JPL PUBLICATION 82-38

A Survey of the Theory of the Earth's Rotation

W. H. Cannon

(NASA-CR-168989) A SURVEY OF THE THEORY OF THE EARTH'S ROTATION (Jet Propulsion Lab.) 87 p HC A05/MF A01 CSCL 08E

N82-25589

Unclas

G3/42 27972

November 1, 1981

NASA

National Aeronautics and
Space Administration

Jet Propulsion Laboratory
California Institute of Technology
Pasadena, California

JPL PUBLICATION 82-38

A Survey of the Theory of the Earth's Rotation

W. H. Cannon

November 1, 1981

National Aeronautics and
Space Administration

Jet Propulsion Laboratory
California Institute of Technology
Pasadena, California

The research described in this publication was carried out by the Jet Propulsion Laboratory, California Institute of Technology, under contract with the National Aeronautics and Space Administration.

Contents

Tables

Figures

Abstract

This report presents a survey of the theory of the earth's rotation and the geophysical phenomena affecting it, with emphasis on polar motion and UT1 variations. The theoretical development in this review begins with first principles and formulates the problem of polar motion and UT1 variations in considerable generality and detail. The treatment includes an analysis of the effects of earth deformations and the solid earth tides.

A Survey of the Theory of the Earth's Rotation

I. Introduction

In recent years the technique of long baseline interferometry (LBI) has demonstrated its ability to carry out measurements of the variations in the earth's rotation rate (UT1) and of the variations in the position of the earth's rotation axis with respect to the earth's crust (polar motion) with an accuracy which is comparable to that obtained by the classical methods using photographic zenith tubes (PZT's). While the classical methods have probably been extended to their maximum capabilities by their present day use, it is expected that continued development of the technique of long baseline interferometry will ultimately yield measurements of these geophysical quantities with accuracies which exceed present day capabilities by an order of magnitude.

This document was prepared in anticipation of these future developments. Its purpose is to comprehensively review the present day dynamical theory of the earth's rotation in order to provide a coherent theoretical basis for the development of future data analysis procedures and software models for the treatment of future high quality long baseline interferometry data. Its purpose then is to serve as a tutorial handbook for workers who will be involved in the process of extracting geophysical information from the interferometry data. In order to enhance its usefulness in this respect the work is presented in considerable detail.

II. Coordinate Systems

The term "axis" will always be associated with a corresponding vector. An axis is a straight line passing through the origin of coordinates in a direction parallel to the associated vector. The point of intersection of an axis with the surface of the earth or the celestial sphere is called a "pole"; in the former case a "terrestrial pole" and in the latter case a "celestial pole." If not explicitly stated which, "terrestrial pole" will be understood.

In establishing a theoretical and operational framework for describing the earth's rotation it is necessary to use two coordinate frames: a space-fixed frame spanned by basis vectors $\hat{E}_1$ $\hat{E}_2$ $\hat{E}_3$ and a body-fixed frame spanned by basis vectors $\hat{e}_1$ $\hat{e}_2$ $\hat{e}_3$. Coordinate systems can be grouped into "geometrical" and "dynamical" classifications according to the nature of their fundamental defining quantities. Hybrid coordinate systems requiring a combination of geometrical and dynamical quantities for their definition are also possible.

The present space-fixed coordinate frame is a dynamical coordinate frame which uses the orbital and equatorial planes of the earth to define $\hat{E}_1$ $\hat{E}_2$ $\hat{E}_3$. The $\hat{E}_3$ axis is parallel to the earth's mean orbital angular momentum vector of 1950.0 and $\hat{E}_1$ is contained by the intersection of the mean orbital and mean equatorial planes of 1950.0 and points toward the ascending node. The $\hat{E}_2$ axis is orthogonal to $\hat{E}_1$ and $\hat{E}_3$ to complete a right-handed coordinate frame. The origin of the space-fixed coordinate frame is placed at the center of mass of the earth, including the oceans and atmosphere. In defining the mean orbital plane of the earth it is necessary to reckon with the fact that the motion of the earth about the sun is continually perturbed by the gravitational attractions of the other bodies of the solar system and that the actual orbit of the earth's center of mass is an irregular and ever varying curve in space (Woolard and Clemence, 1966).

Even in the case of a rigid earth certain difficulties are encountered when operationally defining the celestial equator. The instantaneous celestial equator is usually defined as being contained by a plane which is orthogonal to the earth's instantaneous rotation axis. For a rigid earth the instantaneous rotation axis is defined by the vector sum of the angular rotation rates of polar motion, spin, precession, and nutation and is unobservable by conventional astronomical means. What is observable is a position on the celestial sphere, to be here called the celestial reference pole, located at the center of the quasi-circular diurnal paths of the stars in the sky. The celestial reference pole is in continual motion across the sky owing to the effects of precession and nutation alone. Polar motion changes the latitudes of observatories and not the declination of stars and so does not contribute to the motion of the celestial reference pole on the celestial sphere. The motion of the celestial reference pole across the sky implies the existence of an additional rotation rate (due to precession and nutation) which, along with the rotation rate due to the polar motion, must be added vectorially to the spin to obtain the total earth rotation vector.

The realization of the above dynamically defined space-fixed basis vectors is provided in an implicit manner by the coordinates assigned the stars of the FK4 catalogue whose positions have been measured with reference to the earth's orbital and equatorial planes. The FK4 catalogue contains about 1500 stars and has an internal consistency of $\pm 0\overset{''}{.}15$ arc and an overall internal accuracy of $\pm 0\overset{''}{.}10$ arc (Kolaczek and Weiffenbach, p. 32, 1975). Soon to replace the FK4 catalogue is the FK5 catalogue with about 3000 stars and an internal consistency of $\pm 0\overset{''}{.}10$ arc and an overall accuracy of $\pm 0\overset{''}{.}02$ arc.

Long baseline interferometry is expected to provide the relative positions of roughly 100 compact radio sources with an overall internal accuracy of $\pm 0\overset{''}{.}001$ arc in the near future. This celestial coordinate system will be essentially geometrical and some effort should be dedicated to tying it to the dynamically defined FK4 and FK5 coordinate systems, at least to the level of the errors inherent in the stellar coordinate systems.

The body-fixed coordinate system spanned by basis vectors $\hat{r}_1$ $\hat{r}_2$ $\hat{r}_3$ has its origin at the center of mass of the earth including the oceans and atmosphere. Hence the space-fixed coordinate system and the body-fixed coordinate system share a common origin, and the general linear coordinate transformation relating the two at any particular time consists of a rotation about some axis and a scaling. A relative scaling of unity is maintained by adopting the same unit of length in both systems.

The $\hat{r}_3$ basis vector is defined to be parallel to the mean axis of figure of the earth. The axis of figure corresponds to the principal eigenvector (the eigenvector of maximum eigenvalue) of the earth's inertia tensor. For a deformable earth the body-fixed orientation of the axis of figure is time-dependent since the inertia tensor is time-dependent.

The mean axis of figure is defined as corresponding to the principal eigenvector of the mean inertia tensor.

The $\hat{r}_1$ basis vector is orthogonal to the $\hat{r}_3$ basis vector and contained in the plane of the conventional prime (zero) meridian. The $\hat{r}_2$ basis vector is orthogonal to $\hat{r}_1$ $\hat{r}_3$ and oriented so as to form a right-handed orthogonal triad. The realization of the body-fixed basis vectors $\hat{r}_1$ $\hat{r}_2$ $\hat{r}_3$ is provided implicitly by the coordinates of a set of fixed observatory sites located on the earth's surface. The following is a brief examination of current practices involved in the determination of the terrestrial coordinate frame.

In the presatellite era the geocenter was, to a large extent inaccessible geodetically speaking, and geodetic networks were essentially "local" coordinate systems. The location of the geocenter relative to the earth's surface could be determined in principle by solving the boundary value problem in the theory of gravitational potential. However, the practical realization of this procedure was hampered by the lack of gravity measurements over the oceans, and any attempt based on the limited data available excluded the earth's atmosphere since it was outside the surface over which the measurements were made.

The artificial satellite senses the center of mass of the earth directly since the osculating orbit plane passes through it. The usual procedure is to use satellite tracking data to solve simultaneously for the geocentric position of the tracking station and the spherical harmonic coefficients in the expansion for the earth's gravitational potential. In this manner the location of the geocenter relative to points on the earth's surface can be determined to about ±0.5m – ±1.0m.

Even in the case of a rigid earth, owing to the presence of the fluid portions, the geocenter does not remain fixed relative to the earth's solid surface. The seasonal redistribution of the masses of the oceans and atmosphere and particularly the redistribution of global ground water displace the geocenter relative to the solid earth periodically in a year along a roughly elliptical path with a major axis of the order of 0.5cm (Stolz 1976).

Considering the real deformable earth, the tidal deformations of the solid portion, because of their symmetry, do not displace the geocenter. However, owing to their asymmetry the tidal redistribution of the masses of the oceans does displace the geocenter. Each tidal constituent will displace the geocenter along a curved path relative to the solid earth with a period equal to that of the tidal constituent. Brosche and Sündermann (1977) have shown that the M_2 tidal constituent displaces the geocenter relative to the solid earth around a closed curve of the order of 4 cm in extent with a period of 12.42 hours. The motion of the geocenter due to the M_2 tide is essentially confined to the plane of the equator (the displacement parallel to the polar axis being an order of magnitude smaller) along the directions $\lambda = 45°$E, 135°W.

It can be shown (Goldreich and Toomre, 1969) that for a quasi-rigid, evolving, extended body rotating in the absence of external torques about its axis of largest moment of inertia, the axis of figure is constrained dynamically to coincide with the axis of rotation. However, the earth, being subjected to external torques, differs in some respects from the body being considered by Goldreich and Toomre.

For historical reasons the motion of a rigid body in the absence of external torques is known as "Eulerian motion." In the case of the earth subjected as it is to external torques it can be shown (Woolard, 1953) that the motion of the rotation axis within the body-fixed frame is almost entirely due to the Eulerian motion, with only a small perturbation of the order of 0".02 arc occurring as a result of the luni-solar gravitational torques. Although the elastic yielding of the earth greatly alters the Eulerian motion from what would prevail on a rigid earth, the effect of the luni-solar torques is simply to displace the instantaneous rotation axis in a diurnal circular path of diameter roughly 0".02 arc about the Eulerian position. Thus the mean axis of rotation of the earth coincides with the Eulerian (torque-free) axis of rotation and hence, by the arguments of Goldreich and Toomre, also coincides with the mean axis of figure $\hat{e}_3$.

In general, at any instant the rotation axis is displaced, in the body-fixed frame, from its mean position on the axis of figure. The figure axis of the earth or the $\hat{e}_3$ basis vector can be determined by establishing the mean position of the rotation axis. In practice the procedure of determining the location of the "instantaneous" rotation axis in the body-fixed frame by PZT observations yields the position of the "instantaneous" Eulerian or spin axis. However, the mean position of the Eulerian axis will also serve to determine the earth's figure axis or $\hat{e}_3$.

Figure II-1 (after Smylie and Mansinha, 1971b) illustrates the geometrical relationships pertaining to polar motion. The figure is drawn with reference to the Eulerian or spin axis rather than the rotation axis, since it is with reference to this axis that latitudes on earth are observationally determined.

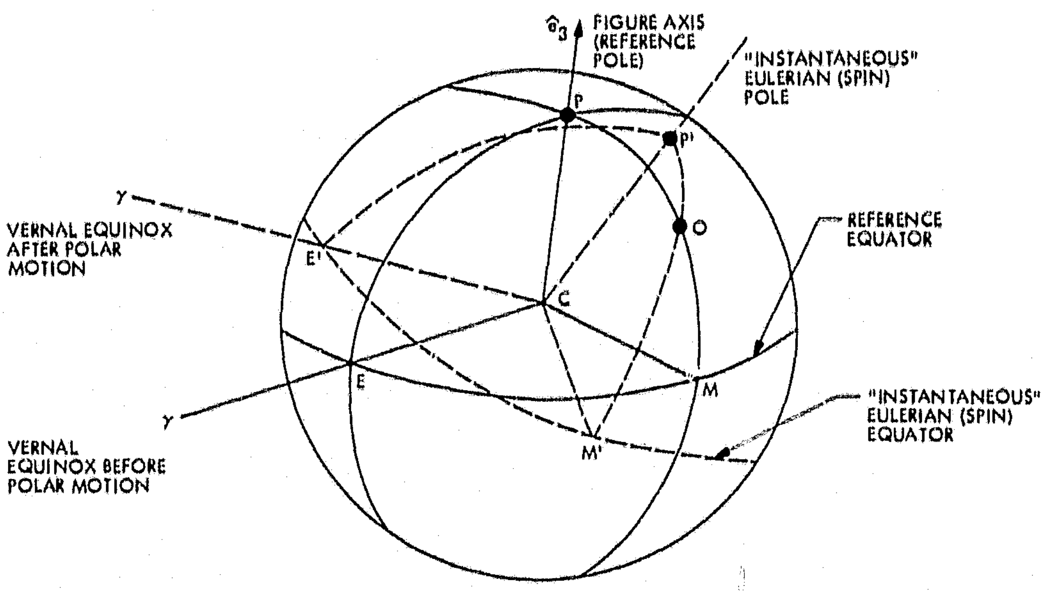

Figure II-1. The geometry of polar motion. For clarity the figure is drawn in a body-fixed frame so that the figure of the earth defined by $\hat{e}_3$ appears fixed in orientation. When viewed in a space-fixed frame, the equator and the spin axis appear fixed in orientation while the figure of the earth is displaced. (After Smylie and Mansinha, 1971b)

In Figure II-1 the point O refers to an observer on the earth's surface or alternately to the body fixed angular coordinates of an interferometer baseline. The point C is the geocenter, the origin of the terrestrial coordinate system. The displacement of the Eulerian or spin pole from P, the axis of figure or reference pole, to P′ changes the latitude of the observer or the declination of the interferometer baseline at O from the angle OCM to the angle OCM′ and changes local sidereal time at O or the sidereal hour angle of the meridian of the interferometer baseline at O from the angle ECM to the angle E′CM′ where E and E′ refer to the subequinox point on the earth's equator before and after polar motion respectively.

There are two systems of reckoning in use today to describe polar motion. One is most widely used by astronomers and corresponds to the usage of the BIH (Bureau International de l'Heure) while the other is most widely used by geophysicists and corresponds to the usage in this document. The geophysicist orients the surface of the earth so that the positive normal points toward the zenith. The use of a right-handed coordinate system then requires that the location of the "instantaneous" Eulerian axis in the body-fixed frame be specified by m_1, the angular displacement of the Eulerian pole parallel to the prime (Greenwich) meridian, and m_2, the angular displacement of the Eulerian pole parallel to the 90°E meridian. This is illustrated in Figure II-2. The net angular displacement of the Eulerian pole from the figure axis is given by angle β, where

$$\beta^2 = m_1^2 + m_2^2 + o(m^4). \qquad \text{(II-1)}$$

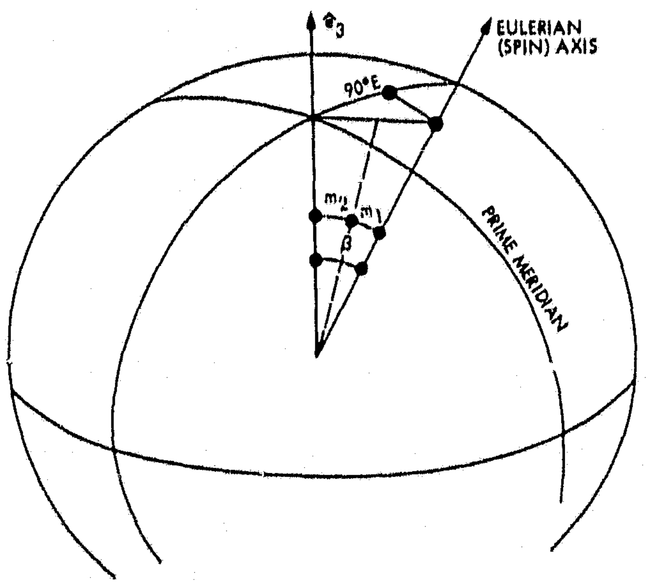

Figure II-2. The geometrical definition of the angles, m_1 m_2 used by geophysicists to indicate the location of the Eulerian pole or spin axis of the earth relative to the CIO (Conventional International Origin). The axis of the CIO is considered to be coincident with $\hat{e}_3$, the figure axis of the earth.

