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The stability of the axially symmetric pendent drop

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Abstract

We analyze the axially symmetric pendent drop as it occurs in three different physical settings: Problem A, constant pressure, fixed circular opening (the siphon); Problem B, constant volume, fixed circular opening (the medicine dropper); Problem C, prescribed volume, constant angle of contact with a horizontal plate. As examples, the following results are verified. For Problem B we show that if the opening is small enough to support a stable pendent drop with a bulge, then as the exposed volume is increased, stable pendent drops with both a neck and a bulge will be formed. For Problem C we show that with increasing volume the profile curves for the family of stable pendent drops will develop an inflection point before instability arises.

Introduction

We first analyze the equilibrium and stability criteria for each of the problems.

Problem A

Here the drop is to protrude downward from a fixed circular opening of radius $\bar{r}$, held at the level $u = \bar{u}$ where u is the vertical coordinate with positive direction upward and $u = 0$ is the zero pressure level of the fluid. (see Figure 1). If X is the exposed body of the fluid and Ω is the liquid-air interface with $A(\Omega)$ its area, the potential energy of the configuration is

$$E_o(\Omega) = \sigma A(\Omega) + \rho g \int_X z\,dV. \tag{1}$$

σ is the surface tension of the liquid-air interface, ρ is the density of the fluid, and g is the gravitational constant. The condition for equilibrium is that the first variation of the potential energy $\delta E_o(\Omega,N) = 0$ for all normal perturbations N of Ω which vanish on the boundary. The Euler Equations yield

$$2H = -ku \quad \text{on } \Omega, \qquad k = \rho g/\sigma . \tag{2}$$

H is the mean curvature of the surface measured so that it is positive at the drop tip. By a suitable scaling we may assume that $k = 1$. The condition for stability is that the second variation be positive for all non-trivial normal perturbations .

$$\delta^2 E_o(\Omega,N) > 0 \quad \text{for all } N \neq 0, \text{ but } N = 0 \text{ on } \delta\Omega. \tag{3}$$

Problem B

As in Problem A the fixed circular opening of radius $\bar{r}$ lies in a horizontal plane, but now the exposed volume is prescribed (see Figure 1). Now the condition for equilibrium is that the first variation of the energy $\delta E_o(\Omega,N) = 0$ for all perturbations N, vanishing on the boundary and for which the first variation of the volume is also zero. By the method of Lagrange multipliers we obtain

$$\delta(E_o + \lambda V)(\Omega,N) = 0 \quad \text{for some constant } \lambda \tag{4}$$

and all normal perturbations N vanishing on the boundary. This yields the condition

$$2H = -ku + \lambda' , \quad \text{where } k = \rho g/\sigma \text{ and } \lambda' \text{ is a constant.} \tag{5}$$

By a vertical translation of coordinates we may take λ' to be zero, reducing (5) to the condition (2), and with the vertical coordinate of the opening at the level $u = \bar{u}$.

The condition for stability is that

$$\delta^2(E_0+\lambda V)(\Omega,N) > 0 \tag{6}$$

for all non-trivial normal perturbations N, which vanish on the boundary and for which the first variation of the volume is zero.

Problem C

The drop is now pendent from a homogeneous horizontal plate. The potential energy is now

$$E(\Omega) = E_0(\Omega) - \sigma\beta|\Sigma| \tag{7}$$

where β is a physical constant and $|\Sigma|$ is the area of contact of the liquid with the plate. Setting the first variation equal to zero for all volume preserving perturbations yields the conditions

$$\begin{aligned} &\text{a)}\quad 2H = -ku + \lambda \quad \text{for some constant } \lambda,\ k = \rho g/\sigma \\ &\text{b)}\quad \cos\alpha = \beta \end{aligned} \tag{8}$$

Here α is the angle of contact of the liquid-air interface with the horizontal plate measured interior to the fluid. Again we may choose $k = 1$, and by a vertical translation of coordinates may set $\lambda = 0$, with the horizontal plate at a level $u = \bar{u}$. Clearly it is necessary that $|\beta| \leq 1$ so that $0 \leq \alpha \leq \pi$. There are no possible pendent drops when $\alpha = \pi$ so that we may consider $0 \leq \alpha < \pi$ ($-1 < \beta \leq 1$). As in Problem B the condition for stability is that the second variation $\delta^2(E + \lambda V)(\Omega,N)$ be positive for all non-trivial normal perturbations for which the first variation of volume is zero. (see Figure 1).

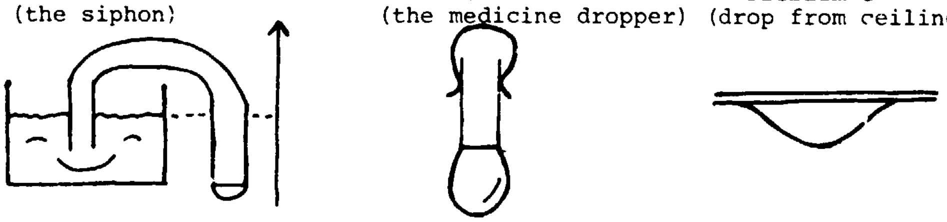

Figure 1. The various drop configurations.

