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# Asymptotic expansions and estimates for the capillary problem

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## Abstract

This paper analyzes the asymptotic properties for small Bond number B of the equilibrium capillary interface interior to a circular cylindrical tube vertically dipped in an infinite reservoir of liquid. (The Bond number B is a dimensionless parameter which is the ratio of gravitational to capillary forces.) The formal expansion in powers of B of the solution to the differential equation describing the equilibrium surface (as can be obtained by standard perturbation methods) is proved to be truly asymptotic—to all orders and uniformly in the variable and parameter $\gamma$, the contact angle.

Sequences of general estimates, in closed form, from above and from below, are also given for the solution and related functions. The $m^{\text{th}}$ term in these sequences are asymptotically exact to order m. An idiosyncrasy of the problem, crucial in obtaining these estimates, is the absolute monotonicity of the structural function of the system in integral form.

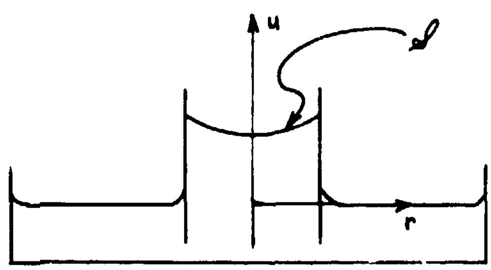


Figure 1.

## Introduction

We consider the classical capillary problem of describing the equilibrium fluid interface $\mathcal{S}$ interior to a circular cylindrical tube vertically dipped into an infinite reservoir of liquid (Fig. 1). Let u(r) be the height (above the level in the outer reservoir) of the surface $\mathcal{S}$ as a function of the distance r to the axis of the tube. Then u(r) is a solution of the following boundary value problem

$$\frac{1}{r}\left(r\,\frac{u_r}{\sqrt{1+u_r^2}}\right)_r = Bu \tag{1}$$

$$u_r(0+) = 0 \tag{2}$$

$$u_r(1-) = \tan\left(\frac{\pi}{2}-\gamma\right). \tag{3}$$

In this formulation, the quantities r, u, $\gamma$, and B are dimensionless; $\gamma$ is the contact angle of $\mathcal{S}$ with the boundary cylinder $(0 \le \gamma \le \pi)$, and $B = \rho g R^2/T > 0$ is the Bond number with $\rho$ the density difference across $\mathcal{S}$, g the gravitational acceleration, R the radius of the cylinder, and T the surface tension. We refer for background to previous papers[3,4,5,6,10] and, in particular to[1,2], for detailed proofs of most of the theorems.

Boundary value problem (1,2,3) has a unique solution; this solution will be studied indirectly by the shooting method. Set v(r,h,B) to be the unique solution of the following initial value problem

$$\frac{1}{r}\left(r\,\frac{v_r}{\sqrt{1+v_r^2}}\right)_r = 2h + Bv \tag{4}$$

$$v(0,h,B) = 0 \tag{5}$$

$$v_r(0,h,B) = 0. \tag{6}$$

Set also $\sigma(r,h,B) = \frac{v_r}{\sqrt{1+v_r^2}}(r,h,B)$ and $s = \sin(\frac{\pi}{2} - \gamma)$. (The parameter $h$ is the mean curvature at the apex of the corresponding surface of revolution and $\sigma$ is the sine of the slope angle of $v$ ) Then there exists a unique $h = h(B,s)$ such that $\sigma(1,h,B) = s$; this is also the unique $h$ such that the function $2h/B + v(\cdot,h,B)$ is the solution of (1,2,3). Consequently, set

$$u(r,B,s) = \frac{2h(B,s)}{B} + v(r,h(B,s),B). \tag{7}$$

For the above existence and uniqueness results, see, for example, [6] or [2].

