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AN EXACT PLANE-STRESS SOLUTION
FOR A CLASS OF PROBLEMS IN
ORTHOTROPIC ELASTICITY

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16. Abstracts

An exact solution for the stress field within a rectangular slab of orthotropic material is found using a two-dimensional Fourier series formulation. The material is required to be in plane stress, with general stress boundary conditions, and the principle axes of the material must be parallel to the sides of the rectangle.

Two load cases similar to those encountered in materials testing are investigated using the solution. The solution method developed is seen to have potential uses in stress analysis of composite structures.

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1. INTRODUCTION

Because of the effects of material properties, structures constructed from orthotropic materials often require extensive analysis, even when they are geometrically simple. These analyses are commonly done using finite elements, boundary collocation, and Fourier analysis.

The finite element method is well-documented in books such as Gallagher [1] and Zienkiewicz [2] and will be dealt with here only briefly. Finite element methods require the division of the region to be analyzed into smaller regions, or "elements." Within each of these regions, the form of an approximate solution for a stress field is assumed. Enforcement of boundary conditions and interelement continuity of displacement then gives a piecewise continuous solution for the stresses over the region.

Similar to the finite element method is boundary collocation, or "point matching," which is described in the papers by Slot and Yalch [3] and Hooke [4]. For this method, an "exact" solution for the stresses in a region of materials is developed by requiring the stress field to match known values at given points on the boundaries. Unlike the finite element method, however, the stress field solution is good over the entire region being analyzed, not merely on one small portion of the region.

Still another way to handle boundary value problems is to specify a solution form that allows the boundary conditions to be met exactly, rather than merely from point to point. This can be done with infinite series of polynomials, but is more often done with Fourier series. It

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is to a form of Fourier series solution that this report is devoted.

Timoshenko and Goodier [5] describe a fairly standard procedure for stress analysis using Fourier series. A Fourier series stress function is specified so as to match the conditions on two opposing boundaries and stresses are found from this function. A similar solution can be performed for the other two boundaries of the rectangular slab and, due to linearity, the stresses from the two solutions can be added together. A problem arises with this method, however. Since the two supposedly stress-free boundaries frequently acquire "residual" stresses from the solution, a new series solution must be formulated to cancel these stresses, i.e., to restore the stress-free condition on the adjacent boundaries. These series, in turn, will leave residuals on the original boundaries, and must in turn be cancelled. The process can be repeated as many times as necessary to reduce the residuals to acceptable magnitudes. In 1944, Pickett [6] proposed a method to effectively "automate" this procedure. By taking a series expansion of the stress function in two dimensions, all boundary conditions can be met simultaneously. Details of the method used are described in Little [7]. Theocaris and Dafermos [8] have used Pickett's method to analyze a thin strip of isotropic material subjected to displacement along a single boundary.

In this report, a solution for the general plane stress problem in a rectangular slab of ortnotropic material is presented. The principal axes of the material are aligned parallel with the sides of the slab. Only stress boundary conditions are treated. However, the inclusion of displacement boundary conditions in the analysis is discussed briefly

in Appendix D. The solution is valid for general boundary stresses, as long as they can be represented by Fourier series. The solution method is an extension and generalization of Pickett's method [6] to allow the consideration of specially orthotropic materials and general stress boundary conditions. The expansion of the capabilities of Pickett's method is made feasible by the existence of high speed computers which can quickly and easily handle large matrices and complicated algebraic expressions.

When this solution is used, no redefinition of a finite element mesh is required to change loading conditions. Also (and probably of the most practical use in the long run), if regions over which solutions of this form are valid can be joined together to form more complicated shapes, they become "exact" elements in a new sort of finite element solution. Accurate results can then be obtained with far fewer elements.

The solution generated is used to work two problems similar to tests used for material property evaluation. The stress fields found are examined for the effects of orthotropy, and possibilities for future use of the solution are discussed.

1.1 Method of Solution

1.1.1 Development of Governing Equations

The equation governing the plane stress distribution in an anisotropic slab is given by Lekhnitskii [9] as

$$S_{22} \frac{\partial^4 \phi}{\partial x^4} - 2S_{26} \frac{\partial^4 \phi}{\partial x^3 \partial y} + (2S_{12} + S_{66}) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} - 2S_{16} \frac{\partial^4 \phi}{\partial x \partial y^3}$$

$$+ S_{11} \frac{\partial^4 \phi}{\partial y^4} = 0 , \tag{1}$$

where the stress function ϕ is defined by the following relations:

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} , \qquad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} , \qquad \text{and } \tau_{xy} = - \frac{\partial^2 \phi}{\partial x \partial y} . \tag{2}$$

The elastic compliances S_{ij} are defined in Jones [10] and occur in the constitutive relations as

$$\begin{aligned}
\varepsilon_x &= S_{11}\sigma_x + S_{12}\sigma_y + S_{16}\tau_{xy} \\
\varepsilon_y &= S_{12}\sigma_x + S_{22}\sigma_y + S_{26}\tau_{xy} \\
\gamma_{xy} &= S_{16}\sigma_x + S_{26}\sigma_y + S_{66}\tau_{xy} .
\end{aligned} \tag{3}$$

Positive values of the stresses σ_x, σ_y, and τ_{xy} are shown in Fig. 1.

If the problem is specialized to describe the stress distribution in a rectangular slab of orthotopic material where principal material axes coincide with the slab boundaries, the compliances S_{16} and S_{26} vanish and the governing equation (1) reduces to

$$S_{22} \frac{\partial^4 \phi}{\partial x^4} + (2S_{12} + S_{66}) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + S_{11} \frac{\partial^4 \phi}{\partial y^4} = 0 . \tag{4a}$$

Using the standard engineering constants, this equation is
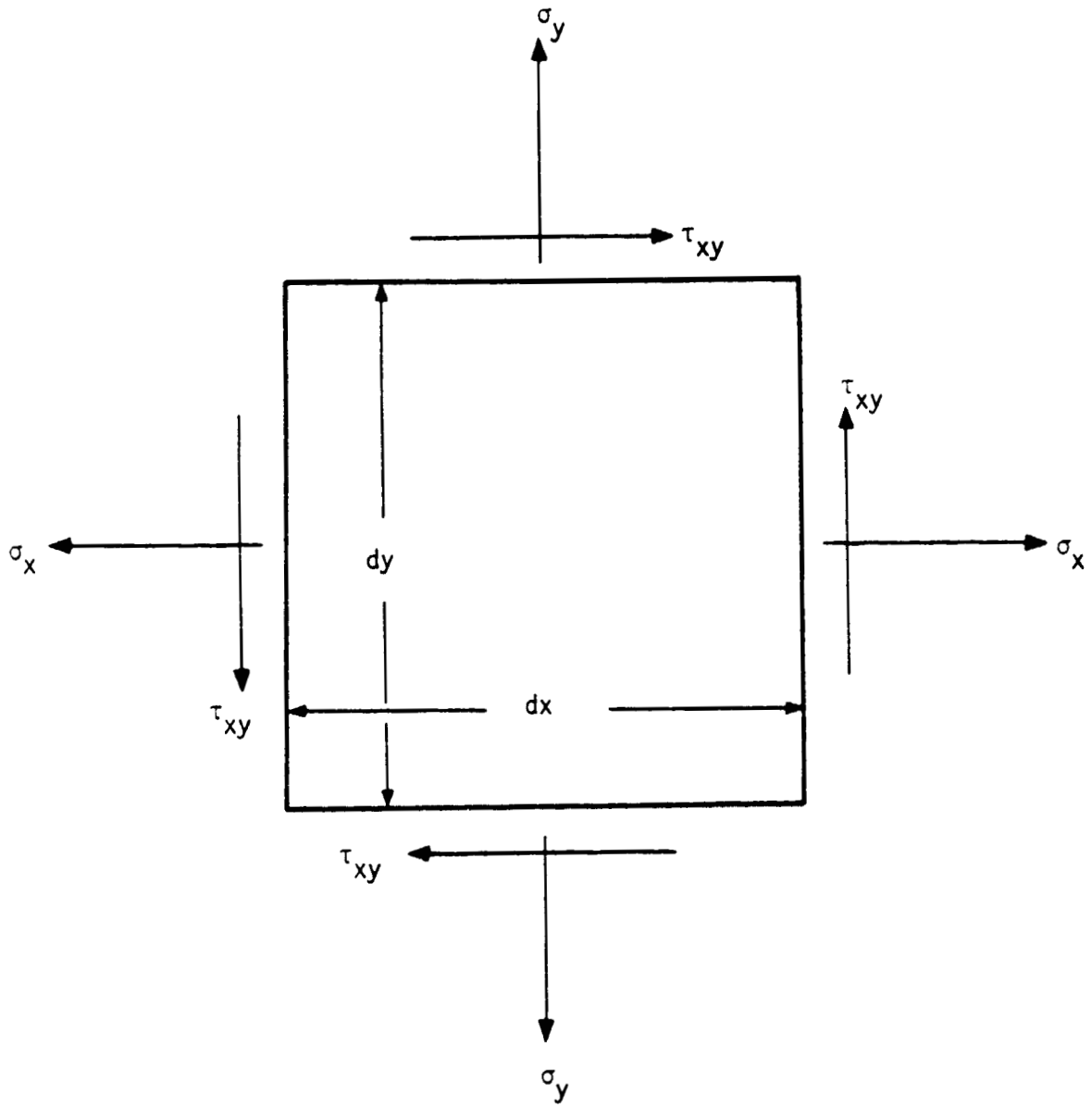
Fig. 1 Sign Convention for Stresses

$$\frac{1}{E_2}\frac{\partial^4\phi}{\partial x^4} + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1}\right)\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{1}{E_1}\frac{\partial^4\phi}{\partial y^4} = 0 \ . \tag{4b}$$

For the special case of isotropic material such that

$$E_1 = E_2 \qquad \text{and} \qquad G_{12} = \frac{E_1}{2(1 + \nu_{12})} \ ,$$

the governing equation reduces to the well-known biharmonic equation for determining the stress function in plane stress:

$$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0 \ . \tag{5}$$

The development of a solution to the orthotropic slab will be restricted to those problems where known stress distributions are applied on the boundaries $x = \pm\ell$ and $y = \pm t$, where ℓ and t are the half-lengths of the slab in the x and y-directions, respectively (see Fig. 2). This restriction leads to the following boundary conditions:

$$\begin{aligned}
\sigma_x(\ell,y) &= r_1(y) & \tau_{xy}(\ell,y) &= r_5(y) \\
\sigma_x(-\ell,y) &= r_2(y) & \tau_{xy}(-\ell,y) &= r_6(y) \\
\sigma_y(x,t) &= r_3(x) & \tau_{xy}(x,t) &= r_7(x) \\
\sigma_y(x,-t) &= r_4(x) & \tau_{xy}(x,-t) &= r_8(x)
\end{aligned} \tag{6}$$

where r_i are the given boundary stress distributions. These functions are not independent but from consideration of overall equilibrium of forces and moments are related as follows:
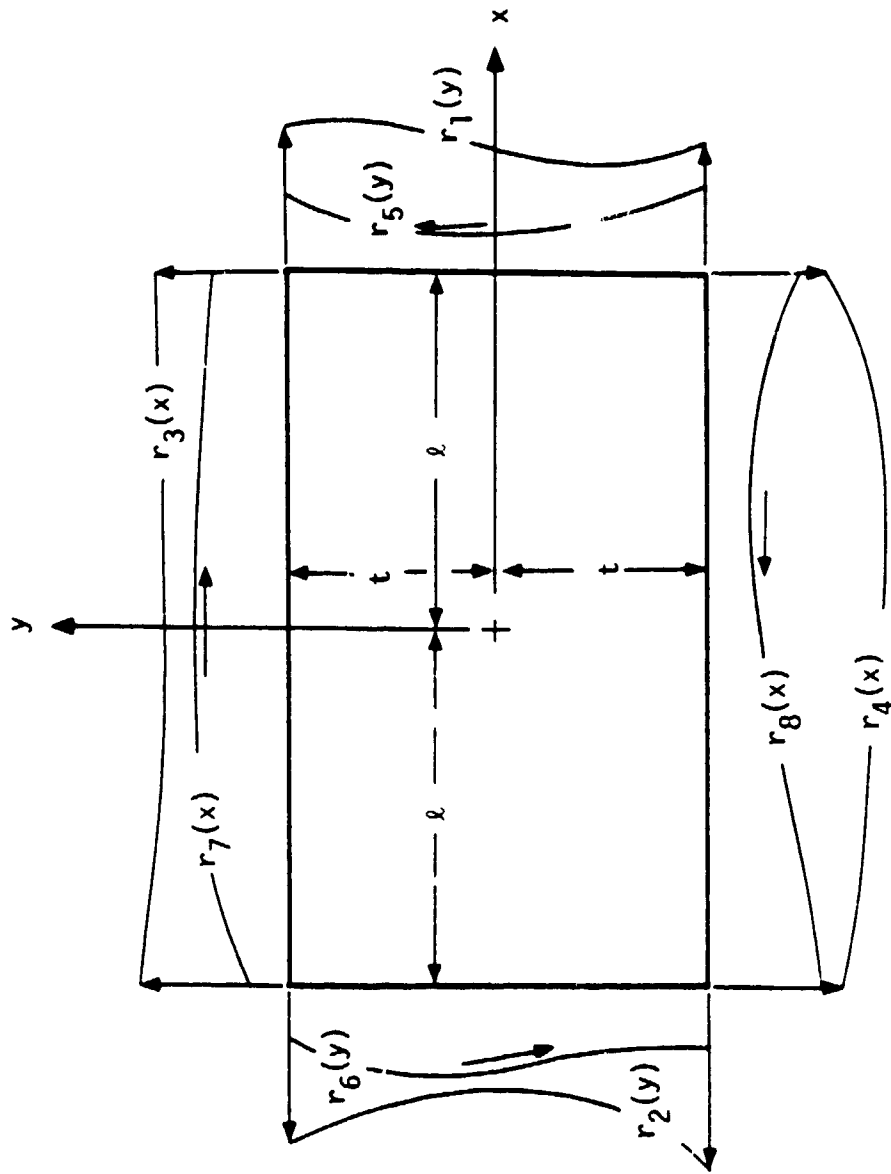
Fig. 2 Geometry of Slab

$$\int_{-t}^{t} [r_1(y) - r_2(y)] \, dy + \int_{-\ell}^{\ell} [r_5(x) - r_6(x)] \, dx = 0$$

$$\int_{-\ell}^{\ell} [r_3(x) - r_4(x)] \, dx + \int_{-t}^{t} [r_7(y) - r_8(y)] = 0 \tag{7}$$

$$\int_{-t}^{t} [r_5(y) + r_6(y)]\ell \, dy - \int_{-\ell}^{\ell} [r_7(x) + r_8(x)]t \, dx$$

$$+ \int_{-t}^{t} [r_2(y) - r_1(y)]y \, dy + \int_{-\ell}^{\ell} [r_3(x) - r_4(x)]x \, dx = 0 \quad .$$

Furthermore, the boundary stress distributions r_i must be expressible as Fourier series.