Description of the Profile curves

Suitably normalized, the differential equation for the profile curve whose surface of revolution represents the liquid-air interface (satisfying (2) with $k = 1$) is

$$\begin{aligned} &\text{a)}\quad r'(s) = \cos\psi & r(0) &= 0 \\ &\text{b)}\quad u'(s) = \sin\psi & u(0) &= u_0 \\ &\text{c)}\quad \psi'(s) = -(\sin\psi/r) - u & \psi(0) &= 0. \end{aligned} \tag{9}$$

The solutions to this system have been carefully studied by many people. In particular, I should mention the work of D.W. Thomson[1], F.Bashforth and J.C. Adams[2], and recently P. Concus and R. Finn[3].

There is a unique solution, $\{r(s,\kappa), u(s,\kappa), \psi(s,\kappa)\}$, to the system satisfying the initial conditions, $r(0,\kappa) = 0$, $u(0,\kappa) = u_0 = -2\kappa$, $\psi(0,\kappa) = 0$, where κ is the mean curvature at the drop tip. The solution exists for all s and all κ and is analytic in both variables. We note that $u \equiv 0$ gives a solution and that reflection of any solution about the r-axis yields another solution. Drops with $u_0 < 0$ represent pendent drops and the solutions corresponding to $u_0 > 0$ represent "emerging" bubbles. We now list other important properties of the family of solutions.

1. For "small" $u_0 < 0$ the solution can be expressed non-parametrically with u as a function of r over the entire positive r-axis, and $u(r) \sim u_0 J_0(r)$ where $J_0(r)$ is the Bessel function of order zero.

2. There is a value u_0^* ($u_0^* \cong -2.5678-$) such that the profile curve with drop tip at u_0^* attains a simultaneous vertical tangent and inflection point at (r_1^*, u_1^*) where $r_1^* \cong .91$ and $u_1^* \cong -1.1$. For $0 < r < r_1^*$ the curve is convex while for r greater than r_1^* the curve may again be expressed non-parametrically $u = u(r)$.

3. For $u_0^* < u_0 \leq 0$ the solutions may be expressed in non-parametric form, $u = u(r)$, for all r.

4. For $u_0 < u_0^*$ the profile curves attain a vertical tangent at a point (r_1, u_1) where $0 < r_1 < r_1^*$ and $u_1 < u_1^*$. The curves form a bulge at this point and r decreases to a value r_2, and forms a neck at (r_2, u_2) where $u_2 < 0$. r_1 and u_1 are increasing functions of u_0 for $u_0 < u_0^*$ with limit $r_1 = 0$ and limit $u_1 = -\infty$ as u_0 approaches $-\infty$.

5. For $u_0 << u_0^*$ the profile curves form a sequence of bulges and necks until it crosses the r-axis with $r'(s)$ and $u'(s)$ both positive from which point on the curves may be expressed non-parametrically $u = u(r)$ out to $r = \infty$.

6. The first inflection point on a profile curve with tip at $u_0 < 0$ occurs at a point $(\hat{r}, \hat{u})$ where $\hat{u} < 0$. $\hat{r}$ and $\hat{u}$ are monotonically increasing functions of u_0 for $u_0 < 0$. If $u_0 < u_0^*$, so that the profile curve has both a neck and a bulge, then the first inflection point lies between the first neck and first bulge.

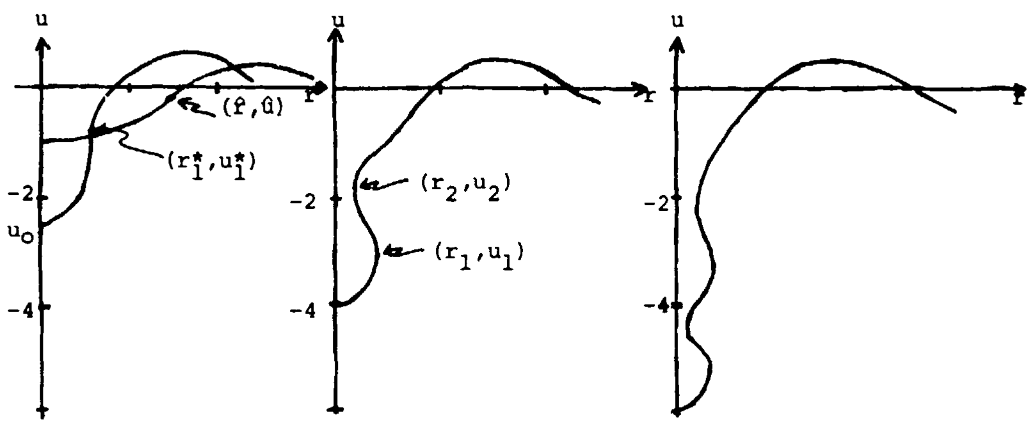

Figure 2. Possible drop configurations

Analysis of stability

Our method for determining the stable configurations for each of the problems proceeds as follows. Take a given profile curve $\{r(s,\kappa), u(s,\kappa), \psi(s,\kappa)\}$ and let $(\bar{r}, \bar{u})$ be a point on the curve, $\bar{r} = r(\bar{s}, \bar{\kappa})$ and $\bar{u} = u(\bar{s}, \bar{\kappa})$. The profile curve from the drop tip to this point generates a possible pendent drop whose exposed volume V, can be calculated.

