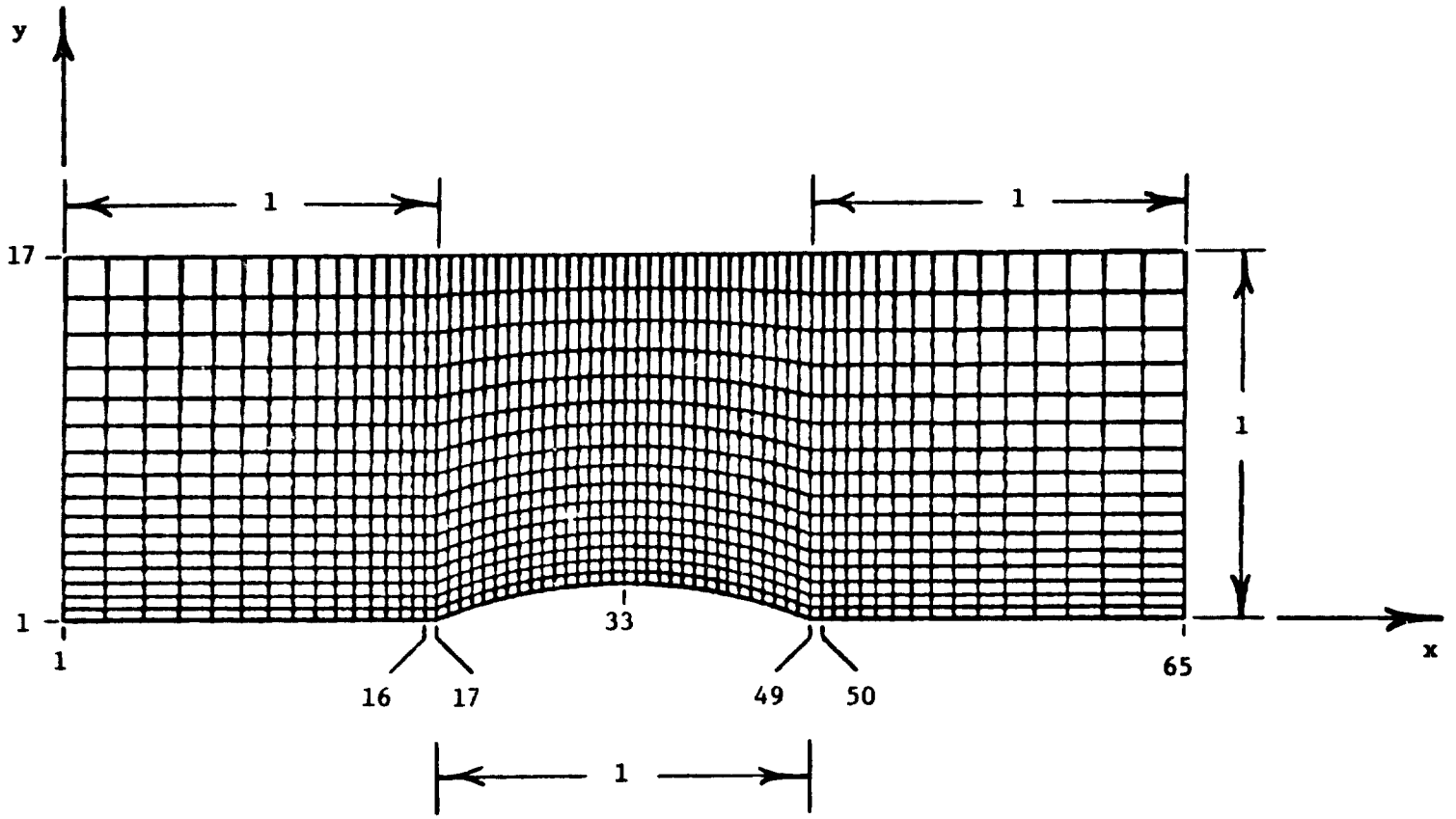
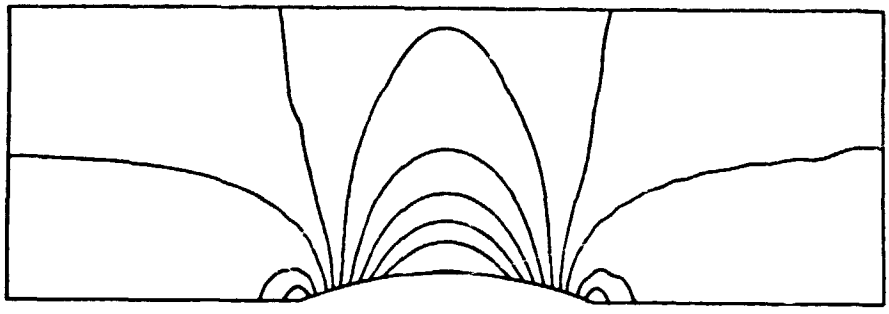


