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DETERMINATION OF EIGENVALUES OF DYNAMICAL SYSTEMS

BY SYMBOLIC COMPUTATION

James C. Howard

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SUMMARY

Problems requiring the determination of eigenvalues occur frequently in the study of dynamical systems. The formulation and processing of the equations describing these systems involve time-consuming matrix operations. Although numerical methods are available, it is more convenient and informative to formulate the relevant matrices and transfer functions symbolically. This can be achieved by the use of symbolic computation, which is a technique whereby algebraic operations, symbolic differentiation, matrix formulation and inversion, etc., can be performed on a digital computer equipped with a formula-manipulation compiler. An example is included that demonstrates the facility with which the system dynamics matrix and the control distribution matrix from the state space formulation of the equations of motion can be processed to obtain eigenvalue loci as a function of a system parameter.

INTRODUCTION

The response of a dynamical system to control inputs may be stable or unstable, depending on the characteristics of the system parameters. An important aspect of stability studies is the determination of the eigenvalues of the system dynamics matrix in the case of a continuous system, and the eigenvalues of the corresponding transition matrix in the case of discrete systems. It is shown that symbolic computation facilitates the determination of the eigenvalues of open-loop systems and the data required to plot stability loci for closed-loop systems.

The locations of the eigenvalues in the complex plane provide the researcher with a clear indication of the stability of the system being considered and the extent to which it is damped. Moreover, an examination of the eigenvalue loci suggests what steps must be taken to ensure a satisfactory response.

The example chosen to demonstrate the technique is a fourth-order system representing the longitudinal response of a DC 8 aircraft to elevator inputs. This simplified system has two dominant modes, one of which is lightly damped and the other well damped. The loci may be used to determine the value of the controlling parameter that satisfies design requirements.

The results were obtained using the MACSYMA Symbolic Manipulation System. This is a large computer-programming system used for performing symbolic as well as numerical mathematical manipulations. It was developed by the Mathlab Group of the MIT Laboratory for Computer Science.

Laplace Transforms and Transfer Functions

State space equations- For a linear, time-invariant dynamical system, which is free from disturbance inputs, the vector-matrix form of the state space equations is (ref. 1)

$$\dot{X} = AX + BU \tag{1}$$

where X is the state vector and U is the input vector. In general, A and B are n×n and n×m matrices, respectively. If the output vector Y is a linear combination of the state vector and the input vector, it may be written as follows:

$$Y = CX + DU \tag{2}$$

Here, C and D are p×n and p×m constant matrices, respectively. The general system described by equations (1) and (2) has n state variables, m inputs, and p outputs. See figure 1.

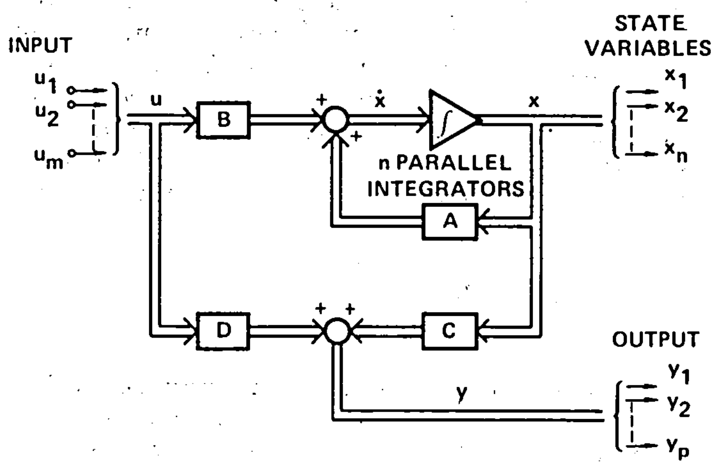

Figure 1.- The state variable formulation for a multi-input, multi-output system.

From equation (1), with zero initial conditions

$$(sI - A)X(s) = BU(s) \tag{3}$$

where X(s) and U(s) are the Laplace transforms of the state vector and the input vector, respectively. The Laplace transform of the state vector is obtained by solving equation (3)

$$X(s) = (sI - A)^{-1}BU(s) \tag{4}$$

From equation (2)

$$Y(s) = CX(s) + DU(s)$$

By assuming that D is a null matrix the output simplifies accordingly, and

$$Y(s) = CX(s) \tag{5}$$

Determination of Eigenvalues in a Practical Case

Open-loop transfer functions- A state space formulation of the equation of motion describing the longitudinal response of a DC 8 aircraft to an elevator input, has the following form (ref. 2).