$$V = \text{volume of drop} = \pi\bar{r}(\bar{r}\bar{u} + 2\sin\bar{\psi}) \tag{10}$$

The volume gives us a fourth function of the parameters s and κ, $V = V(s,\kappa)$. For each of our three problems two of the four functions are prescribed. This generates a mapping from the (s,κ)-plane (the parameter space) into a two-dimensional "control" space. The analysis of this map determines stability for each of the problems.

Stability for problem A

Here the appropriate map is $A(s,\kappa)$ where it is defined by

$$A(s,\kappa) = (\, r(s,\kappa), \quad s,\kappa)) \,. \tag{11}$$

The "control" space for this problem is the (r,u)-plane. It is easily checked that the derivative of A, $DA(s,\kappa)$ is invertible when s equals zero. Let O be the set of all points in the (s,κ)-plane where the derivative is invertible.

Definition. O_S contained in O is that component of O in the parameter space which contains the line $s = 0$.

Theorem 1. Every point $(\bar{s},\bar{\kappa})$ in O_S determines a stable pendent drop for Problem A. (i.e. the drop generated by the profile curve $(r(s,\bar{\kappa}), u(s,\bar{\kappa}))$ for $0 < s < \bar{s}$). Any point outside $\bar{O}_S$ determines an unstable pendent drop for Problem A.

This theorem is essentially classical.

It follows that the "control set" $A(O_S)$, is an open set in the (r,u)-plane. A point $(\bar{r},\bar{u})$ determines a stable configuration for problem A if and only if it is a member of the set $A(O_S)$. We now wish to describe the regions O_s and $A(O_S)$. Since $A(O_S)$ is symmetric about the coordinate axes, we may restrict ourselves to the case $r \geqslant 0$, $u \leqslant 0$

Theorem 2. Choose $\bar{\kappa} > 0$ and consider the profile curve $(r(s,\bar{\kappa}),u(s,\bar{\kappa}))$, $s \geqslant 0$. There is a smallest positive value $\hat{s}$, such that $(s,\bar{\kappa})$ is in O_S for $0 < s < \hat{s}$ while $(\hat{s},\bar{\kappa})$ is on the boundary of O_S. On the interval $0 < s \leqslant \hat{s}$, we have $r'(s) = \cos\psi$ positive so that $0 < \psi < (\pi/2)$.

Therefore the corresponding profile curve $(r(s,\bar{\kappa}),u(s,\bar{\kappa}))$ $0 < s < \hat{s}$ may be expressed in nonparametric form $u = f(r,\bar{\kappa})$, for $0 < r < \hat{r}$ where $\hat{r} = r(\hat{s},\bar{\kappa})$ and $\hat{u} = f(\hat{r},\bar{\kappa})$. The point $(\hat{r},\hat{u})$ lies on the boundary of $A(O_S)$. It is the conjugate point to the drop tip along this profile curve.

Since $r'(s)$ is positive, we may use r as an independent variable rather than s. Points $(\bar{r},\bar{u})$ on the boundary of $A(O_S)$ are determined by the condition that the derivative $D\tilde{A}(r,\kappa)$ be singular where $\tilde{A}(r,\kappa) = (r, f(r,\kappa))$ and $u = f(r,\kappa)$ is the nonparametric representation of the curve. This occurs when $f_\kappa(r,\kappa)$ equals zero. (i.e. the point $(\bar{r},\bar{u})$ is on the envelope Γ_A, of the family of profile curves.

Theorem 3. The first envelope Γ_A, of the family of profile curves $u = f(r,\kappa)$ for $\kappa \geqslant 0$ $(u_o \leqslant 0)$ and r positive, is the graph of a smooth (analytic) function $u = e(r)$ for $0 < r \leqslant \alpha_o$ where α_o is the first positive zero of the zero order Bessel function, $J_o(r)$. This function has the following properties.

$$\text{limit } e(r) = -\infty \text{ as } r \to 0^+ \quad , \quad \text{limit } e(r) = 0 \text{ as } r \to \alpha_o$$

The derivative $e'(r)$ is positive on the interval $0 < r < \alpha_o$ with $e'(\alpha_o) = 0$ and limit $e'(r) = +\infty$ as r approaches zero.

The entire envelope is thus a smooth curve without self-intersections which possesses a cusp only at $(\alpha_o,0)$. (see Figure 3)

Consequences.