FIGURE 2. - Ni's Coarse Grid Scheme

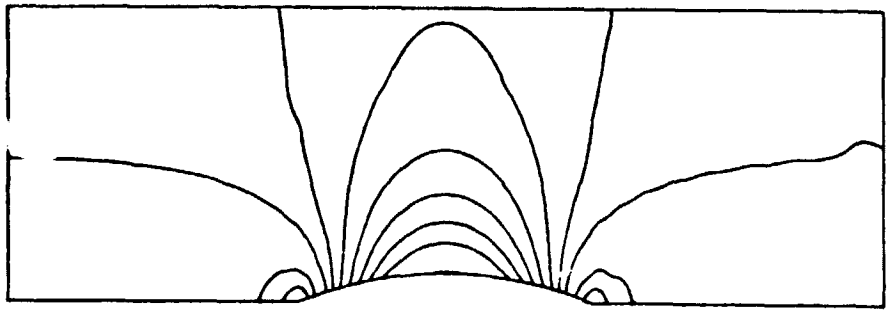


Smallest Grid Spacing : $\Delta x = 0.03125$
$\Delta y = 0.03125$

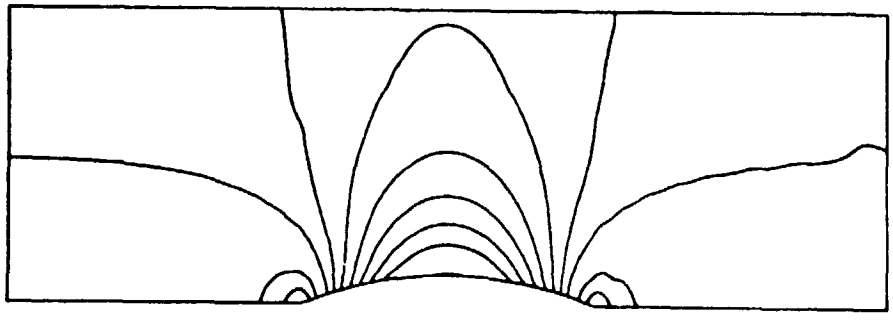
FIGURE 3. - Computational Domain



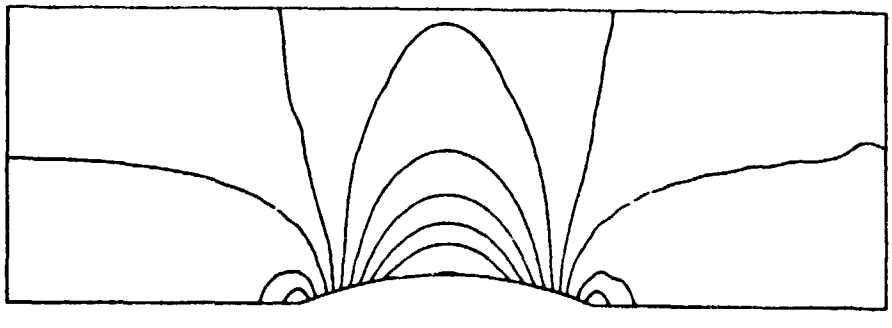
1 Grid



2 Grids



3 Grids



4 Grids