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -A_0 & -A_1 & -A_2 & -A_3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ K \end{bmatrix} U \tag{6}$$

where U is the elevator displacement. In this case, the output Y is

$$Y = C^T X = (B_0 \quad B_1 \quad 1 \quad 0) \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \tag{7}$$

Before transfer functions for the individual states can be obtained, equation (4) requires that the identity matrix, the Laplace operator s, and the system dynamics matrix A, be combined as shown. This operation is described in terms of reproduced computer output as follows:

First, the system dynamics matrix A is entered

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -A_0 & -A_1 & -A_2 & -A_3 \end{bmatrix} = A \tag{8}$$

Next, the computer provides the fourth-order identity matrix I and multiplies it by s

$$\begin{bmatrix} S & 0 & 0 & 0 \\ 0 & S & 0 & 0 \\ 0 & 0 & S & 0 \\ 0 & 0 & 0 & S \end{bmatrix} = sI \tag{9}$$

When equation (8) is subtracted from equation (9) the matrix (sI - A) is obtained

$$\begin{bmatrix} S & -1 & 0 & 0 \\ 0 & S & -1 & 0 \\ 0 & 0 & S & -1 \\ A_0 & A_1 & A_2 & S + A_3 \end{bmatrix} = sI - A \tag{10}$$

Inversion of this matrix yields the coefficient of BU(s) in equation (4). Because of space limitation, the result is printed out in four columns (ref. 3).

$$\left.\begin{array}{l} \text{Col 1} = \begin{bmatrix} \dfrac{S^3 + A_3S^2 + A_2S + A_1}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\ -\dfrac{A_0}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\ -\dfrac{A_0S}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\ -\dfrac{A_0S^2}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \end{bmatrix} \\ \text{Col 2} = \begin{bmatrix} \dfrac{S^2 + A_3S + A_2}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\ \dfrac{S^3 + A_3S^2 + A_2S}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\ -\dfrac{A_1S + A_0}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\ -\dfrac{A_1S^2 + A_0S}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \end{bmatrix} \end{array}\right\} \tag{11}$$

$$
\left.\begin{aligned}
\text{Col 3} &= \begin{bmatrix} \dfrac{S + A_3}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\[2ex] \dfrac{S^2 + A_3S}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\[2ex] \dfrac{S^3 + A_3S^2}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\[2ex] -\dfrac{A_2S^2 + A_1S + A_0}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \end{bmatrix} \\[3ex]
\text{Col 4} &= \begin{bmatrix} \dfrac{1}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\[2ex] \dfrac{S}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\[2ex] \dfrac{S^2}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\[2ex] \dfrac{S^3}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \end{bmatrix}
\end{aligned}\right\} \quad \begin{matrix}(11)\\ \text{Concluded}\end{matrix}
$$

Before equation (4) can be expanded, the control distribution matrix must be entered.

$$
\begin{bmatrix} 0 \\ 0 \\ 0 \\ K \end{bmatrix} = B \tag{12}
$$

Premultiplication of this vector by $(sI - A)^{-1}$ yields the following column vector:

$$
\begin{bmatrix}
\dfrac{K}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\[2ex]
\dfrac{KS}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\[2ex]
\dfrac{KS^2}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\[2ex]
\dfrac{KS^3}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0}
\end{bmatrix} \tag{13}
$$

If zero initial conditions are assumed, the Laplace transform of the input is $U(s)$.

Multiplication of equation (13) by $U(s)$ gives the Laplace transforms of the states. That is:

$$
X(s) = \begin{bmatrix}
\dfrac{KU(S)}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\[2ex]
\dfrac{KSU(S)}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\[2ex]
\dfrac{KS^2U(S)}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\[2ex]
\dfrac{KS^3U(S)}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0}
\end{bmatrix} \tag{14}
$$

Up to this point the only operations undertaken were addition, subtraction, multiplication, and inversion of matrices. These operations require the following simple program statements.

