APPLICATIONS OF PERTURBATION TECHNIQUES TO HEAT-TRANSFER PROBLEMS

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Summary

Two perturbation techniques are applied to two singular perturbation problems in heat transfer to obtain uniformly valid solutions which can serve as benchmarks for the numerical techniques: finite-difference and finite-element techniques. In the first problem, the method of strained parameters coupled with the application of a solvability condition is used to obtain a uniform solution for the problem of unsteady heat conduction in a long nearly circular cylinder. In the second problem, the method of matched asymptotic expansion coupled with Van Dyke's matching principle is used to obtain a uniform solution for the problem of one dimensional conduction-convection heat transfer of a uniform fluid flow.

I. Introduction

The main purpose of this paper is to demonstrate the capabilities of the perturbation techniques in developing approximate closed-form solutions for heat transfer problems involving difficulties which preclude their solutions exactly or require resorting to computational techniques such as finite-difference, finite-element, and panel techniques. These difficulties may be due to nonlinear governing equations, equations with variable coefficients, nonlinear boundaries, and existance of boundary layers near portions of the boundaries.

Although computational techniques in the areas of fluid dynamics, heat transfer, and structures are rapidly advancing and are capable of developing excellent solutions for realistic problems, one always needs bench-mark solutions, if experimental data are not available, to check the developed computer code or to check the accuracy of the computed results. In this regard, a closed form perturbation solution for a simplified problem which retains the same difficulties (weakly nonlinear equations and boundary conditions and weakly irregular boundaries) can best serve this purpose.

Among the perturbation techniques, the straightforward expansion in terms of a parameter in the problem leads to satisfactory results if one is dealing with a regular perturbation problem or if its region of nonuniformity is avoided (ref. 1 and 2). However, for singular perturbation problems, the straightforward expansions yield nonuniform solutions and one has to use other perturbation techniques to obtain uniform solutions (ref. 3-6). Infinite domains in a problem, a small parameter multiplying the highest derivative of the governing equation, type change of a partial differential equation, and existence of singularities are some of the sources of nonuniformities of straightforward solutions.

In this paper, two applications in the area of heat transfer are considered and closed-form uniform perturbation solutions are developed. In the first application, the problem of unsteady heat conduction in a long nearly circular cylinder is considered and a straightforward solution is shown to breakdown. The method of strained parameters coupled with the application of a solvability condition is used to develop a uniform solution.

Similar problems in the areas of duct acoustics (ref. 7) and vibrations (ref. 3) were considered where the methods of multiple scales and the method of strained parameters were used, respectively.

In the second application, the problem of one dimensional conduction-convection heat transfer of a uniform fluid flow in a single channel is considered. For small ratios of conduction to convection heat transfer, the problem is shown to possess a thermal boundary layer where large temperature gradients exist. The method of matched asymptotic expansion coupled with Van Dyke's matching principle is used to develop a uniform solution. This problem was considered in reference 8 for single and merging flows by using the finite-element technique. Steady two-dimensional problems with different locations of the boundary layer can be found in reference 6.

II. Unsteady Heat Conduction in a Long Nearly Circular Cylinder

We consider the two-dimensional unsteady heat conduction in a long cylinder whose cross sectional area is nearly circular. Initially, the cylinder is at temperature $g(r^*, \phi)$ and at any later time the surface is kept at zero temperature. The radius of the cylinder is expressed as

$$r_o^* = R + a \, f(\phi) \quad \text{where} \int_o^{2\pi} f(\phi) \, d\phi = 0 \text{ and } a \ll R \tag{1}$$