FIGURE 4. - Isomach Contours for Subcritical Case

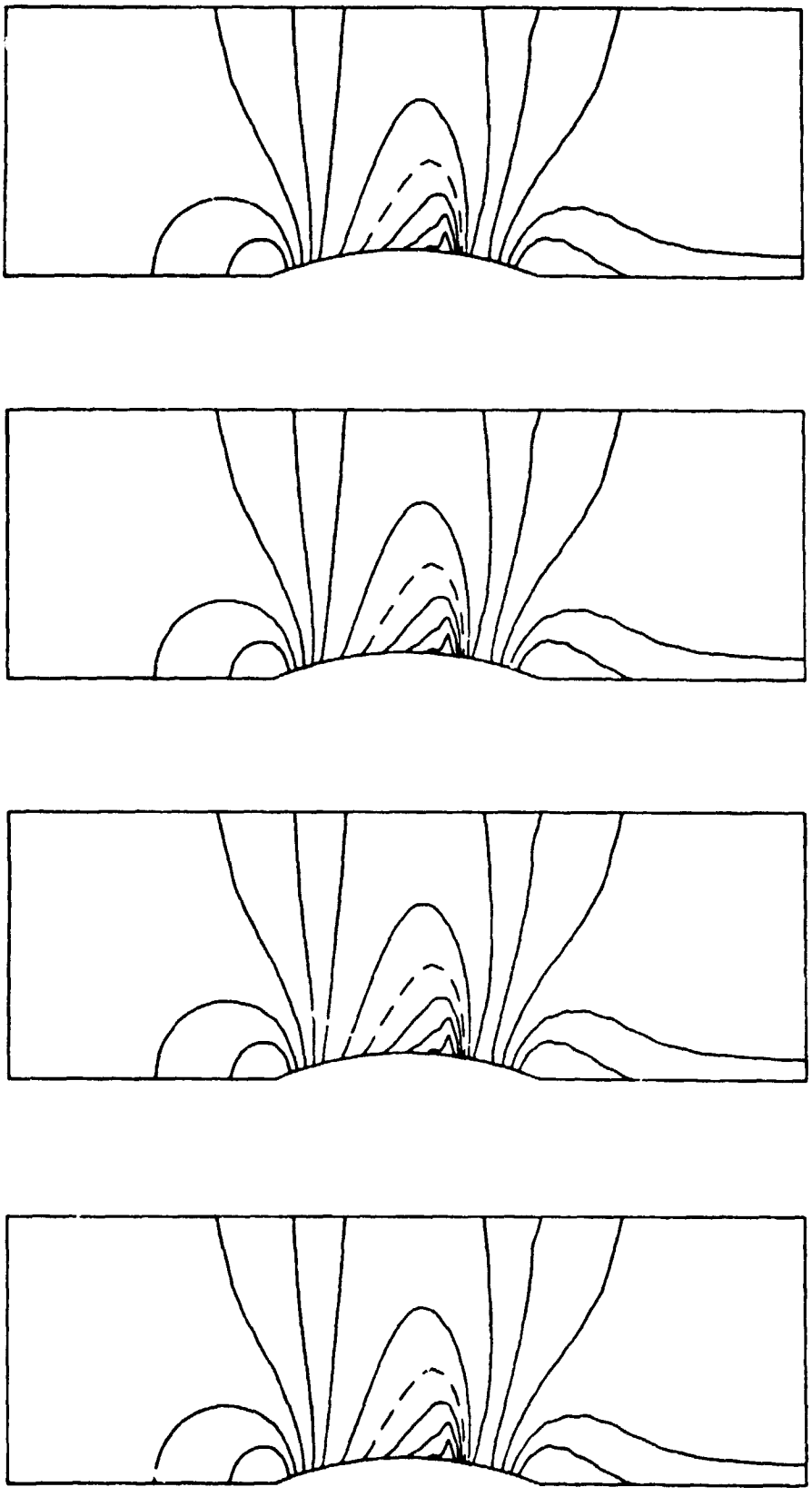


FIGURE 5. - Isomach Contours for Supercritical Case

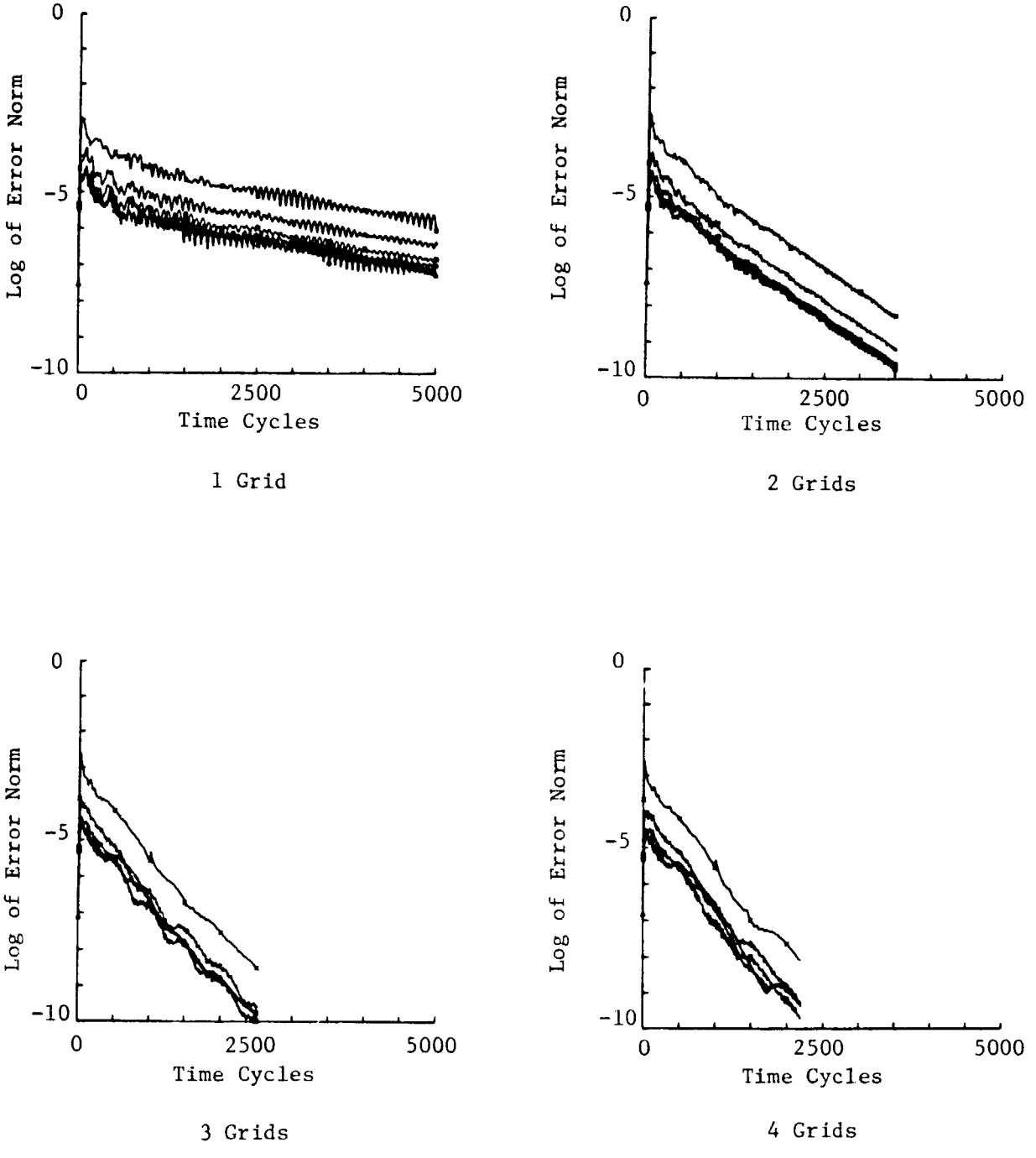