Addition and subtraction of the matrices A and B: A ± B;

Multiplication: A.B;

Inversion of the matrix A: AΛΛ-1;

Two programming steps are required to obtain the Laplace transforms of the individual states. These are:

```
FOR I:1 THRU 4 DO ROW[I] :FIRST (ROW((D7),I))$
FOR I:1 THRU 4 DO (X[I] :ROW[1][1], DISPLAY (X[1]));
```

The resulting Laplace transforms X_i are

$$
\left.
\begin{aligned}
X_1 &= \frac{KU(S)}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\
X_2 &= \frac{KSU(S)}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\
X_3 &= \frac{KS^2U(S)}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\
X_4 &= \frac{KS^3U(S)}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0}
\end{aligned}
\right\} \qquad (15)
$$

The corresponding transfer functions T_i for the single input case are obtained by using the following program statements:

```
FOR I:1  THRU 4 DO T[I]: (X[I]/U(S))$
FOR I:1  THRU 4 DO DISPLAY(T[I]);
```

$$
\left.
\begin{aligned}
T_1 &= \frac{K}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\
T_2 &= \frac{KS}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\
T_3 &= \frac{KS^2}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} \\
T_4 &= \frac{KS^3}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0}
\end{aligned}
\right\} \qquad (16)
$$

The Laplace transform of the output $Y(s)$ is obtained by expanding equation (7).

$$Y(s) = C^T X(s)$$

where

$$C^T = (B_0 \quad B_1 \quad 1 \quad 0)$$

That is

$$Y(s) = C^T(sI - A)^{-1}BU(s)$$

Substitution of the vector (14) in this equation gives the Laplace transform of the output

$$Y(s) = \frac{(KS^2 + B_1KS + B_0K)U(S)}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0}$$

The corresponding transfer function T(s) is

$$T(s) = \frac{KS^2 + B_1KS + B_0K}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0}$$

Open-loop eigenvalues- The eigenvalues of the open-loop system are the roots of the determinant of the matrix (sI - A), or alternatively, the poles of the open-loop transfer function. Subsequent to the formulation of (sI - A), the determinant is obtained as follows:

```
DETERMINANT(%);
```

$$S(S(S(S + A_3) + A_2) + A_1) + A_0$$

The expanded form of this expression is obtained by using

```
EXPAND(%);
```

$$S^4 + A_3S^3 + A_2S^2 + A_1S + A_0$$

The eigenvalues of the open-loop system are the roots of this equation. These can be evaluated by using the root finding program. However, before the root finding program can be implemented, all the coefficients must be replaced by their numerical values. This is easily accomplished by using a substitution routine. The following are values for the landing approach phase of the motion (also discussed in ref. 2).

$B_0 = 0.0323675$ $\qquad$ $A_0 = 0.07006953$

$B_1 = 0.5955$ $\qquad$ $A_1 = 0.09712526$

$A_2 = 2.6813872$

$A_3 = 1.700522$

Substitution of these values in the determinantal equation yields:

$$S^4 + 1.700522S^3 + 2.6813872S^2 + 0.09712526S + 0.07006953$$

Using the notation %I to denote the complex number $\sqrt{-1}$, the roots are

S = 0.163195707 %I - 9.9571822E-3

S = -0.163195707 %I - 9.9571822E-3

S = 1.38386287 %I - 0.84030382

S = -1.38386287 %I - 0.84030382

These are the eigenvalues of the open-loop system. Since the ranges of the closed-loop eigenvalue loci are determined by the zeros of the transfer functions, these must be determined. The only transfer function of interest in the present application is T(s), the zeros of which are the roots of the equation

$$(s^2 + 0.5955s + 0.0323675) = 0$$

which are

$$s = -0.0604999, \quad s = -0.535000$$

It should be noted that in addition to the two finite zeros, there are two zeros at infinity.

Closed-toop transfer functions- In addition to the plant input, closed-loop systems have a reference or system input $R(t)$. See figure 2.

