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The size and shape of equilibrium capillary surfaces

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Abstract

The classical theory of capillarity is concerned largely with size and shape estimates in symmetric asymptotic configurations. Recent developments have led to global results for all symmetric cases, and to new qualitative information on asymptotic properties. Also new stability criteria have been found. It has been discovered that asymmetric situations can lead to behavior that differs strikingly from the symmetric case. When gravity vanishes, capillary surfaces in the accustomed sense may not appear. The question of characterizing those tubes in which surfaces can be found has partially been settled. New progress has been made toward determining the effects of contact angle hysteresis in cases of particular interest.

In 1805, P. S. Laplace (Tr. méc. cél., Suppl. au livre X) introduced the notion of the mean curvature H of a surface and derived for it, in the representation $z = u(x,y)$, the expression

$$2H = \text{div}\, Tu, \qquad \text{with} \qquad Tu \equiv \frac{1}{\sqrt{1+|\nabla u|^2}}\nabla u \tag{1}$$

The context in which this basic contribution appeared was not an abstract study of the geometry of surfaces; it lay instead in his effort to clarify conceptually and describe quantitatively the rise of liquid in a capillary tube. For that problem there holds $2H = \kappa u$, where $\kappa > 0$ is a physical constant, and thus the physical problem is transformed by (1) into an analytical and geometrical one.

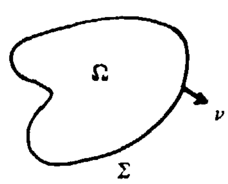

Figure 1

In the same year 1805, T. Young gave a formal reasoning supporting the view that the surface meets the bounding walls in an angle γ depending only on the materials; thus, $\nu \cdot Tu = \cos\gamma$ on the boundary Σ of a section Ω of the tube (Figure 1). Thus one has to solve a nonlinear equation under a nonlinear boundary condition.

For the problem considered, not a single nontrivial explicit solution is known. However, Laplace integrated (1) approximately in the case of a "narrow" circular tube of radius a to obtain the celebrated formula

$$u_0 \sim L(a;\gamma) \equiv 2\,\frac{\cos\gamma}{\kappa a} - \frac{a}{\cos\gamma}\left(1 - \frac{2}{3}\,\frac{1-\sin^3\gamma}{\cos^2\gamma}\right) \tag{2}$$

for the height u_0 on the axis of symmetry (Figure 2).

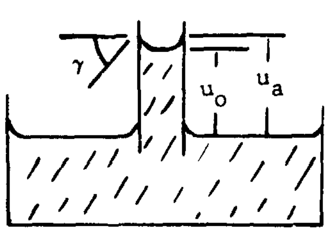

Figure 2

Laplace did not prove (2), nor did he indicate how small a must be in order to achieve a prescribed accuracy. The first proof that (2) is correct was given by D. Siegel (Pacific J. Math., 1980). Later, Finn (ZAMM, 1981) gave a simpler proof with improved error estimates. The method derives from a discovery of Laplace, that the volume of fluid lifted in the tube is given explicitly by $2\pi a\kappa^{-1}\cos\gamma$. The volume is compared with that lifted by certain spherical caps through u_0. One is led to the relations

$$L(a;\gamma) < u_0 < u_0^+ < 2\,\frac{\cos\gamma}{\kappa a} \tag{3}$$

where u_o^+ is the <u>unique</u> nontrivial solution of the equation

$$u_o^4 - \frac{2}{\kappa a} u_o^3 \cos\gamma + \frac{2}{\kappa} u_o^2 - \frac{16}{3\kappa^3 a^2}\left[1 - \left(1 - \frac{\kappa^2 a^2}{4} u_o^2\right)^{3/2}\right] = 0 . \tag{4}$$

Thus, <u>the Laplace formula provides a strict lower bound for</u> u_o.

The method leads also to a new bound for the height u_a at the contact line (Fig. 2)

$$u_a < \frac{2}{\kappa a}\cos\gamma + \frac{a}{\cos\gamma}\left(\frac{1}{1+\sin\gamma} - \frac{1-\sin^3\gamma}{3\cos^2\gamma}\right) . \tag{5}$$

Also a lower bound analogous to the upper bound in (3) can be given.