The astronomer adopts a right-handed coordinate system on the celestial sphere, with the "surface" of the celestial oriented so that the positive normal points toward the earth. As a consequence of this the astronomer is required for consistency to orient the surface of the earth with the positive normal pointing toward the geocenter. The use of a right-handed coordinate system then requires that the location of the "instantaneous" Eulerian axis in the body-fixed frame be specified by x, the angular displacement of the Eulerian pole parallel to the prime (Greenwich) meridian, and y, the angular displacement of the Eulerian pole parallel to the 90°W meridian.

Both systems of reckoning use the CIO or Conventional International Origin as the reference pole defining the figure axis or $\hat{e}_3$. The CIO is the nominal point of 90°N latitude and is defined implicitly by the assigned nominal coordinates of a number of observatories around the world. The CIO was defined initially by the mean latitudes $\bar{\phi}_i$ and longitudes $\bar{\lambda}_i$ $i = 1, 2, \ldots, 5$ of 5 observatories as a result of latitude and longitude observations made at these sites extending over the interval 1900.0 – 1905.0. The CIO defines the $\hat{e}_3$ basis vector, the axis of which makes an angle of $90° - \bar{\phi}_i$ with the verticals of each of the 5 defining observatories.

It can be shown that for an observatory (interferometer baseline) with nominal (referred to the CIO and the Greenwich meridian) geocentric coordinates given by latitude ϕ_0 and east longitude λ_0, the increments to the latitude $\Delta\phi_0$ (OCM′ – OCM in Figure II-1) and to the longitude $\Delta\lambda_0$ (E′CM′ – ECM in Figure II-1) accompanying a displacement m_1 m_2 radians of the Eulerian pole are given by

$$\Delta\phi_0 = m_1 \cos\lambda_0 + m_2 \sin\lambda_0 \qquad \text{(II-2)}$$

$$\Delta\lambda_0 = \tan\phi_0 \,(m_1 \sin\lambda_0 - m_2 \cos\lambda_0). \qquad \text{(II-3)}$$

UT0 is a "raw" measure of Universal Time based on data obtained at a particular site where the assumptions are made throughout the data analysis that

(1) The observatory (interferometer) coordinates are its nominal coordinates referenced to the CIO and the Greenwich meridian.

(2) The earth has been spinning about an axis through the CIO.

UT1 is a "true" measure of Universal Time intended to give the hour angle of the mean sun at Greenwich when the effects of polar motion described above are duly taken into account. UT1 cannot be measured from a single point on the earth's surface since it is impossible to distinguish the effects of polar motion which are to be incorporated into UT1 from the effects of variations in the earth's rotation (spin) rate which are normally incorporated into UT2.

UT0 and UT1 are related by

$$\mathrm{UT0} = \mathrm{UT1} + \frac{86400}{2\pi} \tan\phi_0 \,(m_1 \sin\lambda_0 - m_2 \cos\lambda_0)\,. \tag{II-4}$$

UTC is an atomic time scale broadcast by national time services and maintained continuously by onsite atomic frequency standards at observatory sites. At each of m observatory sites it is possible to observe directly the quantities

$$\mathrm{UT0}_i - \mathrm{UTC} \qquad i = 1, 2, 3 \cdots m$$

$$\phi_i - \phi_{0i} \qquad i = 1, 2, 3 \cdots m$$

where ϕ_i is the "instantaneous" latitude of the ith observatory and ϕ_{0i} the nominal latitude of the ith observatory. These two observables are related to m_1 m_2 and UT1 − UTC by

$$\mathrm{UT0}_i - \mathrm{UTC} = \mathrm{UT1} + \frac{86400}{2\pi} \tan\phi_{0i}\,(m_1 \sin\lambda_{0i} - m_2 \cos\lambda_{0i}) - \mathrm{UTC} \tag{II-5}$$

$$\phi_i - \phi_{0i} = m_1 \cos\lambda_{0i} + m_2 \sin\lambda_{0i} \tag{II-6}$$

where λ_{0i} is the nominal longitude of the ith observatory.

From a large set (of the order of 50) of such observations the BIH solves for the quantities m_1 m_2 and UT1 − UTC by least squares adjustment. This adjustment procedure produces a value of UT1 which is not reduced to the prime (Greenwich) meridian passing through the meridian circle of the transit telescope at Greenwich but to a meridian which is displaced from Greenwich by several milliseconds of time. The meridian to which UT1 is adjusted is called the meridian of the "mean observatory" or the Greenwich Mean Astronomic Meridian. The corrections for the displacement of the mean observatory from Greenwich (usually of the order of 2-5 msec) are published in the Bulletin Horaire of the BIH.

III. Dynamics of Rotating Bodies

Before proceeding to a treatment of the rotational dynamics of the earth we shall review the dynamical theory of rotating bodies generally, arriving finally at the Liouville and Euler equations.

A. Rotational Dynamics of an Assemblage of Particles

We consider an assemblage of N particles whose masses and positions relative to our origin of coordinates are given by m_i and $\mathbf{r}_i$, $i = 1, 2, 3 \cdots N$, respectively. If $\ddot{\mathbf{r}}_i$ is the acceleration of the ith particle relative to the origin of coordinates and if $\ddot{\mathbf{S}}$ is the acceleration of the origin of coordinates relative to inertial space, then Newton's second law applied to the ith particle gives

$$m_i(\ddot{\mathbf{S}} + \ddot{\mathbf{r}}_i) = \mathbf{F}_i + \sum_{\substack{j=1 \\ j\neq i}}^{N} \mathbf{R}_{ji} \tag{III-1}$$

where $\mathbf{F}_i$ is the net force on the ith particle due to all influences external to the assemblage and $\mathbf{R}_{ji}$ is the force on the ith particle due to the jth particle of the assemblage.

Summing over all particles

$$\sum_{i=1}^{N} m_i(\ddot{\mathbf{S}} + \ddot{\mathbf{r}}_i) = \sum_{i=1}^{N} \mathbf{F}_i + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \mathbf{R}_{ji}$$

and denoting

$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i$$

as the total external force on the assemblage and denoting

$$M = \sum_{i=1}^{N} m_i$$

as the total mass of the assemblage gives

$$M\ddot{\mathbf{S}} + \sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i = \mathbf{F} + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \mathbf{R}_{ji}\,. \tag{III-2}$$

Introducing ρ as the position of the center of mass

$$\rho = \frac{\sum_{i=1}^{N} m_i \mathbf{r}_i}{\sum_{i=1}^{N} m_i} \tag{III-3}$$

allows Equation (III-2) to be written as

$$M(\ddot{\mathbf{S}} + \ddot{\rho}) = \mathbf{F} + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \mathbf{R}_{ji} . \tag{III-4}$$

In its "weak" form Newton's third law merely asserts the equality of action and reaction and gives

$$\mathbf{R}_{ji} = -\mathbf{R}_{ij} .$$

Simply stated this implies that the force exerted by the jth particle on the ith particle is equal and opposite to the force exerted by the ith particle on the jth particle. If the weak form of Newton's third law is assumed, then in the double sum

$$\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \mathbf{R}_{ji}$$

the terms cancel in pairs with the conclusion that the net force on the assemblage of particles due to internal action and reaction pairs vanishes.

Returning now to Equation (III-1) and taking the cross product of Equation (III-1) with the vector $\mathbf{r}$ and summing over all the particles gives

$$\sum_{i=1}^{N} m_i \mathbf{r}_i \times \ddot{\mathbf{S}} + \sum_{i=1}^{N} m_i \mathbf{r}_i \times \ddot{\mathbf{r}}_i = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \mathbf{r}_i \times \mathbf{R}_{ji} . \tag{III-5}$$

Assuming nonrelativistic mechanics for which $dm_i/dt = 0$, several terms in Equation (III-5) can be rewritten:

(1) $\sum_{i=1}^{N} m_i \mathbf{r}_i \times \ddot{\mathbf{r}}_i = \frac{d}{dt}\left(\sum_{i=1}^{N} m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i\right)$,

since $\dot{\mathbf{r}}_i \times \dot{\mathbf{r}}_i = 0$

(2) $\sum_{i=1}^{N} m_i \mathbf{r}_i \times \ddot{\mathbf{S}} = M\rho \times \ddot{\mathbf{S}}$,

using Equation (III-3), and if we define:

(3) $\mathbf{L}$ the total angular momentum of the assemblage about the origin of coordinates 0 as

$$\mathbf{L} = \sum_{i=1}^{N} m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i$$

(4) $\mathbf{N}$ the net external torque on the assemblage about the origin of coordinates 0 as

$$\mathbf{N} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i$$

then Equation (III-5) can be written

$$M\rho \times \ddot{\mathbf{S}} + \dot{\mathbf{L}} = \mathbf{N} + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \mathbf{r}_i \times \mathbf{R}_{ji} . \tag{III-6}$$

The last term,

$$\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \mathbf{r}_i \times \mathbf{R}_{ji}$$

in Equation (III-6) represents the net torque on the assemblage resulting from its internal actions and reactions. In order for this term to vanish it is necessary to invoke Newton's third law in its "strong" form; namely, that

$$\mathbf{R}_{ji} = \alpha_{ji} [\mathbf{r}_i - \mathbf{r}_j] \; ; \; \alpha_{ij} = \alpha_{ji} . \tag{III-7}$$

Simply stated this implies that the force exerted by the jth particle on the ith particle is equal and opposite to the force exerted on the ith particle by the jth particle and that these forces act along the line joining the centers of the particles.

If Newton's third law in its strong form is assumed, then

$$\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \mathbf{r}_i \times \mathbf{R}_{ji} = - \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \alpha_{ji} \mathbf{r}_i \times \mathbf{r}_j \qquad \text{(III-8)}$$

and the terms on the RHS of Equation (III-8) cancel in pairs since $\mathbf{r}_i \times \mathbf{r}_j = -\mathbf{r}_j \times \mathbf{r}_i$ and $\alpha_{ji} = \alpha_{ij}$. It follows that the dynamical equation (III-6) reduces to

$$M \boldsymbol{\rho} \times \ddot{\mathbf{S}} + \dot{\mathbf{L}} = \mathbf{N} . \qquad \text{(III-9)}$$

It should be noted that Equation (III-7) is not valid for electrodynamical Lorentz forces acting between charged particle pairs. However, the conclusion of Equation (III-9), which is still valid when the mass assemblage includes charged particles, must be obtained by a more extensive argument than that presented here.

In order for the term $M \boldsymbol{\rho} \times \ddot{\mathbf{S}}$ to vanish it is necessary to choose an origin of coordinates to coincide with the center of mass, in which case $\rho = 0$ and the dynamical equation (III-9) further reduces to

$$\dot{\mathbf{L}} = \mathbf{N} . \qquad \text{(III-10)}$$

In dealing with the dynamics of rotating bodies it is important to realize that the dynamical equation governing rotation only assumes this simple form of (III-10) when expressed in a coordinate system whose origin coincides with the center of mass of the body. The coordinate system need not be an inertial frame in that $\ddot{\mathbf{S}}$ need not vanish to achieve this simplification. The simplification occurs because ρ vanishes. However, while the origin of the coordinate system may be accelerating arbitrarily with respect to inertial space, the coordinate system itself cannot be rotating, for nowhere in our dynamics have we allowed for this.

The angular momentum of a particle of mass m reckoned in a nonrotating frame with origin O relative to which the particle has position $\mathbf{r}$ is defined

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad \text{(III-11)}$$

where

$$\mathbf{p} = m\dot{\mathbf{r}} \qquad \text{(III-12)}$$

is the linear momentum of the particle relative to the origin of coordinates. If we now consider a second coordinate frame sharing the same origin O as the inertial frame but rotating relative to it with an angular velocity ω (defined as usual in the right-hand sense), then the velocity of the particle relative to an observer at rest in the rotating frame denoted $d\mathbf{r}_{rot}/dt$ is related to the velocity of the particle relative to an observer at rest in the inertial frame by the kinematical relationship

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}_{rot}}{dt} + \boldsymbol{\omega} \times \mathbf{r} . \qquad \text{(III-13)}$$

It follows from these considerations that the angular momentum of the particle may be equivalently written

$$\mathbf{L} = \mathbf{r} \times m \left(\frac{d\mathbf{r}_{rot}}{dt} + \boldsymbol{\omega} \times \mathbf{r} \right) . \qquad \text{(III-14)}$$

In considering an assemblage of particles of masses m and position vectors $\mathbf{r}_i$ $i = 1, 2, 3, \cdots, N$, we can express the total angular momentum of the assemblage as

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{L}_i$$

where

$$\mathbf{L}_i = \mathbf{r}_i \times m_i \left(\frac{d\mathbf{r}_{i\,rot}}{dt} + \boldsymbol{\omega} \times \mathbf{r}_i \right) \qquad \text{(III-15)}$$

is the angular momentum of the ith particle.

Introducing $\mathbf{V}_i$ where

$$\mathbf{V}_i = \frac{d\mathbf{r}_{i\,rot}}{dt} \qquad \text{(III-16)}$$

is the velocity of the ith particle *relative to the rotating coordinate frame*, it is possible to write the total angular momentum of the assemblage of particles as

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{V}_i + \sum_{i=1}^{N} m_i \left[\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) \right] . \qquad \text{(III-17)}$$

Introducing $\mathbf{p}_i$ where

$$\mathbf{p}_i = m_i \mathbf{V}_i$$

is the linear momentum of the ith particle *relative to the rotating coordinate frame*, we can rewrite Equation (III-17) as

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i + \sum_{i=1}^{N} m_i \left[\mathbf{r}_i \times (\omega \times \mathbf{r}_i)\right] . \quad \text{(III-18)}$$

Introducing $\mathbf{h}$ where

$$\mathbf{h} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i \qquad \text{(III-19)}$$

is the "relative angular momentum" of the assemblage of particles, that is, the angular momentum of the assemblage *relative to the rotating coordinate frame*, then Equation (III-18) can be written

$$\mathbf{L} = \mathbf{h} + \sum_{i=1}^{N} m_i \left[\mathbf{r}_i \times (\omega \times \mathbf{r}_i)\right] . \qquad \text{(III-20)}$$

Using the triple vector product expansion

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\,\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\,\mathbf{C}$$

the total angular momentum of the assemblage of mass points can be written

$$\mathbf{L} = \mathbf{h} + \sum_{i=1}^{N} m_i \left[(\mathbf{r}_i \cdot \mathbf{r}_i)\,\omega - (\mathbf{r}_i \cdot \omega)\,\mathbf{r}_i\right] . \qquad \text{(III-21)}$$

Considering for the moment only the kth component of this vector

$$L_k = h_k + \sum_{i=1}^{N} m_i \left[(\mathbf{r}_i \cdot \mathbf{r}_i)\,\omega_k - (\mathbf{r}_i \cdot \omega)(\mathbf{r}_i)_k\right] . \qquad \text{(III-22)}$$

Now using the Einstein summation and range convention

$$\mathbf{r}_i \cdot \omega = (\mathbf{r}_i)_j\, \omega_j$$

and

$$\mathbf{r}_i \cdot \mathbf{r}_i = r_i^2$$

and so

$$L_k = h_k + \sum_{i=1}^{N} m_i \left[r_i^2\, \omega_k - (\mathbf{r}_i)_j\, \omega_j\, (\mathbf{r}_i)_k\right]$$

which can be written as

$$L_k = h_k + \sum_{i=1}^{N} m_i \left[r_i^2\, \delta_{kj} - (\mathbf{r}_i)_j\, (\mathbf{r}_i)_k\right] \omega_j \qquad \text{(III-23)}$$

where δ_{kj} is the Kronecker delta defined by

$$\delta_{kj} = \begin{cases} 0 & k \neq j \\ 1 & k = j \end{cases} .$$

The quantity I_{kj} defined by

$$I_{kj} = \sum_{i=1}^{N} m_i \left[r_i^2\, \delta_{kj} - (\mathbf{r}_i)_k\, (\mathbf{r}_i)_j\right] \qquad \text{(III-24)}$$

is a second order tensor (although we have not proved its tensor character) referred to as the inertia tensor. The inertia tensor is clearly symmetric in the indices k, j, and its six independent elements consist of the six independent second-order moments of the mass distribution of the assemblage of particles.