I. The mapping $A(s,\kappa)$ is a diffeomorphism of O_S onto its image $A(O_S)$.

II. For $(\bar{r},\bar{u})$ in $A(O_S)$ where $\bar{r} < r_1^\dagger$, the profile curve for the stable pendent drop is convex. For $\bar{r}$ near α_o ($\bar{r} < \alpha_o$) and $(\bar{r},\bar{u})$ in $A(O_S)$, the profile curve will contain an inflection point, and so the stable pendent drop loses convexity. (Figure 3)

III. There are no "inaccessible" stable pendent drops for Problem A. The vertical line $r = \bar{r}$ ($\bar{r}$ less than α_o) intersects $A(O_S)$ in a connected interval. Thus the stable pendent drop corresponding to the point $(\bar{r},\bar{u})$ may be reached from the stable zero pressure solution ($u \equiv 0$) corresponding to the point $(\bar{r},0)$ in $A(O_S)$ merely by increasing the pressure $(p = -u)$ continuously from 0 to $-\bar{u}$. (see Figure 3.)

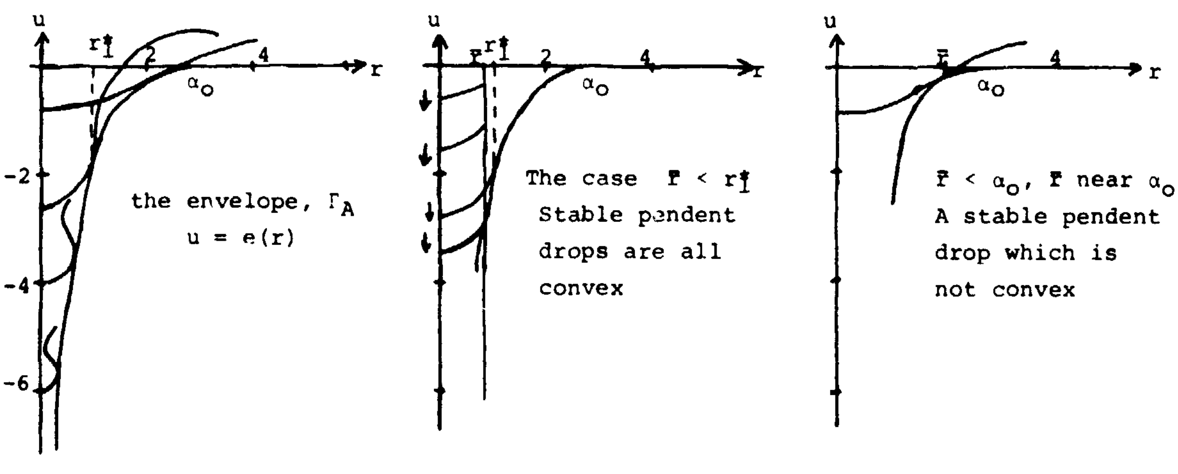

Figure 3. The envelope Γ_A, stable configurations for problem A

Stability for problem B.

The constraints are now the radius of the tube r, and the exposed volume V. Thus the control space is the (r,V)-plane and we are led to study the mapping $B(s,\kappa)$ from the parameter space to the control space defined by

$$B(s,\kappa) = (r(s,\kappa), V(s,\kappa)) \tag{12}$$

As in problem A we let O be the open set in the (s,κ)-plane where the derivative $DB(s,\kappa)$, is invertible. However, since $B(0,\kappa) = (0,0)$ the line $s = 0$ lies outside the set O.

Fix $\bar{\kappa}$ and consider the curve $B(s,\bar{\kappa})$ for positive s. There exists a smallest positive value s_B, such that the derivative $DB(s,\bar{\kappa})$ is invertible for $0 < s < s_B$ but singular when $s = s_B$. Let (r_B, u_B) be the corresponding point on the profile curve $(r(s,\bar{\kappa}),u(s,\bar{\kappa}))$, where $r_B = r(s_B,\bar{\kappa})$ and $u_B = u(s_B,\bar{\kappa})$. It is a classical result that if $(\bar{r},\bar{u})$ is a point on the profile curve prior to (r_B,u_B) then the corresponding pendent drop generated by the profile curve up to $(\bar{r},\bar{u})$ is "symmetrically" stable for problem B, while if the profile segment contains the point (r_B,u_B) then the generated drop is unstable for problem B.

Definition. We call the point (r_B,u_B) the "Volume-constrained" conjugate point on the profile curve relative to the drop tip.

Note. The axisymmetric pendent drop is said to be symmetrically stable if the second variation $\delta^2(E_O + \lambda V)(\Omega,N)$ is positive for all non-trivial symmetric normal perturbations N, of Ω which vanish on the boundary and for which the first variation of volume is zero. If the profile curve can be expressed non-parametrically in the form $r = r(u)$, then symmetric stability implies stability.[4] In this case we observe that the angle of inclination ψ, must be non-negative. However, if the angle of inclination becomes negative on some portion of the profile curve (the corresponding drop is of re-entrant type), then the drop is unstable for problem B due to a non-symmetric perturbation. This fact was noted by D.H. Michael and P.G. Williams [4]. For an alternative discussion see reference 5.