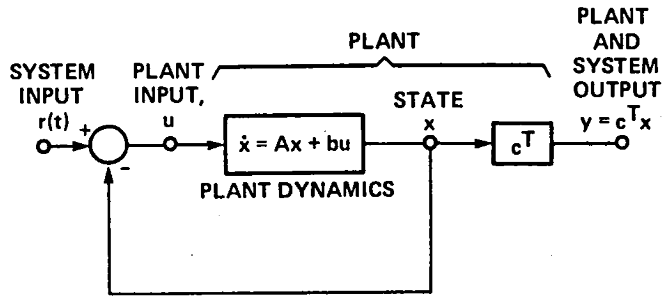

Figure 2.- Single-input, single-output state variable feedback system.

For this reason, equation (1) will be rewritten as follows:

$$\dot{X} = AX + B(\ddot{U} + B_1\dot{U} + B_0U)$$

This formulation of the state equations requires that equation (7) be modified accordingly. That is

$$Y = C^TX = (1 \quad 0 \quad 0 \quad 0)\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

If initial conditions are assumed to be zero, the corresponding Laplace transform equations are

$$sX(s) = AX(s) + B(s^2 + B_1s + B_0)U(s)$$

therefore

$$X(s) = (sI - A)^{-1}B(s^2 + B_1 s + B_0)U(s)$$

The Laplace transform of the system input $R(s)$ is related to the Laplace transforms of the plant input $U(s)$ and the state $X(s)$ by the equation

$$U(s) = R(s) - kX(s)$$

However, since the intention here is not to emphasize the control system aspect of the problem, but rather to demonstrate the feasibility of using symbolic computation to find the eigenvalues of dynamical systems, this equation will be simplified as follows:

$$U(s) = R(s) - X(s)$$

Substitution in the preceding equation yields the Laplace transform of the closed-loop state

$$X(s) = \left[I + (s^2 + B_1 s + B_0)(sI - A)^{-1}B\right]^{-1}(s^2 + B_1 s + B_0)(sI - A)^{-1}BR(s) \tag{17}$$

where I is a (4×4) identity matrix.

To permit inversion of the matrix

$$\left[I + (S^2 + B_1 s + B_0)(sI - A)^{-1}B\right] \tag{18}$$

the (4×1) column vector B must be converted to a (4×4) matrix as follows:

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ K & 0 & 0 & 0 \end{bmatrix} \tag{19}$$

With this modification, the required Laplace transforms are obtained. First, equation (19) is multiplied by $(s^2 + B_1 s + B_0)$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ K(S^2 + B_1 S + B_0) & 0 & 0 & 0 \end{bmatrix} \tag{20}$$

Premultiplication of matrix (20) by matrix (11) yields the following matrix:

$$
\begin{bmatrix}
\dfrac{K(S^2 + B_1S + B_0)}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} & 0 & 0 & 0 \\[2ex]
\dfrac{KS(S^2 + B_1S + B_0)}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} & 0 & 0 & 0 \\[2ex]
\dfrac{KS^2(S^2 + B_1S + B_0)}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} & 0 & 0 & 0 \\[2ex]
\dfrac{KS^3(S^2 + B_1S + B_0)}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} & 0 & 0 & 0
\end{bmatrix} \tag{21}
$$

When the identity matrix is added to matrix (21), the expanded form of matrix (18) is obtained.

$$
\begin{bmatrix}
\dfrac{K(S^2 + B_1S + B_0)}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} + 1 & 0 & 0 & 0 \\[2ex]
\dfrac{KS(S^2 + B_1S + B_0)}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} & 1 & 0 & 0 \\[2ex]
\dfrac{KS^2(S^2 + B_1S + B_0)}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} & 0 & 1 & 0 \\[2ex]
\dfrac{KS^3(S^2 + B_1S + B_0)}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} & 0 & 0 & 1
\end{bmatrix} \tag{22}
$$

Before equation (17) can be expanded, the matrix (22) must be inverted. The inverted form of this matrix is

$$
\begin{bmatrix}
\dfrac{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0}{S^4 + A_3S^3 + (K + A_2)S^2 + (B_1K + A_1)S + B_0K + A_0} & 0 & 0 & 0 \\[2ex]
\dfrac{KS^3 + B_1KS^2 + B_0KS}{S^4 + A_3S^3 + (K + A_2)S^2 + (B_1K + A_1)S + B_0K + A_0} & 1 & 0 & 0 \\[2ex]
\dfrac{KS^4 + B_1KS^3 + B_0KS^2}{S^4 + A_3S^3 + (K + A_2)S^2 + (B_1K + A_1)S + B_0K + A_0} & 0 & 1 & 0 \\[2ex]
\dfrac{KS^5 + B_1KS^4 + B_0KS^3}{S^4 + A_3S^3 + (K + A_2)S^2 + (B_1K + A_1)S + B_0K + A_0} & 0 & 0 & 1
\end{bmatrix} \tag{23}
$$