FIGURE 6. - Convergence Histories for Subcritical Case

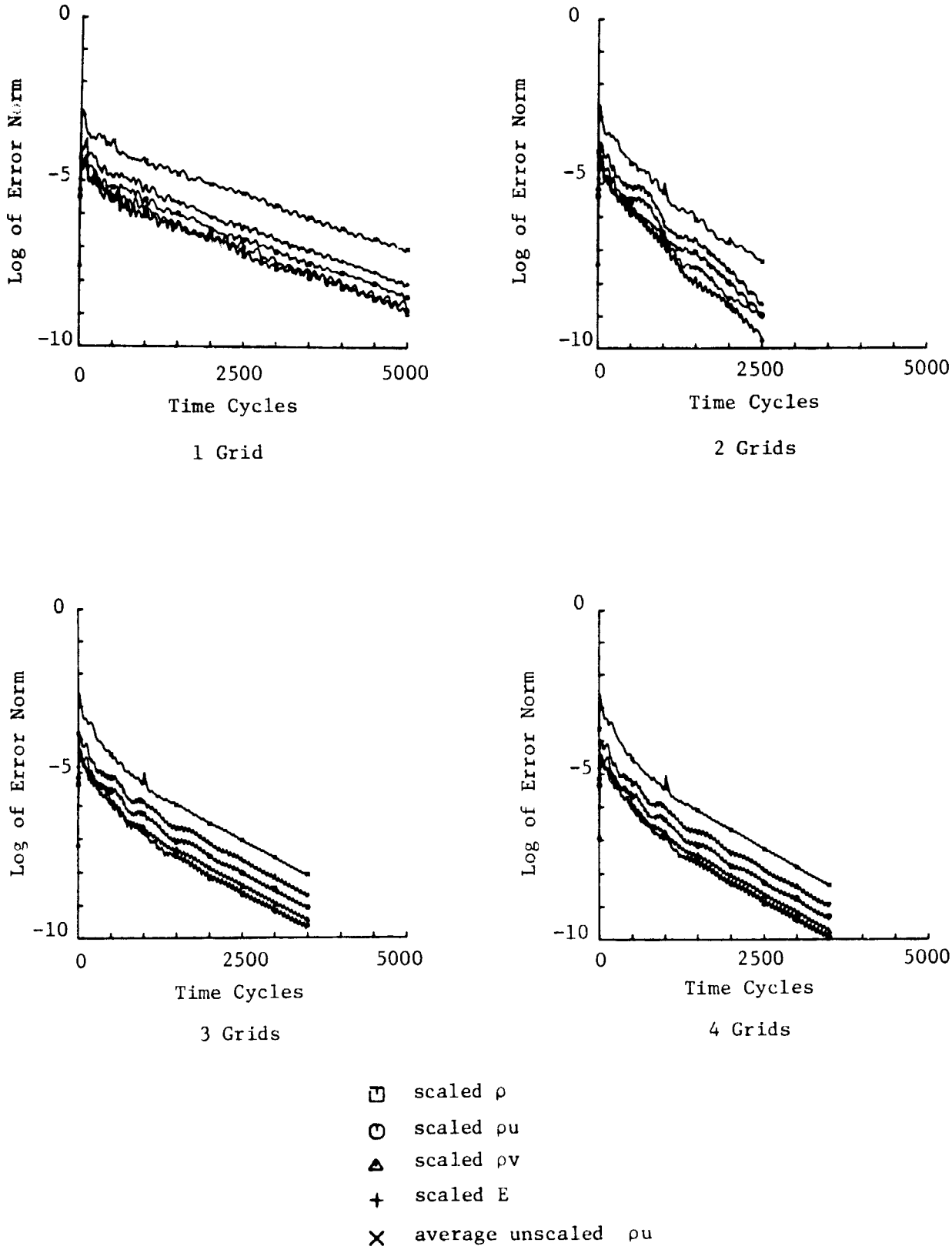


FIGURE 7. - Convergence Histories for Supercritical Case