Dimensionless quantities are introduced by using the mean radius of the cylinder R, the characteristic temperature T_c, and the time R^2/α (α is the thermal diffusivity) as reference quantities. The dimensionless form of the problem is given by

$$\tilde{\theta}_{rr} + \frac{1}{r} \tilde{\theta}_r + \frac{1}{r^2} \tilde{\theta}_{\phi\phi} = \tilde{\theta}_t \tag{2}$$

$$\tilde{\theta}(r_o, \phi, t) = 0 \qquad \text{on } r_o = 1 + \varepsilon \, f(\phi) \text{ and } \int_o^{2\pi} f(\phi) \, d\phi = 0 \tag{3}$$

$$\tilde{\theta}(r, \phi, \, o) = g(r, \phi) \tag{4}$$

The parameter ε is a small quantity characterizing the small deviation of the cross sectional area from the circular shape. The temporal variation is separated by assuming a solution of the form

$$\tilde{\theta}(r, \phi, t) = \theta(r, \phi) \, e^{-\beta^2 t} \tag{5}$$

Substituting equation (5) into equations (2) and (3), we obtain

$$\theta_{rr} + \frac{1}{r} \theta_r + \frac{1}{r^2} \theta_{\phi\phi} + \beta^2 \theta = 0 \tag{6}$$

$$\theta(r_o, \phi) = 0 \qquad \text{on } r_o = 1 + \varepsilon \, f(\phi) \tag{7}$$

Equation (6) is the Helmholtz equation. Although equation (6) is linear, the problem is not separable because the boundary condition, in the present form, is not separable. Since ε is a small parameter, one can expand $\theta(r, \phi; \varepsilon)$ in the form of a power series in terms of ε as follows

$$\theta(r, \phi; \varepsilon) = \theta_o \, (r, \phi) + \varepsilon \, \theta_1 \, (r, \phi) + ----- \tag{8}$$

In equation (8), only two terms are considered and hence a first-order solution is intended. Since ε appears in the argument of θ, equation (7), one needs to extract ε from this argument so that the process of equating coefficients of like powers of ε can correctly be accomplished. Therefore, the boundary condition of equation (11) is expanded around $r = 1$ using a Taylor-series expansion. This process is well known as the "transfer of the boundary condition." Thus, we get

$$\theta(r_0, \phi) = \theta(1, \phi) + \varepsilon \, f(\phi) \, \theta_r(1, \phi) + \text{----} = 0 \tag{9}$$

Substituting equation (8) into equations (6) and (9) and equating coefficients of like powers of ε, we obtain the following two sets of problems:

$0(\varepsilon^\circ)$ - Problem

$$\theta_{0rr} + \frac{1}{r} \theta_{0r} + \frac{1}{r^2} \theta_{0\phi\phi} + \beta^2 \theta_0 = 0 \tag{10}$$

$$\theta_0(1, \phi) = 0 \tag{11}$$

$0(\varepsilon)$ - Problem

$$\theta_{1rr} + \frac{1}{r} \theta_{1r} + \frac{1}{r^2} \theta_{1\phi\phi} + \beta^2 \theta_1 = 0 \tag{12}$$

$$\theta_1(1, \phi) = - f(\phi) \, \theta_{0r}(1, \phi) \tag{13}$$

In the perturbation expansion used above, we note that only the dependent variable θ is expanded in terms of the small parameter ε. Such a perturbation method is called a "straightforward-perturbation method." Straightforward expansions break down when we deal with singular perturbation problems. Next, it is shown that the straightforward expansion breaks down for this problem.

The solution of the $0(\varepsilon^\circ)$ problem is obtained by using the method of separation of variables. The solution is found as

$$\theta_0 = J_n(k_{nm} r)(A_{nm} e^{in\phi} + \bar{A}_{nm} e^{-in\phi}) \tag{14}$$

where $k_{nm}(=\beta)$ are the zeros of the $J_n(\beta) = 0$, and A_{nm} are complex constants. Setting $\beta^2 = k^2_{nm}$ in equation (12) and substituting equation (14) into

equation (13), we get

$$\theta_{1rr} + \frac{1}{r} \theta_{1r} + \frac{1}{r^2} \theta_{1\phi\phi} + k_{nm}^2 \, \theta_1 = 0 \tag{15}$$

$$\theta_1(1,\phi) = - k_{nm} \, J_n'(k_{nm}) \, f(\phi) \, (A_{nm} e^{in\phi} + \overline{A}_{nm} e^{-in\phi}) \tag{16}$$