Definition. O_S is the subset of O consisting of all points (s,κ) where $0 < s < s_B$ where s_B depends on κ. It follows that (s,κ) determines a stable configuration if it is in O_S and an unstable configuration if it lies outside of $\bar{O}_S$.

We now wish to describe O_S and its image $B(O_S)$ contained in the "control space", the (r,V)-plane.

Theorem 4. O_S is a connected open set in the (s,κ)-plane bounded on the left by the line $s = 0$, and on the right by an analytic curve γ, which is the graph of a positive analytic function $s_B = \sigma(\kappa)$.

Therefore O_S is an open set in the parameter space and $B(O_S)$ is an open set in the control space.

Theorem 5. Let (r_B,u_B) be the volume-constrained conjugate point on the profile curve $(r(s,\kappa),u(s,\kappa))$. At the point (r_B,u_B) the derivative, $r'(s)$ is positive. The point (r_B,u_B) is located between the first and second inflection points on the profile curve. If the profile curve possesses a bulge (and hence a neck) then (r_B,u_B) is located above the neck. As κ approaches zero the point (r_B,u_B) approaches the point $(\alpha_1,0)$, where α_1 is a root of the equation $rJ_O(r) + 2J_O'(r) = 0$. (see Figure 4)

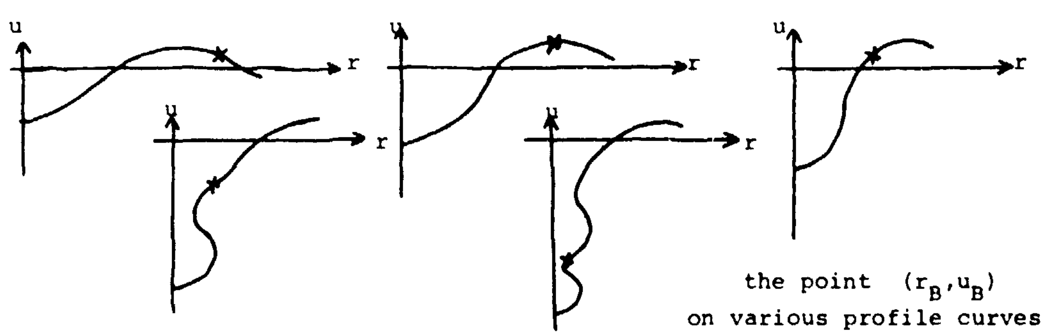

Figure 4. Locations of the volume-constrained conjugate point

By Theorem 4 the curve γ is an analytic arc parameterized by κ. Its image $B(\gamma)$ is a parameterized curve in the (r,V)-plane and, as in problem A, it is the envelope, Γ_B, of the family of curves $(r(s,\kappa),V(s,\kappa))$. Thus Γ_B may be expressed in the form $(r(\kappa),V(\kappa))$ where $r(\kappa) = r(\sigma(\kappa),\kappa)$ and $V(\kappa) = V(\sigma(\kappa),\kappa)$ are analytic functions of κ. Furthermore

$$\text{limit } (r(\kappa),V(\kappa)) = (0,0) \text{ as } \kappa \to \infty, \quad \text{limit } (r(\kappa),V(\kappa)) = (\alpha_1,0) \text{ as } \kappa \to 0.$$

By Theorem 5 we know that a given curve $B(s,\bar{\kappa})$, touches the envelope at a point where $r'(s)$ is positive. Thus, in a neighborhood of this point, (r_B,V_B), we may express the curves $B(s,\kappa)$ non-parametrically in the form $V = g(r,\kappa)$. If the envelope is smooth it will be tangent to the family of curves $V = g(r,\kappa)$, and itself would have a non-parametric representation $V = G(r)$. A point on the envelope of the family of curves $V = g(r,\kappa)$, is determined by the condition $g_\kappa(r,\kappa) = 0$ while the condition for smoothness is that $g_{\kappa\kappa}(r,\kappa) \neq 0$. Since the envelope $\Gamma_B = B(\gamma)$ is a parameterized analytic curve it will be smooth except perhaps at isolated points where the derivatives $r'(\kappa)$ and $V'(\kappa)$ both vanish. At such points the possibility of a cusp arises. One such cusp occurs at $(\alpha_1,0)$.

Conjecture. That part of the envelope Γ_B which lies in the half-space $V > 0$ is a smooth curve which may be expressed non-parametrically in the form $V= G(r)$, $0 < r < \alpha_1$, with $G(0) = G(\alpha_1) = 0$ and $G'(\alpha_1) = 0$. There is a single value r^* where $G'(r^*) = 0$.

Computer calculations [6] strongly indicate that the conjecture is valid, but a complete proof of this is lacking at present. (see Figure 5)

Figure 5. The curves $B(s,\kappa)$ and their envelope, Γ_B.