Instead of the initial value problem (4,5,6), we shall use an equivalent integral system for the pair $(\sigma,v)$, namely,

$$\sigma(r,h,B) = hr + \frac{B}{r}\int_0^r \rho v(\rho,h,B)d\rho \tag{8}$$

$$v(r,h,B) = \int_0^r g(\sigma(\rho,h,B))d\rho \tag{9}$$

where the structural function $g$ is given by

$$g(y) = \frac{y}{\sqrt{1-y^2}} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} y^{2n+1}, \quad \text{for } |y| < 1. \tag{10}$$

Note that $g$ is <u>absolutely monotonic</u> on the interval $(0,1)$, i.e., has nonnegative derivatives on all orders on $(0,1)$. This fact is essential in obtaining the estimates below.

The approach will consist in first obtaining asymptotic results (resp. estimates) for the solution $\sigma(r,h,B)$, $v(r,h,B)$ of the initial value problem (in integral form), then deriving from these results similar ones for $h(B,s)$ and then $u(r,B,s)$.

<u>Absolute monotonicity and analyticity of $\sigma$ and $v$</u>

Since (4) is singular at $r = 0$, we need to show that the pair $(\sigma,v)$ is locally analytic at $r = 0$ and depends analytically on the parameters $h$ and $B$. This can be done in a number of ways (cf., for example, Wente[11]); alternatively[1], it can be observed that the "one dimensional solution" $(\bar{\sigma},\bar{v})$ (i.e., the profile curve of the capillary surface between two vertical parallel plates) is a majorant for $(\sigma,v)$. Since $(\bar{\sigma},\bar{v})$ is solution of a regular initial value problem, we conclude, by the method of majorants, that

<u>Proposition 1</u>: For every $\eta,\beta > 0$, there exists $\rho > 0$ such that the functions $\sigma(r,h,B)$ and $v(r,h,B)$ are analytic in the domain $|r| < \rho$, $|h| < \eta$, $|B| < \beta$ in $\mathbb{C}^3$.

Now the absolute monotonicity of $g$ on $(0,1)$ yields that $\sigma(r,h,B)$ and $v(r,h,B)$ are absolutely monotonic in all three variables for positive values of $r$, $h$, and $B$. This can be checked by power series substitution into system (8,9). Thus the triple power series expansion of $\sigma$ and $v$ at $(0,0,0)$ has nonnegative coefficients. This last fact together with Pringsheim's Theorem below implies that this expansion is convergent on the maximal interval $0 < r < \rho = \rho(h,B)$ where $\sigma(\cdot,h,B)$ and $v(\cdot,h,B)$ can be continued as a solution of integral system (8,9). Since $\sigma(\cdot,h,B)$ and $v(\cdot,h,B)$ are monotonic and bounded, we obtain convergence up to the boundary of the disc of convergence.

<u>Proposition 2</u>: The power series expansions of $\sigma(\cdot,h,B)$ and $v(\cdot,h,B)$ at $r = 0$ converge <u>absolutely</u> and uniformly in the closed convex disc $|r| \le \rho(h,B)$. Moreover $0 < \rho(h,B) < \infty$ and, at $r = \rho(h,B)$, $\sigma(r,h,B) = 1$.

<u>Corollary 3</u>: The triple power series expansions of $\sigma$ and $v$ at $(r,h,B) = (0,0,0)$ converge <u>absolutely</u> and uniformly in the closed domain

$$D = \{(r,h,B) \in \mathbb{C}^3: |r| \le \rho(|h|,|B|)\}. \tag{11}$$

<u>Pringsheim's Theorem</u>: Let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \tag{12}$$

and $R$ the radius of convergence of power series (12). Suppose $c_n \geq 0$, for all $n \geq 0$. Then $z = R$ is a singular point of $f$.

## Asymptotic expansions

Theorem 4: For each $m \geq 0$, the series expansions in powers of $B$ of the functions $h(B,s)$, $\sigma(r,h(B,s),B)$, $v(r,h(B,s),B)$, and $u(r,B,s)$ are asymptotic to order $m$ uniformly in $r$ and $s$ over the entire range $0 \leq r \leq 1$, $-1 \leq s \leq 1$, as $B \to 0$.