Premultiplication of matrix (21) by matrix (23) yields the matrix

$$\begin{bmatrix} \frac{KS^2 + B_1KS + B_0K}{S^4 + A_3S^3 + (K + A_2)S^2 + (B_1K + A_1)S + B_0K + A_0} & 0 & 0 & 0 \\ \frac{KS^3 + B_1KS^2 + B_0KS}{S^4 + A_3S^3 + (K + A_2)S^2 + (B_1K + A_1)S + B_0K + A_0} & 0 & 0 & 0 \\ \frac{KS^4 + B_1KS^3 + B_0KS^2}{S^4 + A_3S^3 + (K + A_2)S^2 + (B_1K + A_1)S + B_0K + A_0} & 0 & 0 & 0 \\ \frac{KS^5 + B_1KS^4 + B_0KS^3}{S^4 + A_3S^3 + (K + A_2)S^2 + (B_1K + A_1)S + B_0K + A_0} & 0 & 0 & 0 \end{bmatrix} \tag{24}$$

Formulation of the reference input vector which has only one component $R_1(s)$ and premultiplication by matrix (24) completes the expansion of equation (17), and gives the column vector of closed-loop Laplace transforms for the individual states.

$$\begin{bmatrix} R_1(S) \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{25}$$

$$\begin{bmatrix} \frac{(KS^2 + B_1KS + B_0K)R_1(S)}{S^4 + A_3S^3 + (K + A_2)S^2 + (B_1K + A_1)S + B_0K + A_0} \\ \frac{(KS^3 + B_1KS^2 + B_0KS)R_1(S)}{S^4 + A_3S^3 + (K + A_2)S^2 + (B_1K + A_1)S + B_0K + A_0} \\ \frac{(KS^4 + B_1KS^3 + B_0KS^2)R_1(S)}{S^4 + A_3S^3 + (K + A_2)S^2 + (B_1K + A_1)S + B_0K + A_0} \\ \frac{(KS^5 + B_1KS^4 + B_0KS^3)R_1(S)}{S^4 + A_3S^3 + (K + A_2)S^2 + (B_1K + A_1)S + B_0K + A_0} \end{bmatrix} \tag{26}$$

The program statements used to obtain the Laplace transforms of the individual open-loop states can be used in this case also. The components of the closed-loop vector are

$$\left.\begin{aligned}
X_1 &= \frac{(KS^2 + B_1KS + B_0K)R_1(S)}{S^4 + A_3S^3 + (K + A_2)S^2 + (B_1K + A_1)S + B_0K + A_0} \\
X_2 &= \frac{(KS^3 + B_1KS^2 + B_0KS)R_1(S)}{S^4 + A_3S^3 + (K + A_2)S^2 + (B_1K + A_1)S + B_0K + A_0} \\
X_3 &= \frac{(KS^4 + B_1KS^3 + B_0KS^2)R_1(S)}{S^4 + A_3S^3 + (K + A_2)S^2 + (B_1K + A_1)S + B_0K + A_0} \\
X_4 &= \frac{(KS^5 + B_1KS^4 + B_0KS^3)R_1(S)}{S^4 + A_3S^3 + (K + A_2)S^2 + (B_1K + A_1)S + B_0K + A_0}
\end{aligned}\right\} \tag{27}$$