Equation 4 and the boundary conditions of Eqn. (6) fully describe the governing set of equations to determine the distribution of stress in a rectangular slab of dimensions ℓ by t subject to any stress distribution on the boundary.

2. SOLUTION TECHNIQUE

A general solution of Eqn. (4) and (6) can be expressed as the trigonometric expansion

$$\phi = c_1 x^2 + c_2 xy + c_3 y^2 + c_4 x^2 y + c_5 xy^2$$

$$+ \sum_{n=0}^{\infty} [f_n(x) \cos(\alpha_n y) + g_n(y) \cos(\beta_n x)]$$

$$+ \sum_{n=1}^{\infty} [h_n(x) \sin(\alpha_n y) + k_n(y) \sin(\beta_n x)],$$

where the polynomial terms are necessary in the solution procedure that follows to assure a general solution set, and where

$$\alpha_n = \frac{n\pi}{t} \qquad \text{and} \qquad \beta_n = \frac{n\pi}{\ell} .$$

The functions $f_n(x)$, $g_n(y)$, $h_n(x)$, and $k_n(y)$ can be determined from the set of equations found upon substitution of the solution of Eqn. (8) into Eqn. (4). The result is a set of fourth-order ordinary differential equations with constant coefficients in x for $f_n(x)$ and $h_n(x)$ and a corresponding set of equations in y for $g_n(y)$ and $k_n(y)$. These equations admit of solutions of the form

$$f_n(x) = F_n e^{sx} \tag{9}$$

which upon substitution in the ordinary differential equations all give the characteristic equation

$$s^4 - As^2 + B = 0 , \tag{10}$$

where

$$A = E_2[\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1}] \qquad \text{and} \qquad B = \frac{E_2}{E_1} . \tag{10}$$

The roots to the characteristic equation are functions of material properties only. There are three possible solution forms; these are

1) roots are real, in pairs --- $s = \pm u, \pm v$ ($u, v > 0$, $u \neq v$)

2) roots are real and equal in pairs --- $s = \pm u$ ($u > 0$)

3) roots are complex --- $s = \pm u \pm iv$ ($u, v > 0$; $i = \sqrt{-1}$) .

These cases are now explored separately.

CASE 1

When the roots are real in pairs, case 1, the functions $f_n(x)$, $g_n(y)$, $h_n(x)$, and $k_n(y)$ have the form

$$f_n(x) = a_{1n}e^{u\alpha_n x} + a_{2n}e^{v\alpha_n x} + a_{3n}e^{-u\alpha_n x} + a_{4n}e^{v\alpha_n x} ,$$

$$g_n(y) = a_{5n}e^{u\beta_n y} + a_{6n}e^{v\beta_n y} + a_{7n}e^{-u\beta_n y} + a_{8n}e^{-v\beta_n y} ,$$

$$\tag{11}$$

$$h_n(x) = a_{9n}e^{u\alpha_n x} + a_{10n}e^{v\alpha_n x} + a_{11n}e^{-u\alpha_n x} + a_{12n}e^{-v\alpha_n x} ,$$

$$k_n(y) = a_{13n}e^{u\beta_n y} + a_{14n}e^{v\beta_n y} + a_{15n}e^{-u\beta_n v} + a_{16n}e^{-v\beta_n y} .$$

To take advantage of symmetries which might occur in the formulation of specific problems, the solution form of Eqn. (11) is recast in terms of even and odd functions to yield

$$f_n(x) = [SS1_n \cosh(u\alpha_n x) + AS1_n \sinh(u\alpha_n x)]/\cosh(u\alpha_n \ell)$$
$$+ [SS3_n \cosh(v\alpha_n x) + AS3_n \sinh(v\alpha_n x)]/\cosh(v\alpha_n \ell) \ ,$$

$$g_n(y) = [SS2_n \cosh(u\beta_n y) + AS2_n \sinh(u\beta_n y)]/\cosh(u\beta_n t)$$
$$+ [SS4_n \cosh(v\beta_n y) + AS4_n \sinh(v\beta_n y)]/\cosh(v\beta_n t) \ ,$$

$$\tag{12}$$

$$h_n(x) = [SA1_n \cosh(u\alpha_n x) + AA1_n \sinh(u\alpha_n x)]/\cosh(u\alpha_n \ell)$$
$$+ [SA3_n \cosh(v\alpha_n x) + AA3_n \sinh(v\alpha_n x)]/\cosh(v\alpha_n \ell) \ ,$$

$$k_n(y) = [SA2_n \cosh(u\beta_n y) + AA2_n \sinh(u\beta_n y)]/\cosh(u\beta_n t)$$
$$+ [SA4_n \cosh(v\beta_n y) + AA4_n \sinh(v\beta_n y)]/\cosh(v\beta_n t) \ .$$

The terms $\cosh(u\alpha_n \ell)$, $\cosh(v\alpha_n \ell)$, $\cosh(u\beta_n t)$, and $\cosh(v\beta_n t)$ are inserted to avoid numerical instability in computation for large values of n, and the coefficients ($SS1_n$, $AS2_n$, etc.) are named in such a way as to denote states of symmetry in x and y. The first letter in each coefficient describes the symmetry with respect to the x-axis while the second letter denotes symmetry with respect to the y-axis ("S" denotes symmetry, while "A" denotes anti-symmetry). Roots corresponding to case (1) occur in the analysis of highly unidirectional materials such as fiber-reinforced composites. In fact, any orthotropic material in which either of the normal moduli E_1 and E_2 is significantly larger than the shear modulus G_{12} will fall into this class.

CASE (2)

When the roots are real and equal in pairs, as occurs for isotropic materials, the functions $f_n(x)$, $g_n(y)$, $h_n(x)$, and $k_n(y)$ similarly have the forms

$$f_n(x) = [SS1_n \cosh(u\alpha_n x) + SS3_n \, u\alpha_n x \, \sinh(u\alpha_n x)$$
$$+ AS1_n \, \sinh(u\alpha_n x) + AS3_n \, u\alpha_n x \, \cosh(u\alpha_n x)]/\cosh(u\alpha_n \ell) \, ,$$

$$g_n(y) = [SS2_n \cosh(u\beta_n y) + SS4_n \, u\beta_n y \, \sinh(u\beta_n y)$$
$$+ AS2_n \, \sinh(u\beta_n y) + AS4_n \, u\beta_n y \, \cosh(u\beta_n y)]/\cosh(u\beta_n t)$$

$$(13)$$

$$h_n(x) = [SA1_n \cosh(u\alpha_n x) + SA3_n \, u\alpha_n x \, \sinh(u\alpha_n x)$$
$$+ AA1_n \, \sinh(u\alpha_n x) + AA3_n \, u\alpha_n x \, \cosh(u\alpha_n x)]/\cosh(u\alpha_n \ell)$$

$$k_n(y) = [SA2_n \cosh(u\beta_n y) + SA4_n \, u\beta_n y \, \sinh(u\beta_n y)$$
$$+ AA2_n \, \sinh(u\beta_n y) + AA4_n \, u\beta_n y \, \cosh(u\beta_n y)]/\cosh(u\beta_n t) \, .$$

As with the previous case, the coefficients denote symmetry conditions.

CASE 3

When the roots are complex, case (3), the functions $f_n(x)$, $g_n(y)$, $h_n(x)$, and $k_n(y)$ have the forms

$$f_n(x) = [(SS1_n \cos(v\alpha_n x) + AS1_n \sin(v\alpha_n x)) \cosh(u\alpha_n x)$$
$$+ (SS3_n \cos(v\alpha_n y) + AS3_n \sin(v\alpha_n x)) \sinh(u\alpha_n x)]/\cosh(u\alpha_n \ell) \, ,$$

$$g_n(y) = [(SS2_n \cos(v\beta_n y) + AS2_n \sin(v\beta_n y)) \cosh(u\beta_n y)$$
$$+ (SS4_n \cos(v\beta_n y) + AS4_n \sin(v\beta_n y)) \sinh(u\beta_n y)]/\cosh(u\beta_n t) \, ,$$

$$(14)$$

$$h_n(x) = [(SA1_n \cos(v\alpha_n x) + AA1_n \sin(v\alpha_n x)) \cosh(u\alpha_n x)$$
$$+ (SA3_n \cos(v\alpha_n x) + AA3_n \sin(v\alpha_n x)) \sinh(u\alpha_n x)]/\cosh(u\alpha_n \ell)$$

$$k_n(y) = [(SA2_n \cos(v\beta_n y) + AA2_n \sin(v\beta_n y)) \cosh(u\beta_n y)$$
$$+ (SA4_n \cos(v\beta_n y) + AA4_n \sin(v\beta_n y)) \sinh(u\beta_n y)]/\cosh(u\beta_n t) \, .$$

As with the previous two cases, the coefficients denote symmetry conditions. While case (3) is not fully developed in this report, it is nevertheless quite significant. Any material with large shear modulus G_{12} (as compared to the normal moduli E_1 and E_2) fits into this case. For instance, graphite/epoxy in unidirectional form has material properties [11]

$$E_1 = 19.2 \times 10^6 \text{ pxi}$$
$$E_2 = 1.56 \times 10^6 \text{ psi}$$
$$G_{12} = 0.82 \times 10^6 \text{ psi}$$
$$\nu_{12} = 0.238$$

and falls into case (1). However, using this material in angle-ply laminates having the stacking sequence $[\pm\theta]_{\text{symmetric}}$ raises the shear modulus while lowering the maximum normal modulus, and gives properties corresponding to case (3) when θ lies in the range

$$25.1° \leq \theta \leq 64.9°.$$

Since laminates of this form are used frequently in composite structures, this latter case obviously is important. The purpose of this paper is to describe a general technique for solution and the general development of equations will be limited to cases (1) and (2). The procedure for incorporation of material systems requiring case (3) solutions follows directly using the same development as described for the first two cases.

Reformulation of the Stress Function

The equation for the stress function ϕ, Eqn. (8), can be recast to take advantage of even and odd functions as follows:

$$\phi = \phi_{SS} + \phi_{SA} + \phi_{AS} + \phi_{AA} \tag{15}$$

where

$$\phi_{SS} = \frac{1}{2} R_{xss} y^2 + \frac{1}{2} R_{yss} x^2 + \sum_{n=0}^{\infty} X_{nss} \cos(\alpha_n y) + \sum_{n=0}^{\infty} Y_{nss} \cos(\beta_n x)$$

$$\phi_{SA} = \frac{1}{2} R_{ysa} x^2 y + \sum_{n=1}^{\infty} X_{nsa} \sin(\alpha_n y) + \sum_{n=0}^{\infty} Y_{nsa} \cos(\beta_n x)$$

$$\phi_{AS} = \frac{1}{2} R_{xas} x y^2 + \sum_{n=0}^{\infty} X_{nas} \cos(\alpha_n y) + \sum_{n=1}^{\infty} Y_{nas} \sin(\beta_n x)$$

$$\phi_{AA} = R_{aa} xy + \sum_{n=1}^{\infty} X_{naa} \sin(\alpha_n y) + \sum_{n=1}^{\infty} Y_{naa} \sin(\beta_n X) .$$

In these expressions, the terms X_{nss}, X_{nsa}, X_{nas}, and X_{naa} are functions of x, while the terms Y_{nss}, Y_{nsa}, Y_{nas}, and Y_{naa} are functions of y. The forms of these functions depend on the roots to the governing differential Eqn. (4), as discussed previously, and are presented later. The terms R_{xss}, R_{yss}, R_{ysa}, R_{xas}, and R_{aa} are arbitrary constants.

Using the definition of the stress function (Eqn. (2)), stresses are derived from the stress function by differentiation. These stresses are given in terms of the stress function in Appendix A.