The size of a capillary tube is best measured in terms of the nondimensional parameter $B = \kappa a^2$. (For a water-air interface on the earth's surface, $\kappa \sim 29$.) If $B \leq 1$, (3) and (5) yield quite precise estimates. For larger B, one writes the equation in the parametric form

$$\frac{dr}{d\psi} = \frac{r\cos\psi}{\kappa r u - \sin\psi} , \qquad \frac{du}{d\psi} = \frac{r\sin\psi}{\kappa r u - \sin\psi} \tag{6}$$

in terms of the inclination angle ψ of a vertical section of the solution surface. (6) can be integrated approximately to obtain a hierarchy of estimates, valid for all B and asymptotically exact both for small and large B (Finn, Moscow Math. Soc., vol. dedicated to Vekua, 1978; Siegel, Pacific J. Math., 1980; Finn, Pacific J. Math., 1980). We mention the results

$$\sqrt{\frac{p+1}{p}}\sqrt{\frac{2}{\kappa}(1-\cos\psi) + \frac{p+1}{4}u_o^2} < u < 2\,\frac{\sin\psi}{\kappa r} + \sqrt{\frac{2}{\kappa}(1-\cos\psi) - \frac{\sin^2\psi}{\kappa^2 r^2} + \frac{u_o^2}{2}} \tag{7}$$

with $p = \sqrt{1 + \frac{\kappa r^2}{1+\cos\psi}}$. These (and other related) formulas yield the first general estimates valid in the range $1 < B < 10$. They also have remarkable monotonicity properties, which lead to precise estimates for the meniscus height $q = u_a - u_o$.

Brulois (Dissertation, Stanford University, 1981) has given a formal iterative procedure leading to an arbitrarily good upper bound for u_o.

The above methods can be modified and extended to apply also to the problem of the "sessile" liquid drop, and they lead to general estimates for the parameters describing its shape (Figure 3). Here the "physical" prescribed data are in general the volume V and γ, rather than a and γ as above. It turns out there is a "reciprocity" between the two problems, becoming arbitrarily exact for small and for large B (Finn, Pacific J. Math., 1980).

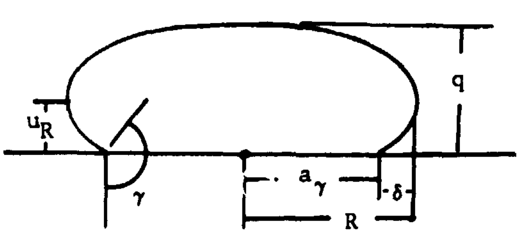

Figure 3

If $V \to 0$ the drop tends asymptotically to a spherical cap; however, its behavior near the wetted surface changes strikingly, depending on whether $\gamma = \pi$ or $\gamma \neq \pi$. We set $\mathcal{B} = \kappa \mathcal{P}^2$ where $\mathcal{P}$ is the radius of a ball of volume V, and write

$$\lambda^3(\gamma) = \frac{3}{4\sin^3\gamma}\int_0^\gamma \sin^3\theta \, d\theta .$$

Then if $\gamma \neq \pi$ there holds

$$\lim_{\mathcal{B}\to 0} \frac{\mathcal{B}}{B} = \lambda^2(\gamma) \tag{8}$$

while if $\gamma = \pi$ we find

$$\lim_{\mathcal{B} \to 0} \frac{\mathcal{B}^2}{B} = \frac{3}{2} . \tag{9}$$

Thus, the rate of decrease of wetted surface is nonuniform in contact angle. If $\gamma = \pi$ the drop rests--for small $\mathcal{B}$ --on a negligibly small surface (Figure 4). It seems likely that this surface acts as a point of support about which the drop can rotate rigidly when disturbed slightly, thus establishing new points of contact with the supporting plane and leading to a kind of "rolling" instability (Finn, J. Reine Angew. Math., to appear).