The corresponding closed-loop transfer functions, which are defined by (C39), and the activating program statements are

```
(C39) FOR  I:1 THRU 4 DO T[I]:  (X[I]/R(S))$

(C40) FOR  I:1 THRU 4 DO DISPLAY(T[I]);
```

$$T_1 = \frac{KS^2 + B_1KS + B_0K}{S^4 + A_3S^3 + (K + A_2)S^2 + (B_1K + A_1)S + B_0K + A_0}$$

$$T_2 = \frac{KS^3 + B_1KS^2 + B_0KS}{S^4 + A_3S^3 + (K + A_2)S^2 + (B_1K + A_1)S + B_0K + A_0}$$

$$T_3 = \frac{KS^4 + B_1KS^3 + B_0KS^2}{S^4 + A_3S^3 + (K + A_2)S^2 + (B_1K + A_1)S + B_0K + A_0}$$

$$T_4 = \frac{KS^5 + B_1KS^4 + B_0KS^3}{S^4 + A_3S^3 + (K + A_2)S^2 + (B_1K + A_1)S + B_0K + A_0}$$

The determinant of the matrix (22) is obtained by using the program statement

```
DETERMINANT (22);
```

$$\frac{K(S^2 + B_1S + B_0)}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} + 1$$

The closed-loop eigenvalues are the roots of the determinant $|(22)| = 0$

$$\frac{K(S^2 + B_1S + B_0)}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} + 1 = 0$$

that is, the roots of

$$\frac{S^4 + A_3S^3 + (K + A_2)S^2 + (B_1K + A_1)S + B_0K + A_0}{S^4 + A_3S^3 + A_2S^2 + A_1S + A_0} = 0$$

The roots of this equation which are the roots of the numerator are seen to equal the poles of the closed-loop transfer functions T_i. Because the locations of the closed-loop poles in the complex plane determine the stability of the dynamical system being considered, these will be plotted as a function of the parameter K. As in the open-loop case, the eigenvalues can be determined by using the root finding program. However, before the program can be implemented, the coefficients must be replaced by their numerical values. These have been introduced by the use of the previously described substitution routine. Subsequent to the introduction of the numerical values, the root finding program was used to compute a set of close-loop poles. The range of values obtained was sufficient to permit the plotting of the loci for the two dominant modes of motion. The tabulated results are shown in table 1, and the eigenvalue loci are plotted in figures 3 and 4.

TABLE 1.- CLOSED-LOOP EIGENVALUES

K	Phugoid mode	Short period mode
0	-0.00996 ±j 0.1632	-0.8403 ±j 1.3839
1.0	-.0962 ±j .1457	-.7541 ±j 1.6710
2.0	-.1467 ±j .1014	-.7035 ±j 1.9343
3.0	-.1779 ±j .0258	-.6723 ±j 2.1722
3.1	-.16695 ; -.1938	-.6699 ±j 2.1948
3.2	-.1510 ; -.2145	-.6675 ±j 2.2171
3.3	-.1424 ; -.2276	-.6652 ±j 2.2392
3.4	-.1362 ; -.2382	-.6631 ±j 2.2611
3.5	-.13123 ; -.2474	-.6610 ±j 2.2829
3.6	-.1271 ; -.2555	-.6589 ±j 2.3044
3.8	-.1205 ; -.2698	-.6551 ±j 2.3470
4.0	-.1154 ; -.2821	-.6515 ±j 2.3888
10.0	-.0775 ; -.4215	-.6007 ±j 3.4188
20.0	-.0686 ; -.4750	-.5785 ±j 4.6567
50.0	-.0637 ; -.5101	-.5634 ±j 7.1892
100.0	-.0621 ; -.5224	-.5580 ±j 10.0839
1,000.0	-.0607 ; -.5337	-.5531 ±j 31.6494
10,000.0	-.0605 ; -.5349	-.5526 ±j 100.0084
100,000.0	-.0605 ; -.5349	-.5525 ±j 316.2304