If the envelope is a smooth curve, then it follows that the map $B(s,\kappa)$ is a diffeomorphism of O_S onto its image $B(O_S)$. In this case any vertical line $r = \bar{r}$ in the control space would intersect $B(O_S)$ in a connected interval. The stable pendent drop corresponding to $(\bar{r},\bar{V})$ is accessible from the flat drop $u \equiv 0$, corresponding to the point $(\bar{r},0)$ in the control space, through a smooth one-parameter family of stable pendent drops of increasing volume and fixed radius for the aperture until a maximum volume is reached. This is the proceedure used by E. Pitts in his paper. [7]

If the envelope were not smooth and contained cusps, then the possibility arises that the mapping $B(s,\kappa)$ is not a one-to-one map of O_S onto its image, or that for some $\bar{r}$ the intersection of the line $r = \bar{r}$ with the set $B(O_S)$ is not connected. In either case there would exist stable pendent drops corresponding to some control value $(\bar{r},\bar{V})$ which could not be connected to $(\bar{r},0)$ in the manner described above.

If we follow the "usual" proceedure of trying to describe those drops which are accessible from the initial drop $u \equiv 0$, corresponding to the point $(\bar{r},0)$ then we have the theorem:

Theorem 6. (a) If $\bar{r} < r^*_1$ then as we increase the volume from zero there will be produced a one-parameter family of stable pendent drops for Problem B. Through an initial range of volumes $0 < V < V_1(\bar{r})$, the profile curves will be convex and the drops will develop a bulge. When the exposed volume reaches $V_1(\bar{r})$, the profile curve will develop an inflection point at the edge of the drop. With increasing volumes the drops lose convexity, but before the limit of stability is reached pendent drops possessing both a neck and a bulge will evolve.

(b) For $\bar{r} > \beta_1$ where $J_1(\beta_1) = 0$ the "drop" $u \equiv 0$ is unstable for problem B due to non-symmetric perturbations. For $\bar{r} < \beta_1$ the drop $u \equiv 0$ is stable and with increasing volume the profile curves for the family of stable pendent drops will develop an inflection point before the limit of stability is reached.

(c) For any radius $\bar{r}$, drop height increases monotonically with volume throughout the range of stability.

The result (a) of Theorem 6 was observed in the limiting case of small drop with narrow necks by A.K. Chesters [8].

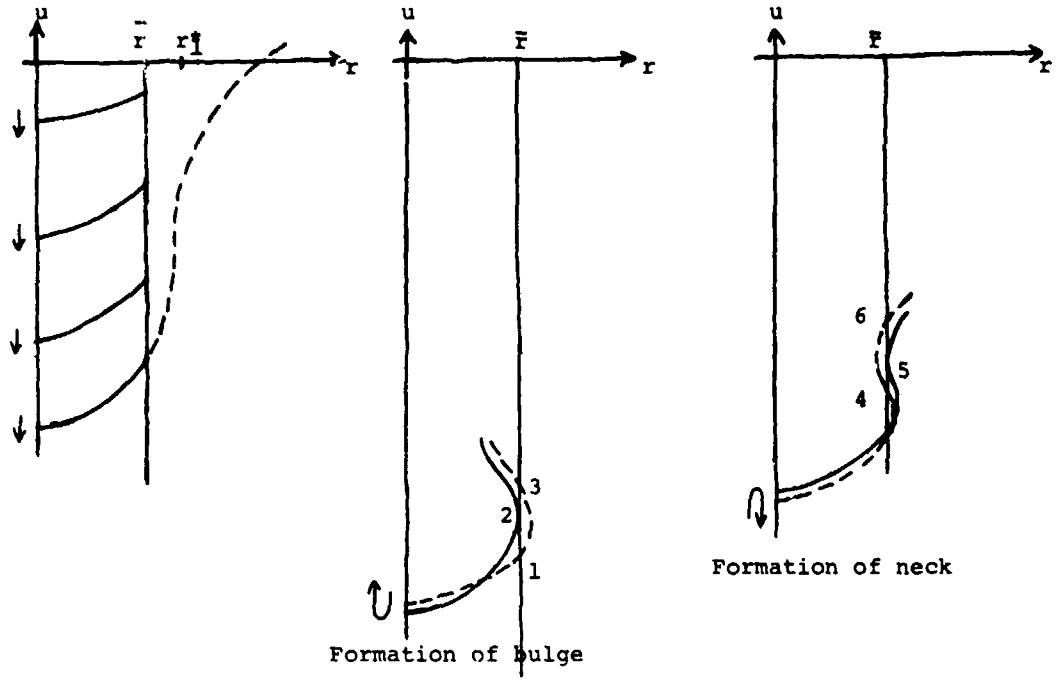

Figure 6. Drop formation for Problem B

Stability for problem C.