With this result we can write the kth component of the angular momentum of the assemblage of particles as

$$L_k = I_{kj}\, \omega_j + h_k \qquad \text{(III-25)}$$

or in coordinate free notation

$$\mathbf{L} = \tilde{I} \cdot \omega + \mathbf{h} . \qquad \text{(III-26)}$$

To avoid possible confusion it should be emphasized that the vector ω was introduced in Equation (III-13) as the angular velocity of a rotating coordinate frame and has *nothing* to do with the angular rotation rate (if any) of the mass assemblage.

It is assumed without loss of generality that at some instant during the rotation the basis vectors spanning the nonrotating frame and the basis vectors spanning the rotating frame coincide. The instant is called the "moment of coincidence."

Fixing attention on the moment of coincidence permits us to compare components of the vector equation (III-13) and the tensor equation (III-24) as well as their time derivatives as represented in the rotating and nonrotating coordinate frames. At the moment of coincidence the basis vectors of the rotating and nonrotating coordinate frames coincide and such comparisons are mathematically permissible.

It should be emphasized that the inertia tensor $\tilde{I}$ consists of the second-order mass moments of the mass distribution taken about the coordinate axes *at the moment of coincidence*. If the mass distribution remains fixed relative to the nonrotating coordinate frame, then clearly the body is not rotating and $\mathbf{L} = 0$. However $\tilde{I} \cdot \omega$ is still nonzero but is cancelled exactly by $\mathbf{h}$. If the mass distribution remains fixed relative to the rotating frame then clearly the body is rotating and $\mathbf{L} \neq 0$. However, $\tilde{I} \cdot \omega$ is exactly the same value as in the nonrotating case! This time, however, $\mathbf{h} = 0$.

There is in general only one circumstance under which the vector ω is to be identified with the "rotation rate of a body" and that is the case wherein a rotating coordinate frame is found such that $\mathbf{h} = 0$. In this case

$$\mathbf{L} = \tilde{I} \cdot \omega \, . \qquad \text{(III-27)}$$

However, ω is *still* the rotation rate of the coordinate system, but the above relationship occurs only for a unique value of ω, which can then be defined as "the rotation rate of the body." This unique rotating coordinate system can always be found for the case of rigid bodies by fixing the rotating coordinate axes relative to the rigid body itself.

This discussion illuminates the essential kinematic nature of the term $\tilde{I} \cdot \omega$ appearing in Equation (III-26). The magnitude of $\tilde{I} \cdot \omega$ can be changed at will by a change of coordinates. It is the sum of $\tilde{I} \cdot \omega$ plus $\mathbf{h}$ which has dynamical significance and yields the quantity $\mathbf{L}$. The term $\tilde{I} \cdot \omega$ only has dynamical significance if the rotating coordinate system is referenced or "attached" in some way to the rotating body.

B. Rotational Dynamics of Extended Deformable Bodies

Passing from the case of an assemblage of particles to a continuous, extended and deformable mass distribution it is necessary to introduce a mass density distribution function $\rho(\mathbf{r}, t)$ which will in general be time-dependent. The mass density distribution function need not be a differentiable or even a continuous function of position as the body may possess internal density discontinuities. The extended deformable body occupies the time-dependent volume $V(t)$ bounded by the time-dependent enclosing surface $S(t)$. The summation over individual particles is replaced by an integration over the volume $V(t)$.

Once again we consider two coordinate frames sharing a common origin O, one coordinate frame rotating and one coordinate frame nonrotating. The angular velocity of the rotating frame is defined by the vector ω. At the moment of coincidence of the two coordinate frames the velocity vector $\dot{\mathbf{r}}$ relative to the nonrotating frame is related to the velocity vector $d\mathbf{r}_{rot}/dt$ relative to the rotating frame by the kinematical relationship

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}_{rot}}{dt} + \omega \times \mathbf{r} \qquad \text{(III-28)}$$

It will be convenient to introduce $\mathbf{v}(\mathbf{r}, t)$ as the velocity vector relative to the rotating coordinate system and $\nu(\mathbf{r}, t)$ as the velocity vector relative to the nonrotating coordinate system. Thus we have

$$\mathbf{v}(\mathbf{r}, t) = \frac{d\mathbf{r}_{rot}}{dt} \qquad \text{(III-29)}$$

$$\nu(\mathbf{r}, t) = \dot{\mathbf{r}} \qquad \text{(III-30)}$$

and so at the moment of coincidence

$$\nu(\mathbf{r}, t) = \mathbf{v}(\mathbf{r}, t) + \omega \times \mathbf{r} \, . \qquad \text{(III-31)}$$

1. Dynamics of rotation without the inertia tensor. The total angular momentum of the extended deformable body is

$$\mathbf{L} = \int_V \rho(\mathbf{r}, t) \, [\mathbf{r} \times \nu(\mathbf{r}, t)] \, dV \qquad \text{(III-32)}$$

and if the origin of coordinates is at the center of mass of the body then the dynamical equation governing the rotation is given by Equation (III-10) as

$$\dot{\mathbf{L}} = \mathbf{N} \tag{III-33}$$

where, as before, the dot "·" indicates a time derivative taken with respect to the nonrotating coordinate system.

In establishing the time derivative of the integral in Equation (III-32) the deformable nature of the body must be explicitly recognized. To do this we consider two instants of time t and $t + dt$ and the increment $d\mathbf{L}$ to the angular momentum which occurs in the interval dt.

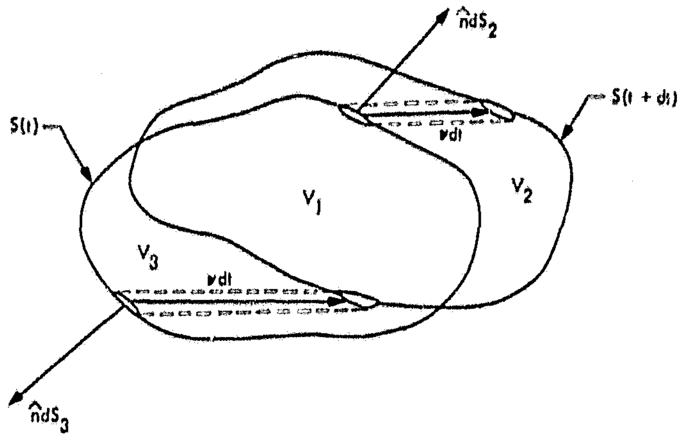

Figure III-1. The mathematical constructions used to evaluate the time derivative of integral quantities whose values depend on an integral taken throughout the volume of a body whose shape is changing with time.

$$d\mathbf{L} = \int_{V(t+dt)} \rho(\mathbf{r}, t+dt)\,[\mathbf{r} \times \nu(\mathbf{r}, t+dt)]\, dV$$
$$- \int_{V(t)} \rho(\mathbf{r}, t)\,[\mathbf{r} \times \nu(\mathbf{r}, t)]\, dV.$$

In evaluating the above integrals we have adopted an Eulerian viewpoint. The vector $\mathbf{r}$ refers to a fixed position in the nonrotating coordinate frame.

The volumes $V(t + dt)$ and $V(t)$ are related by

$$V(t + dt) = V_1 + V_2$$
$$V(t) = V_1 + V_3$$

where, as shown in Figure III-1 (after Prager, 1973),

(1) V_1 is the volume common to both $V(t + dt)$ and $V(t)$.

(2) V_2 is the volume swept out in the interval dt by those portions S_2 of the bounding surface $S(t)$ whose velocity ν (relative to the nonrotating frame) has a component parallel to the positive direction of the unit outward normal $\hat{n}$. S_2 refers to that portion of S for which $\hat{n} \cdot \nu > 0$.

(3) V_3 is the volume swept out in the interval dt by those portions S_3 of the bounding surface $S(t)$ whose velocity ν (relative to the nonrotating frame) has a component parallel to the negative direction of the unit outward normal $\hat{n}$. S_3 refers to that portion of S for which $\hat{n} \cdot \nu < 0$.

And so we have

$$d\mathbf{L} = \int_{V_1+V_2} \rho(\mathbf{r}, t+dt)\,[\mathbf{r} \times \nu(\mathbf{r}, t+dt)]\, dV$$
$$- \int_{V_1+V_3} \rho(\mathbf{r}, t)\,[\mathbf{r} \times \nu(\mathbf{r}, t)]\, dV$$

or

$$d\mathbf{L} = \int_{V_1} \{\rho(\mathbf{r}, t+dt)\,[\mathbf{r} \times \nu(\mathbf{r}, t+dt)]$$
$$- \rho(\mathbf{r}, t)\,[\mathbf{r} \times \nu(\mathbf{r}, t)]\,\}\, dV_1$$
$$+ \int_{V_2} \rho(\mathbf{r}, t+dt)\,[\mathbf{r} \times \nu(\mathbf{r}, t+dt)]\, dV_2$$
$$- \int_{V_3} \rho(\mathbf{r}, t)\,[\mathbf{r} \times \nu(\mathbf{r}, t)]\, dV_3\,.$$

It is clear from Figure III-1 that

$$dV_2 = \hat{n}dS_2 \cdot \nu dt$$

$$dV_3 = -\hat{n}dS_3 \cdot \nu dt$$

and that the volume integrals over volume V_2 V_3 can be replaced, to first order in infinitesimals, by surface integrals over the regions S_2 S_3 respectively

$$\mathbf{dL} = \int_{V_1} \{\rho(\mathbf{r}, t+dt)\,[\mathbf{r} \times \nu(\mathbf{r}, t+dt)]$$

$$- \rho(\mathbf{r}, t)\,[\mathbf{r} \times \nu(\mathbf{r}, t)]\}\, dV_1$$

$$+ \iint_{S_2} \rho(\mathbf{r}, t+dt)\,[\mathbf{r} \times \nu(\mathbf{r}, t+dt)]\, \hat{n}\, dS_2 \cdot \nu dt$$

$$+ \iint_{S_3} \rho(\mathbf{r}, t)\,[\mathbf{r} \times \nu(\mathbf{r}, t)]\, \hat{n} dS_3 \cdot \nu dt\,. \quad \text{(III-34)}$$

We now make the approximation that since dt is an infinitesimal time increment whose magnitude is going to be allowed to shrink in the limit to a vanishingly small quantity, we can set

$$V_1 = V(t)$$

and write

$$\mathbf{dL} = \int_{V(t)} \frac{1}{dt} \{\rho(\mathbf{r}, t+dt)\,[\mathbf{r} \times \nu(\mathbf{r}, t+dt)]$$

$$- \rho(\mathbf{r}, t)\,[\mathbf{r} \times \nu(\mathbf{r}, t)]\}\, dV dt$$

$$+ \iint_{S_2(t)} \rho(\mathbf{r}, t)\,[\mathbf{r} \times \nu(\mathbf{r}, t)]\, \hat{n}\, dS_2 \cdot \nu dt$$

$$+ \iint_{S_3(t)} \rho(\mathbf{r}, t)\,[\mathbf{r} \times \nu(\mathbf{r}, t)]\, \hat{n}\, dS_3 \cdot \nu dt\,. \quad \text{(III-35)}$$

The entire area of the bounding surface $S(t)$ is divided into regions which

(1) lie in S_2,

(2) lie in S_3,

(3) lie neither in S_2 nor S_3.

It follows from the definition of S_2 and S_3 that for the latter classification which lie neither in S_2 or S_3 we necessarily have $\hat{n} ds \cdot \nu dt = 0$. It follows that including these regions in the surface integrals will not alter the value of the integrals. Consequently we may write

$$\mathbf{dL} = \int_{V(t)} \frac{1}{dt} \{\rho(\mathbf{r}, t+dt)\,[\mathbf{r} \times \nu(\mathbf{r}, t+dt)]$$

$$- \rho(\mathbf{r}, t)\,[\mathbf{r} \times \nu(\mathbf{r}, t)]\}\, dV dt$$

$$+ \iint_{S(t)} \rho(\mathbf{r}, t)\,[\mathbf{r} \times \nu(\mathbf{r}, t)]\, \hat{n}\, dS \cdot \nu dt$$

from which it follows that

$$\dot{\mathbf{L}} = \int_{V(t)} \frac{1}{dt} \{\rho(\mathbf{r}, t+dt)\,[\mathbf{r} \times \nu(\mathbf{r}, t+dt)]$$

$$- \rho(\mathbf{r}, t)\,[\mathbf{r} \times \nu(\mathbf{r}, t)]\}\, dV$$

$$+ \iint_{S(t)} \rho(\mathbf{r}, t)\,[\mathbf{r} \times \nu(\mathbf{r}, t)]\, \nu(\mathbf{r}, t) \cdot \hat{n}\, dS\,. \quad \text{(III-36)}$$

In Equation (III-36) we recognize the Eulerian time derivative taken with respect to the nonrotating frame

$$\frac{\partial \ell(\mathbf{r}, t)}{\partial t} = \frac{1}{dt} \{\rho(\mathbf{r}, t+dt)\,[\mathbf{r} \times \nu(\mathbf{r}, t+dt)]$$

$$- \rho(\mathbf{r}, t)\,[\mathbf{r} \times \nu(\mathbf{r}, t)]\} \quad \text{(III-37)}$$

where $\ell(\mathbf{r}, t)$ given by

$$\ell(\mathbf{r}, t) = \rho(\mathbf{r}, t)\,[\mathbf{r} \times \nu(\mathbf{r}, t)] \quad \text{(III-38)}$$

is an angular momentum density field defined also relative to the nonrotating frame.