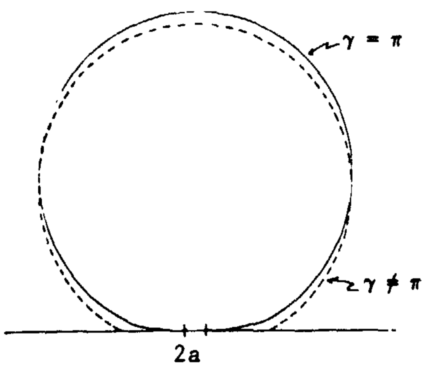

Figure 4

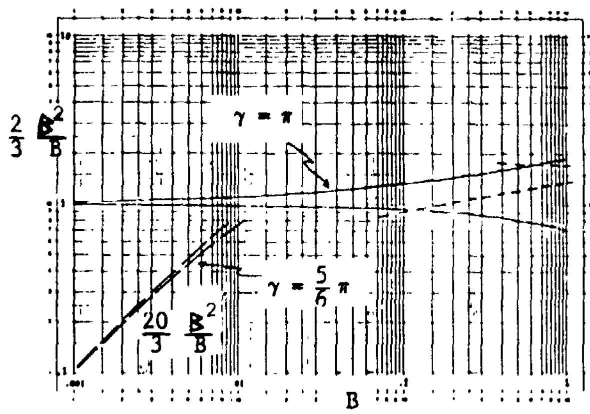

Figure 5

The nonuniformity is illustrated in Figure 5, which shows--on logarithmic scale--upper and lower bounds for the expression in (9) when $\gamma = \pi$, and for ten times that expression when $\gamma = \frac{5}{6}\pi$.

For large drops, one finds the exact asymptotic relation for the"overhang"

$$\lim_{\mathcal{B} \to \infty} \sqrt{\kappa}(R - a) = \sqrt{2} - \log(1 + \sqrt{2}) - 2 \cos \frac{\gamma}{2} + \log \cot \frac{\gamma}{4} . \tag{10}$$

If $\gamma = \pi$, this relation simplifies to

$$\lim_{\mathcal{B} \to \infty} \sqrt{\kappa}(R - a) = \sqrt{2} - \log(1 + \sqrt{2}) . \tag{11}$$

Also, R, a can be estimated in terms of $\mathcal{B}$.

The behavior of liquid in a capillary tube with asymmetric section Ω can differ in striking ways from what happens with a circular section. For surfaces of the form $z(x,y)$ general estimates can be obtained by comparison with symmetric surfaces, using maximum principles that are idiosyncratic for the equation. An important distinction between these principles and the classical ones for elliptic equations is that the comparison on the boundary need be prescribed only up to a set of Hausdorff measure zero. The distinction has as consequence the following result (Concus and Finn, Acta Math., 1974):

Let $u(x,y)$ be a capillary surface over a section Ω which contains the intersection of a ball B_δ of radius δ and a wedge of opening 2α (Figure 6). Then if $\alpha + \gamma \geq \pi/2$, there holds $u < \frac{2}{\kappa\delta} + \delta$; if $\alpha + \gamma < \pi/2$, then $u \to \infty$ at V. Thus the solutions depend discontinuously on the boundary data. Figure 7 shows a "kitchen sink" experiment that exhibits the discontinuity for water in a wedge formed by two plastic plates.

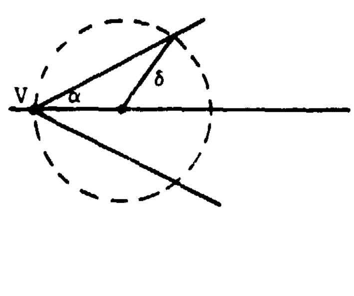

Figure 6

Figure 7

Figure 8

The problem was studied further by L. Simon (Pacific J. Math., 1980) who proved that if $\alpha + \gamma > \pi/2$, $\alpha < \pi/2$, then $u(x,y)$ is differentiable up to V. In independent work, N. Korevaar (Pacific J. Math., 1980) found the surprising result that if $\alpha > \pi/2$, there exist solutions that are bounded and discontinuous at V.