Proof (outline): By symmetry, we may restrict ourselves to $s \geq 0$. Recall that $h(B,s)$ is the unique solution of

$$\sigma(1,h,B) = s. \tag{13}$$

Now $\sigma(1,h,0) = h$ and $\sigma(1,\cdot,\cdot)$ is analytic at each point $(h,0)$ where $h$ is in the complex disc $|h| < 1$. Since $\sigma_h(1,h,0) = 1$, the implicit function theorem implies that $h(B,s)$ is analytic at each point $(0,s)$ where $s$ is in the complex disc $|s| < 1$. A compactness argument shows that, for each $s_0$ with $0 < s_0 < 1$, there exists $B_0 > 0$ such that $h(B,s)$ is analytic in $|B| \leq B_0$, $|s| \leq s_0$ and

$$h(B,s) = \sum_{n=0}^{\infty} B^n \xi_n(s) \quad \text{where} \quad \xi_n(s) = \frac{1}{n!}\frac{\partial^n h}{\partial B^n}(0,s). \tag{14}$$

This yields the asymptotic statement for $h(B,s)$ away from $s = 1$, i.e., contact angle $\gamma = 0$.

The neighborhood of $s = 1$ requires a special treatment and the use of the parametric system for the profile curve (parametrized by arclength). The corresponding function $\bar{h}(B,\alpha)$ where $\alpha = \frac{\pi}{2} - \gamma$ is shown to be analytic in a neighborhood of $(0,\frac{\pi}{2})$ by the implicit function theorem. Thus $h(B,s) = \bar{h}(B,\arcsin s)$. We conclude, since the function arcsine is continuous at $s = 1$.

The other three functions are handled in a similar way. QED.

In particular, $\xi_n(1) = \lim_{s\to 1-} \xi_n$ exists. This settles in the negative the problem of possible nonuniformity as $s \nearrow 1$ (cf. [1]).

In Tables 1, 2, 3, and 4 below, we give the first few coefficient-functions of each expansion as can be obtained by standard perturbation techniques. Previously Laplace[7], Poisson[8], Rayleigh[9], and Concus[3] computed formally these asymptotic expansions to various orders. We should also mention that Siegel[10] had proved recently the first term of those expansions to be truly asymptotic.

First we need

$$\sigma(r,h,B) = \sum_{n=0}^{\infty} (Br^2)^n \lambda_n(hr) \tag{15}$$

$$v(r,h,B) = r\sum_{n=0}^{\infty} (Br^2)^n \mu_n(hr) \tag{16}$$

(where the series converge in $D$) and recurrence formulas for the functions $\lambda_n$ and $\mu_n$

$$\lambda_0(x) = x \tag{17}$$

$$\lambda_n(x) = \frac{1}{x^{2n+1}}\int_0^x \xi^{2n}\mu_{n-1}(\xi)\,d\xi \tag{18}$$

$$\mu_n(x) = \frac{1}{x^{2n+1}}\int_0^x \xi^{2n}\sum_{p=0}^{n} g^{(p)}(\xi)\sum \frac{(\lambda_1^{\ell_1}\cdots\lambda_n^{\ell_n})(\xi)}{\ell_1!\cdots\ell_n!}\,d\xi \tag{19}$$

where the second summation in (19) is for $\ell_1 + \cdots + \ell_n = p$ and $\ell_1 + 2\ell_2 + \cdots + n\ell_n = n$.