For Case (1), roots real and in pairs, the functions X_{nss}, Y_{nsa}, etc. are given by

$$X_{nss} = SS1_n \cosh(u\alpha_n x)/\cosh(u\alpha_n \ell) + SS3_n \cosh(v\alpha_n x)/\cosh(v\alpha_n \ell)$$

$$X_{nsa} = SA1_n \cosh(u\alpha_n x)/\cosh(u\alpha_n \ell) + SA3_n \cosh(v\alpha_n x)/\cosh(v\alpha_n \ell)$$

$$X_{nas} = AS1_n \sinh(u\alpha_n x)/\cosh(u\alpha_n \ell) + AS3_n \sinh(v\alpha_n x)/\cosh(v\alpha_n \ell)$$

$$X_{naa} = AA1_n \sinh(u\alpha_n x)/\cosh(u\alpha_n \ell) + AA3_n \sinh(v\alpha_n x)/\cosh(v\alpha_n \ell) \qquad (16)$$

$$Y_{nss} = SS2_n \cosh(u\beta_n y)/\cosh(u\beta_n t) + SS4_n \cosh(v\beta_n y)/\cosh(v\beta_n t)$$

$$Y_{nsa} = SA2_n \sinh(u\beta_n y)/\cosh(u\beta_n t) + SA4_n \sinh(v\beta_n y)/\cosh(v\beta_n t)$$

$$Y_{nas} = AS2_n \cosh(u\beta_n y)/\cosh(u\beta_n t) + AS4_n \cosh(v\beta_n y)/\cosh(v\beta_n t)$$

$$Y_{naa} = AA2_n \sinh(u\beta_n y)/\cosh(u\beta_n t) + AA4_n \sinh(v\beta_n y)/\cosh(v\beta_n t) .$$

For case (2), roots real and equal in pairs, these functions are given by

$$X_{nss} = [SS1_n \cosh(u\alpha_n x) + SS3_n u\alpha_n x \sinh(u\alpha_n x)]/\cosh(u\alpha_n \ell)$$

$$X_{nsa} = [SA1_n \cosh(u\alpha_n x) + SA3_n u\alpha_n x \sinh(u\alpha_n x)]/\cosh(u\alpha_n \ell)$$

$$X_{nas} = [AS1_n \sinh(u\alpha_n x) + AS3_n u\alpha_n x \cosh(u\alpha_n x)]/\cosh(u\alpha_n \ell)$$

$$X_{naa} = [AA1_n \sinh(u\alpha_n x) + AA3_n u\alpha_n x \cosh(u\alpha_n x)]/\cosh(u\alpha_n \ell) \qquad (17)$$

$$Y_{nss} = [SS2_n \cosh(u\beta_n y) + SS4_n u\beta_n y \sinh(u\beta_n y)]/\cosh(u\beta_n t)$$

$$Y_{nsa} = [SA2_n \cosh(u\beta_n y) + AS4_n u\beta_n y \sinh(u\beta_n y)]/\cosh(u\beta_n t)$$

$$Y_{nas} = [AS2_n \cosh(u\beta_n y) + AS4_n u\beta_n y \sinh(u\beta_n y)]/\cosh(u\beta_n t)$$

$$Y_{naa} = [AA2_n \sinh(u\beta_n y) + AA4_n u\beta_n y \cosh(u\beta_n y)]/\cosh(u\beta_n t) .$$

Finally, for case (3), complex roots, these functions are given by

$$X_{nss} = [SS1_n \cos(v\alpha_n x)\cosh(u\alpha_n x) + SS3_n \sin(v\alpha_n x)\sinh(u\alpha_n x)]/\cosh(u\alpha_n \ell)$$

$$X_{nsa} = [SA1_n \cos(v\alpha_n x)\cosh(u\alpha_n x) + SA3_n \sin(v\alpha_n x)\sinh(u\alpha_n x)]/\cosh(u\alpha_n \ell)$$

$$X_{nas} = [AS1_n \sin(v\alpha_n x)\cosh(u\alpha_n x) + AS3_n \cos(v\alpha_n x)\sinh(u\alpha_n x)]/\cosh(u\alpha_n \ell)$$

$$X_{naa} = [AA1_n \sin(v\alpha_n x)\cosh(u\alpha_n x) + AA3_n \cos(v\alpha_n x)\sinh(u\alpha_n x)]/\cosh(u\alpha_n \ell)$$

$$Y_{nss} = [SS2_n \cos(v\beta_n y)\cosh(u\beta_n y) + SS4_n \sin(v\beta_n y)\sinh(u\beta_n y)]/\cosh(u\beta_n t)$$

$$Y_{nsa} = [SA2_n \sin(v\beta_n y)\cosh(u\beta_n y) + SA4_n \cos(v\beta_n y)\sinh(u\beta_n y)]/\cosh(u\beta_n t)$$

$$Y_{nas} = [AS2_n \cos(v\beta_n y)\cosh(u\beta_n y) + AS4_n \sin(v\beta_n y)\sinh(u\beta_n y)]/\cosh(u\beta_n t)$$

$$Y_{naa} = [AA2_n \sin(v\beta_n y)\cosh(u\beta_n y) + AA4_n \cos(v\beta_n y)\sinh(u\beta_n y)]/\cosh(u\beta_n t) .$$

$$(18)$$

Formulation of Boundary Conditions

The stress function ϕ as expressed by Eqn. (15)-(18) is composed of three types of functions which contain unknown coefficients. These are the polynomials ($\frac{1}{2} R_{xss} y^2$, etc.), the even series functions (X_{nss}, X_{nsa}, Y_{nss}, Y_{nas}), and the odd series functions (X_{nas}, X_{naa}, Y_{nsa}, Y_{naa}). Each of the series functions contains two unknown coefficients, so the even series functions contain eight ($SS1_n$, $SS3_n$, ----, $AS2_n$, $AS4_n$) for each value of n, and the odd series functions contain eight ($AS1_n$, $AS3_n$, ----, $AA2_n$, $AA4_n$) for each value of n. Thus, 16 unknown coefficients must be found for each value of n in the series, and another five must be found to account for the polynomial terms (R_{xss}, R_{yss}, R_{ysa}, R_{xas}, R_{aa}).

To obtain the solution for the coefficients in Eqns. (15)-(18), the boundary conditions (Eqn. (6)) are expanded in general Fourier series, with the form

$$\sigma_x(\ell,y) = \frac{1}{2t} C_{1_0} + \sum^{\infty} \frac{1}{t}[C_{1_n} \cos(\alpha_n y) + S_{1_n} \sin(\alpha_n y)] \qquad (19)$$

where

$$C_{1_0} = \int_{-t}^{t} r_1(y)\, dy$$

$$C_{1_n} = \int_{-t}^{t} r_1(y) \cos(\alpha_n y) \, dy$$

$$S_{1_n} = \int_{-t}^{t} r_1(y) \sin(\alpha_n y) \, dy \ .$$

Similar expansions hold for the other applied boundary stresses. See Appendix B for these expansions and their Fourier coefficients.

To take advantage of possible symmetries which might occur with the stresses applied on the boundaries, boundary conditions are obtained by equating the sums and differences of the applied stresses (given in Appendix B) at the boundaries to the corresponding sums and differences of the general expressions for stress (see Appendix A) evaluated at the boundaries. The resulting equations are then manipulated to extract expressions for the arbitrary coefficients in the following fashion. For example, on the boundaries $x = \pm \ell$,

$$r_1(y) - r_2(y) = \sigma_x(\ell,y) - \sigma_x(-\ell,y) \tag{20}$$

Substituting for $r_1(y)$ and $r_2(y)$ expanded in Fourier series (as given in Appendix B) and for σ_x as given in Appendix A and evaluated at the boundaries, this equation becomes

$$\frac{1}{2t} C_{1-2_0} + \frac{1}{t} \sum_{n=1}^{\infty} [C_{1-2_n} \cos(\alpha_n y) + S_{1-2_n} \sin(\alpha_n y)]$$

$$= 2R_{xas} - 2 \sum_{n=0}^{\infty} \alpha_n^2 X_{nas}(\ell) \cos(\alpha_n y) \tag{21}$$

$$- 2 \sum_{n=1}^{\infty} \alpha_n^2 X_{naa}(\ell) \sin(\alpha_n y)$$

where terms of the form C_{1-2_p} are a shorthand notation for $(C_{1_p} - C_{2_p})$. Recall that terms of the form $X_{nas}(\ell)$ are functions involving unknown coefficients such as $AS1_n$, evaluated at ℓ. To relate the unknown coefficients to the known Fourier coefficients of the applied stresses, both sides of the equation are multiplied by either $\sin(\alpha_n y)$ or $\cos(\alpha_n y)$ and integrated along the boundary from $-t$ to t (for stresses on the adjacent boundaries, the boundary condition equation is multiplied by $\sin(\beta_n x)$ or $\cos(\beta_n x)$ and integrated from $-\ell$ to ℓ). Continuing this process for the remaining boundary condition relations,

$$
\begin{aligned}
r_1(y) + r_2(y) &= \sigma_x(\ell,y) + \sigma_x(-\ell,y) \\
r_3(x) + r_4(x) &= \sigma_y(x,t) - \sigma_y(x,-t) \\
r_3(x) + r_4(x) &= \sigma_y(x,t) + \sigma_y(x,-t) \\
r_5(y) - r_6(y) &= \tau_{xy}(\ell,y) - \tau_{xy}(-\ell,y) \\
r_5(y) + r_6(y) &= \tau_{xy}(\ell,y) + \tau_{xy}(-\ell,y) \\
r_7(x) - r_8(x) &= \tau_{xy}(x,t) - \tau_{xy}(x,-t) \\
r_7(x) + r_8(x) &= \tau_{xy}(x,t) + \tau_{xy}(x,-t) \ ,
\end{aligned}
\tag{22}
$$

the following set of equations result for the unknown coefficients with all material cases:

$$
SS1_p + (E1A_p) \sum_{n=1}^{\infty} (E1B_{pn})SS2_n = (E1A_p)\{(E1C_p) + \sum_{n=1}^{\infty} (E1D_{pn})\}
$$

$$
(F1A_p) \sum_{n=1}^{\infty} (F1B_{pn})SS1_n + SS2_p = (F1A_p)\{(F1C_p) + \sum_{n=1}^{\infty} (F1D_{pn})\}
$$

$$SA1_p + (E2A_p) \sum_{n=1}^{\infty} (E2B_{pn})SA2_n = (E2A_p)\{ E2C_p) + \sum_{n=1}^{\infty} (E2D_{pn})\}$$

$$(F2A_p) \sum_{n=1}^{\infty} (F2B_{pn})SA1_n + SA2_p = (F2A_p)\{(F2C_p) + \sum_{n=1}^{\infty} (F2D_{pn})\}$$

$$AS1_p + (E3A_p) \sum_{n=1}^{\infty} (E3B_{pn})AS2_n = (E3A_p)\{(E3C_p) + \sum_{n=1}^{\infty} (E3D_{pn})\}$$

$$(F3A_p) \sum_{n=1}^{\infty} (F3B_{pn})AS1_n + AS2_p = (F3A_p)\{(F3C_p) + \sum_{n=1}^{\infty} (F3D_{pn})\}$$

$$AA1_p + (E4A_p) \sum_{n=1}^{\infty} (E4B_{pn})AA2_n = (EA4_p)\{(E4C_p) + \sum_{n=1}^{\infty} (E4D_{pn})\}$$

$$(F4A_p) \sum_{n=1}^{\infty} (F4B_{pn})AA1_n + AA2_p = (F4A_p)\{(F4C_p) + \sum_{n=1}^{\infty} (F4D_{pn})\} \quad .$$

(23)

Furthermore, equations of the following type also result (these examples are for Case 1 material):

$$SS3_p = \frac{1}{v \, \tanh(v\alpha_p\ell)}\left[\frac{S_{5-6p}}{2\alpha_p^2 t} - SS1_p u \, \tanh(u\alpha_p\ell)\right]$$

$$SS4_p = \frac{1}{v \, \tanh(v\beta_p t)}\left[\frac{S_{7-8p}}{2\beta_p^2 \ell} - SS2_p u \, \tanh(u\beta_p t)\right]$$

$$SA3_p = \frac{-1}{v \, \tanh(v\alpha_p\ell)}\left[\frac{C_{5-6p}}{2\alpha_p^2 t} + SA1_p u \, \tanh(u\alpha_p\ell)\right]$$

$$SA4_p = -\left[SA2_p \frac{\tanh(u\beta_p t)}{\tanh(v\beta_p t)} + \frac{C_{3-4\nu}}{2\beta_p^2 \ell \, \tanh(v\beta_p t)}\right]$$

(24)

$$AS3_p = \frac{-1}{\tanh(v\alpha_p \ell)} \left[AS1_p \tanh(u\alpha_p \ell) + \frac{C_{1-2p}}{2\alpha_p^2 t} \right]$$

$$AS4_p = \frac{-1}{v \tanh(v\alpha_p t)} \left[\frac{C_{7-8p}}{2\beta_p^2 \ell} + AS2_p u \tanh(u\beta_p t) \right]$$

$$AA3_p = \frac{-1}{\tanh(v\alpha_p \ell)} \left[AA1_p \tanh(u\alpha_p \ell) + \frac{S_{1-2p}}{2\alpha_p^2 t} \right]$$

$$AA4_p = \frac{-1}{\tanh(v\beta_p t)} \left[AA2_p \tanh(u\beta_p t) + \frac{S_{3-4p}}{2\beta_p^2 \ell} \right]$$

For a complete definition of the terms in Eqn. (23), as well as for the Case 2 expressions corresponding to Eqn. (24), see Appendix C.

The Eqns. (23) can be paired in matrix form according to symmetry conditions as, for example,

$$\left[\begin{array}{c|c} I & E1 \\ \hline F1 & 1 \end{array} \right] \left\{ \begin{array}{c} SS1 \\ \hline SS2 \end{array} \right\}_n = \left\{ \begin{array}{c} P1A \\ \hline P1B \end{array} \right\}_n \qquad n = 1,2, --- \qquad (25a)$$

where, after partitioning, [I] is the identity matrix, [E1] and [F1] are square matrices dependent on material properties and slab geometry; $\{P1A\}_n$ and $\{P1B\}_n$ are column vectors dependent on material properties, slab geometry, and boundary conditions; and $\{SS1\}_n$ and $\{SS2\}_n$ are column vectors containing the unknown constant coefficients of the series stress function. Similarly,

$$\left[\begin{array}{c|c} I & E2 \\ \hline F2 & I \end{array} \right] \left\{ \begin{array}{c} SA1 \\ \hline SA2 \end{array} \right\}_n = \left\{ \begin{array}{c} P2A \\ \hline P2B \end{array} \right\}_n \qquad n = 1,2, --- \qquad (25b)$$

$$\begin{bmatrix} I & \vdots & E3 \\ \text{-----} & \vdots & \text{-----} \\ F3 & \vdots & I \end{bmatrix} \left\{ \begin{array}{c} AS1 \\ \text{-----} \\ AS2 \end{array} \right\}_n = \left\{ \begin{array}{c} P3A \\ \text{---} \\ P3B \end{array} \right\}_n \qquad n = 1,2 \text{ ---} \qquad (25c)$$

$$\begin{bmatrix} I & \vdots & E4 \\ \text{-----} & \vdots & \text{-----} \\ F4 & \vdots & I \end{bmatrix} \left\{ \begin{array}{c} AA1 \\ \text{-----} \\ AA2 \end{array} \right\}_n = \left\{ \begin{array}{c} P4A \\ \text{---} \\ P4B \end{array} \right\}_n \qquad n = 1,2, \text{ ---} \qquad (25d)$$

Each set of Eqn. (24) may be solved independently; for example, solving for $\{SS1\}_n$ and $\{SS2\}_n$ gives

$$SS2_n = \left[[I] - [F1][E1] \right]^{-1} \{P1B\} - [F1]\{P1A\}_n$$
$$SS1_n = P1A - [E1]\{SS2\}_n .$$

(26)

Using the unknown coefficients determined from the above matrix solutions, the remaining unknown coefficients can be found from Eqn. (24) for Case 1 or from the corresponding equations for Case 2 material.

The only terms which are still unknown at this point are the coefficients of the polynomial terms in the expression for the stress function (Eqn. (15)). These are found in similar fashion to the above using the boundary relationships of Eqns. (20) and (22) and n = 0.