ZEROS : -0.0605; -0.535 ; ±∞

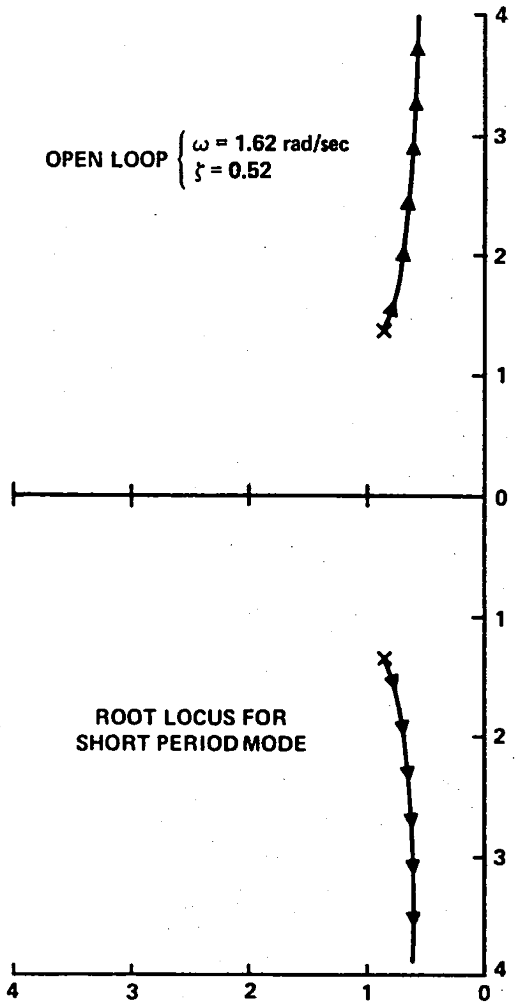

Figure 3.- Root locus for short-period mode.

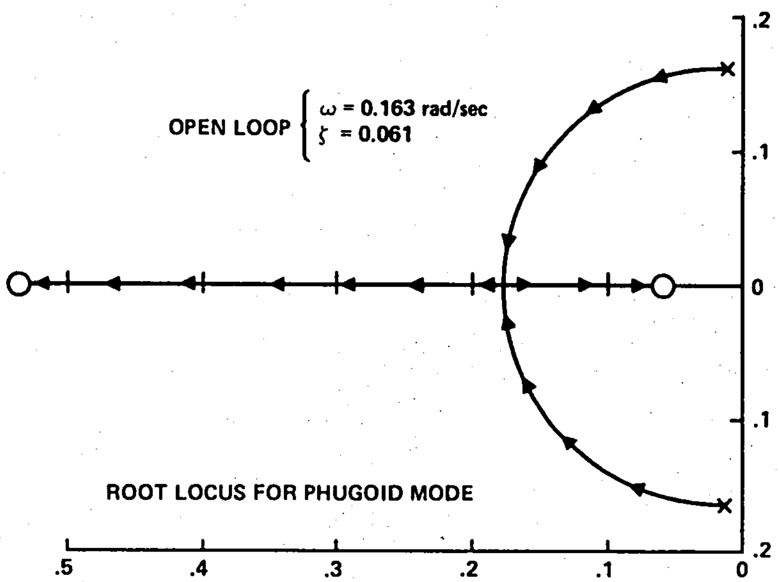

Figure 4.- Root locus for phugoid mode.

CONCLUSIONS

It has been demonstrated that the technique of symbolic computation can be used to facilitate the formulation and processing of the equations of motion of dynamical systems. The manual formulation of these equations, which involves a variety of matrix operations, including matrix inversion, is time consuming and subject to human error. An example is included which demonstrates the facility with which the system dynamics matrix and the control distribution matrix from the state space formulation of the equations of motion can be processed to obtain eigenvalue loci as a function of a system parameter. The example describes a single-input, single-output state variable feedback system. It should be emphasized, however, that the technique is not limited to the study of systems of this type, but can be applied with equal facility to the study of more general systems.

REFERENCES

1. Gupta, C. S.; and Masdorff, L.: Fundamentals of Automatic Control. John Wiley & Sons, Inc., 1970.

2. Powell, J. D.; and Akhter, M.: Multi-Calculation Rate Simulations. Joint Institute for Aeronautics and Acoustics, Department of Aeronautics and Astronautics, Stanford University, Nov. 1977.

3. Howard, J. C.: Practical Applications of Symbolic Computation. IPC Science and Technology Press, Limited, 1980.

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16. Abstract

Problems requiring the determination of eigenvalues occur frequently in the study of dynamical systems. The formulation and processing of the equations describing these systems involve time-consuming matrix operations. Although numerical methods are available, it is more convenient and informative to formulate the relevant matrices and transfer functions symbolically. This can be achieved by the use of symbolic computation, which is a technique whereby algebraic operations, symbolic differentiation, matrix formulation and inversion, etc., can be performed on a digital computer equipped with a formula-manipulation compiler. An example is included that demonstrates the facility with which the system dynamics matrix and the control distribution matrix from the state space formulation of the equations of motion can be processed to obtain eigenvalue loci as a function of a system parameter.

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