The control parameters are now the angle of inclination ψ, and the volume V, giving

$$C(s,\kappa) = (\psi(s,\kappa), V(s,\kappa)) \tag{13}$$

as our mapping from the parameter space to the control space. Again we let O be the set of all points (s,κ) where the derivative $DC(s,\kappa)$, is invertible and we observe that the line $s = 0$ does not meet the set O since $C(0,\kappa) = (0,0)$. For each $\bar{\kappa}$, there is a positive value s_C depending on $\bar{\kappa}$, such that $DC(s,\bar{\kappa})$ is invertible for $0 < s < s_C$ but is singular at s_C. Let O_S be the set of all points in O of the form $(s,\bar{\kappa})$ where $0 < s < s_C$.

Theorem 7. O_S is an open simply-connected set bounded on the left by the line $s = 0$ and on the right by a curve γ_C which is the graph of a positive analytic function $s_C = \sigma_C(\kappa)$ where limit $\sigma_C(\kappa)$ is zero as κ becomes infinite.

Definition. For a given profile curve $(r(s,\bar{\kappa}), u(s,\bar{\kappa}))$ $s \geq 0$, the volume-constrained focal point for problem C is the point (r_C,u_C) on the curve with $r_C = r(s_C,\bar{\kappa})$, $u_C = u(s_C,\bar{\kappa})$.

If a profile curve is to generate a physically meaningful configuration for problem C, it is necessary that the angle of inclination ψ, be non-negative along the segment of the profile curve generating the drop. Otherwise the drop would intersect the face. This eliminates from consideration the re-entrant drops. Therefore we let O_S' be the set of those points $(\bar{s},\bar{\kappa})$ in O_S for which the angle of inclination is positive, $0 < s < \bar{s}$.

Theorem 8. The profile curve segment corresponding to any member of O_S' generates a stable configuration for problem C. If the point $(\bar{s},\bar{\kappa})$ lies outside the closure of the set O_S', then the generated drop is unstable for problem C.

Remark: This result which was essentially "classical" for problems A and B, is somewhat more difficult for problem C.

In other words, let (r_C,u_C) be the volume-constrained focal point for problem C on some profile curve. Suppose that (r_C,u_C) comes before the point where the angle of inclination is zero. If $(\bar{r}.\bar{u})$ is a point on the profile curve prior to (r_C,u_C) then the corresponding pendent drop is stable for problem C, while if $(\bar{r},\bar{u})$ comes after (r_C,u_C) then the resulting drop is unstable.

Theorem 9. The volume-constrained focal point (r_C, u_C) for a given profile curve lies between the first and second inflection points. It comes ahead of the volume-constrained conjugate point for problem B, (r_B, u_B).

Theorem 10. The set $C(O_S')$ is symmetric about the line $\psi = 0$. It is bounded on the left by the line $\psi = 0$. The rest of $C(O_S')$ is bounded by $C(\gamma_C)$ where γ_C is the curve described in Theorem 7.

As in problem B, the set $C(\gamma_C)$ is the envelope Γ_C, of the family of curves $(\psi(s,\kappa), V(s,\kappa))$. By Theorem 9 each curve in the family will touch the envelope at a point where $\psi'(s)$ is negative. Therefore in a neighborhood of the touching point each of these curves may be expressed non-parametrically $V = h(\psi,\kappa)$. The envelope is determined by the condition $h_\kappa(\psi,\kappa) = 0$. It will be a smooth curve if $h_{\kappa\kappa}(\psi,\kappa) \neq 0$. If the angle ψ is positive, then $dV/d\psi = h_\psi(\psi,\kappa) = V_s/\psi_s$ is negative, and where it is smooth, the envelope will be the graph of a decreasing function.

Conjecture. That part of the envelope Γ_C, which lies in the first quadrant of the (ψ, V)-plane is the graph of a smooth function $V = V(\psi)$ $0 < \psi < \pi$, with $V'(0) = 0$, $V'(\psi)$ negative for $0 < \psi < \pi$, and $V'(\pi) = 0$.

Computer calculations strongly support the conjecture.[6] If the conjecture is true, then (as in problem B) the map C would be a diffeomorphism of O_S' onto its image, and the intersection of a vertical line $\psi = \bar\psi$, with $C(O_S')$ would be a connected interval. This would imply that as we move vertically along the line $\psi = \bar\psi$ from $(\bar\psi, 0)$ to $(\bar\psi, V_{max})$ in the control space, the corresponding drops would generate the entire family of stable pendent drops for problem C. If the conjecture were not true, then as was the case for problem B, the procedure just described would fail to pick up some stable pendent drops for Problem C. (see figure 7 .)

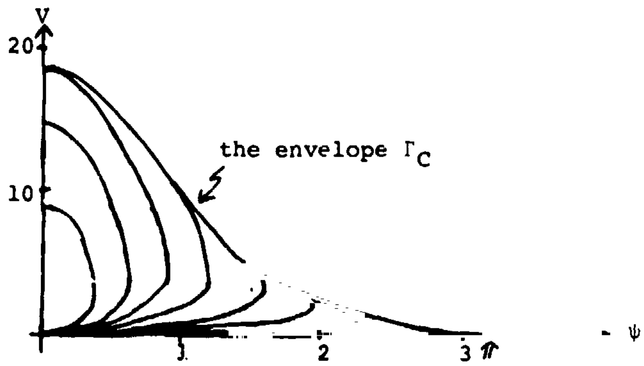

Figure 7. The curves $C(s,\kappa)$ and their envelope Γ_C

The following theorem identifies those stable pendent drops that are accessible from drops of very small volume.