The polynomial terms R_{ysa} and R_{xas} are then given by

$$R_{ysa} = \frac{C_{3-4_0}}{4\ell t}$$

$$R_{xas} = \frac{C_{1-2_0}}{4\ell t}$$

(27)

for both materials (1) and (2). The polynomials R_{xss}, R_{yss}, and R_{aa} are dependent, not only on material properties, but also on the solutions for the coefficients $SS1_n$, $SS2_n$, etc. For material case 1, roots real and in pairs, these terms are given by

$$R_{xss} = \frac{1}{4t} C_{1+2_0} - \sum_{n=1}^{\infty} (-1)^n \frac{\beta_n}{t} [SS2_n u \tanh(u\beta_n t) + SS4_n v \tanh(v\beta_n t)]$$

$$R_{yss} = \frac{1}{4\ell} C_{3+4_0} - \sum_{n=1}^{\infty} (-1)^n \frac{\alpha_n}{\ell} [SS1_n u \tanh(u\alpha_n \ell) + SS3_n v \tanh(v\alpha_n \ell)] \tag{28}$$

$$R_{aa} = - \frac{C_{5+6_0}}{4t} - \sum_{n=1}^{\infty} (-1)^n \frac{\beta_n}{t} [AA2_n \tanh(u\beta_n t) + AA4_n \tanh(v\beta_n t)] .$$

For material Case 2, roots real and equal in pairs, these polynomials are given by

$$R_{xss} = \frac{1}{4t} C_{1+2_0} - \sum_{n=1}^{\infty} (-1)^n \frac{u\beta_n}{t} [SS2_n \tanh(u\beta_n t)$$

$$+ SS4_n \{\tanh(u\beta_n t) + u\beta_n t\}]$$

$$R_{yss} = \frac{1}{4\ell} C_{3+4_0} - \sum_{n=1}^{\infty} (-1)^n \frac{u\alpha_n}{\ell} [SS1_n \tanh(u\alpha_n \ell) \tag{29}$$

$$+ SS3_n \{\tanh(u\alpha_n \ell) + u\alpha_n \ell\}]$$

$$R_{aa} = - \frac{C_{5+6_0}}{4t} - \sum_{n=1}^{\infty} (-1)^n \frac{\beta_n}{t} [AA2_n \tanh(u\beta_n t) + AA4_n u\beta_n t] .$$

Using Eqns. (23) through   ) and Appendix C all of the unknown coefficients can be solved. Then, using the equations given in Appendix A, the stresses at any point in the material can be found.

3. COMPUTER IMPLEMENTATION

To solve the equations described previously, a computer program was
written and programmed on a Control Data Corporation Cyber 173 computer
at the Langley Research Center of the National Aeronautics and Space
Administration. A brief description of this program follows.

After receiving input of material properties and the Fourier series
coefficients for the boundary stresses, the program calculates the roots
of the characteristic equation from the material properties and decides
which solution form to use. The subroutine corresponding to that form
is then called to solve for the unknown coefficients. The various
symmetry cases are solved one at a time by computing the elements of
matrices E and F as required with the values indicated in Appendix C.
A matrix manipulation subroutine is then called to perform a set of
operations common to all symmetry cases. At the end of these operations,
the coefficients for the symmetry cases in question are put into the
proper storage locations and the matrix routine returns control to the
routine which fills the matrices. The next symmetry case is then solved
in the same fashion, until all symmetry cases are done. Any single type
of symmetry can be considered (to the exclusion of all others), or the
program can solve all four types in turn, thus solving a general
(asymetric) loading case. A check on the accuracy of the solution is
given by printing out the input boundary conditions (found by summing
the input Fourier series) in columns beside the corresponding output
boundary conditions (found by summing the "exact" solution on the
boundaries). This feature can also be used to check for errors in input.

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Finally, the locations for which stress values are desired are checked, stresses are calculated at these locations, and a subroutine is called to print or plot these stresses in useful form.

Several problems arise in implementing the computer code. It has already been pointed out that terms based on the maximum values of x and y and on the value of n are included in the solution. These terms are included to prevent terms involving hyperbolic functions from becoming too large and destroying the accuracy of the series (by requiring the differences of large numbers to be much smaller than the numbers themselves). For plates with high or low aspect ratios (ℓ/t greater than about 4 or less than about 0.25), the arguments of terms such as cosh ($u\alpha_n\ell$) become too large for the computer to handle when n becomes large (n approximately equal to 50). Thus, the solutions for these plates require that fewer terms of the series solution be taken. It is possible that this may lead to unacceptable accuracy if the loading is such as to require a large number of terms for convergence of its Fourier series representation.

Because of the nature of Fourier series expansions, convergence problems might occur for some types of loadings at the corners of the slab.

3.1 Sample Problem Solutions

Two practical sample cases were chosen as examples of potential applications of this solution procedure. The sample cases presented were selected for their similarity to actual problems encountered in materials testing and to demonstrate the wide applicability of the

program developed.

Solutions were generated using 35 and 50 terms in the series formulations. Plots were generated taking cuts at constant x/ℓ equal to 0.0, 0.2, 0.4, 0.6, and 0.8 with $-t \leq y \leq t$. In all cases, $t = 1.0$, while $\ell = 1.0$, 2.0, 4.0 and 6.0. Thus the slabs analyzed had aspect ratios A.R. $= \ell/t = 1.0$, 2.0, 4.0 and 6.0.

The analysis also involved two different materials. One of these was isotropic with the properties of steel:

$$E_1 = E_2 = 30.0 \times 10^6 \text{ psi}$$
$$G_{12} = 12.0 \times 10^6 \text{ psi}$$
$$\nu_{12} = 0.25 \ .$$

The other material was orthotropic, having the properties of unidirectional graphite/epoxy from [11]:

$$E_1 = 10.2 \times 10^6 \text{ psi}$$
$$E_2 = 1.56 \times 10^6 \text{ psi}$$
$$G_{12} = 0.82 \times 10^6 \text{ psi}$$
$$\nu_{12} = 0.238 \ .$$

The loadings, to be referred to subsequently as the "tensile coupon" and "rail shear" cases, are described as follows.

3.1.1 "Tensile Coupon: Load Case

The "tensile coupon" test has the following boundary conditions:

$$\tau_{xy}(x,\pm t) = 0 \text{ for } -\ell \le x \le -(0.95)\ell$$

$$-(0.90)\ell \le x \le (0.09)\ell$$

$$(0.95)\ell \le x \le \ell$$

$$= \pm(20)t/\ell \text{ for } -(0.95)\ell \le x \le -(0.90)\ell$$

$$= \pm(20)t/\ell \text{ for } (0.90)\ell \le x \le (0.95)\ell$$

$$\tau_{xy}(\pm\ell,y) = 0$$

$$\sigma_x(\pm\ell,y) = 0$$

$$\sigma_y(x,\pm t) = 0 .$$

These boundary conditions (see Fig. 3) are chosen to give a very crude approximation of the stresses on a rectangular coupon clamped in a test rig and being pulled for the determination of tensile strength. The non-zero values of boundary shear are selected to give an average normal stress σ_x of magnitude 1.0 when the applied stresses have diffused completely.

3.1.2 "Rail Shear" Load Case

The "rail shear" test has the boundary conditions

$$\tau_{xy}(x,\pm t) = 1.0 \text{ for } -(0.80)\ell \le x \le (0.80)\ell$$

$$= 0 \quad \text{for } -\ell \le x \le -(0.80)\ell, \ (0.80)\ell \le x \le \ell$$

$$\sigma_y(x,\pm t) = \frac{3t}{(0.64)\ell^2} x \text{ for } -(0.80)\ell \le x \le (0.80)\ell$$

$$= 0 \text{ for } -\ell \le x \le -(0.80)\ell, \ (0.80)\ell \le x \le \ell$$

$$\tau_{xy}(\pm\ell,y) = 0$$
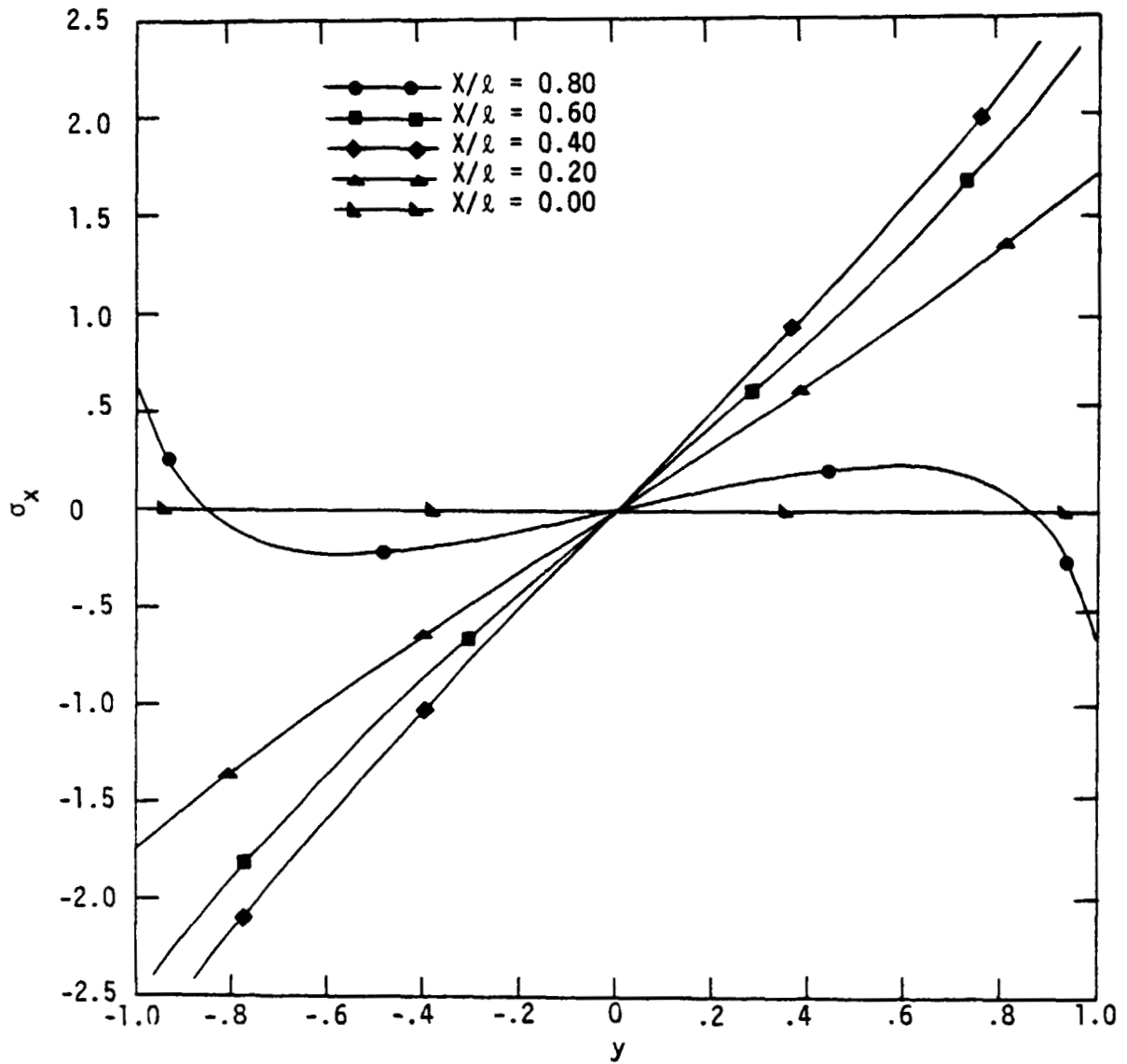
$$\sigma_x(\pm\ell,y) = 0 .$$

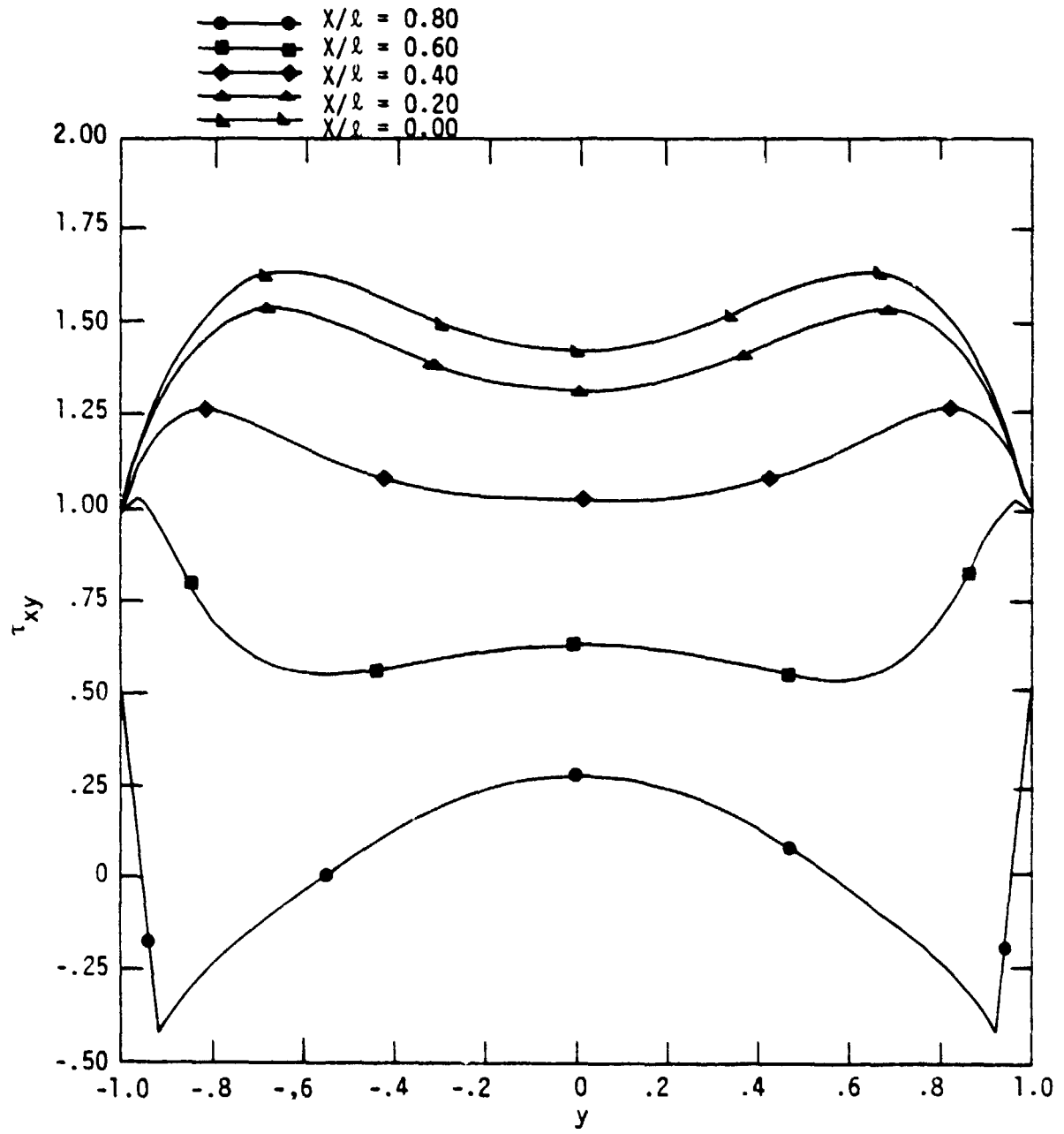Fig. 12   Plot of σ_x for Orthotropic "Rail Shear Specimen" of A.R. = 6