The solution of the $O(\epsilon)$ problem is obtained by expanding each of θ_1 and $f(\phi)$ in a Fourier series as

$$\theta_1(r,\phi) = \sum_{-\infty}^{\infty} G_t(r) \, e^{it\phi} \tag{17}$$

$$f(\phi) = \sum_{-\infty}^{\infty} f_p \, e^{ip\phi} \tag{18}$$

where $f_p = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \, e^{-ip\phi}$ and $f_0 = 0$ according to the condition in equation (3). Substituting equation (17) and (18) into equations (15) and (16), multiplying the results by $\exp(-is\phi)$ and integrating from 0 to 2π, we get

$$G_s'' + \frac{1}{r} G_s' + (k_{nm} - \frac{s^2}{r^2}) \, G_s = 0 \tag{19}$$

$$G_s(1) = - k_{nm} \, J_n'(k_{nm}) \, [A_{nm} \, f_{s-n} + \overline{A}_{nm} \, f_{s+n}] \tag{20}$$

If $s \neq n$, equations (19) and (20) have a unique solution since the only solution for the corresponding homogeneous problem is the trivial solution. If $s = n$, we obtain

$$G_n'' + \frac{1}{r} G_n' + (k_{nm} - \frac{n^2}{r^2}) \, G_n = 0 \tag{21}$$

$$G_n(1) = - k_{nm} \, f_{2n} \, \overline{A}_{nm} \, J_n'(k_{nm}) \tag{22}$$

we note that the corresponding homogeneous problem has a nontrivial solution and hence the inhomogeneous problem will not have a solution unless a solvability condition is satisfied. However, a solvability condition does not exist and hence there is no solution to equations (21) and (22). Therefore, the solution of the $O(\epsilon)$ problem breaks down.

The reason behind this trouble is due to the straightforward method used here. To obtain a uniform solution, we use the method of strained parameters. In this method, we expand β, in addition to the expansion given by equation (8), as follows

$$\beta = \beta_o + \varepsilon \, \beta_1 + \text{---}$$ (23)

where β_1 is to be determined in the course of the solution. Substituting equations (8) and (23) into equation (6), substituting equation (8) into equation (9), and equating coefficients of like powers of ε, we again obtain two sets of problems. The $O(\varepsilon^o)$ problem is the same as that given by equations (10) and (11) with the exception of replacing β^2 by β_o^2. The $O(\varepsilon)$ problem is given by

$$\theta_{1rr} + \frac{1}{r} \theta_{1r} + \frac{1}{r^2} \theta_{1\phi\phi} + \beta_o^2 \, \theta_1 = - \, 2 \, \beta_o \beta_1 \, \theta_o$$ (24)

and by equation (13).

Substituting equation (14) into equation (24) and replacing β_o by k_{nm}, we get

$$\theta_{1rr} + \frac{1}{r} \theta_{1r} + \frac{1}{r^2} \theta_{1\phi\phi} + k_{nm}^2 \, \theta_1 = - \, 2 \, k_{nm} \, \beta_1 \, J_n(k_{nm}r)(A_{nm}e^{in\phi} + \overline{A}_{nm}e^{-in\phi})$$

(25)

The boundary condition is still given by equation (16).

The solution of equations (25) and (16) is obtained by substituting equations (17) and (18) into equations (25) and (16), multiplying the result by $\exp(-is\phi)$ and integrating from 0 to 2π. Again, we obtain two cases corresponding to $s \neq n$ and $s = n$. The former case is the same as that given by equations (19) and (20) in which a unique solution exists. In the latter case, we have

$$G_n^{''} + \frac{1}{r} G_n^{'} + (k_{nm}^2 - \frac{n^2}{r^2}) \, G_n = - \, 2 \, k_{nm} \, \beta_1 \, A_{nm} \, J_n(k_{nm}r)$$ (26)

$$G_n(1) = - \, k_{nm} \, f_{2n} \, \overline{A}_{nm} \, J_n^{'}(k_{nm})$$ (22)