Theorem 11. (a) For any angle of contact $\bar\psi$, $0 < \bar\psi < \pi$, stable drops of small volume are convex and resemble spherical caps. These drops are generated by profile curves whose tip is at u_o, where u_o is large and negative. At a certain positive volume V_1, where V_1 depends on $\bar\psi$, the profile curve generating the drop will develop an inflection point at its edge. This drop is stable. As the volume is increased, further stable pendent drops are formed, and the inflection point on the profile curves will move to the interior. With increasing volume the limit of stability will be reached before a second inflection point appears.

(b) If $\bar\psi = 0$ all profile curves corresponding to pendent drops of positive volume contain an inflection point. Drops of small volume correspond to small values for u_o at the drop tip. As u_o is decreased stable pendent drops of increasing volume are formed. The drop generated by that profile curve whose tip is at $u_o = u_o^* = -2.5678-$ is unstable. (Computer results indicate that the stable pendent drop of maximum volume occurs with $u_o = -1.6$ with a volume of 18.4)

(c) For any angle of contact drop height increases monotonically with volume throughout the range of stability.

Remark. For example, if the angle of contact $\psi = (\pi/2)$, it follows that with increasing volume and before the point of instability is reached, pendent drops containing both a neck and a bulge will appear.

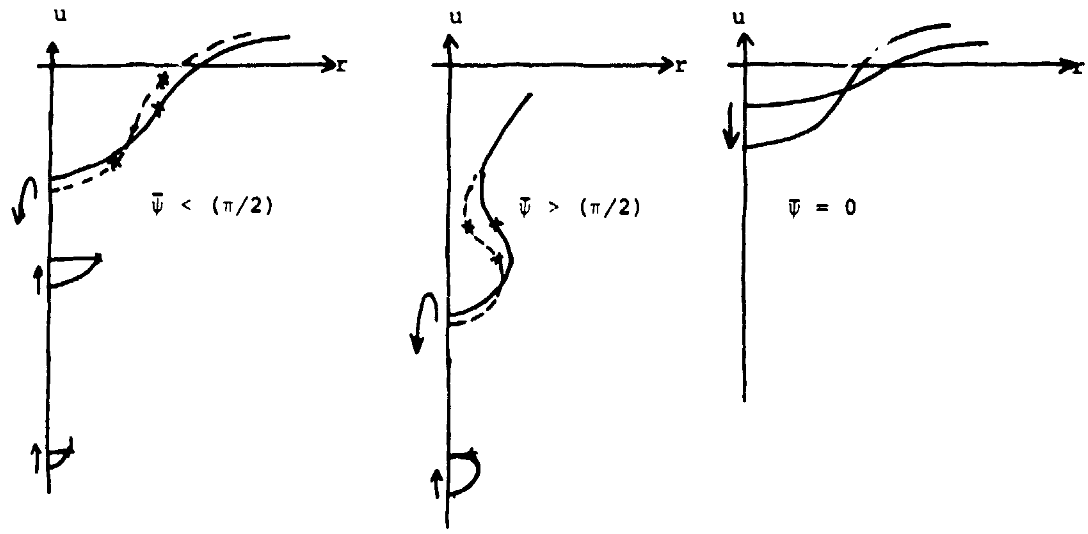

Figure 8. Drop formation for problem C

Justification of the stability criterion.

As noted earlier, any re-entrant drop (drop for which the angle of inclination becomes negative) is necessarily unstable. Otherwise a stable symmetric pendent drop represents a strong local minimum of energy for any of the problems discussed. A nice proof is based of the method of H.A. Schwarz in his proof of the isoperimetric property of the sphere. For example, relative to Problem C we can show the following result.

Theorem 12. Let (X,Ω) be a stable pendent drop for problem C with exposed volume $\bar{V}$, and angle of contact $\bar{\Psi}$. Consider any other pendent drop (Y,S) whose contained volume is $\bar{V}$. For each horizontal plane P, below the supporting plane, let $A(P)$ be the cross-sectional area of $Y \cap P$, and let $V(P)$ be the volume of the drop lying below the plane P. Let $r(P)$ be the radius of the circle whose area is $A(P)$. Suppose that the pair $(r(P),V(P))$ determines a stable pendent drop for problem B and for every plane P. Then the energy $E(\Omega)$ (see (7)) is less than or equal to $E(S)$, with equality only if $(X,\Omega) = (Y,S)$.

The method of proof is to first symmetrize (Y,S) producing a new drop of less energy and the same volume, which we then compare to the given stable drop.

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