Fig. 13 Plot of τ_{xy} for Isotropic "Rail Shear Specimen" of A.R. = 1
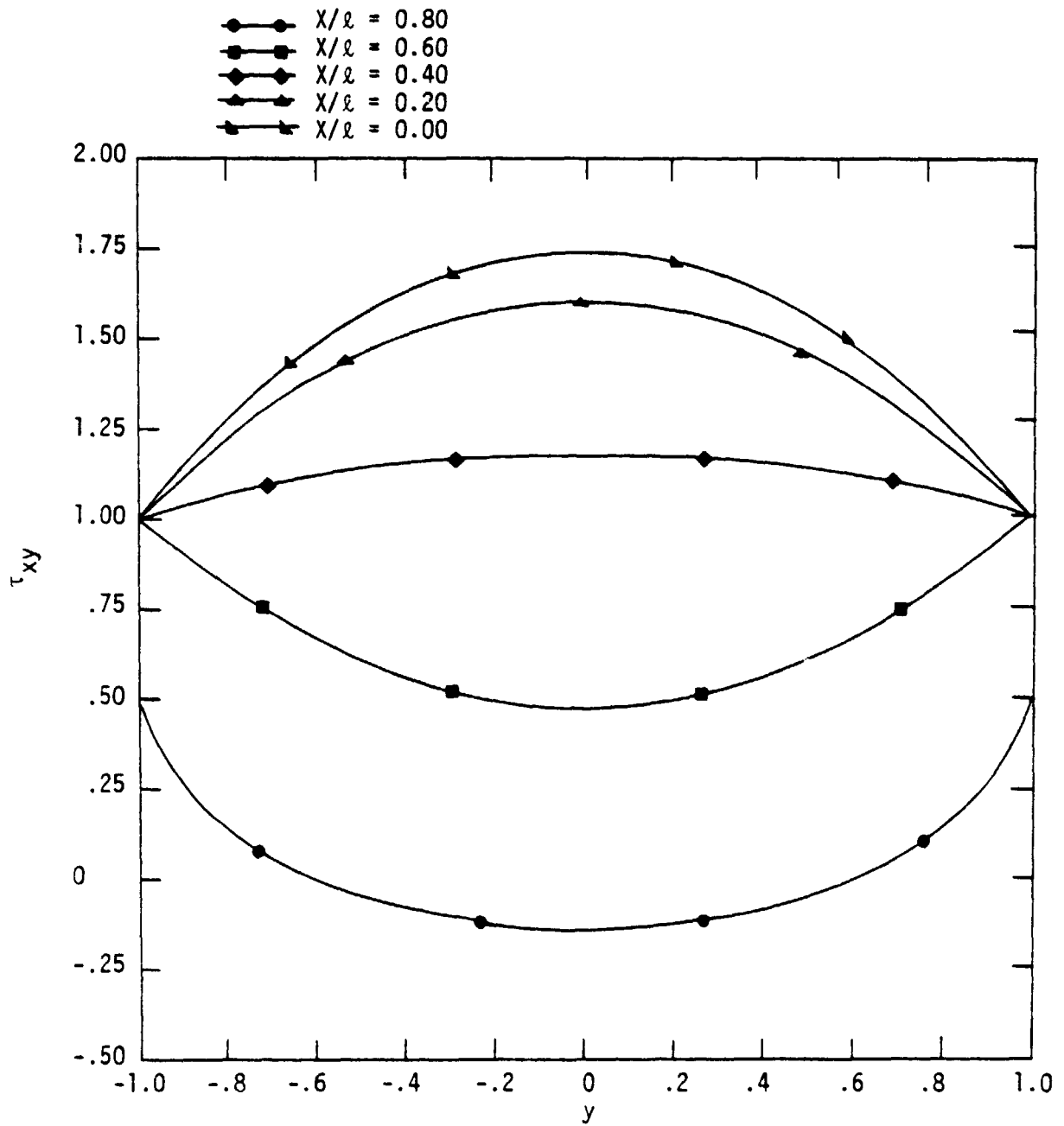
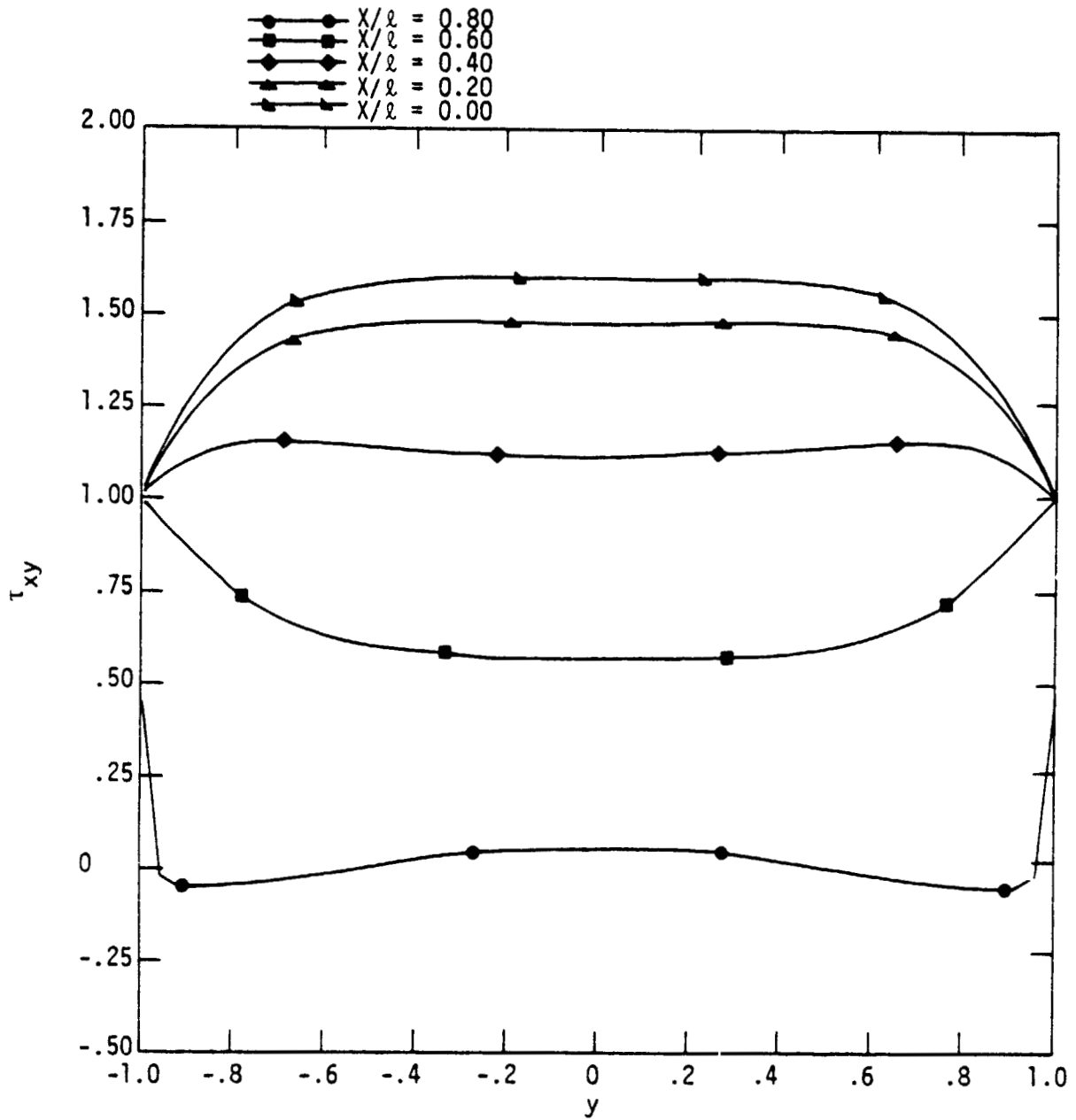Fig. 14   Plot of τ_{xy} for Isotropic "Rail Shear Specimen" of A.R. = 6

Fig. 15 Plot of τ_{xy} for Orthotropic "Rail Shear Specimen" of A.R. = 1

"rail shear" specimen. All specimens showed essentially identical behavior, with only the magnitude of the stress changing. This is an aspect ratio effect, since higher magnitudes of σ_y are required to satisfy moment equilibrium for lower aspect ratio specimens.

Aspect ratio provides most of the variation in normal stress σ_x distribution for the "rail shear" tests. The distributions of σ_x in orthotropic and isotropic plates have nearly identical forms and magnitudes for a given aspect ratio, though the maximum magnitude of σ_x is slightly lower for orthotropic materials at low aspect ratios. At high aspect ratios, the difference in magnitudes due to material properties is negligible. See Figs. 11 and 12 for σ_x plots for an isotropic plate of aspect ratio of 1.0 and an orthotropic plate of aspect ratio 6.0, respectively.

Figures 13 and 14 show the shear distribution in isotropic plates of aspect ratios 1.0 and 6.0, respectively. A comparison of these plots serves to illustrate the smoothing of the curves for τ_{xy} as aspect ratio increases in both isotropic and orthotropic plates. Orthotropic plates of low aspect ratio, however, tended to display τ_{xy} behavior similar to isotropic plates of slightly higher aspect ratio (see Fig. 15). The material properties were of little consequence to the shear stress field at higher aspect ratios. At the edge of the stress boundary ($x/\ell = 0.8$), the behavior of $y = \pm t$ shows a value of $\tau_{xy} = 0.5$ instead of the anticipated value of 1.0. This is a result of the fact that Fourier series give average values at discontinuities, so the applied boundary stress input at this point is, in fact, $\tau_{xy} = 0.5$, rather than 1.0.

3.3 Conclusions and Recommendations

A method for solving exactly the stress field in a slab of ortho-
tropic material was developed. Tests were run to determine the useful-
ness of the method. The procedure suggested by Pickett [6] shows consi-
derable promise for use in the analysis of both isotropic and ortho-
tropic materials. Accurate results were obtained with fairly low run
times.

One of the expected benefits of the method proved to be nonexistent.
It was originally anticipated that all rectangular membranes could be
analyzed with similar accuracy, but this was found not to be true. The
inability of the computer to handle large arguments of hyperbolic sine
and cosine functions limits the number of terms which can be taken in
the series solution. For plates with very high aspect ratios, this
limitation can become critically small and destroy the accuracy of the
solution.

The first and most important task for future researchers using
Pickett's method is to develop the solution for Case 3, involving complex
eigenvalues for the governing differential equation (Eq. (4)). Due to
the significance of materials which fall into this case (for example a
$[\pm45°]_{symmetric}$ laminate of graphite/epoxy) the inclusion of this
solution is important. Once this is done, a thorough study of convergence,
accuracy, and computing cost for Pickett's method needs to be undertaken.
These parameters should be compared to similar parameters for one or
more finite element programs.

Assuming that the suggested research proceeds with no major difficulties, a displacement solution should be formulated following the approach suggested in Appendix D. This would allow the solution of more general problems by allowing completely general boundary formulations.

Once a displacement solution has been developed, a method for joining one of these rectangular slabs to another needs to be developed. The slab for which the displacement solution holds could then be turned into a finite element, capable of deforming compatibly with its neighbors at every point of their common boundaries. For the time being, this joining of solutions must necessarily involve a minimal number of elements, due to computer limitations. However, more powerful computers already being developed make the concept of "exact" finite elements appear feasible in the very near future, and the technology required for them needs to be developed now.

An attractive feature of using Pickett's method for finite elements is that only a few elements may be required even when high local stess gradients exist. Presently, large numbers of elements are required for accurate solutions in such regions.

References

1. Gallagher, R. H., Finite Element Analysis: Fundamentals. Prentice-Hall, Englewood Cliffs, NJ, 1975.

2. Zienkiewicz, u. C., The Finite Element Method. McGraw-Hill, London, 1977.

3. Slot, T. and Yalch, J. P., "Stress Analysis of Plane Perforated Structures by Point-Wise Matching of Boundary Conditions," Nuclear Engineering and Design, Vol. 4, 1966.

4. Hooke, C. J., "Numerical Solution of Plane Elastostatic Problems by Point Matching," Journal of Strain Analysis, Vol. 3, No. 2, 1968.

5. Timoshenko, S. P. and Goodier, J. N., Theory of Elasticity. McGraw-Hill, New York, 1970.

6. Pickett, G., "Application of the Fourier Method to the Solution of Certain Boundary Problems in the Theory of Elasticity," Journal of Applied Mechanics, Vol. 14, 1944.

7. Little, R. W., Elasticity. Prentice-Hall, Englewood Cliffs, NJ, 1973.

8. Theocaris, P. S. and Dafermos, K. "The Elastic Strip Under Mixed Boundary Conditions," Journal of Applied Mechanics, Vol. 31, 1964.

9. Lekhnitskii, S. G. Anisotropic Plates. Gordon and Breach Science Publishers, New York, 1968.

10. Jones, R. M., Mechanics of Composite Materials. Scripta Book Co., Washington D.C., 1975.

11. Nagarkar, A. P. and Herakovich, C. T., "Nonlinear Temperature Dependent Failure Analysis of Finite Width Composite Laminates," VPI-E-/9-36, 1979.