As we mentioned before, the problem given by equations (26) and (22) has a

solution if and only if a solvability condition exists. To obtain the solvability condition, we write equation (26) in the self-adjoint form by multiplying both sides of the equation by r. Next, we multiply both sides by the adjoint u(r) and integrate the result by parts over the range of r. Thus, we get

$$\int_0^1 G_n [(ru')' + (k_{nm}^2 r - \frac{n^2}{r})u] \, dr + [r u G_n' - r u' G_n]_0^1$$

$$= - 2 k_{nm} \beta_1 A_{nm} \int_0^1 r u J_n(k_{nm}r) \, dr \qquad (27)$$

The adjoint equation is obtained by setting the coefficient of G_n to zero. Thus, we obtain

$$(r u')' + (k_{nm}^2 r - \frac{n^2}{r}) u = 0 \qquad (28)$$

The adjoint boundary conditions are obtained by choosing

$$u(1) = 0 \qquad \text{and } u(0) < \infty \qquad (29)$$

in equation (27). Equation (27) reduces to

$$u'(1) G_n(1) = 2 k_{nm} \beta_1 A_{nm} \int_0^1 r u J_n (k_{nm}r) \, dr \qquad (30)$$

The solution of the adjoint problem, equations (28) and (29), is given by

$$u(r) = J_n(k_{nm}r) \qquad (31)$$

Substituting equations (22) and (31) into equation (30), and performing the integration on the right hand side, we obtain the equation defining β_1 as

$$\beta_1 = - k_{nm} f_{2n} \overline{A}_{nm}/A_{nm} \qquad (32)$$

since f_{2n}, $\overline{A}_{nm}$ and A_{nm} are complex constants, we assume

$$f_{2n} = b_{2n} e^{i\gamma_{2n}}$$

$$\qquad (33)$$

$$A_{nm} = \frac{1}{2} a_{nm} e^{i\lambda_{nm}} \quad , \quad \overline{A}_{nm} = \frac{1}{2} a_{nm} e^{-i\lambda_{nm}}$$

Substituting equation (33) into equation (32), and equating the real and imaginary parts, we get

$$\lambda_{nm} = \frac{1}{2} \gamma_{2n} \quad \text{or} \quad \frac{1}{2} (\gamma_{2n} - \pi) \tag{34}$$

$$\beta_1 = -k_{nm} b_{2n} \quad \text{or} \quad k_{nm} b_{2n} \tag{35}$$

Substituting equation (34) into equation (33) and substituting the result into equation (4), we get

$$\theta_0^{(1)} = a_{nm} J_n(k_{nm}r) \cos(n\phi + \gamma_{2n}/2) \tag{36}$$

and

$$\theta_0^{(2)} = a_{nm} J_n(k_{nm}r) \sin(n\phi + \gamma_{2n}/2) \tag{37}$$

Substituting equation (35) into equation (23), we get

$$\beta^{(1)} = k_{nm} - \varepsilon k_{nm} b_{2n} \tag{38}$$

$$\beta^{(2)} = k_{nm} + \varepsilon k_{nm} b_{2n} \tag{39}$$

Substituting equation (36) into equation (8), substituting this result and equation (38) into equation (5), repeating the same process with equations (37) and (39), and forming a linear combination of the two, we obtain

$$\theta(r,\phi,t) = \sum_{n,m} a_{nm} J_n(k_{nm}r) \{\cos(n\phi + \gamma_{2n}/2) \operatorname{Exp}[-k_{nm}(k_{nm} - 2\varepsilon b_{2n})t]$$

$$+ c_{nm} \sin(n\phi + \gamma_{2n}/2) \operatorname{Exp}[-k_{nm}(k_{nm} + 2\varepsilon b_{2n})t]\} + \text{----} \tag{40}$$

The constants a_{nm} and c_{nm} are found from the initial condition.