Consequently Equation (III-36) can be written

$$\dot{\mathbf{L}} = \int_{V(t)} \frac{\partial \ell}{\partial t}\, dV + \iint_{S(t)} \ell \nu \cdot \hat{n}\, dS\,. \quad \text{(III-39)}$$

The quantity $\mathfrak{L}\mathbf{v}$ appearing in Equation (III-39) and given by

$$\mathfrak{L}\nu = \rho(\mathbf{r}, t)\,[\mathbf{r} \times \nu\,(\mathbf{r}, t)]\,\nu\,(\mathbf{r}, t) \qquad \text{(III-40)}$$

is a second-order tensor. It is an angular momentum flux density and shall be denoted $\widetilde{\mathscr{L}}(\mathbf{r}, t)$.

$$\widetilde{\mathscr{L}}(\mathbf{r}, t) = \rho(\mathbf{r}, t)[\mathbf{r} \times \nu\,(\mathbf{r}, t)]\,\nu(\mathbf{r}, t)\,. \qquad \text{(III-41)}$$

The elements $\mathscr{L}_{ij}$ of the tensor density $\widetilde{\mathscr{L}}$ are

$$\mathscr{L}_{ij} = \rho\,(\mathbf{r}, t)\,[\mathbf{r} \times \nu\,(\mathbf{r}, t)]_i\,\nu_j\,(\mathbf{r}, t)$$

and so Equation (III-39) can be written

$$\dot{\mathbf{L}} = \int_{V(t)} \frac{\partial \mathfrak{L}}{\partial t}\,dV + \iint_{S(t)} \widetilde{\mathscr{L}} \cdot \hat{n}\,dS\,. \qquad \text{(III-42)}$$

We may now use Gauss's theorem to convert the surface integral in Equation (III-42) into a volume integral

$$\iint_{S(t)} \widetilde{\mathscr{L}} \cdot \hat{n}\,dS = \int_{V(t)} \nabla \cdot \widetilde{\mathscr{L}}\,dV$$

where

$$\nabla \cdot \widetilde{\mathscr{L}} = \frac{\partial \mathscr{L}_{ij}}{\partial x_j} \qquad \text{(III-43)}$$

which then gives

$$\dot{\mathbf{L}} = \int_{V(t)} \left(\frac{\partial \mathfrak{L}}{\partial t} + \nabla \cdot \widetilde{\mathscr{L}} \right) dV\,. \qquad \text{(III-44)}$$

It can be shown directly that the integrand of Equation (III-44) can be written

$$\frac{\partial \mathfrak{L}}{\partial t} + \nabla \cdot \widetilde{\mathscr{L}} = \frac{\partial \rho}{\partial t}(\mathbf{r} \times \nu) + \rho\,(\mathbf{r} \times \dot{\nu})$$

$$+ \nabla \cdot (\rho\nu)\,(\mathbf{r} \times \nu) + \rho\nu \cdot \nabla\,(\mathbf{r} \times \nu)\,. \qquad \text{(III-45)}$$

Collecting terms in Equation (III-45) gives

$$\frac{\partial \mathfrak{L}}{\partial t} + \nabla \cdot \widetilde{\mathscr{L}} = \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\nu)\right](\mathbf{r} \times \nu)$$

$$+ \rho\,(\mathbf{r} \times \dot{\nu}) + \rho\nu \cdot \nabla\,(\mathbf{r} \times \nu)$$

and by invoking the equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\nu) = 0 \qquad \text{(III-46)}$$

This reduces to

$$\frac{\partial \mathfrak{L}}{\partial t} + \nabla \cdot \widetilde{\mathscr{L}} = \rho\,(\mathbf{r} \times \nu) + \rho\nu \cdot \nabla\,(\mathbf{r} \times \nu) \qquad \text{(III-47)}$$

and Equation (III-44) assumes a final form

$$\dot{\mathbf{L}} = \int_{V(t)} \left[\rho\,(\mathbf{r} \times \dot{\nu}) + \rho\nu \cdot \nabla\,(\mathbf{r} \times \nu)\right] dV\,. \qquad \text{(III-48)}$$

The RHS of the governing equation (III-32) represents the torque acting on the extended deformable body. The net torque on the body arises as a result of the combined actions of a system of body forces $\mathbf{f}$ and surface stresses $\widetilde{S}$. $\mathbf{N}$ can be expressed quite generally as

$$\mathbf{N}(t) = \int_{V(t)} \mathbf{r} \times \mathbf{f}\,dV + \iint_{S(t)} \widetilde{S} \cdot \hat{n}\,dS \qquad \text{(III-49)}$$

where it is understood that in general both $\mathbf{f}$ and S are time-dependent fields.

The complication which must be borne in mind when applying Equation (III-49) is that usually the vector field $\mathbf{f}$ representing the system of body forces and the tensor field S representing the system of surface stresses are referenced to the material of the rotating body. When expressed as vectors and tensors in the nonrotating coordinate system the body forces will in general appear as the time-dependent vector field $\mathbf{f}(\mathbf{r}, t)$ and the surface stresses will in general appear as the time-dependent tensor field $\widetilde{S}(\mathbf{r}, t)$, where $\mathbf{r}$ refers to a fixed position in the nonrotating frame, even if, when viewed by an observer at rest relative to the body, these fields have no time dependence. Thus in the nonrotating frame

$$\mathbf{N}(t) = \int_{V(t)} \mathbf{r} \times \mathbf{f}(\mathbf{r},t)\, dV + \iint_{S(t)} \tilde{S}(\mathbf{r},t) \cdot \hat{n}\, dS \tag{III-50}$$

The bounding surface S defining the volume V of "the body" can be drawn arbitrarily. Material density $\rho(\mathbf{r}, t)$ excluded from the volume is regarded as an external medium not belonging to "the body." In such cases the external medium can in general act on "the body" by a system of induced body forces and surface stresses.

2. Dynamics of rotation with the inertia tensor. The inertia tensor can be introduced into the dynamics by the kinematical expedient of introducing an arbitrary vector ω such that the velocity field $\nu(\mathbf{r}, t)$ is kinematically decomposed into

$$\nu(\mathbf{r}, t) = \mathbf{v}(\mathbf{r}, t) + \omega \times \mathbf{r}$$

which when substituted into Equation (III-32) gives the total angular momentum as

$$\mathbf{L} = \int_{V(t)} \rho(\mathbf{r},t)\, [\mathbf{r} \times \mathbf{v}(\mathbf{r},t) + \mathbf{r} \times (\omega \times \mathbf{r})]\, dV \tag{III-51}$$

where $\mathbf{v}(\mathbf{r}, t)$ is a "residual" velocity field (which may or may not be small depending on the choice of ω) defined as a function of position $\mathbf{r}$ in the nonrotating frame.

Introducing $\mathbf{h}$ the "residual" angular momentum vector

$$\mathbf{h} = \int_{V(t)} \rho(\mathbf{r},t)\, [\mathbf{r} \times \mathbf{v}(\mathbf{r},t)]\, dV \tag{III-52}$$

we may write Equation (III-51) as

$$\mathbf{L} = \int_{V(t)} \rho(\mathbf{r},t)\, [\mathbf{r} \times (\omega \times \mathbf{r})]\, dV + \mathbf{h} \tag{III-53}$$

By a set of manipulations identical to those carried out in Equations (III-20) through (III-24) it is possible to show that Equation (III-53) may be written as

$$\mathbf{L} = \tilde{I} \cdot \omega + \mathbf{h}\,. \tag{III-54}$$

Here the inertia tensor I is given by

$$\tilde{I} = \int_{V(t)} \rho(\mathbf{r},t)\, [r^2 \tilde{1} - \mathbf{r}\,\mathbf{r}]\, dV \tag{III-55}$$

where $\tilde{1}$ is the unit tensor

$$\tilde{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{III-56}$$

and where

$$\begin{aligned} r^2 &= \mathbf{r} \cdot \mathbf{r} \\ \mathbf{r}\,\mathbf{r} &= r_k r_j\,. \end{aligned} \tag{III-57}$$

Substituting Equation (III-54) into the dynamical equation governing rotation, Equation (III-33) gives

$$\dot{\tilde{I}} \cdot \omega + \tilde{I} \cdot \dot{\omega} + \dot{\mathbf{h}} = \mathbf{N} \tag{III-58}$$

where the indicated time derivatives are reckoned relative to the nonrotating frame of reference.

Once again we are faced with the problem of establishing the time derivative with respect to the nonrotating frame of quantities, namely $\tilde{I}$ and $\mathbf{h}$, which depend on volume integrals carried out in the nonrotating frame.

We proceed in the same manner as before by considering two instants of time t and $t + dt$ and the increments $d\mathbf{h}$ and $d\tilde{I}$ to the relative angular momentum and inertia tensor respectively in the interval dt.

$$d\mathbf{h} = \int_{V(t+dt)} \rho\,(\mathbf{r}, t + dt)\, [\mathbf{r} \times \mathbf{v}\,(\mathbf{r}, t + dt)]\, dV$$

$$- \int_{V(t)} \rho\,(\mathbf{r}, t)\, [\mathbf{r} \times \mathbf{v}\,(\mathbf{r}, t)]\, dV \tag{III-59}$$

$$d\tilde{I} = \int_{V(t+dt)} \rho\,(\mathbf{r}, t + dt)\, [r^2 \tilde{1} - \mathbf{r}\,\mathbf{r}]\, dV$$

$$- \int_{V(t)} \rho\,(\mathbf{r}, t)\, [r^2 \tilde{1} - \mathbf{r}\,\mathbf{r}]\, dV \tag{III-60}$$

can be written as

$$d\mathbf{h} = \int_{V(t)} \frac{1}{dt} \{ \rho(\mathbf{r}, t+dt) [\mathbf{r} \times \mathbf{v}(\mathbf{r}, t+dt)]$$

$$- \rho(\mathbf{r}, t) [\mathbf{r} \times \mathbf{v}(\mathbf{r}, t)] \} dV dt$$

$$+ \iint_{S(t)} \rho(\mathbf{r}, t) [\mathbf{r} \times \mathbf{v}(\mathbf{r}, t)] \hat{n}\, dS \cdot \nu\, dt \tag{III-61}$$

$$d\widetilde{T} = \int_{V(t)} \frac{1}{dt} \{ \rho(\mathbf{r}, t+dt) - \rho(\mathbf{r}, t) \} [r^2 \widetilde{I} - \mathbf{r}\mathbf{r}]\, dV\, dt$$

$$+ \iint_{S(t)} \rho(\mathbf{r}, t) [r^2 \widetilde{I} - \mathbf{r}\mathbf{r}]\, \hat{n}\, dS \cdot \nu\, dt. \tag{III-62}$$

From this it follows that

$$\dot{\mathbf{h}} = \int_{V(t)} \frac{\partial \boldsymbol{h}}{\partial t} dV + \int_{S(t)} \widetilde{\mathcal{H}} \cdot \hat{n}\, dS \tag{III-63}$$

where $\boldsymbol{h}(\mathbf{r}, t)$ is a residual angular momentum density given by

$$\boldsymbol{h}(\mathbf{r}, t) = \rho(\mathbf{r}, t) [\mathbf{r} \times \mathbf{v}(\mathbf{r}, t)] \tag{III-64}$$

and where $\widetilde{\mathcal{H}}$ is a residual angular momentum flux density given by

$$\widetilde{\mathcal{H}}(\mathbf{r}, t) = \rho(\mathbf{r}, t) [\mathbf{r} \times \mathbf{v}(\mathbf{r}, t)]\, \boldsymbol{\nu}(\mathbf{r}, t). \tag{III-65}$$

The quantity $\widetilde{\mathcal{H}}$ is a second-order tensor density whose elements are $\mathcal{H}_{ij}$ given by

$$\mathcal{H}_{ij} = \rho(\mathbf{r}, t) [\mathbf{r} \times \mathbf{v}(\mathbf{r}, t)]_i\, \nu_j(\mathbf{r}, t). \tag{III-66}$$

It also follows from this that

$$\dot{\widetilde{I}} = \int_{V(t)} \frac{\partial \rho(\mathbf{r}, t)}{\partial t} (r^2 \widetilde{I} - \mathbf{r}\mathbf{r})\, dV + \iint_{S(t)} \widetilde{\mathcal{J}} \cdot \hat{n}\, dS \tag{III-67}$$

where $\widetilde{\mathcal{J}}(\mathbf{r}, t)$ is an inertia flux density given by

$$\widetilde{\mathcal{J}}(\mathbf{r}, t) = \rho(\mathbf{r}, t) (r^2 \widetilde{I} - \mathbf{r}\mathbf{r})\, \nu(\mathbf{r}, t). \tag{III-68}$$

$\widetilde{\mathcal{J}}(\mathbf{r}, t)$ is a third-order tensor density with elements $\mathcal{J}_{ijk}$ given by

$$\mathcal{J}_{ijk} = \rho(\mathbf{r}, t) (r^2 \delta_{ij} - r_i r_j)\, \nu_k(\mathbf{r}, t). \tag{III-69}$$

We may now use Gauss's theorem to convert the surface integrals in Equations (III-63) (III-67) into volume integrals to obtain

$$\dot{\mathbf{h}} = \int_{V(t)} \left\{ \frac{\partial h}{\partial t} + \nabla \cdot \widetilde{\mathcal{H}} \right\} dV \tag{III-70}$$

$$\dot{\widetilde{I}} = \int_{V(t)} \left\{ \frac{\partial \rho(\mathbf{r}, t)}{\partial t} [r^2 \widetilde{I} - \mathbf{r}\mathbf{r}] + \nabla \cdot \widetilde{\mathcal{J}} \right\} dV \tag{III-71}$$

where

$$\nabla \cdot \widetilde{\mathcal{H}} = \frac{\partial \mathcal{H}_{ij}}{\partial x^j} \tag{III-72}$$

$$\nabla \cdot \widetilde{\mathcal{J}} = \frac{\partial \mathcal{J}_{ijk}}{\partial x^k} \tag{III-73}$$

3. Discussion. A theoretical description of the rotational dynamics of an extended, generally deformable body has been presented both with and without the introduction of the inertia tensor. The velocity of the material of the body and the deformation of the body with respect to time relative to the system of coordinates is accommodated, in the first place by the introduction of $\widetilde{\mathcal{P}}(\mathbf{r}, t)$, the tensor density flux of absolute angular momentum, and in the second place by the introduction of $\widetilde{\mathcal{H}}(\mathbf{r}, t)$ and $\widetilde{\mathcal{J}}(\mathbf{r}, t)$, the tensor density fluxes of residual angular momentum and inertia respectively.

Although this development illustrates the theoretical tools necessary to handle problems of this sort, the choice of a nonrotating frame of reference in which to describe the dynamics of rotating bodies is generally a poor one. This fact can be illustrated by considering the case of the rotation of a rigid body. Even in this simple case the tensor fields $\widetilde{\mathcal{P}}$, $\widetilde{\mathcal{H}}$, and $\widetilde{\mathcal{J}}$ do not in general vanish although $\widetilde{\mathcal{H}}$ can be made to vanish by an appropriate choice of $\boldsymbol{\omega}$.

The term $\tilde{I} \cdot \dot{\omega}$ appearing in Equation (III-58) has in general only kinematical significance. Its value can be changed at will by a change in the choice of ω. In particular it can be made to vanish by choosing ω as constant. It is the combination of $\tilde{I} \cdot \dot{\omega}$ plus $\dot{\mathbf{h}}$ which has true dynamical significance. Effects not appearing in one will appear in the other.

For the case of rigid bodies and quasirigid bodies the rotational dynamics can be greatly simplified by transforming to a rotating frame of reference. Such a transformation is not accomplished by simply introducing the inertia tensor into the dynamics as in Equation (III-55), for this tensor is still defined by moments of the mass distribution taken about the nonrotating coordinate axes. The transformation to a rotating frame of reference is accomplished by the Liouville equation in which the necessary time derivatives are taken with reference to a rotating system of coordinates.