APPENDIX A

STRESSES IN TERMS OF THE STRESS FUNCTION

Using Equation 15 and the definition of the stress function (Equation 2), the following relations for stresses can be defined:

$$\sigma_x = \sigma_{xss} + \sigma_{xsa} + \sigma_{xas} + \sigma_{xaa}$$

$$\sigma_y = \sigma_{yss} + \sigma_{ysa} + \sigma_{yas} + \sigma_{yaa}$$

$$\tau_{xy} = \tau_{xyss} + \tau_{xysa} + \tau_{xyas} + \tau_{xyaa} , \tag{A-1}$$

where

$$\sigma_{xss} = \frac{\partial^2}{\partial y^2} \phi_{ss}$$

$$\tau_{xyas} = - \frac{\partial^2}{\partial x \partial y} \phi_{as} , \text{ etc.}$$

With these terms defined, the following component stresses for σ_x result:

$$\sigma_{xss} = R_{xss} - \sum_{n=0}^{\infty} \alpha_n^2 X_{nss} \cos(\alpha_n y) + \sum_{n=0}^{\infty} Y_{nss}'' \cos(\beta_n x)$$

$$\tag{A-2}$$

$$\sigma_{xsa} = - \sum_{n=1}^{\infty} \alpha_n^2 X_{nsa} \sin(\alpha_n y) + \sum_{n=0}^{\infty} Y_{nsa}'' \cos(\beta_n x)$$

$$\sigma_{xas} = R_{xas} x - \sum_{n=0}^{\infty} \alpha_n^2 X_{nas} \cos(\alpha_n y) + \sum_{n=1}^{\infty} Y_{nas}'' \sin(\beta_n x)$$

$$\sigma_{xaa} = - \sum_{n=1}^{\infty} \alpha_n^2 X_{naa} \sin(\alpha_n y) + \sum_{n=1}^{\infty} Y_{naa}'' \sin(\beta_n x)$$

Similarly, for σ_y:

$$\sigma_{yss} = R_{yss} + \sum_{n=0}^{\infty} X''_{nss}\cos(\alpha_n y) - \sum_{n=0}^{\infty} \beta_n^2 Y_{nss}\cos(\beta_n x)$$

$$\sigma_{ysa} = R_{ysa}y + \sum_{n=0}^{\infty} X''_{nsa}\sin(\alpha_n y) - \sum_{n=0}^{\infty} \beta_n^2 Y_{nsa}\sin(\beta_n x)$$

$$\sigma_{yas} = \sum_{n=0}^{\infty} X''_{nas}\cos(\alpha_n y) - \sum_{n=1}^{\infty} \beta_n^2 Y_{nas}\sin(\beta_n x)$$

$$\sigma_{yaa} = \sum_{n=1}^{\infty} X''_{naa}\sin(\alpha_n y) - \sum_{n=1}^{\infty} \beta_n^2 Y_{naa}\sin(\beta_n x) \ .$$

(A-3)

Finally, for τ_{xy}:

$$\tau_{xyss} = \sum_{n=1}^{\infty} \alpha_n X'_{nss}\sin(\alpha_n y) + \sum_{n=1}^{\infty} \beta_n Y'_{nss}\sin(\beta_n x)$$

$$\tau_{xysa} = -R_{ysa}x - \sum_{n=1}^{\infty} \alpha_n X'_{nsa}\cos(\alpha_n y) + \sum_{n=1}^{\infty} \beta_n Y'_{nsa}\sin(\beta_n x)$$

$$\tau_{xyas} = -R_{xas}y + \sum_{n=1}^{\infty} \alpha_n X'_{nas}\sin(\alpha_n y) - \sum_{n=1}^{\infty} \beta_n Y'_{nas}\cos(\beta_n x)$$

$$\tau_{xyaa} = -R_{aa} - \sum_{n=1}^{\infty} \alpha_n X'_{naa}\cos(\alpha_n y) - \sum_{n=1}^{\infty} \beta_n Y'_{naa}\cos(\beta_n x).$$

(A-4)

In all of the above expressions,

$$X' = \frac{d}{dx} X$$

$$X'' = \frac{d^2}{dx^2} X$$

$$Y' = \frac{d}{dy} Y$$

$$Y'' = \frac{d^2}{dy^2} Y$$

The X and Y Terms are dependent on material properties and are defined in Equations 16-18.

APPENDIX B

FOURIER EXPANSIONS OF BOUNDARY CONDITIONS

The boundary conditions of Equation 6 can be expanded in general Fourier series. These expansions are given below.

$$\sigma_x(\ell,y) = \frac{1}{2t}C_{1_0} + \sum_{n=1}^{\infty} \frac{1}{t} C_{1_n}\left[\cos(\alpha_n y) + S_{1_n}\sin(\alpha_n y)\right]$$

$$C_{1_0} = \int_{-t}^{t} r_1(y)dy$$

$$C_{1_n} = \int_{-t}^{t} r_1(y)\cos(\alpha_n y)dy$$

$$S_{1_n} = \int_{-t}^{t} r_1(y)\sin(\alpha_n y)dy$$

$$\sigma_x(-\ell,y) = \frac{1}{2t} C_{2_0} + \sum_{n=1}^{\infty} \frac{1}{t} \left[C_{2_n}\cos(\alpha_n y) + S_{1_n}\sin(\alpha_n y)\right]$$

$$C_{2_0} = \int_{-t}^{t} r_2(y)dy$$

$$C_{2_n} = \int_{-t}^{t} r_2(y)\cos(\alpha_n y)dy$$

$$S_{2_n} = \int_{-t}^{t} r_2(y)\sin(\alpha_n y)dy$$

$$\sigma_y(x,t) = \frac{1}{2\ell} C_{3_0} + \sum_{n=1}^{\infty} \frac{1}{\ell} \left[C_{3_n}\cos(\beta_n x) + S_{3_n}\sin(\beta_n x)\right]$$

$$C_{3_0} = \int_{-\ell}^{\ell} r_3(x)dx$$

$$C_{3_n} = \int_{-\ell}^{\ell} r_3(x)\cos(\beta_n x)dx$$

$$S_{3_n} = \int_{-\ell}^{\ell} r_3(x)\sin(\beta_n x)dx$$

$$\sigma_y(x,-t) = \frac{1}{2\ell} C_{4_\circ} + \sum_{n=1}^{\infty} \frac{1}{\ell} \left[C_{4_n}\cos(\beta_n x) + S_{4_n}\sin(\beta_n x) \right]$$

$$C_{4_\circ} = \int_{-\ell}^{\ell} r_4(x)dx$$

$$C_{4_n} = \int_{-\ell}^{\ell} r_4(x)\cos(\beta_n x)dx$$

$$S_{4_n} = \int_{-\ell}^{\ell} r_4(x)\sin(\beta_n x)dx$$

$$\tau_{xy}(\ell,y) = \frac{1}{2t} C_{5_\circ} + \sum_{n=1}^{\infty} \frac{1}{t} \left[C_{5_n}\cos(\alpha_n y) + S_{5_n}\sin(\alpha_n y) \right]$$

$$C_{5_\circ} = \int_{-t}^{t} r_5(y)dy$$

$$C_{5_n} = \int_{-t}^{t} r_5(y)\cos(\alpha_n y)dy$$

$$S_{5_n} = \int_{-t}^{t} r_5(y)\sin(\alpha_n y)dy$$

$$\tau_{xy}(-\ell,y) = \frac{1}{2t} \left[C_{6_\circ} + \sum_{n=1}^{\infty} \frac{1}{t} C_{6_n}\cos(\alpha_n y) + S_{6_n}\sin(\alpha_n y) \right]$$

$$C_{6_\circ} = \int_{-t}^{t} r_6(y)dy$$

$$C_{6_n} = \int_{-t}^{t} r_6(y)\cos(\alpha_n y)dy$$

$$S_{6_n} = \int_{-t}^{t} r_6(y)\sin(\alpha_n y)dy$$

$$\tau_{xy}(x,t) = \frac{1}{2\ell} C_{7_o} + \sum_{n=1}^{\infty} \frac{1}{\ell} \left[C_{7_n} \cos(\beta_n x) + S_{7_n} \sin(\beta_n x) \right]$$

$$C_{7_o} = \int_{-\ell}^{\ell} r_7(x) dx$$

$$C_{7_n} = \int_{-\ell}^{\ell} r_7(x) \cos(\beta_n x) dx$$

$$S_{7_n} = \int_{-\ell}^{\ell} r_7(x) \sin(\beta_n x) dx$$

$$\tau_{xy}(x,-t) = \frac{1}{2\ell} C_{8_o} + \sum_{n=1}^{\infty} \frac{1}{\ell} \left[C_{8_n} \cos(\beta_n x) + S_{8_n} \sin(\beta_n x) \right]$$

$$C_{8_o} = \int_{-\ell}^{\ell} r_8(x) dx$$

$$C_{8_n} = \int_{-\ell}^{\ell} r_8(x) \cos(\beta_n x) dx$$

$$S_{8_n} = \int_{-\ell}^{\ell} r_8(x) \sin(\beta_n x) dx$$

Due to linearity, expressions of the form

$$\sigma_x(\ell,y) + \sigma_x(-\ell,y)$$

$$= \frac{1}{2t} (C_{1_o} + C_{2_o}) + \sum_{n=1}^{\infty} \frac{1}{t} \left[(C_{1_n} + C_{2_n}) \cos(\alpha_n y) + (S_{1_n} + S_{2_n}) \sin(\alpha_n y) \right]$$

and

$$\sigma_x(\ell,y) - \sigma_x(-\ell,y)$$

$$= \frac{1}{2t} (C_{1_o} - C_{2_o}) + \sum_{n=1}^{\infty} \frac{1}{t} \left[(C_{1_n} - C_{2_n}) \cos(\alpha_n y) + (S_{1_n} - S_{2_n}) \sin(\alpha_n y) \right]$$

are valid. These forms are also useful in the solution technique of this
paper, so the following shorthand expressions are introduced:

$$C_{1+2_n} = C_{1_n} + C_{2_n} \qquad\qquad C_{1-2_n} = C_{1_n} - C_{2_n}$$

$$S_{1+2_n} = S_{1_n} + S_{2_n} \qquad\qquad S_{1-2_n} = S_{1_n} - S_{2_n}$$

$$C_{3+4_n} = C_{3_n} + C_{4_n} \qquad\qquad C_{3-4_n} = C_{3_n} - C_{4_n}$$

$$S_{3+4_n} = S_{3_n} + S_{4_n} \qquad\qquad S_{3-4_n} = S_{3_n} - S_{4_n}$$

$$C_{5+6_n} = C_{5_n} + C_{6_n} \qquad\qquad C_{5-6_n} = C_{5_n} - C_{6_n}$$

$$S_{5+6_n} = S_{5_n} + S_{6_n} \qquad\qquad S_{5-6_n} = S_{5_n} - S_{6_n}$$

$$C_{7+8_n} = C_{7_n} + C_{8_n} \qquad\qquad C_{7-8_n} = C_{7_n} - C_{8_n}$$

$$S_{7+8_n} = S_{7_n} + S_{8_n} \qquad\qquad S_{7-8_n} = S_{7_n} - S_{8_n}$$

Using these,

$$\sigma_x(\ell,y) + \sigma_x(-\ell,y)$$

$$= \frac{1}{2t} C_{1+2_0} + \sum_{n=1}^{\infty} \frac{1}{t} \left[C_{1+2_n} \cos(\beta_n y) + S_{1+2_n} \sin(\beta_n y) \right] ,$$

$$\tau_{xy}(x,t) - \tau_{xy}(x,-t)$$

$$= \frac{1}{2\ell} C_{7-8_0} + \sum_{n=1}^{\infty} \frac{1}{t} \left[C_{7-8_n} \cos(\beta_n x) + S_{7-8_n} \sin(\beta_n x) \right] ,$$

and so forth.

APPENDIX C

EXPRESSIONS FOR SOLUTION TERMS

The matrix expressions given by Equations 23 and 25, along with the explicit expressions for solution terms as in Equation 24, are the backbone of the solution technique presented in this paper. Each of the three material cases provides a distinct set of terms for Equation 23 as well as distinct sets of equations with the forms of Equations 24, 27, 28 and 29. The terms and equations corresponding to material cases (1) and (2) are given below:

Case 1: Roots Real and in Pairs ($\pm u$, $\pm v$; $u, v > 0$; $u \neq v$)

Symmetric x, symmetric y (SS):

$$SS1_p + (E1A_p) \sum_{n=1}^{\infty} (E1B_{pn}) SS2_n = (E1A_p)\{(E1C_p) + \sum_{n=1}^{\infty} (E1D_{pn})\}$$

$$(F1A_p) \sum_{n=1}^{\infty} (F1B_{pn}) SS1_n + SS2_p = (F1A_p)\{(F1C_p) + \sum_{n=1}^{\infty} (F1D_{pn})\}$$

where

$$E1A_p = \frac{1}{2\alpha_p^2 t \left[1 - \frac{u\tanh(u\alpha_p \ell)}{v\tanh(v\alpha_p \ell)}\right]}$$

$$E1B_{pn} = -4(-1)^{n+p}\beta_n^3 \left[\frac{u^3}{[(u\beta_n)^2 + \alpha_p^2]} - \frac{uv^2}{[(v\beta_n)^2 + \alpha_p^2]}\right] \tanh(u\beta_n t)$$

$$E1C_p = -\left[\frac{S_{5-6p}}{v\tanh(v\alpha_p \ell)} + C_{1+2p}\right]$$

$$E1D_{pn} = 2(-1)^{n+p}\beta_n \frac{v^2}{\ell[(v\beta_n\ell)^2 + \alpha_p^2]} S_{7-8n}$$

$$F1A_p = \frac{1}{2\beta_p^2\ell\left[1 - \dfrac{utanh(u\beta_p t)}{vtanh(u\beta_p t)}\right]}$$

$$F1B_{pn} = -4(-1)^{n+p}\alpha_n^3 \left[\frac{u^3}{[(u\alpha_n)^2 + \beta_p^2]} - \frac{uv^2}{[(v\alpha_n)^2 + \beta_p^2]}\right] tanh(u\alpha_n\ell)$$

$$F1C_p = -\left[\frac{S_{7-8p}}{vtanh(v\beta_p t)} + C_{3+4p}\right]$$

$$F1D_{pn} = 2(-1)^{n+p}\alpha_n \frac{v^2}{t[(v\alpha_n)^2 + \beta_p^2]} S_{5-6n}$$