Table 1. First few $\lambda_n$

$$\lambda_0(x) = x$$

$$\lambda_1(x) = \frac{1}{6} \frac{1+2\sqrt{1-x^2}}{\left(1+\sqrt{1-x^2}\right)^2} x$$

$$\lambda_2(x) = \frac{1}{6x^3}\left[-\frac{x^2}{\left(1+\sqrt{1-x^2}\right)^2} + \log \frac{1+\sqrt{1-x^2}}{2}\right]$$

$$\lambda_3(x) = \frac{1}{x^7}\left[\frac{1}{72}\frac{1}{\sqrt{1-x^2}} - \frac{13}{36} + \frac{11}{24}\sqrt{1-x^2} - \frac{1}{18}(1-x^2) - \frac{1}{18}(1-x^2)^{\frac{3}{2}}\right.$$
$$\left. - \frac{1}{12}(1+\sqrt{1-x^2})(5-3\sqrt{1-x^2}) \log \frac{1+\sqrt{1-x^2}}{2}\right]$$

Table 2. First few $\mu_n$

$$\mu_0(x) = \frac{1}{x}(1-\sqrt{1-x^2}) = \frac{x}{1+\sqrt{1-x^2}}$$

$$\mu_1(x) = \frac{1}{6x^3}\left[\frac{1}{\sqrt{1-x^2}} - 1 + 2 \log \frac{1+\sqrt{1-x^2}}{2}\right]$$

$$\mu_2(x) = \frac{1}{x^5}\left[\frac{1}{72}\frac{1}{\left(1-x^2\right)^{3/2}} - \frac{1}{24}\frac{1}{\sqrt{1-x^2}} - \frac{5}{36} + \frac{1}{6}\sqrt{1-x^2}\right.$$
$$\left. + \frac{1-3\sqrt{1-x^2}}{6\sqrt{1-x^2}} \log \frac{1+\sqrt{1-x^2}}{2}\right]$$

Table 3. First few $\xi_n$

$$\xi_0 = \lambda_0$$

$$\xi_1 = -\lambda_1$$

$$\xi_2 = \lambda_1\lambda_1' - \lambda_2$$

$$\xi_3 = \lambda_1\lambda_2' + \lambda_2\lambda_1' - \lambda_1\lambda_1'^2 - \frac{1}{2}\lambda_1^2\lambda_1'' - \lambda_3$$

Table 4. Expansions for the BVP

$$\sigma(r,h(B,s),B) = sr + B[r^2\lambda_1(sr) - \lambda_1(s)] + B^2[r\lambda_1(s)\lambda_1'(s) - r^3\lambda_1(s)\lambda_1'(sr) + r^4\lambda_2(sr) - r\lambda_2(s)] + \cdots$$

$$v(r,h(B,s),B) = r\mu_0(sr) + B[r^3\mu_1(sr) - r^2\lambda_1(s)\mu_0'(sr)] + B^2[r^2\lambda_1(s)\lambda_1'(s)\mu_0'(sr) - r^2\lambda_2(s)\mu_0'(sr) - \frac{1}{2}r^3\lambda_1^2(s)\mu_0''(sr) - r^4\lambda_1(s)\mu_1'(sr) + r^5\mu_2(sr)] + \cdots$$

$$u(r,B,s) = \frac{2h(B,s)}{B} + v(r,h(B,s),B)$$

## Estimates

The coefficient-functions $\lambda_n$ and $\mu_n$ are odd, analytic in the unit disc, and absolutely monotonic in the interval $(0,1)$. This yields the following lower estimates by truncation of the series (15) and (16), for each $m \geq 0$,

$$\sigma(r,h,B) \geq \sum_{n=0}^{m} (Br^2)^n\lambda_n(hr) \tag{20}$$

$$v(r,h,B) \geq r\sum_{n=0}^{\infty} (Br^2)^n\mu_n(hr) \tag{21}$$

for $(r,h,B) \in D^+ = D \cap [0,\infty)^3$.