C. The Liouville and Euler Equations

To obtain the Liouville equation we begin with the dynamical equation governing earth rotation expressed in a nonrotating center of mass coordinate system

$$\dot{\mathbf{L}} = \mathbf{N}. \qquad \text{(III-74)}$$

We then consider a rotating center of mass coordinate system whose rotation rate relative to the nonrotating coordinates is given by the angular velocity vector ω. At the moment of coincidence we can relate the time derivative $\dot{\mathbf{L}}$ taken with respect to the nonrotating frame to time derivative $d\mathbf{L}/dt$ taken with respect to the rotating frame by the kinematical relationship

$$\dot{\mathbf{L}} = \frac{d\mathbf{L}}{dt} + \omega \times \mathbf{L}, \qquad \text{(III-75)}$$

The angular momentum of the rotating body can be expressed as

$$\mathbf{L} = \tilde{I} \cdot \omega + \mathbf{h} \qquad \text{(III-76)}$$

Equations (III-74) (III-75) (III-76) together yield

$$\frac{d}{dt}(\tilde{I} \cdot \omega + \mathbf{h}) + \omega \times (\tilde{I} \cdot \omega + \mathbf{h}) = \mathbf{N} \qquad \text{(III-77)}$$

which is the Liouville equation first obtained by Liouville in 1858.

At this point in the analysis the elements of the tensor $\tilde{I}$ and the components of the vector $\mathbf{h}$ are by definition (Equations III-9 and III-24) reckoned relative to the basis vectors of the rotating coordinate frame and in particular the appearance of the operator "d/dt" rather than the operator "$\cdot$" on the LHS of Equation (III-77) indicates that we are to differentiate these quantities with respect to the basis vectors of the rotating frame. That is, we are to consider the rate at which the components of $\tilde{I}$ and $\mathbf{h}$ are changing with time when projected onto the basis vectors of the rotating coordinate frame. By the same token the components of the torque $\mathbf{N}$ appearing on the RHS of Equation (III-77) must also be given with respect to the basis vectors of the rotating coordinate frame. This aspect of the Liouville equation is discussed at some length in Munk and MacDonald (1960 pp 12-14).

Written out in full

$$\frac{d\tilde{I}}{dt} \cdot \omega + \tilde{I} \cdot \frac{d\omega}{dt} + \frac{d\mathbf{h}}{dt} + \omega \times \tilde{I} \cdot \omega + \omega \times \mathbf{h} = \mathbf{N} \qquad \text{(III-78)}$$

For a generally deformable body we can use the previous arguments to show that in the rotating frame

$$\frac{d\mathbf{h}}{dt} = \int_{V(t)} \frac{\partial h}{\partial t} dV + \iint_{S(t)} \tilde{\mathcal{H}}_{rot} \cdot \hat{n}\, dS, \qquad \text{(III-79)}$$

and

$$\frac{d\tilde{I}}{dt} = \int_{V(t)} \frac{\partial \rho(\mathbf{r}, t)}{\partial t} (r^2 \tilde{I} - \mathbf{r}\,\mathbf{r})\, dV + \iint_{S(t)} \tilde{\mathcal{I}}_{rot} \cdot \hat{n}\, dS,$$

where $\mathbf{r}$ now refers to a fixed position in the rotating frame, and as before the relative angular momentum density $h(\mathbf{r}, t)$ is given by

$$h(\mathbf{r}, t) = \rho(\mathbf{r}, t)\, [\mathbf{r} \times \mathbf{v}(\mathbf{r}, t)] \qquad \text{(III-81)}$$

The rotating frame tensor density fluxes $\tilde{\mathcal{H}}_{rot}$ and $\tilde{\mathcal{I}}_{rot}$ are by definition obtained from their nonrotating frame equivalents $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{I}}$ by the replacement of $\nu(\mathbf{r}, t)$ with $\mathbf{v}(\mathbf{r}, t)$.

Thus

$$\tilde{\mathcal{H}}_{rot}(\mathbf{r}, t) = \rho(\mathbf{r}, t)\, [\mathbf{r} \times \mathbf{v}(\mathbf{r}, t)]\, \mathbf{v}(\mathbf{r}, t) \qquad \text{(III-82)}$$

$$\tilde{\mathcal{I}}_{rot}(\mathbf{r}, t) = \rho(\mathbf{r}, t) [r^2 \tilde{I} - \mathbf{r}\,\mathbf{r}]\, \mathbf{v}(\mathbf{r}, t). \qquad \text{(III-83)}$$

Using Gauss's theorem in Equations (III-79), (III-80) gives

$$\frac{dh}{dt} = \int_{V(t)} \left(\frac{\partial h}{dt} + \nabla \cdot \widetilde{\mathcal{H}}_{rot}\right) dV \qquad \text{(III-84)}$$

$$\frac{d\widetilde{I}}{dt} = \int_{V(t)} \left[\frac{\partial \rho}{\partial t}(r^2\, \widetilde{I} - \mathbf{r}\,\mathbf{r}) + \nabla \cdot \widetilde{\mathcal{I}}_{rot}\right] dV. \qquad \text{(III-85)}$$

In the case of a rigid body ω can be chosen so that $\mathbf{v}(\mathbf{r}, t)$ vanishes with the consequence that:

$$\frac{\partial \rho}{\partial t} = 0$$

$$h = 0$$

$$\widetilde{\mathcal{H}}_{rot} = 0$$

$$\widetilde{\mathcal{I}}_{rot} = 0$$

The maintenance of the above four conditions implies that the coordinate system remains in corotation with the rigid body or that the coordinate system is "attached" to the rigidly rotating body. When expressed in the corotating coordinate system the dynamical equation governing the rotation of a rigid body reduces to

$$\widetilde{I} \cdot \frac{d\omega}{dt} = \mathbf{N} - \omega \times \widetilde{I} \cdot \omega. \qquad \text{(III-86)}$$

The quantity $-\omega \times \widetilde{I} \cdot \omega$ is called the gyroscopic torque and vanishes if the rotation axis coincides with any of the principal axes of inertia. To prove this we consider the inertia tensor expressed in the principal axes coordinate system

$$\widetilde{I} = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$$

and the rotation vector expressed in the same coordinate system is

$$\omega = \omega_1\, e_1 + \omega_2\, e_2 + \omega_3\, e_3$$

Then

$$\begin{aligned} \omega \times \widetilde{I} \cdot \omega = {} & \omega_2 \omega_3 (C-B) \hat{e}_1 \\ & + \omega_1 \omega_3 (A-C)\, \hat{e}_2 \\ & + \omega_1 \omega_2 (B-A) \hat{e}_3. \end{aligned} \qquad \text{(III-87)}$$

If ω lies along a principal axis of inertia then *two* components of $\omega_1\ \omega_2\ \omega_3$ must vanish and so $\omega \times \widetilde{I} \cdot \omega$ vanishes.

If Equation (III-86) is expressed in the principal axes coordinate system it becomes

$$\begin{aligned} A \frac{d\omega_1}{dt} \hat{e}_1 + B \frac{d\omega_2}{dt} \hat{e}_2 + C \frac{d\omega_3}{dt} = {} & [N_1 - \omega_2 \omega_3 (C-B)]\, \hat{e}_1 \\ & + [N_2 - \omega_1 \omega_3 (A-C)]\, \hat{e}_2 + [N_3 - \omega_1 \omega_2 (B-A)]\, \hat{e}_3 \end{aligned} \qquad \text{(III-88)}$$

This is Euler's equation for the dynamics of rigidly rotating bodies obtained by Euler in 1765.

A comparison of the Liouville equation, Equation (III-78), valid in the rotating frame with Equation (III-58), its counterpart valid in the nonrotating frame procedures

$$\dot{\widetilde{I}} \cdot \omega + \widetilde{I} \cdot \dot{\omega} + \dot{\mathbf{h}} = \mathbf{N} \qquad \text{(III-58)}$$

$$\frac{d\widetilde{I}}{dt} \cdot \omega + \widetilde{I} \cdot \frac{d\omega}{dt} + \frac{d\mathbf{h}}{dt} = \mathbf{N} - \omega \times \mathbf{L} \qquad \text{(III-78)}$$

where we have used

$$\mathbf{L} = \widetilde{I} \cdot \omega + \mathbf{h}$$

in Equation (III-78).

D. Kinetic Energy of a Rotating, Extended, Deformable Body

The total kinetic energy of a rotating, extended, deformable body is T where

$$T = \int_V 1/2\, |\, \dot{\mathbf{S}} + \dot{\mathbf{r}}\, |^2\, dm \qquad \text{(III-89)}$$

where the integral is taken over the entire volume of the body under consideration. In Equation (III-89) $\mathbf{S}$ is the velocity of the origin of coordinates relative to inertial space and $\mathbf{r}$ is the velocity of the mass element dm relative to the origin of coordinates.

Now

$$|\dot{\mathbf{S}} + \dot{\mathbf{r}}|^2 = |\dot{\mathbf{S}}|^2 + |\dot{\mathbf{r}}|^2 + 2\, \dot{\mathbf{S}} \cdot \dot{\mathbf{r}} \qquad \text{(III-90)}$$

and so the total kinetic energy can be written

$$T = \int_V 1/2\,|\dot{\mathbf{S}}|^2\,dm + \int_V 1/2\,|\dot{\mathbf{r}}|^2\,dm + \int_V \dot{\mathbf{S}}\cdot\dot{\mathbf{r}}\,dm. \tag{III-91}$$

The vector $\dot{\mathbf{S}}$ is common to all the mass elements in V and can pass through the integrals

$$T = 1/2\,|\dot{\mathbf{S}}|^2 \int_V dm + 1/2 \int_V |\dot{\mathbf{r}}|^2\,dm + \dot{\mathbf{S}}\cdot\int_V \dot{\mathbf{r}}\,dm. \tag{III-92}$$

The position of the center of mass of the body relative to our origin of coordinates is denoted by $\boldsymbol{\rho}$ where

$$\boldsymbol{\rho} = \frac{\int_V \mathbf{r}\,dm}{\int_V dm} = \frac{1}{M}\int_V \mathbf{r}\,dm \tag{III-93}$$

and where M is the total mass of the body.

It follows from Equation (III-92) that

$$\int_V \dot{\mathbf{r}}\,dm = M\dot{\boldsymbol{\rho}} \tag{III-94}$$

and using this result in Equation (III-92) the expression for the total kinetic energy of the body becomes

$$T = 1/2\,M\,|\dot{\mathbf{S}}|^2 + M\,\dot{\mathbf{S}}\cdot\dot{\boldsymbol{\rho}} + 1/2\int_V |\dot{\mathbf{r}}|^2\,dm. \tag{III-95}$$

Equation (III-95) illustrates how the total kinetic energy of a rotating extended deformable body can be decomposed into a translational kinetic energy associated with the motion of the center of mass relative to inertial space denoted T_{trans} and a rotational kinetic energy associated with velocity of rotation of the body about its center of mass denoted T_{rot}. Thus we may rewrite Equation (III-95) as

$$T = T_{trans} + T_{rot} \tag{III-96}$$

where

$$T_{trans} = 1/2\,M\,|\dot{\mathbf{S}}|^2 + M\,\dot{\mathbf{S}}\cdot\dot{\boldsymbol{\rho}} \tag{III-97}$$

and

$$T_{rot} = 1/2\int_V |\dot{\mathbf{r}}|^2\,dm. \tag{III-98}$$

If the origin of our coordinate system is placed at the center of mass of the body then $\boldsymbol{\rho} = 0$ and the translational kinetic energy reduces to

$$T_{trans} = 1/2\,M\,|\dot{\mathbf{S}}|^2 \tag{III-99}$$

where $\dot{\mathbf{S}}$ is now the velocity of the center of mass of the body relative to inertial space.

We shall not concern ourselves further with the properties of the translational kinetic energy but shall investigate in some detail the properties of the rotational kinetic energy.

The rotation vector $\boldsymbol{\omega}$ for a rotating, deformable extended body with fluid regions can be unambiguously defined as the angular rotation vector of the mean body axes frame in which the relative angular momentum $\mathbf{h}$ vanishes. (Note that there is no need for $\boldsymbol{\omega}$ to be parallel to any of the three body-fixed basis vectors of the mean body axes frame.) If $d\mathbf{r}_{rot}/dt$ is the velocity of a mass element relative to an observer fixed at position $\mathbf{r}$ in the rotating mean body axes frame then $d\mathbf{r}_{rot}/dt$ is related to $\dot{\mathbf{r}}$, the velocity of the mass element relative to an observer fixed in inertial space, by the formula

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}_{rot}}{dt} + \boldsymbol{\omega}\times\mathbf{r} \tag{III-100}$$

and so the rotational kinetic energy T_{rot} can be expressed as

$$T_{rot} = 1/2\int_V \left|\frac{d\mathbf{r}_{rot}}{dt} + \boldsymbol{\omega}\times\mathbf{r}\right|^2 dm$$

which becomes

$$T_{rot} = 1/2\int_V \left|\frac{d\mathbf{r}_{rot}}{dt}\right|^2 dm + \int_V \frac{d\mathbf{r}_{rot}}{dt}\cdot(\boldsymbol{\omega}\times\mathbf{r})\,dm + 1/2\int_V |\boldsymbol{\omega}\times\mathbf{r}|^2\,dm \tag{III-101}$$

Using the vector identities:

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) \equiv \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$

(2) $|\mathbf{A}\times\mathbf{B}|^2 = (\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{B})$

(3) $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) \equiv \mathbf{A}\cdot[\mathbf{B}\times(\mathbf{C}\times\mathbf{D})]$

Equation (III-101) can be written

$$T_{rot} = 1/2 \int_V |\mathbf{v}|^2 \, dm + \omega \cdot \int_V (\mathbf{r} \times \mathbf{v}) \, dm$$

$$+ 1/2 \, \omega \cdot \int_V [\mathbf{r} \times (\omega \times \mathbf{r})] \, dm \qquad \text{(III-102)}$$

where we have made use of Equation (III-29).

Defining the mass element dm as

$$dm = \rho(\mathbf{r}, t) \, dV$$

and using Equations (III-52), (III-53) and (III-55), Equation (III-102) can be written as

$$T_{rot} = 1/2 \int_V |\mathbf{v}|^2 \, dm + \omega \cdot \mathbf{h} + 1/2 \, \omega \cdot \tilde{I} \cdot \omega \qquad \text{(III-103)}$$

It is of interest to obtain an expression for the time derivative of the rotational kinetic energy T_{rot} of an extended deformable body. Since T_{rot} is a scalar quantity we can conclude that its time derivative can be taken relative to a rotating or nonrotating frame of reference with identical results. If we persist with the convention of using dot "•" to indicate a time derivative taken with respect to the nonrotating frame and d/dt to indicate a time derivative taken with respect to a rotating frame then

$$\dot{T}_{rot} \equiv \frac{dT_{rot}}{dt} \qquad \text{(III-104)}$$

where

$$\dot{T}_{rot} = 1/2 \int_V \dot{|\mathbf{v}|^2} \, dm + \dot{\omega} \cdot \mathbf{h} + \omega \cdot \dot{\mathbf{h}} + 1/2 \, \dot{\omega} \cdot \tilde{I} \cdot \omega$$

$$+ 1/2 \, \omega \cdot \dot{\tilde{I}} \cdot \omega + 1/2 \, \omega \cdot \tilde{I} \cdot \dot{\omega} \qquad \text{(III-105)}$$

$$\frac{dT_{rot}}{dt} = 1/2 \frac{d}{dt} \int_V |\mathbf{v}|^2 \, dm + \frac{d\omega}{dt} \cdot \mathbf{h} + \omega \cdot \frac{d\mathbf{h}}{dt}$$

$$+ 1/2 \frac{d\omega}{dt} \cdot \tilde{I}\omega + 1/2 \, \omega \cdot \frac{d\tilde{I}}{dt} \cdot \omega$$

$$+ 1/2 \, \omega \cdot \tilde{I} \cdot \frac{d\omega}{dt} \qquad \text{(III-106)}$$

are respectively the space-fixed and body-fixed expressions for the rate of change of kinetic energy.