Then, using these coefficients,

$$SS3_p = \frac{1}{vtanh(v\alpha_p\ell)}\left[\frac{S_{5-6p}}{2\alpha_p^2 t} - SS1_p utanh(u\alpha_p\ell)\right]$$

$$SS4_p = \frac{1}{vtanh(v\beta_p t)}\left[\frac{S_{7-8p}}{2\beta_p^2\ell} - SS2_p utanh(u\beta_p t)\right]$$

$$R_{xss} = \frac{1}{4t} C_{1+2_\circ} - \sum_{n=1}^{\infty} (-1)^n \frac{\beta_n}{t}[SS2_n utanh(u\beta_n t) + SS4_n vtanh(v\beta_n t)]$$

$$R_{yss} = \frac{1}{4\ell} C_{3+4_\circ} - \sum_{n=1}^{\infty} (-1)^n \frac{\alpha_n}{\ell}[SS1_n utanh(u\alpha_n\ell) + SS3_n vtanh(v\alpha_n\ell)] .$$

Symmetric x, antisymmetric y (SA):

$$SA1_p + (E2A_p) \sum_{n=1}^{\infty} (E2B_{pn})SA2_n = (E2A_p) \{(E2C_p) + \sum_{n=1}^{\infty} (E2D_{pn})\}$$

$$(F2A_p) \sum_{n=1}^{\infty} (F2B_{pn})SA1_n + SA2_p = (F2A_p) \{(F2C_p) + \sum_{n=1}^{\infty} (F2D_{pn})\}$$

where

$$E2A_p = \frac{1}{2\alpha_p^2 t \left[1 - \frac{u\tanh(u\alpha_p \ell)}{v\tanh(v\alpha_p \ell)}\right]}$$

$$E2B_{pn} = 4(-1)^{p+n}\alpha_p \left[\frac{(u\beta_n)^2}{[(u\beta_n)^2 + \alpha_p^2]} - \frac{(v\beta_n)^2}{[(v\beta_n)^2 + \alpha_p^2]}\right] \tanh(u\beta_n t)$$

$$E2C_p = \frac{C_{5-6p}}{v\tanh(v\alpha_p \ell)} - S_{1+2p}$$

$$E2D_{pn} = 4(-1)^{p+n}\alpha_p \frac{(v\beta_n)^2}{[(v\beta_n)^2 + \alpha_p^2]} \frac{C_{3-4n}}{2\beta_n^2 \ell}$$

$$F2A_p = \frac{1}{2\beta_p^2 \ell \left[u - v\frac{\tanh(u\beta_p t)}{\tanh(v\beta_p t)}\right]}$$

$$F2B_{pn} = 4(-1)^{p+n}u\alpha_n^2\beta_p\tanh(u\alpha_n \ell)$$

$$\left[\frac{1}{[(u\alpha_n)^2 + \beta_p^2]} - \frac{1}{[(v\alpha_n)^2 + \beta_p^2]}\right]$$

$$F2C_p = S_{7+8p} + \frac{v}{\tanh(v\beta_p t)} C_{3-4p} - (-1)^p \frac{C_{3-4o}}{\beta_p t}$$

$$F2D_{pn} = 2(-1)^{p+n} \frac{\beta_p}{t[(v\alpha_n)^2 + \beta_p^2]} C_{5-6n}$$

Further SA relations are

$$SA3_p = \frac{-1}{v\tanh(v\alpha_p \ell)} \left[\frac{C_{5-6p}}{2\alpha_p^2 t} + SA1_p u\tanh(u\alpha_p \ell)\right]$$

$$SA4_p = - SA2_p \left[\frac{\tanh(u\beta_p t)}{\tanh(v\beta_p t)} + \frac{C_{3-4p}}{2\beta_p^2 \ell \tanh(v\beta_p t)} \right]$$

$$R_{ysa} = \frac{C_{3-4o}}{4\ell t}$$

Antisymmetric x, Symmetric y (AS):

$$AS1_p + (E3A_p) \sum_{n=1}^{\infty} (E3B_{pn})AS2_n = (E3A_p) \{(E3C_p) + \sum_{n=1}^{\infty} (E3D_{pn})\}$$

$$(F3A_p) \sum_{n=1}^{\infty} (F3B_{pn})AS1_n + AS2_p = (F3A_p) \{(F3C_p) + \sum_{n=1}^{\infty} (F3D_{pn})\}$$

where

$$E3A_p = \frac{1}{2\alpha_p^2 t \left[u - v \dfrac{\tanh(u\alpha_p \ell)}{\tanh(v\alpha_p \ell)} \right]}$$

$$E3B_{pn} = 4(-1)^{p+n} u\beta_n^2 \alpha_p \tanh(u\beta_n t) \left[\frac{1}{[(u\beta_n)^2 + \alpha_p^2]} - \frac{1}{[(v\beta_n)^2 + \alpha_p^2]} \right]$$

$$E3C_p = S_{5+6p} + \frac{v}{\tanh(v\alpha_p \ell)} C_{1-2p} - (-1)^p \frac{C_{1-2o}}{\alpha_p \ell}$$

$$E3D_{pn} = 2(-1)^{p+n} \frac{\alpha_p}{\ell[(v\beta_n)^2 + \alpha_p^2]} C_{7-8n}$$

$$F3A_p = \frac{1}{2\beta_p^2 \ell \left[1 - \dfrac{u\tanh(u\beta_p t)}{v\tanh(v\beta_p t)} \right]}$$

$$F3B_{pn} = 4(-1)^{p+n} \beta_p \left[\frac{(u\alpha_n)^2}{[(u\alpha_n)^2 + \beta_p^2]} - \frac{(v\alpha_n)^2}{[(v\alpha_n)^2 + \beta_p^2]} \right] \tanh(u\alpha_n \ell)$$

$$F3C_p = \frac{C_{7-8p}}{v\tanh(v\beta_p t)} - S_{3+4p}$$

$$F3D_{pn} = 2(-1)^{p+n}\beta_p \frac{(v\alpha_n)^2}{[(v\alpha_n)^2 + \beta_p^2]} \frac{C_{1-2n}}{\alpha_n^2 t} \ .$$

Also,

$$AS3_p = \frac{-1}{\tanh(v\alpha_p \ell)} \left[AS1_p \tanh(u\alpha_p \ell) + \frac{C_{1-2p}}{2\alpha_p^2 t} \right]$$

$$AS4_p = \frac{-1}{v\tanh(v\beta_p t)} \left[\frac{C_{7-8p}}{2\beta_p^2 \ell} + AS2_p u\tanh(u\beta_p t) \right]$$

$$R_{xas} = \frac{C_{1-2o}}{4\ell t} \ .$$

Antisymmetric x, antisymmetric y (AA):

$$AA1_p + (E4A_p) \sum_{n=1}^{\infty} (E4B_{pn})AA2_n = (EA4_p) \{(E4C_p) + \sum_{n=1}^{\infty} (E4D_{pn})\}$$

$$(F4A_p) \sum_{n=1}^{\infty} (F4B_{pn})AA1_n + AA2_p = (F4A_p) \{(F4C_p) + \sum_{n=1}^{\infty} (F4D_{pn})\}$$

where

$$E4A_p = \frac{1}{2\alpha_p^2 t \left[u - v \frac{\tanh(u\alpha_p \ell)}{\tanh(v\alpha_p \ell)} \right]}$$

$$E4B_{pn} = 4(-1)^{p+n}\beta_n^3 \left[\frac{u^2}{[(u\beta_n)^2 + \alpha_p^2]} - \frac{v^2}{[(v\beta_n)^2 + \alpha_p^2]} \right] \tanh(u\beta_n t)$$

$$E4C_p = \frac{v}{\tanh(v\alpha_p \ell)} S_{1-2p} - C_{5+6p}$$

$$E4D_{pn} = 2(-1)^{p+n} \frac{(v\beta_n)^2}{\beta_n \ell [(v\beta_n)^2 + \alpha_p^2]} S_{3-4n}$$

$$F4A_p = \frac{1}{2\beta_p^2 \ell \left[u - v \frac{\tanh(u\beta_p t)}{\tanh(v\beta_p t)} \right]}$$

$$F4B_{pn} = 4(-1)^{p+n}\alpha_n^3 \left[\frac{u^2}{[(u\alpha_n)^2 + \beta_p^2]} - \frac{v^2}{[(v\alpha_n)^2 + \beta_p^2]} \right] \tanh(u\alpha_n \ell)$$

$$F4C_p = \frac{v}{\tanh(v\beta_p t)} S_{3-4p} - C_{7+8p}$$

$$F4D_{pn} = 2(-1)^{p+n} \frac{(v\alpha_n)^2}{\alpha_n t [(v\alpha_n)^2 + \beta_p^2]} S_{1-2n} .$$

Also,

$$AA3_p = \frac{-1}{\tanh(v\alpha_p \ell)} \left[AA1_p \tanh(u\alpha_p \ell) + \frac{S_{1-2p}}{2\alpha_p^2 t} \right]$$

$$AA4_p = \frac{-1}{\tanh(v\beta_p t)} \left[AA2_p \tanh(u\beta_p t) + \frac{S_{3-4p}}{2\beta_p^2 \ell} \right]$$

$$R_{aa} = -\frac{C_{5+6o}}{4t} - \sum_{n=1}^{\infty} (-1)^n \frac{\beta_n}{t} [AA2_n \tanh(u\beta_n t) + AA4_n \tanh(v\beta_n t)] .$$

Case 2: Roots Real and Equal in Pairs (±u, v > 0)

Symmetric x, symmetric y (SS):

$$SS1_p + (E1A_p) \sum_{n=1}^{\infty} (E1B_{pn})SS2_n = (E1A_p) \{(E1C_p) + \sum_{n=1}^{\infty} (E1D_{pn})\}$$

$$(F1A_p) \sum_{n=1}^{\infty} (F1B_{pn})SS1_n + SS2_p = (F1A_p) \{(F1C_p) + \sum_{n=1}^{\infty} (F1D_{pn})\}$$

where

$$E1A_p = \cfrac{1}{2\alpha_p^2 t \left[1 - \cfrac{u\alpha_p \ell \tanh^2 u\alpha_p \ell}{\tanh(u\alpha_p \ell) + u\alpha_p \ell} \right]}$$

$$E1B_{pn} = 4(-1)^{p+n} \cfrac{(u\beta_n)^3}{[(u\beta_n)^2 + \alpha_p^2]} \tanh(u\beta_n t) \left[\cfrac{E11_{pn}}{[\tanh(u\beta_n t) + u\beta_n t]} - 1 \right]$$

$$E1C_p = - \left[C_{1+2p} - \cfrac{\alpha_p \ell \tanh(u\alpha_p \ell)}{[\tanh(u\alpha_p \ell) + u\alpha_p \ell]} S_{5-6p} \right]$$

$$E1D_{pn} = 2(-1)^{p+n} \cfrac{(u\beta_n)^3}{[(u\beta_n)^2 + \alpha_p^2]} \cfrac{E11_{pn}}{u\beta_n^2 \ell [\tanh(u\beta_n t) + u\beta_n t]} S_{7-8n}$$

$$E11_{pn} = \left(2 - \cfrac{[(u\beta_n)^2 - \alpha_p^2]}{[(u\beta_n)^2 + \alpha_p^2]} \right) \tanh(u\beta_n t) + u\beta_n t$$

$$F1A_p = \cfrac{1}{2\beta_p^2 \ell \left[1 - \cfrac{u\beta_n t \tanh^2 u\beta_p}{\tanh(u\beta_p t) + u\beta_p t} \right]}$$

$$F1B_{pn} = 4(-1)^{p+n} \cfrac{(u\alpha_n)^3}{[(u\alpha_n)^2 + \beta_p^2]} \tanh(u\alpha_n \ell) \left[\cfrac{F11_{pn}}{\tanh(u\alpha_n \ell) + u\alpha_n \ell} - 1 \right]$$

$$F1C_p = - \left[C_{3+4p} - \cfrac{\beta_p t \tanh(u\beta_p t)}{[\tanh(u\beta_p t) + u\beta_p t]} S_{7-8p} \right]$$

$$F1D_{pn} = 2(-1)^{p+n} \cfrac{(u\alpha_n)^3}{[(u\alpha_n)^2 + \beta_p^2]} \cfrac{F11_{pn}}{u\alpha_n^2 t [\tanh(u\alpha_n \ell) + u\alpha_n \ell]} S_{5-6n}$$

$$F11_{pn} = \left(2 - \frac{[(u\alpha_n)^2 - \beta_p^2]}{[(u\alpha_n)^2 + \beta_p^2]}\right) \tanh(u\alpha_n \ell) + u\alpha_n \ell$$