Using the fact that the right hand side of (20) is increasing in $h$ and setting $h = h(B,s)$ and $r = 1$ in (20), we get the upper estimate

$$h(B,s) \leq \hat{h}_m(B,s) \tag{22}$$

where $\hat{h}_m(B,s)$ is the unique nonnegative solution of equation

$$\sum_{n=0}^{m} B^n\lambda_n(h) = s. \tag{23}$$

Estimates on the other side are obtained by induction on $m$ from integral system (8,9), using Picard's method of successive approximations and a careful estimation of each iterate. The proof uses strongly the absolute monotonicity of the function $g$. For each $m \geq 0$, we get the upper estimates

$$\sigma(r,h,B) \leq \sum_{n=0}^{m-1} (Br^2)^n\lambda_n(hr) + (Br^2)^m\lambda_m(\sigma(r,h,B)) \tag{24}$$

$$v(r,h,B) \leq r\sum_{n=0}^{m-1} (Br^2)^n\mu_n(hr) + r(Br^2)^m\mu_m(\sigma(r,h,B)) \tag{25}$$

for $(r,h,B) \in D^+$. Using the fact that the right hand side of (24) is increasing in $h$ and setting $h = h(B,s)$ and $r = 1$ in (24), we get, for $m \geq 1$, the lower estimate

$$h(B,s) \geq \check{h}_m(B,s) \tag{26}$$

where $\check{h}_m(B,s)$ is

(i) 0, if $B^m \geq s/\lambda_m(s)$,

(ii) the unique nonnegative solution of equation

$$\sum_{n=0}^{m-1} B^n \lambda_n(h) + B^m \lambda_m(s) = s, \tag{27}$$

if $B^m \leq s/\lambda_m(s)$.

Now, setting $h = h(B,s)$ in (20) and (21) and using (26),

$$\sigma(r,h(B,s),B) \geq \sum_{n=0}^{m} (Br^2)^n \lambda_n(r\check{h}_m(B,s)) \tag{28}$$

$$v(r,h(B,s),B) \geq r \sum_{n=0}^{m} (Br^2)^n \mu_n(r\check{h}_m(B,s)). \tag{29}$$

Similarly, setting $h = h(B,s)$ in (24) and (25) and using (22),

$$\sigma(r,h(B,s),B) \leq \sum_{n=0}^{m-1} (Br^2)^n \lambda_n(r\hat{h}_m(B,s)) + (Br^2)^m \lambda_m(sr) \tag{30}$$

$$v(r,h(B,s),B) \leq r \sum_{n=0}^{m-1} (Br^2)^n \mu_n(r\hat{h}_m(B,s)) + r(Br^2)^m \mu_m(sr). \tag{31}$$

Estimates (22,26,28,29,30,31) are valid for $m \geq 1$, $B \geq 0$, $0 \leq s \leq 1$, and $0 \leq r \leq 1$; they are all asymptotically exact to order $m$, as $B \to 0$.

Some of these estimates had been obtained previously by Finn[4,5] and Siegel[10].

## References


1. Brulois, F., Ph.D. dissertation, Stanford Univ. 1981.
2. Brulois, F., "Estimates for axially symmetric capillary surfaces," to appear.
3. Concus, P., "Static menisci in a vertical right circular cylinder," J. Fluid Mech., Vol. 34, pp. 482-495. 1968.
4. Finn, R., "The sessile liquid drop I: Symmetric case," Pacific J. Math., Vol. 88, pp. 541-587. 1980.
5. Finn, R., "On the Laplace formula and the meniscus height for a capillary surface," to appear.
6. Johnson, W. E., and L. M. Perko, "Interior and exterior boundary value problems from the theory of the capillary tube," Arch. Rat. Mech. Anal., Vol. 29, pp. 125-143. 1968.
7. Laplace, P. S., Traité de mécanique céleste, Tome 4, Supplément au Livre X, "Sur l'action capillaire," Bauthier-Villars, Paris 1806.
8. Poisson, S.D., Nouvelle théorie de l'action capillaire, Bachelier, Paris 1831.
9. Rayleigh, J. W. S., "On the theory of the capillary tube," Proc. Roy. Soc. London, Ser. A 92, pp. 184-195. 1915.
10. Siegel, D., "Height estimates for capillary surfaces," Pacific J. Math., Vol. 88, pp. 471-515. 1980.
11. Wente, H. C., "The stability of the axially symmetric pendant drop," Pacific J. Math., Vol. 88, pp. 421-470. 1980.