Concus and Finn (Math. Z., 1976) showed there exist sections Ω, $\Omega' \subset \Omega$, such that Ω raises more fluid over Ω' than Ω' does. The problem was studied further by Finn (Vekua volume, l.c.) who gave general conditions under which this behavior will or will not occur. Siegel (Pacific J. Math., l.c.) gave another condition under which the "smaller" tube must raise a larger volume over its section.

For a capillary tube in outer space (zero gravity), solutions of the problem as posed do not in general exist. At a corner, as in Figure 6, there can be no solution when $\alpha+\gamma < \pi/2$ (Concus and Finn, Acta Math., 1974). Physically, the fluid flows out along the corner, to infinity or to the top of the container, whichever comes first. For a regular polygon the above condition is best possible: if $\alpha + \gamma \geqslant \pi/2$ a lower spherical cap yields an explicit bounded solution. Figure 8 shows the results of an experiment conducted in the NASA drop tower in Cleveland and verifying the predicted behavior.

For a general section Ω it appears to be not easy to find existence criteria. The case $\gamma = 0$ was studied by Chen (Pacific J. Math., 1980), who gave a simple geometric sufficiency condition. For general γ, Finn (Manuscripta Math., 1979) reduced the question to that of properties of vector fields over Ω. Applying the results to polygonal domains, he found that in a parallelogram of arbitrary side ratio a solution exists if and only if $\alpha+\gamma \geqslant \pi/2$ at the smaller vertex angle 2α. Thus, a solution exists in any rectangle if $\gamma \geqslant \pi/4$. However, the existence can fail for any $\gamma \neq \pi/2$, in trapezoids obtained from rectangles by arbitrarily small deformations.

This behavior was clarified recently by Finn (Indiana Univ. Math. J., to appear), who showed that a solution surface exists if and only if there is no subarc Γ of a semicircle of radius $R_\gamma = \frac{\Omega}{\Sigma \cos \gamma}$, meeting Σ in angles γ as indicated in Figure 9, for which

$$\Phi(\Gamma) \equiv \Gamma - \Sigma^* \cos \gamma + \frac{1}{R_\gamma} \Omega^* \leqslant 0 . \qquad (12)$$

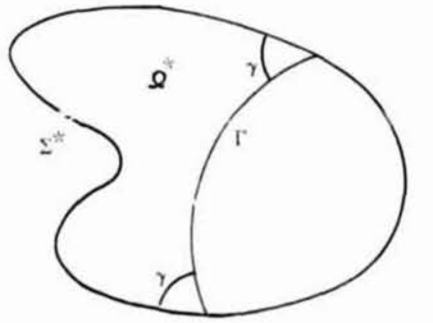

Figure 9

Here the lengths and areas Σ, Ω, $\cdots$ are as indicated in Figures 1, 9.

Consider a situation in which $\Phi(\Gamma) = 0$, and in which there is no Γ for which $\Phi(\Gamma) < 0$. Let $\gamma_j \downarrow \gamma$. Then there is a corresponding sequence of solution surfaces with boundary angle γ_j, tending to a solution with boundary angle γ on $\Omega \setminus \Omega^*$, and tending to infinity on Γ and throughout Ω^*. The solution is asymptotic at Γ to a vertical cylinder of radius R_γ. The cylinder acts as a barrier across which the solution surface cannot be extended.

The behavior just described actually occurs in a trapezoidal section. Also, letting the smaller base $\to 0$ while the nonparallel sides meet (at V) in a fixed angle 2α, Γ will tend to V while $\gamma \to (\pi/2) - \alpha$; thus, the above angle theorem appears as a limiting case.

Gerhardt (Pacific J. Math., 1980) considered tubes closed at the bottom and partially filled with liquid. He showed there always exist energy minimizing solutions (with or without gravity) which may have the value $z = 0$ on part of the base. In this region, the solutions appear to admit the physical interpretation of a thin film covering the base.