Since the inertia tensor is symmetric it can easily be shown that

$$\dot{\omega} \cdot \tilde{I} \cdot \omega = \omega \cdot \tilde{I} \cdot \dot{\omega}. \qquad \text{(III-107)}$$

$$\frac{d\omega}{dt} \cdot \tilde{I} \cdot \omega = \omega \cdot \tilde{I} \cdot \frac{d\omega}{dt}, \qquad \text{(III-108)}$$

and so Equations (III-105) (III-106) can be written

$$\dot{T}_{rot} = 1/2 \int_V \dot{|\mathbf{v}|^2} \, dm + \dot{\omega} \cdot \mathbf{h} + \omega \cdot \dot{\mathbf{h}} + \omega \cdot \tilde{I} \cdot \dot{\omega}$$

$$+ 1/2 \, \omega \cdot \dot{\tilde{I}} \cdot \omega \qquad \text{(III-109)}$$

$$\frac{dT_{rot}}{dt} = 1/2 \frac{d}{dt} \left(\int_V |\mathbf{v}|^2 \, dm \right) + \frac{d\omega}{dt} \cdot \mathbf{h} + \omega \cdot \frac{d\mathbf{h}}{dt}$$

$$+ \omega \cdot \tilde{I} \cdot \frac{d\omega}{dt} + 1/2 \, \omega \cdot \frac{d\tilde{I}}{dt} \cdot \omega \qquad \text{(III-110)}$$

Adding and subtracting $1/2 \, \omega \cdot \dot{\tilde{I}} \cdot \omega$ and $1/2 \, \omega \cdot d\tilde{I}/dt \cdot \omega$ in Equations (III-109) (III-110) gives

$$\dot{T}_{rot} = 1/2 \int_V \dot{|\mathbf{v}|^2} \, dm + \dot{\omega} \cdot \mathbf{h} + \omega \cdot (\dot{\tilde{I}} \cdot \omega + \tilde{I} \cdot \dot{\omega} + \dot{\mathbf{h}})$$

$$- 1/2 \, \omega \cdot \dot{\tilde{I}} \cdot \omega \qquad \text{(III-111)}$$

$$\frac{dT_{rot}}{dt} = 1/2 \frac{d}{dt} \left(\int_V |\mathbf{v}|^2 \, dm \right) + \frac{d\omega}{dt} \cdot \mathbf{h} + \omega \cdot \left(\frac{d\tilde{I}}{dt} \cdot \omega \right.$$

$$\left. + \tilde{I} \cdot \frac{d\omega}{dt} + \frac{d\mathbf{h}}{dt} \right) - 1/2 \, \omega \cdot \frac{d\tilde{I}}{dt} \cdot \omega. \qquad \text{(III-112)}$$

Using Equations (III-58) (III-78) in Equations (III-111) and (III-112) reduces them to

$$\dot{T}_{rot} = 1/2\int_V \dot{|\mathbf{v}|^2}\, dm + \dot{\omega}\cdot\mathbf{h} + \omega\cdot\mathbf{N} - 1/2\,\omega\cdot\dot{\tilde{I}}\cdot\omega \tag{III-113}$$

$$\frac{dT_{rot}}{dt} = 1/2\frac{d}{dt}\left(\int_V |\mathbf{v}|^2 dm\right) + \frac{d\omega}{dt}\cdot\mathbf{h}$$

$$+\,\omega\cdot(\mathbf{N} - \omega\times\mathbf{L}) - 1/2\,\omega\cdot\frac{d\tilde{I}}{dt}\cdot\omega \tag{III-114}$$

Now

$$\omega\cdot\omega\times\mathbf{L} = 0$$

and so Equation (III-114) reduces

$$\frac{dT_{rot}}{dt} = 1/2\frac{d}{dt}\left(\int_V |\mathbf{v}|^2\, dm\right)$$

$$+\frac{d\omega}{dt}\cdot\mathbf{h} + \omega\cdot\mathbf{N} - 1/2\,\omega\cdot\frac{d\tilde{I}}{dt}\cdot\omega \tag{III-115}$$

The kinematical relationship for vector time derivatives taken in the rotating and nonrotating frames gives

$$\dot{\omega} = \frac{d\omega}{dt} + \omega\times\omega$$

and since

$$\omega\times\omega = 0$$

we have

$$\dot{\omega} \equiv \frac{d\omega}{dt}\,. \tag{III-116}$$

The time derivative of the rotation vector is the same whether viewed from the rotating or nonrotating frame.

If we consider the case of a *rigid body*, then ω can be chosen so that $\mathbf{v} = 0$ and $\mathbf{h} = 0$ and Equations (III-113) (III-115) reduce to

$$\dot{T}_{rot} = \omega\cdot\mathbf{N} - 1/2\,\omega\cdot\dot{\tilde{I}}\cdot\omega\,, \tag{III-117}$$

$$\frac{dT_{rot}}{dt} = \omega\cdot\mathbf{N} - 1/2\,\omega\cdot\frac{d\tilde{I}}{dt}\cdot\omega \tag{III-118}$$

where the vector $\mathbf{N}$ in Equation (III-117) represents the components of the torque as seen in the space-fixed frame and the vector $\mathbf{N}$ in Equation (III-118) represents the components of the torque as seen in the body-fixed frame. If one of these vectors is constant the other is generally time-dependent.

For a rigidly rotating body with ω chosen that $\mathbf{V} = 0$ and $\mathbf{h} = 0$ it follows that $dI/dt = 0$ and so the two expressions reduce to

$$\dot{T}_{rot} = \omega\cdot\mathbf{N} - 1/2\,\omega\cdot\dot{\tilde{I}}\cdot\omega \tag{III-119}$$

$$\frac{dT_{rot}}{dt} = \omega\cdot\mathbf{N}\,. \tag{III-120}$$

Equation (III-120) is recognized as the familiar work theorem of rotating rigid bodies.

In applying Equations (III-113) or (III-115) to the case of a generally deformable rotating body it is necessary to establish the time derivative with respect to the space-fixed and body-fixed coordinate frames respectively of the time dependent integral

$$\int_{V(t)} |\mathbf{v}|^2\, dm = \int_{V(t)} \rho(\mathbf{r},t)\,|\mathbf{v}(\mathbf{r},t)|^2\, dV. \tag{III-121}$$

It can be shown that in the case of the space-fixed time derivative

$$1/2\int_{V(t)} \rho(\mathbf{r},t)\,\dot{|\mathbf{v}(\mathbf{r},t)|^2}\, dV = \int_{V(t)} \frac{\partial e}{\partial t}\, dV + \iint_{S(t)} \mathbf{E}\cdot\hat{n}\, dS \tag{III-122}$$

where $e(\mathbf{r},t)$ is a relative kinetic energy density

$$e(\mathbf{r},t) = 1/2\,\rho(\mathbf{r},t)\,|\mathbf{v}(\mathbf{r},t)|^2 \tag{III-123}$$

and where $\mathbf{E}(\mathbf{r}, t)$ is a relative kinetic energy flux density measured in the space-fixed frame.

$$\mathbf{E}(\mathbf{r}, t) = 1/2\, \rho(\mathbf{r}, t)\, |\mathbf{v}(\mathbf{r}, t)|^2\, \mathbf{v}(\mathbf{r}, t)\,. \qquad \text{(III-124)}$$

In the case of the body-fixed time derivative it can be shown that

$$1/2 \frac{d}{dt} \int_{V(t)} \rho(\mathbf{r}, t)\, |\mathbf{v}(\mathbf{r} t)|^2\, dV = \int_{V(t)} \frac{\partial e}{\partial t}\, dV$$

$$+ \iint_{S(t)} \mathbf{E}_{rot} \cdot \hat{n}\, dS \qquad \text{(III-125)}$$

where $\mathbf{E}_{rot}$ $(\mathbf{r}, t)$ is a relative kinetic energy flux density measured in the rotating body-fixed frame

$$\mathbf{E}_{rot}(\mathbf{r}, t) = 1/2\, \rho(\mathbf{r}, t)\, |\mathbf{v}(\mathbf{r}, t)|^2\, \mathbf{v}(\mathbf{r}, t)\,. \qquad \text{(III-126)}$$

Using Gauss's theorem, Equations (III-122) (III-125) can be written:

$$1/2 \int_{V(t)} |\dot{\mathbf{v}}|^2\, dm = \int_{V(t)} \left(\frac{\partial e}{\partial t} + \nabla \cdot \mathbf{E} \right) dV\,, \qquad \text{(III-127)}$$

$$1/2 \frac{d}{dt} \int_{V(t)} |\mathbf{v}|^2\, dm = \int_{V(t)} \left(\frac{\partial e}{\partial t} + \nabla \cdot \mathbf{E}_{rot} \right) dV. \qquad \text{(III-128)}$$

This completes the discussion on the general question of the rotational dynamics of extended deformable bodies. We will now turn our attention to the application of the theoretical tools developed here to the question of the rotation of the earth.

IV. The Liouville Equation and the Dynamics of Earth Rotation

With the exception of its fluid portions which include the oceans, atmosphere, ground water and the liquid outer core, the rest of the "solid" earth is so nearly rigid that departures of the actual earth from a rigid body may be incorporated into the dynamical theory of the earth's rotation by a perturbation scheme.

In the zeroth order approximation the earth is an axially symmetric rigid body rotating with uniform angular velocity Ω about an axis coincident with the axis of figure $\hat{e}_3$. In a coordinate frame which is corotating with the earth about the axis of figure the zeroth order inertia tensor $\mathcal{T}^0$ is given by

$$\mathcal{T}^0 = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix} \qquad \text{(IV-1)}$$

and the zeroth order rotation vector is given by Ω^0 where

$$\Omega^0 = \Omega \hat{e}_3\,. \qquad \text{(IV-2)}$$

Henceforth when we speak of the inertia tensor of the earth we shall mean the inertia tensor of the earth as measured in the body-fixed coordinate frame unless we explicitly state otherwise.

The inertia tensor of the "real" earth is $\mathcal{T}$ where

$$\mathcal{T} = \mathcal{T}^0 + \mathcal{T}^1 + \mathcal{T}^2 + \mathcal{T}^3 + \cdots \qquad \text{(IV-3)}$$

and the instantaneous rotation vector of the body-fixed coordinate frame is

$$\omega = \Omega^0 + \Omega \mathbf{m}^1 + \Omega \mathbf{m}^2 + \Omega \mathbf{m}^3 + \cdots \qquad \text{(IV-4)}$$

where the perturbation terms numbered $1, 2, 3, \cdots$ appearing in these expressions are the result of a variety of perturbing geophysical phenomena.

We shall find it convenient to use the notation

$$\mathcal{T} = \mathcal{T}^0 + \tilde{\mathcal{T}} \qquad \text{(IV-5)}$$

$$\omega = \Omega^0 + \Omega \mathbf{m} = \Omega \hat{e}_3 + \Omega \mathbf{m} \qquad \text{(IV-6)}$$

where the tensor $\tilde{\mathcal{T}}$ has elements

$$\tilde{\mathcal{T}} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \qquad \text{(IV-7)}$$

and is given by the sum

$$\tilde{\mathcal{T}} = \mathcal{T}^1 + \mathcal{T}^2 + \mathcal{T}^3 + \cdots \qquad \text{(IV-8)}$$

and where the vector **m** has components

$$\mathbf{m} = m_1 \hat{e}_1 + m_2 \hat{e}_2 + m_3 \hat{e}_3 \qquad \text{(IV-9)}$$

and is given by the sum

$$\mathbf{m} = \mathbf{m}^1 + \mathbf{m}^2 + \mathbf{m}^3 + \cdots . \qquad \text{(IV-10)}$$

In general both $\widetilde{r}$ and **m** are time-dependent perturbations.

The angular momentum of the real earth is given by **L** where

$$\mathbf{L} = \widetilde{I} \cdot \boldsymbol{\omega} + \mathbf{h} . \qquad \text{(IV-11)}$$

In the above equation ω refers to the total instantaneous angular rotation rate, including the effects of precession, nutation, and spin, of the body-fixed coordinate frame. In general ω is a time-dependent vector. The physical definition of the vector ω is implied by the physical definition of the body-fixed basis vectors $\hat{e}_1\ \hat{e}_2\ \hat{e}_3$ and their orientation or rather their rate of change of orientation in inertial space.

Observationally however the situation is more complicated. The measurement of ω is accomplished by combining data from a set of observers scattered over the earth's surface and attached to the earth's crust at various points. The solid earth and in particular its crust is continuously deformed by tides and other geophysical processes and is also the subject of large-scale systematic geotectonic motions. The question of the physical measurement of a unique vector ω, conforming to its definition, and derivable from a set of terrestrial observations from scattered positions on the earth's surface, becomes somewhat problematic at the level of ultrahigh precision measurements. We shall consider this problem later in this work and for now will proceed on the assumption that a unique vector ω is an observable quantity and that this observable ω conforms to the definition offered in the context of this theory.

While the body-fixed coordinate system is corotating "with the crust" in some uniquely definable sense the presence of the fluid portions of the earth, namely the oceans, atmosphere and fluid core, will contribute to a nonzero value for the vector **h**. Also contributing to **h** will be those portions of the solid earth which, as a result of tectonic processes, are in motion relative to the body-fixed coordinate frame.

It follows that while **h** does not vanish in the body-fixed frame of reference the quantity $|\mathbf{h}|/|\mathbf{L}|$ is very small. To first order in small quantities the angular momentum vector **L** has components in the rotating coordinate frame

$$L_1 = A\Omega m_1 + r_{13}\,\Omega\, h_1$$

$$L_2 = A\Omega m_2 + r_{23}\,\Omega + h_2$$

$$L_3 = C\Omega\,[1 + m_3] + r_{33}\,\Omega + h_3 . \qquad \text{(IV-12)}$$

To appreciate the nature of the approximations being made by retaining only the first-order terms in our theory it is useful to recall (Munk and MacDonald 1960) that the total relative angular momentum in the zonal circulation of the earth's atmosphere is of the order of 10^{33} gm cm^2 sec^{-1} and that of the earth's oceans is of the order of 10^{32} gm cm^2 sec^{-1}, whereas the angular momentum of the rotating earth is roughly 6×10^{40} gm cm^2 sec^{-1}. It follows that $|\mathbf{h}|/|\mathbf{L}| \sim 10^{-8}$ for the atmosphere and $\sim 10^{-9}$ for the oceans.

Furthermore it has been shown (Smylie and Mansinha 1971a; Mansinha, Smylie, and Chapman 1979) that the changes in the earth's products and moments of inertia r_{ij} resulting from the Chilean earthquake of 1960 and the Alaskan earthquake of 1964 are of the order of 10^{35} gm cm^2, which is to be compared to the earth's moments of inertia $C \sim A \sim 10^{44}$ gm cm^2. Thus even for the largest of mass movements in the solid earth $r_{ij}/C \sim r_{ij}/A \sim 10^{-9}$.

The Liouville equation governing the dynamics of earth rotation expressed in a rotating body-fixed frame of reference is given by Equation (III-78) as

$$\frac{d\widetilde{I}}{dt} \cdot \boldsymbol{\omega} + \widetilde{I} \cdot \frac{d\boldsymbol{\omega}}{dt} + \frac{d\mathbf{h}}{dt} + \boldsymbol{\omega} \times \widetilde{I} \cdot \boldsymbol{\omega} + \boldsymbol{\omega} \times \mathbf{h} = \mathbf{N} \qquad \text{(IV-13)}$$

Substituting Equations (IV-5), (IV-6) and (IV-11) into Equation (IV-13) and retaining only terms which are first order in small quantities, we obtain the perturbed Liouville equation governing earth rotation which in component form becomes

$$N_1 = A\Omega \frac{dm_1}{dt} + \Omega \frac{dr_{13}}{dt} + \frac{dh_1}{dt} + \Omega^2 (C - A) m_2$$

$$- \Omega^2 r_{23} - \Omega h_2 \qquad \text{(IV-14)}$$

$$N_2 = A\Omega \frac{dm_2}{dt} + \Omega \frac{dr_{23}}{dt} + \frac{dh_2}{dt} - \Omega^2 (C - A) m_1$$

$$+ \Omega^2 r_{13} + \Omega h_1 \qquad \text{(IV-15)}$$

$$N_3 = C\Omega \frac{dm_3}{dt} + \Omega \frac{dr_{33}}{dt} + \frac{dh_3}{dt} \qquad \text{(IV-16)}$$

In obtaining Equations (IV-14) – (IV-16) we have made use of the fact that A, C, Ω are constants and their time derivatives vanish. Equations (IV-14) – (IV-16) are expressed in the rotating coordinate frame and so the components $N_1\ N_2\ N_3$ of the impressed torque on the earth must be expressed in the rotating frame as well.