III. One Dimensional Conduction-Convection Heat-Transfer in a Uniform Flow

We consider the one dimensional convective diffusion equation modified by a convective surface loss term. This equation has been used to model the far-field behavior of thermal regime for single and merging fluid flows. Here, only the problem of single channel flows with specified temperatures at the upstream and downstream boundaries is considered. The governing equation of the average temperature $T(x^*)$ is given by

$$-k\, A\, T'' + \rho\, C\, A\, u\, T' + h\, p(T - T_e) = 0 \tag{1}$$

The boundary conditions are

$$T(0) = T_1 \tag{2}$$

$$T(L) = T_2 \tag{3}$$

In equations (1) and (2), k is the coefficient of thermal conductivity in the flow direction, A is the flow cross-sectional area, ρ is the fluid density, C is the fluid specific heat, u is the flow average velocity, h is the convection heat exchange coefficient, p is the convection perimeter, T_e is the convection exchange temperature, and L is the channel length.

Dimensionless parameters are introduced by using the pipe length L, and the temperature difference $T_1 - T_e$ as reference quantities. The dimensionless form of the problem is given by

$$-(1/P_e)\, y'' + y' + (N_u/P_e)\, y = 0 \tag{4}$$

$$y(0) = 1 \ , \ y(1) = \theta_2 \tag{5}$$

where $P_e = \rho C u L/k$ is the Peclet number, $N_u = hd/k$ is the Nusselt number, d = 4A/b is the hydraulic diameter, and $y(x) = [T(x) - T_e]/(T_1 - T_e)$. For large Peclet numbers (small ratio of conduction to convection heat transfer) we let $1/P_e = \varepsilon$, where ε is now a small parameter. Moreover, we assume $N_u/P_e = b = O(1)$.

Thus, we obtain the following problem describing the spatial variation of the temperature:

$$\varepsilon\, y'' - y' - b\, y = 0 \tag{6}$$

$$y(0) = 1 \ , \ y(1) = \theta_2 \tag{7}$$

Equations (6) and (7) describe a typical boundary-layer problem where a small parameter multiplies the highest derivative. This problem can success-fully be treated by using any of the several suitable perturbation techniques, namely, the method of matched asymptotic expansion, the method of multiple scales, and the method of composite expansions, among others. In this paper, we develop a uniformly valid solution by using the method of matched asymptotic expansion.

Since the coefficient of y' is negative in the interval $0 \leq x \leq 1$, the boundary layer exists at the boundary $x = 1$. Next, we develop outer and inner solutions and match them by using Van Dyke's matching principle to obtain a composite solution which is uniformly valid everywhere.

Outer Solution:

The outer solution $y^{\circ}(x;\varepsilon)$ is expressed in the form

$$y^{\circ}(x;\varepsilon) = y_0(x) + \varepsilon y_1(x) + \text{----} \tag{8}$$

Dropping the second boundary condition of equation (7), substituting equation (8) into equation (6) and into the first boundary condition of equation (7), and equating coefficients of like powers of ε, we obtain the following prob-lems:

$O(\varepsilon^{\circ})$ Problem

$$y_0' + b \, y_0 = 0 \tag{9}$$

$$y_0(o) = 1 \tag{10}$$

$O(\varepsilon^1)$ Problem

$$y_1' + b \, y_1 = y_0'' \tag{11}$$

$$y_1(0) = 0 \tag{12}$$

The outer solution of equations (9)-(12) is given by

$$y^{\circ} = e^{-bx}(1 + \varepsilon b^2 x) + O(\varepsilon^2) \tag{13}$$

Inner Solution:

To develop an inner solution valid near the boundary layer, we drop the boundary condition at $x = 0$ and introduce the following stretching transformation for the inner variable

$$\xi = \frac{1-x}{\varepsilon} \tag{14}$$

In terms of the inner variable, the inner solution $y^i(\xi;\varepsilon)$ is governed by

$$y^{i''} + y^{i'} - \varepsilon b y^i = 0 \tag{15}$$

$$y^i(0) = \theta_2 \tag{16}$$

Next, we expand the inner solution in the form

$$y^i(\xi;\varepsilon) = Y_0(\xi) + \varepsilon Y_1(\xi) + \cdots \tag{17}$$

Substituting equation (17) into equations (15) and (16) and equating coefficients of like powers of ε, we obtain the following problems:

$O(\varepsilon^0)$ Problem

$$Y_0'' + Y_0' = 0 \tag{18}$$

$$Y_0(0) = \theta_2 \tag{19}$$

$O(\varepsilon^1)$ Problem

$$Y_1'' + Y_1' = b Y_0 \tag{20}$$

$$Y_1(0) = 0 \tag{21}$$

The inner solution of equations (18)-(21) is given by

$$y^i = \theta_2 + A_0(1 - e^{-\xi}) + \varepsilon\{b \xi[\theta_2 + A_0(1 + e^{-\xi})] + A_1(1 - e^{-\xi})\} + O(\varepsilon^2) \tag{22}$$

We note that the outer solution is completely known while the inner solution contains the two unknown coefficients A_0 and A_1. They are determined by

applying Van Dyke's matching principle to the inner and outer solutions. The principle states that:

the m-term inner expansion of (the n-term outer expansion) = the n-term outer expansion of (the m-term inner expansion)

where m and n are any two integers. To find the left hand side, we write the n-term outer expansion in terms of the inner variable, equation (14), expand the functions of ε keeping the inner variable fixed, and keep m-terms of the resulting expansions. An opposite procedure is applied to the right hand side. For m = n = 2, we obtain

$$A_0 = e^{-b} - \theta_2, \quad A_1 = b^2 e^{-b} \tag{23}$$

Substituting equation (23) into equation (22), we obtain the inner solution as

$$y^i = \theta_2 + (e^{-b} - \theta_2)(1 - e^{-\xi}) + \varepsilon\{b\ \xi[\theta_2 + (e^{-b} - \theta_2)(1 + e^{-\xi})] + b^2 e^{-b}(1-e^{-\xi})\}$$
$$+ 0(\varepsilon^2) \tag{24}$$

Next, we express the composite expansion in the form

$$y^c = y^o + y^i - (y^o)^i \tag{25}$$

Substituting equations (13) and (24) and the result of the left hand side of Van-Dyke's matching principle in equation (25), we obtain the composite solution as

$$y^c = (1 + \varepsilon\ b^2 x)e^{-bx} + [(1-b+bx)(\theta_2-e^{-b}) - \varepsilon\ b^2 e^{-b}]e^{-(1-x)/\varepsilon} + 0(\varepsilon^2) \tag{26}$$

It should be noted that an exact solution exists for equations (6) and (7) which is given by

$$y = [(e^{m_2} - \theta_2)e^{m_1 x} - (e^{m_1} - \theta_2)e^{m_2 x}]/(e^{m_2} - e^{m_1}) \tag{27}$$

where $m_{1,2} = [1 \pm \varepsilon(1 = 4\ \varepsilon\ b)^{1/2}]/2\ \varepsilon$ \hfill (28)

The solution of the unsteady flow problem described by

$$-k\ A\ T_{x^*x^*} + \rho\ C\ A\ U\ T_{x^*} + h\ p(T-T_e) + \rho\ C\ A\ T_{t^*} = 0 \tag{29}$$

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$$T(0,t^*) = T_1 \, , \; T(L,t^*) = T_2 \, , \; T(x^*,t^*) = T_0 \qquad (30)$$

is an easy extension to the steady solution given by equation (26). The problem is divided into steady and transient problems. The steady problem with the inhomogeneous boundary conditions is already considered. Upon assuming an exponential time-decay solution, the transient problem with homogeneous boundary conditions reduces to an eigenvalue problem coupled with a boundary layer. A uniform solution of the problem is obtained by using the method of matched asymptotic expansions (as shown before) coupled with strained eigenvalues which are expressed as a power series of ε.

IV. Concluding Remarks

The methods of strained parameters and matched asymptotic expansions are successfully used to obtain closed-form perturbation solutions for heat-transfer problems with irregular boundaries and with boundary layers, respectively. The techniques given here are applicable to a large class of similar problems where various geometrical shapes and three-dimensional dependence are considered. Nevertheless, such solutions serve as bench marks for the computational techniques. Moreover, useful solutions can be obtained by combining a computational technique with a perturbation technique in a certain problem. The computational technique is used to solve the zeroth-order problem while the perturbation technique is used to solve the higher-order problems.

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