A drop hanging from a horizontal plane (Figure 10) behaves very differently from the sessile drop. The solution section is uniquely determined by the height u_0 at the vertex and

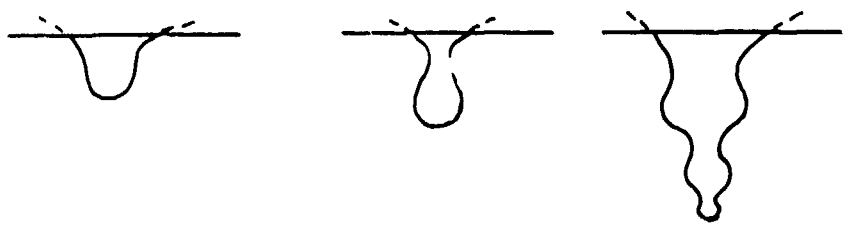

Figure 10

consists, for any u_0, of a curve that can be continued analytically to infinity without limit sets or double points (Concus and Finn, Philos. Trans. Roy. Soc., 1979). There exists also a singular solution $v(r) \sim -(\sqrt{\kappa}\, r)^{-1}$ (Concus and Finn, Invent. Math., 1975; Huh, Dissertation, Dept. Chem. Eng., University of Minnesota, 1969). It is conjectured that as $u_0 \to -\infty$, the "drop" solutions tend, uniformly in compacta, to $v(r)$. A proof of a somewhat weaker result appears in Concus and Finn (Philos. Trans. Roy. Soc., l.c.).

Conditions for stability of the pendent drop have been given by E. Pitts, by Michael and by others. Most recently, the problem was treated in full generality by Wente (Pacific J. Math., 1980). Wente showed in particular that the occurrence of an inflection in the meridional section need not preclude stability.

The reasoning of Young on the constancy of γ is based on a hypothesis that all material forces are central. In the presence of resistive forces the behavior can be very different. Finn and Shinbrot consider a drop of liquid on a horizontal surface, with γ initially determined as in the Young theory. If liquid is now slowly added, the wetted surface may remain constant while the angle γ increases. If resistance is very large, then continued addition of liquid will eventually lead to a value $\gamma > \pi$, which is physically impossible as then the drop would penetrate the supporting plane. It follows that a geometrically imposed instability must occur when γ increases past π, forcing the wetted surface to increase. It can be shown (Finn, J. Reine Angew. Math., l.c.) that an upper bound for the critical $\mathfrak{B}$ is determined as the unique solution of the relation

$$\mathfrak{B}^3 - \frac{3}{2}\mathfrak{B}\, B - \frac{9}{4} B^2 = 0 \,. \tag{13}$$

Finn and Shinbrot interpret the above behavior by postulating a resistance force whose area density F is potential, $F = -\nabla\varphi$, and which is formally equivalent to a distribution of linear density φ directed normally on Σ. They then apply that interpretation to the more complicated situation of a drop on an inclined plane, intially under zero gravity and meeting the plane in the (Young) angle γ_0, and then subjected to slowly increasing gravity (Figure 11). Under hypotheses, that φ depends only on the pressure at the interface, and that the effect can be separated into a radial "squishing" term as occurs for the horizontal plate and a "sliding" term due to the inclination, they are led to a relation of the form (for small $\mathfrak{B}$)

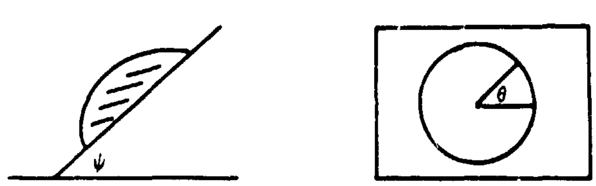

Figure 11

$$\cos \gamma = \cos \gamma_o + \varepsilon(\psi) + \alpha \sin \psi \sin \theta - \beta \sin^2\psi \sin^2\theta . \tag{14}$$

Here α, β are constants, α is explicitly known and of order $\mathfrak{k}$, and ε is decreasing in ψ. β has order $\mathfrak{k}^2$, ε has order $\mathfrak{k}$ if $\psi < \pi/2$ and order $\mathfrak{k}^2$ if $\psi = \pi/2$. Again a geometrically imposed instability appears, and in fact does so for surprisingly small values of $\mathfrak{k}$.