Equations (IV-14) – (IV-16) are "separable" in the sense that the quantities $m_3\ r_{33}$ and h_3 appear exclusively confined to the equation for N_3. This is a consequence, in part, of the restriction to a first-order theory and does not occur in a second-order expansion of the Liouville equation. This means in effect that the effects of the torque N_3 can be treated separately from the effects of the torques $N_1\ N_2$. The set of equations (IV-14) – (IV-16) decouples into what is usually described as a pair of equations governing polar motion or "wobble" and involving only the $m_1\ m_2$ perturbations to the rotation vector ω and a single equation governing UT1 and involving only the m_3 perturbation to the rotation vector ω.

Multiplying Equation (IV-15) by i, where $i^2 = -1$, and adding it to Equation (IV-14) gives the complex wobble equation:

$$\begin{aligned} N_1 + iN_2 &= A\Omega \frac{d}{dt}(m_1 + im_2) + \Omega \frac{d}{dt}(r_{13} + ir_{23}) \\ &+ \frac{d}{dt}(h_1 + ih_2) + \Omega^2 (C - A)(m_2 - im_1) \\ &- \Omega^2 (r_{23} - ir_{13}) - \Omega (h_2 - ih_1) . \end{aligned} \qquad \text{(IV-17)}$$

Recognizing that

$$\begin{aligned} m_2 - im_1 &= -i(m_1 + im_2) \\ r_{23} - ir_{13} &= -i(r_{13} + ir_{23}) \\ h_2 - ih_1 &= -i(h_1 + ih_2) \end{aligned} \qquad \text{(IV-18)}$$

and introducing the complex quantities

$$\begin{aligned} \overline{m} &= m_1 + im_2 \\ \overline{r} &= r_{13} + ir_{23} \\ \overline{h} &= h_1 + ih_2 \\ \overline{N} &= N_1 + iN_2 \end{aligned} \qquad \text{(IV-19)}$$

the complex wobble equation becomes

$$\overline{N} = A\Omega \frac{d\overline{m}}{dt} + \Omega \frac{d\overline{r}}{dt} + \frac{d\overline{h}}{dt} - i\,[\Omega^2 (C - A)\overline{m} - \Omega^2 \overline{r} - \Omega \overline{h}] \qquad \text{(IV-20)}$$

which together with

$$N_3 = C\Omega \frac{dm_3}{dt} + \Omega \frac{dr_{33}}{dt} + \frac{dh_3}{dt} \qquad \text{(IV-21)}$$

constitute the governing equations for earth rotation. With some simple manipulations these equations can be written as

$$\frac{d\overline{m}}{dt} - i\,\frac{C - A}{A}\,\Omega \overline{m} = \frac{1}{A\Omega}\left[\overline{N} - \Omega \frac{d\overline{r}}{dt} - \frac{d\overline{h}}{dt} - i\,(\Omega^2 \overline{r} + \Omega \overline{h})\right] \qquad \text{(IV-22)}$$

and

$$\frac{dm_3}{dt} = \frac{1}{C\Omega}\left(N_3 - \Omega \frac{dr_{33}}{dt} - \frac{dh_3}{dt}\right). \qquad \text{(IV-23)}$$

Written in this way Equations (IV-22) and (IV-23) appear explicitly as equations governing changes on the earth rotation vector ω including both polar motion (wobble) and UT1. The RHS of Equations (IV-22) and (IV-23) appear as forcing functions in the dynamics of earth rotation ω and are often referred to as the geophysical excitation functions for polar motion and UT1 fluctuations.

It is possible in principle to use our present knowledge of geophysical processes to model the excitation functions and hence predict polar motion and UT1 from these equations. However, our ability to do this successfully at this time is limited by a general lack of accurate information regarding the character of the geophysical excitation functions.

It seems most scientifically productive at this point to reverse the above argument and set about to accurately measure $m_1\ m_2\ m_3$ by long baseline interferometry or other methods with the objective of learning more about the geophysical excitation functions. Since these functions reflect the effects of atmospheric and oceanic circulation, external gravitational torques, dislocations due to earthquake faulting, fluid motions in the core, electromagnetic coupling between the core and mantle, changes in sea level, changes in ground water content and other important geodynamical phenomena, this program should hold great potential for scientific discovery.

Equations (IV-22) and (IV-23) are referred to the rotating body-fixed basis vectors $\hat{e}_1\hat{e}_2\hat{e}_3$, and so all quantities appearing in them must also be expressed relative to these basis vectors. If $N_1\ N_2\ N_3$ are the components of a gravitational torque whose magnitude and direction are fixed in inertial space, then when expressed relative to $\hat{e}_1\ \hat{e}_2\ \hat{e}_3$ the torque components $N_1\ N_2\ N_3$ are varying periodically with period $2\pi/|\omega|$.

Although we have referred, and will continue to refer, to the vector ω as the "rotation vector of the earth" it should be borne in mind that ω is in fact the rotation vector of a geophysical coordinate system and strictly speaking has only kinematical significance. The unique "rotation vector of the earth" is a vector $\mathbf{R}$ for which the earth's angular momentum $\mathbf{L}$ is expressed as

$$\mathbf{L} = \overleftrightarrow{I}\cdot\mathbf{R}\,. \tag{IV-24}$$

We can obtain a relationship between $\mathbf{R}$ and ω by decomposing the relative angular momentum $\mathbf{h}$ as

$$\mathbf{h} = \overleftrightarrow{I}\cdot\delta\omega\,. \tag{IV-25}$$

From this we can see that

$$\mathbf{R} = \omega + \delta\omega\,. \tag{IV-26}$$

but since $|\delta\omega|/|\omega| \ll 1$ we have

$$\mathbf{R} \simeq \omega\,. \tag{IV-27}$$

Equation (IV-23) may be written

$$\frac{dm_3}{dt} = \frac{1}{C\Omega}\,\frac{d}{dt}\left[\int_0^t N_3(t')\,dt' - \Omega r_{33} - h_3\right]. \tag{IV-28}$$

The latter equation allows us to integrate the equation for $m_3(t)$ to obtain

$$m_3(t) = \frac{1}{C\Omega}\left[\int_0^t N_3(t')\,dt' - \Omega r_{33} - h_3\right] + m_3(0) \tag{IV-29}$$

Equation (IV-29) expresses the variations of UT1 as a function of time.

The quantities $\overline{h}$ and $\overline{r}$ appearing in the equation for polar motion (IV-22) as well as the quantities h_3 and r_{33} appearing in the equation for UT1 (IV-28) depend on volume integrals defined in the body-fixed frame. The rigorous definition of their time derivatives for the case of a generally deformable body are given by Equations (III-84) and (III-85) respectively. However, later in this work we shall examine some useful approximate methods for calculating these time derivatives which treat the earth as a rigid body with fluid portions.

V. Rotational Dynamics of an Axially Symmetric Rigid Earth

Although the earth is in reality a deformable body, its approximation to a rigid body is sufficiently good that considerable insight into the earth's rotational dynamics can be obtained by examining solutions to the dynamical equations governing earth rotation in their zeroth approximation namely the special case of a rigid axially symmetric earth.

A. Eulerian (Force Free) Motion of an Axially Symmetric Rigid Earth

In 1765 Euler investigated the dynamics of rigidly rotating bodies in the absence of external torques. Such motion has come to be known as "Eulerian motion." We shall investigate the Eulerian motion of the earth from the point of view of a body-fixed coordinate frame and a space fixed coordinate frame.

1. Eulerian motion of the earth in a body-fixed frame. The rotational dynamics of a rigid earth in the absence of any geophysical excitation is governed by

$$\frac{d\overline{m}}{dt} - i\,\frac{C-A}{A}\,\Omega\,\overline{m} = 0 \tag{V-1}$$

$$\frac{dm_3}{dt} = 0 \tag{V-2}$$

which are obtained from Equations (IV-22) and (IV-23) by setting the excitation function to zero.

These equations can be integrated directly to give

$$m_3(t) = \mathit{constant} \tag{V-3}$$

and

$$\overline{m}(t) = \exp i \left(\frac{C-A}{A} \Omega t + \overline{\theta} \right) \qquad \text{(V-4)}$$

where $\overline{\theta}$ is a complex constant of integration,

$$\overline{\theta} = \theta^R + i\,\theta^I \, . \qquad \text{(V-5)}$$

If we introduce the angular rotation rate σ_r where

$$\sigma_r = \frac{C-A}{A} \Omega \qquad \text{(V-6)}$$

we see that Equations (V-4) and (V-5) yield

$$\overline{m}(t) = \exp\left[-\theta^I + i\,(\sigma_r t + \theta^R)\right] \, . \qquad \text{(V-7)}$$

Setting

$$\begin{aligned} \beta_e &= \exp - \theta^I \\ \sigma_r t_0 &= -\theta^R \end{aligned} \qquad \text{(V-8)}$$

and recalling that

$$\overline{m}(t) = m_1(t) + i\, m_2(t) \qquad \text{(V-9)}$$

we have the solution for the rotational dynamics of a rigid earth in the absence of geophysical excitation

$$\begin{aligned} m_1(t) &= \beta_e \cos \sigma_r (t - t_0) \\ m_2(t) &= \beta_e \sin \sigma_r (t - t_0) \end{aligned}$$

or

$$\overline{m}(t) = \beta_e \, e^{i\sigma_r (t-t_0)} \qquad \text{(V-10)}$$

and

$$m_3(t) = \textit{constant} \, .$$

From Equations (IV-2) and (IV-6) we have

$$\omega = \Omega \left[m_1(t)\hat{e}_1 + m_2(t)\hat{e}_2 + (1 + m_3(t))\hat{e}_3 \right] \qquad \text{(V-11)}$$

and so one way of viewing this solution is to see that it corresponds to a constant angular rotation rate of magnitude $\Omega(1 + m_3)$ about the axis of figure $\hat{e}_3$ combined with time-dependent angular rotation rates of magnitude $\Omega m_1(t)$ $\Omega m_2(t)$ about the $\hat{e}_1$ and $\hat{e}_2$ axes respectively.

A more instructive way of viewing this solution is to consider the rotation vector $\boldsymbol{\omega}$ in terms of its magnitude and direction in the body-fixed coordinate frame rather than in terms of its components in the body-fixed coordinate frame. The magnitude of the rotation vector is given by $\omega = (\boldsymbol{\omega} \cdot \boldsymbol{\omega})^{1/2}$ where

$$\omega = (\boldsymbol{\omega} \cdot \boldsymbol{\omega})^{1/2} = \Omega\,(1 + 2m_3) \qquad \text{(V-12)}$$

to first order in small quantities. The direction of the rotation vector is specified by the angles $m_1(t)$ $m_2(t)$.

For a positive value of σ_r we see that this solution corresponds to the uniform circular motion of the axis of rotation about the axis of figure in a prograde or west to east direction. The axis of rotation moves within the earth on a body-fixed cone whose axis coincides with the figure axis $\hat{e}_3$ and whose apex angle is $2\beta_e$. The rotation axis completes one revolution about the figure axis in a period $2\pi/\sigma_r$. This geometry is illustrated in Figures V-1 and V-2.

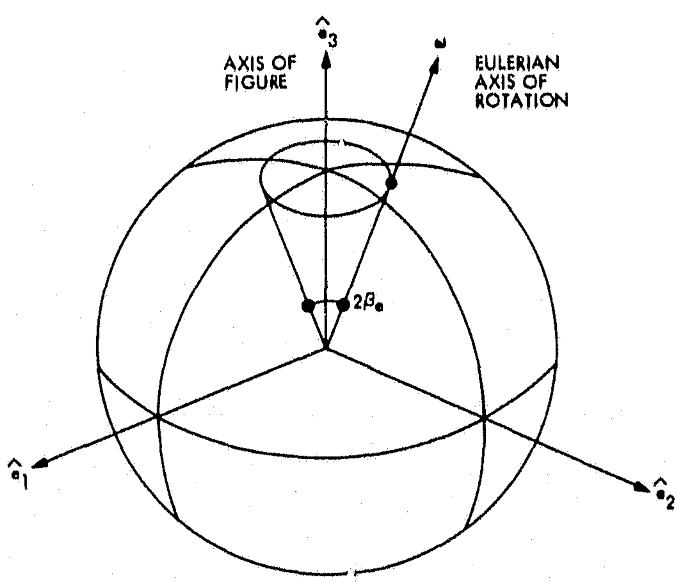

Figure V-1. The Eulerian (torque-free) polar motion for the case of an axially symmetric rigid earth. The Eulerian axis of rotation is confined to the surface of a geometric cone of apex angle $2\beta_e$ aligned with the figure axis.

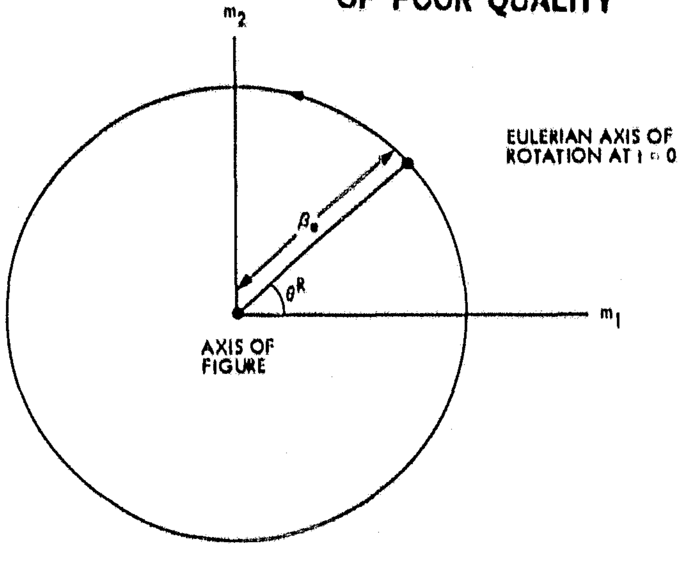

Figure V-2. A polar view of Eulerian (torque-free) polar motion. The Eulerian pole is seen to move at a uniform rate in a prograde sense around the figure axis at a constant angular distance β_e.

From Stacey (1977) we have

$$C = 8.0378 \times 10^{44} \text{ gm cm}^2$$

$$A = 8.0115 \times 10^{44} \text{ gm cm}^2$$

and so

$$\frac{C-A}{A} = 0.0032828 = \frac{1}{304.6}$$

and since $\sigma_r = (C - A/A)\Omega$ and $2\pi/\Omega$ corresponds to an interval of one mean sidereal day we see that the axis of rotation completes one revolution about the axis of figure in 304.6 sidereal days. In the case of a *rigid earth* its angular rate around the figure axis is $2\pi/304.6$ radians per mean sidereal day.