Also,

$$SS3_p = \frac{1}{\tanh(u\alpha_p \ell) + u\alpha_p \ell} \left[\frac{1}{2u\alpha_p^2 t} S_{5-6p} - SS1_p \tanh(u\alpha_p \ell)\right]$$

$$SS4_p = \frac{1}{\tanh(u\beta_p t) + u\beta_p t} \left[\frac{1}{2u\beta_p^2 \ell} S_{7-8p} - SS2_p \tanh(u\beta_p t)\right]$$

$$R_{xss} = \frac{1}{4t} C_{1+2o}$$

$$- \sum_{n=1}^{\infty} (-1)^n \frac{{}^{\prime\prime}\beta_n}{t} [SS2_n \tanh(u\beta_n t)$$

$$+ SS4_n \{\tanh(u\beta_n t) + u\beta_n t\}]$$

$$R_{yss} = \frac{1}{4\ell} C_{3+4o}$$

$$- \sum_{n=1}^{\infty} (-1)^n \frac{u\alpha_n}{\ell} [SS1_n \tanh(u\alpha_n \ell)$$

$$+ SS3_n \{\tanh(u\alpha_n \ell) + u\alpha_n \ell\}] .$$

Symmetric x, antisymmetric y (SA):

$$SA1_p + (E2A_p) \sum_{n=1}^{\infty} (E2B_{pn})SA2_n = (E2A_p) \{(E2C_p) + \sum_{n=1}^{\infty} (E2D_{pn})\}$$

$$(F2A_p) \sum_{n=1}^{\infty} (F2B_{pn})SA1_n + SA2_p = (F2A_p) \{(F2C_p) + \sum_{n=1}^{\infty} (F2D_{pn})\}$$

where

$$E2A_p = \cfrac{1}{2\alpha_p^2 t \left[1 - \cfrac{u\alpha_p \ell \ \tanh^2(u\alpha_p \ell)}{[\tanh(u\alpha_p \ell) + u\alpha_p \ell]} \right]}$$

$$E2B_{pn} = 4 \ E21_{pn} \ \tanh(u\beta_n t)$$

$$E2C_p = \frac{\alpha_p \ell \ \tanh(u\alpha_p \ell)}{[\tanh(u\alpha_p \ell) + u\alpha_p \ell]} C_{5-6p} - S_{1+2p}$$

$$E2D_{pn} = 2 \left[(-1)^{p+n} \frac{(u\beta_n)^2 \alpha_p}{[(u\beta_n)^2 + \alpha_p^2]} - E21_{pn} \right] \frac{C_{3-4n}}{\beta_n^2 \ell}$$

$$E21_{pn} = (-1)^{p+n} \frac{(u\beta_n)^2 \alpha_p}{[(u\beta_n)^2 + \alpha_p^2]} \ \frac{2}{u\beta_n t} \left(\frac{(u\beta_n)^2}{[(u\beta_n)^2 + \alpha_p^2]} - 1 \right) \ \tanh(u\beta_n t)$$

$$F2A_p = \frac{t}{2\beta_p \ell \{u\beta_p t - [1 + u\beta_p t \ \tanh(u\beta_p t)] \tanh(u\beta_p t)\}}$$

$$F2B_{pn} = 4(-1)^{p+n} \frac{u\alpha_n^2 \beta_p}{[(u\alpha_n)^2 + \beta_p^2]} \ \tanh(u\alpha_n \ell) \left[1 - \frac{F21_{pn}}{[\tanh(u\alpha_n \ell) + u\alpha_n \ell]} \right]$$

$$F2C_p = S_{7+8p} + \frac{(-1)^p}{\beta_p t} C_{5-6o} + \frac{[1 + u\beta_p t \ \tanh(u\beta_p t)]}{\beta_p t} C_{3-4p}$$

$$F2D_{pn} = 2 \ (-1)^{p+n} \frac{\beta_p}{t[(u\alpha_n)^2 + \alpha_p^2]} \ \frac{F21_{pn}}{[\tanh(u\alpha_n \ell) + u\alpha_n \ell]} \ C_{5-6n}$$

$$F21_{pn} = \tanh(u\alpha_n \ell) - \frac{2(u\alpha_n)^2}{[(u\alpha_n)^2 + \beta_p^2]} \tanh(u\alpha_n \ell) + u\alpha_n \ell \ .$$

Also,

$$SA3_p = \frac{-1}{[\tanh(u\alpha_p \ell) + u\alpha_p \ell]} \left[SA1_p \tanh(u\alpha_p \ell) + \frac{C_{5-6p}}{2u\alpha_p^2 t} \right]$$

$$SA4_p = \frac{-1}{u\beta_p t} \left[SA2_p \tanh(u\beta_p t) + \frac{C_{3-4p}}{2\beta_p^2 \ell} \right]$$

$$R_{ysa} = \frac{C_{3-4o}}{4\ell t} \ .$$

Antisymmetric x, symmetric y (AS):

$$AS1_p + (E3A_p) \sum_{n=1}^{\infty} (E3B_{pn})AS2_n = (E3A_p) \{(E3C_p) + \sum_{n=1}^{\infty} (E3D_{pn})\}$$

$$(F3A_p) \sum_{n=1}^{\infty} (F3B_{pn})AS1_n + AS2_p = (F3A_p) \{(F3C_p) + \sum_{n=1}^{\infty} F3D_{pn})\}$$

where

$$E3A_p = \frac{\ell}{2\alpha_p t\{u\alpha_p \ell - [1 + u\alpha_p \ell \ \tanh(u\alpha_p \ell)]\tanh(u\alpha_p \ell)\}}$$

$$E3B_{pn} = 4(-1)^{p+n} \frac{u\beta_n^2 \alpha_p}{[(u\beta_n)^2 + \alpha_p^2]} \tanh(u\beta_n t) \left[1 - \frac{E31_{pn}}{[\tanh(u\beta_p t) + u\beta_n t]} \right]$$

$$E3C_p = S_{5+6p} + \frac{(-1)^p}{\alpha_p \ell} C_{7-8o} + \frac{1 + u\alpha_p \ell \ \tanh(u\alpha_p \ell)}{\alpha_p \ell} C_{1-2p}$$

$$E30_{pn} = 2(-1)^{p+n} \frac{\alpha_p}{\ell[(u\beta_n)^2 + \alpha_p^2]} \frac{E31_{pn}}{[\tanh(u\beta_n t) + u\beta_n t]} C_{7-8n}$$

$$E31_{pn} = \tanh(u\beta_n t) - \frac{2(u\beta_n)^2}{[(u\beta_n)^2 + \alpha_p^2]} \tanh(u\beta_n t) + u\beta_n t$$

$$F3A_p = \frac{1}{2\beta_p^2\ell \left[1 - \frac{u\beta_p t \tanh^2(u\beta_p t)}{[\tanh(u\beta_p t) + u\beta_p t]}\right]}$$

$$F3B_{pn} = 4 F31_{pn} \tanh(u\alpha_n \ell)$$

$$F3C_p = \frac{\beta_p t \tanh(u\beta_p t)}{[\tanh(u\beta_p t) + u\beta_p t]} C_{7-8p} - S_{3+4p}$$

$$F3D_{pn} = \frac{2}{\alpha_n^2 t} \left[(-1)^{p+n} \frac{(u\alpha_n)^2 \beta_p}{[(u\alpha_n)^2 + \beta_p^2]} - F31_{pn}\right] C_{1-2n}$$

$$F31_{pn} = 2(-1)^{p+n} \frac{u\alpha_n \beta_p}{\ell[(u\alpha_n)^2 + \beta_p^2]} \left[\frac{(u\alpha_n)^2}{[(u\alpha_n)^2 + \beta_p^2]} - 1\right] \tanh(u\alpha_n \ell)$$

Also,

$$AS3_p = \frac{-1}{u\alpha_p \ell} \left[AS1_p \tanh(u\alpha_p \ell) + \frac{C_{1-2p}}{2\alpha_p^2 t}\right]$$

$$AS4_p = \frac{-1}{\tanh(u\beta_p t) + u\beta_p t} \left[AS2_p \tanh(u\beta_p t) + \frac{C_{7-8p}}{2u\beta_p^2 \ell}\right]$$

$$R_{xas} = \frac{C_{1-2o}}{4\ell t}.$$

Antisymmetric x, antisymmetric y (AA):

$$AA1_p + (E4A_p) \sum_{n=1}^{\infty} (E4B_{pn})AA2_n = (E4A_p) \{(E4C_p) + \sum_{n=1}^{\infty} (E4D_{pn})\}$$

$$(F4A_p) \sum_{n=1}^{\infty} (F4B_{pn})AA1_n + AA2_p = (F4A_p) \{(F4C_p) + \sum_{n=1}^{\infty} (F4D_{pn})\}$$

where

$$E4A_p = \frac{1}{2u\alpha_p^2 t \left[1 - \frac{(1 + u\alpha_p \ell \, \tanh(u\alpha_p \ell))}{u\alpha_p \ell} \tanh(u\alpha_p \ell) \right]}$$

$$E4B_{pn} = 4(-1)^{p+n} \frac{(u\beta_n)^2}{[(u\beta_n)^2 + \alpha_p^2]} \beta_n \tanh(u\beta_n t) \quad 1 - \frac{E41_{pn}}{u\beta_n t}$$

$$E4C_p = \frac{1 + u\alpha_p \ell \, \tanh(u\alpha_p \ell)}{\alpha_p \ell} S_{1-2p} - C_{5+6p}$$

$$E4D_{pn} = 2(-1)^{p+n} \frac{(u\beta_n)^2}{[(u\beta_n)^2 + \alpha_p^2]} \frac{E41_{pn}}{u\beta_n^2 \ell t} S_{3-4n}$$

$$E41_{pn} = u\beta_n t + \left[1 - \frac{[(u\beta_n)^2 - \alpha_p^2]}{[(u\beta_n)^2 + \alpha_p^2]} \right] \tanh(u\beta_n t)$$

$$F4A_p = \frac{1}{2u\beta_p^2 \ell \left[1 - \frac{[1 + u\beta_p t \, \tanh(u\beta_p t)]}{u\beta_p t} \tanh(u\beta_p t) \right]}$$

$$F4B_{pn} = 4(-1)^{p+n} \frac{(u\alpha_n)^2 \alpha_n}{[(u\alpha_n)^2 + \beta_p^2]} \tanh(u\alpha_n \ell) \left[1 - \frac{F41_{pn}}{u\alpha_n \ell} \right]$$

$$F4C_p = \frac{1 + u\beta_p t\ \tanh(u\beta_p t)}{u\beta_p t}\ S_{3-4p} - C_{7+8p}$$

$$F4D_{pn} = 2(-1)^{p+n}\ \frac{(u\alpha_n)^2}{[(u\alpha_n)^2 + \beta_p^2]}\ \frac{F41_{pn}}{u\alpha_n^2 \ell t}\ S_{1-2n}$$

$$F41_{pn} = u\alpha_n \ell + \left[1 - \frac{[(u\alpha_n)^2 - \beta_p^2]}{[(u\alpha_n)^2 + \beta_p^2]}\right]\ \tanh(u\alpha_n \ell)$$

Also,

$$AA3_p = \frac{-1}{u\alpha_p \ell}\ \left[AA1_p\ \tanh(u\alpha_p \ell) + \frac{S_{1-2p}}{2\alpha_p^2 t}\right]$$

$$AA4_p = \frac{-1}{u\beta_p t}\ \left[AA2_p\ \tanh(u\beta_p t) + \frac{S_{3-4p}}{2\beta_p^2 \ell}\right]$$

$$R_{aa} = -\frac{C_{5+6o}}{4t} - \sum_{n=1}^{\infty} (-1)^n\ \frac{\beta_n}{t}\ [AA2_n\ \tanh(u\beta_n t) + AA4_n\ u\beta_n t]\ .$$

APPENDIX D

INCLUSION OF DISPLACEMENT BOUNDARY CONDITIONS IN THE ANALYSIS

To include displacement boundary conditions in the exact solution analysis, the problem is reformulated using the displacements u (for the x-direction) and v (for the y-direction) as the variables of interest. All of the assertions made earlier as to the character of the problem (plane stress, rectangular slab, etc.) are still considered to hold true.

At this point, the normal and shear strains are defined by the relations

$$\varepsilon_x = u,_x \ , \quad \varepsilon_y = v,_y \ , \quad \gamma_{xy} = u,_y + v,_x \ . \tag{D-1}$$

Also, the stiffness version of Hooke's Law is given as

$$\sigma_x = Q_{11}\varepsilon_x + Q_{12}\varepsilon_y + Q_{16}\gamma_{xy}$$

$$\sigma_y = Q_{12}\varepsilon_x + Q_{22}\varepsilon_y + Q_{26}\gamma_{xy} \tag{D-2}$$

$$\tau_{xy} = Q_{16}\varepsilon_x + Q_{26}\varepsilon_y + Q_{66}\gamma_{xy} \ ,$$

where the stiffnesses Q_{ij} are as defined in Jones [10]. Specializing the problem to orthotropic materials having their principal axes aligned with the x- and y- directions of the slab makes

$$Q_{16} = Q_{26} = 0 \ .$$

The equilibrium equations then become

$$Q_{11} \, u,_{xx} + Q_{66} \, u,_{yy} + (Q_{12} + Q_{66}) \, v,_{xy} = 0 \qquad (D-3)$$

$$Q_{66} \, v,_{xx} + Q_{22} \, v,_{yy} + (Q_{12} + Q_{66}) \, u,_{xy} = 0 \; . \qquad (D-4)$$

Differentiating equation D-3 with respect to x gives

$$Q_{11} \, u,_{xxx} + Q_{66} \, u,_{xyy} + (Q_{12} + Q_{66}) \, v,_{xxy} = 0 \; . \qquad (D-5)$$

Differentiating equation D-4 with respect to y gives

$$Q_{66} \, v,_{xxy} + Q_{22} \, v,_{yyy} + (Q_{12} + Q_{66}) \, u,_{xyy} = 0$$

or

$$v,_{xxy} = - \frac{Q_{22}}{Q_{66}} \, v,_{yyy} - \frac{(Q_{12} + Q_{66})}{Q_{66}} \, u,_{xyy} \; . \qquad (D-6)$$

Combining equations D-5 and D-6 gives

$$Q_{11} \, u,_{xxx} + Q_{66} \, u,_{xyy}$$
$$- \frac{(Q_{12} + Q_{66})}{Q_{66}} \, [Q_{22} \, v,_{yyy} + (Q_{12} + Q_{66}) \, u,_{xyy}] = 0. \qquad (D-7)$$

Differentiating equation D-3 twice with respect to y and rearranging gives

$$v,_{xyyy} = - \frac{Q_{11}}{(Q_{12} + Q_{66})} \, u,_{xxyy} - \frac{Q_{66}}{(Q_{12} + Q_{66})} \, u,_{yyyy} \; . \qquad (D-8)$$

This can be combined with equation D-7 and manipulated to give

$$u,_{xxxx} + A \, u,_{xxyy} + B \, u,_{yyyy} = 0 \; , \qquad (D-9)$$

where

$$A = \frac{Q_{22}}{Q_{66}} - \frac{Q_{12}^2}{Q_{11}Q_{66}} - \frac{2Q_{12}}{Q_{11}} = E_2\left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1}\right)$$

$$B = \frac{Q_{22}}{Q_{11}} = \frac{E_2}{E_1} .$$

Similar manipulations also yield

$$v,_{xxxx} + A \ v,_{xxyy} + B \ v,_{yyyy} = 0 \tag{D-10}$$

with A and B defined as above.

At this point, the equations D-9 and D-10 have the same form as equation 4, with u and v, respectively, in place of ϕ.

With suitably complete solution forms for u and v, equations D-3 and D-4 can be used to relate the coefficients of one of the solution forms to the coefficients of the other. This done, the solution becomes a problem in u or v alone. Next, any stress boundary conditions must be expressed in terms of displacements. Following the same basic path as the development for stress boundary conditions then leads to the displacement form of the solution.