2. **Eulerian motion of the earth in a space-fixed frame.** The analysis of the earth's rotational dynamics from the point of view of a space-fixed frame in the special case of force free motion does not require knowledge of the coordinate transformation equations relating the body-fixed frame to the space-fixed frame. This is because in the special case of force free motion the angular momentum vector **L** provides us with an invariant direction in inertial space as a consequence of the conservation of angular momentum.

Now in general

$$\mathbf{L} = \mathcal{T} \cdot \boldsymbol{\omega} + \mathbf{h} \qquad \text{(V-13)}$$

and in the special case of a rigid earth we can choose a set of rotating body-fixed coordinates such that $\mathbf{h} = 0$ and so we can write

$$\mathbf{L} = \mathcal{T} \cdot \boldsymbol{\omega} \qquad \text{(V-14)}$$

for an appropriate choice of ω corresponding to "the rotation rate of the earth."

Using Equations (IV-1) and (IV-11) we have

$$\mathbf{L} = A\Omega m_1 \hat{e}_1 + A\Omega m_2 \hat{e}_2 + C\Omega(1 + m_3)\hat{e}_3 \qquad \text{(V-15)}$$

which can be written

$$\mathbf{L} = A\Omega m_1 \hat{e}_1 + A\Omega m_2 \hat{e}_2 + A\Omega(1 + m_3)\hat{e}_3$$

$$+ (C - A)\,\Omega\,(1 + m_3)\,\hat{e}_3 \,. \qquad \text{(V-16)}$$

Using Equations (IV-11) and (V-6) in (V-16) gives

$$\mathbf{L} = A\boldsymbol{\omega} + A\sigma_r(1 + m_3)\hat{e}_3 \qquad \text{(V-17)}$$

or finally, neglecting the small term in $\sigma_r m_3$,

$$\mathbf{L} = A(\boldsymbol{\omega} + \sigma_r \hat{e}_3) \,. \qquad \text{(V-18)}$$

From the results of Equation (V-18) it is clear that **L**, $A\boldsymbol{\omega}$, $A\sigma_r\hat{e}_3$ form a closed vector triangle and are hence coplanar. Since $|\boldsymbol{\omega}| > |\sigma_r|$ these vectors can be represented as shown in Figure V-3.

The angle β_e is the displacement of the rotation vector $\boldsymbol{\omega}$ from the axis of figure $\hat{e}_3$ and the angle γ is the displacement of the rotation vector $\boldsymbol{\omega}$ from the axis of angular momentum **L**. The angle γ represents a motion of the rotation axis in space and appears as a "nutation." Such a motion of the rotation axis in space in the absence of externally applied torques is called "Eulerian nutation" since it is associated with the Eulerian (force free) motion of a rigid body. This Eulerian nutation has been called "sway" by some authors to distinguish it from "forced nutation" which arises as a result of impressed torques. The term "free nutation" has also been used to denote Eulerian nutation.

Observationally Eulerian nutation and forced nutation are difficult to separate. Physically, however, they are quite distinct as Eulerian nutation *does not* displace the angular momentum vector in space and forced nutation *does* displace the angular momentum vector in space.

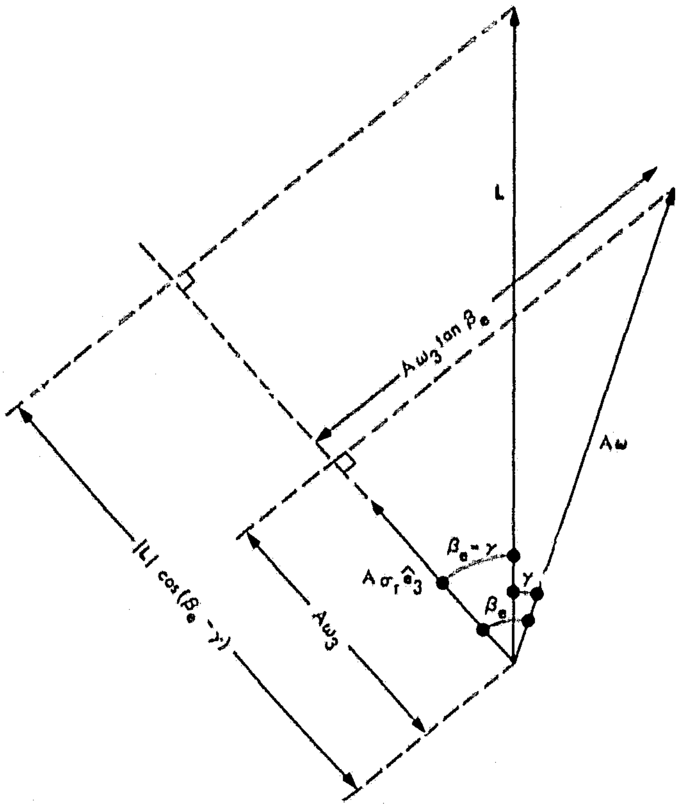

Figure V-3. The geometrical relationship between the earth's figure axis $\hat{e}_3$, Eulerian rotation axis ω, and angular momentum axis L for a rigid axially symmetric earth with moments of inertia A, A, and C. The angles β_e and γ are greatly exaggerated and not drawn to relative scale. (After M. G. Rochester, unpublished research notes.)

The following analysis allows us to deduce a relationship between the Eulerian polar motion described by the angle β_e and the Eulerian nutation described by the angle γ.

From the constructions shown in Figure V-3 (M.G. Rochester, unpublished research notes) we can deduce:

(1) By Pythagoras's theorem

$$|\omega| = [\omega_3^2 + (\omega_3 \tan\beta_e)^2]^{1/2} \qquad \text{(V-19)}$$

(2) By the definition of $\cos\beta_e$

$$|\omega| = \omega_3 \sec\beta_e \qquad \text{(V-20)}$$

(3) By the projection of $\mathbf{L}$ onto $\hat{e}_3$

$$\frac{\mathbf{L}\cdot\hat{e}_3}{A} = \frac{|\mathbf{L}|}{A}\cos(\beta_e - \gamma)\,. \qquad \text{(V-21)}$$

Using Equation (V-18) as an expression for $\mathbf{L}$ we have

$$\frac{\mathbf{L}\cdot\hat{e}_3}{A} = \omega_3 + \sigma_r \qquad \text{(V-22)}$$

and

$$\left|\frac{\mathbf{L}}{A}\right| = (\omega\cdot\omega + 2\sigma_r\omega\cdot\hat{e}_3 + \sigma_r^2\hat{e}_3\cdot\hat{e}_3)^{1/2}\,. \qquad \text{(V-23)}$$

Since

$$\sigma_r^2\,\hat{e}_3\cdot\hat{e}_3 = \sigma_r^2$$

$$2\sigma_r\,\omega\cdot\hat{e}_3 = 2\sigma_r\,\omega_3$$

and by Equation (V-19)

$$\omega\cdot\omega = |\omega|^2 = \omega_3^2 + \omega_3^2\tan^2\beta_e$$

we can rewrite Equation (V-23) as

$$\left|\frac{\mathbf{L}}{A}\right| = (\omega_3^2 + \omega_3^2\tan^2\beta_e + 2\sigma_r\omega_3 + \sigma_r^2)^{1/2}$$

which reduces to

$$\left|\frac{\mathbf{L}}{A}\right| = [(\omega_3 + \sigma_r)^2 + \omega_3^2\tan^2\beta_e]^{1/2}\,. \qquad \text{(V-24)}$$

From Equations (V-21) and (V-22) we have

$$\frac{\left|\frac{\mathbf{L}}{A}\right|^2}{(\omega_3+\sigma_r)^2} = \sec^2(\beta_e - \gamma)$$

and using the identity $\sec^2(\beta_e - \gamma) = \tan^2(\beta_e - \gamma) + 1$ we obtain

$$\tan^2(\beta_e - \gamma) = \frac{\left|\frac{\mathbf{L}}{A}\right|^2}{(\omega_3+\sigma_r)^2} - 1\,. \qquad \text{(V-25)}$$

Substituting Equation (V-24) into Equation (V-25) gives

$$\tan^2(\beta_e - \gamma) = \frac{(\omega_3+\sigma_r)^2 + \omega_3^2\tan^2\beta_e}{(\omega_3+\sigma_r)^2} - 1$$

which reduces to

$$\tan(\beta_e - \gamma) = \frac{\omega_3}{(\omega_3 + \sigma_r)} \tan\beta_e \, . \qquad \text{(V-26)}$$

Using the standard trigonometric formula this can be written as

$$\frac{\tan\beta_e - \tan\gamma}{1 + \tan\beta_e \tan\gamma} = \frac{\omega_3}{(\omega_3 + \sigma_r)} \tan\beta_e$$

or

$$\frac{\tan\gamma}{\tan\beta_e} = \frac{\sigma_r}{\omega_3}\left(\frac{1}{1 + \tan^2\beta_e + \dfrac{\sigma_r}{\omega_3}}\right)$$

and finally

$$\frac{\tan\gamma}{\tan\beta_e} = \frac{\sigma_r}{\omega_3}\left(\frac{\sigma_r}{\omega_3} + \sec^2\beta_e\right)^{-1} . \qquad \text{(V-27)}$$

Since $\sigma_r << \omega_3$ and γ, β_e are small angles this exact relationship can be approximated very well by

$$\frac{\gamma}{\beta_e} \cong \frac{\sigma_r}{\omega_3} .$$

We see that the Eulerian nutation in space is roughly σ_r/ω_3 times the wobble amplitude on earth.

For a rigid earth:

$$\sigma_r = \frac{2\pi}{304.6} \text{ radians per sidereal day,}$$

$$\omega_3 = 2\pi \text{ radians per sidereal day,}$$

and so

$$\gamma_r \cong \frac{1}{304.6}\beta_e$$

where γ_r denotes the amplitude of the Eulerian nutation on a *rigid earth.* For a maximum value of $2\beta_e \approx 0''.40$ arc we have

$$2\gamma_r \approx 1''.30 \times 10^{-3} \text{ arc} .$$

However, in the case of the "real" earth, elastic yielding of the mantle lengthens the period of the wobble to roughly 435 days. For the actual earth

$$\gamma \cong \frac{1}{435}\beta_e$$

and has a maximum value of roughly

$$2\gamma \approx 0''.93 \times 10^{-3} \text{ arc} .$$

B. Poinsot Geometrical Description of Eulerian Motion of an Axially Symmetric Rigid Earth

The famous construction of Poinsot is a general method of geometrically describing Eulerian (torque-free) motion of a rigid body without having to integrate the governing dynamical equations. Since the Poinsot construction provides a complete description of the motion and since the integration of the dynamical equations generally involves the use of elliptic integrals, the Poinsot construction is quite useful as well as elegant.

The general method of application of the Poinsot construction is given in Goldstein (1950, pp. 159-161). The approach adopted in this work will be that of M.G. Rochester (unpublished research notes). We shall compute the time derivatives of the rotation vector $\boldsymbol{\omega}$, both with respect to a space-fixed frame in which the invariant angular momentum vector **L** provides the reference direction, and with respect to the body-fixed frame with reference directions provided by the body-fixed basis vectors $\hat{e}_1$ $\hat{e}_2$ $\hat{e}_3$. The result we seek can then be obtained by appealing to Equation (III-116) and equating these two time derivatives.

We have seen in Figure V-3 that for the case of the earth executing Eulerian motion the rotation axis ω and the angular momentum vector **L** are inclined at angular β_e and $\beta_e - \gamma$ respectively to the axis of figure $\hat{e}_3$ and that all three vectors ω, **L**, $\hat{e}_3$ lie in the same plane. Since the angle γ between ω and **L** is a constant of the motion, ω can only be incremented by the motion in the direction of the unit vector $\hat{\xi}$ which is orthogonal to both ω and **L**. Also since **L** is a constant of the motion we can calculate the space-fixed time derivative of ω, denoted according to our convention by $\dot{\omega}$, by referring to **L** as an invariant space fixed vector.

As shown in Figure V-4, in an interval of time dt the increment $d\omega$ to ω is in the direction of $\hat{\xi}$, a unit vector orthogonal to both **L** and ω

$$\hat{\xi} = \frac{\mathbf{L} \times \omega}{|\mathbf{L} \times \omega|} . \qquad \text{(V-28)}$$

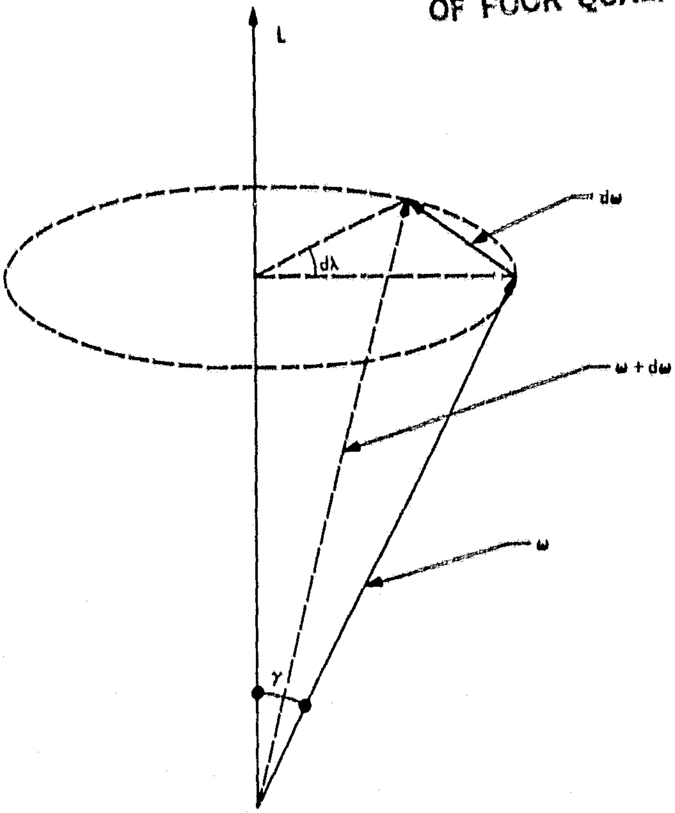

Figure V-4. The dynamical quantities used to obtain the Poinsot construction for the case of Eulerian motion on a rigid axis symmetric earth.

The increment $d\omega$ has magnitude given by

$$|d\omega| = |\omega| \sin\gamma \, d\lambda \tag{V-29}$$

where $d\lambda$ is the increment in the interval dt to the azimuth, reckoned in a space-fixed frame, of the plane containing $\mathbf{L}$ and ω. Combining the direction and magnitude of $d\omega$ we have from Equations (V-28) and (V-29)

$$d\omega = |d\omega| \hat{\xi}$$

$$d\omega = |\omega| \sin\gamma \frac{\mathbf{L} \times \omega}{|\mathbf{L} \times \omega|} d\lambda. \tag{V-30}$$

By definition

$$\sin\gamma = \frac{|\mathbf{L} \times \omega|}{|\mathbf{L}| \; |\omega|} \tag{V-31}$$

which when substituted into Equation (V-30) gives

$$d\omega = \frac{\mathbf{L} \times \omega}{|\mathbf{L}| \, |\omega|} |\omega| \, d\lambda . \tag{V-32}$$

By dividing both sides of Equation (V-32) with the time increment dt we have the final result,

$$\dot{\omega} = \frac{\mathbf{L} \times \omega}{|\mathbf{L}| \, |\omega|} |\omega| \, \dot{\lambda} \tag{V-33}$$

where, by our convention, the dot "·" denotes a time derivative taken with respect to a space-fixed frame.

To compute the time derivative of ω with respect to the body-fixed frame, denoted by our convention as $d\omega/dt$, we begin with

$$\omega = \Omega \left[m_1 \hat{e}_1 + m_2 \hat{e}_2 + (1 + m_3) \hat{e}_3 \right] \tag{V-34}$$

and so

$$\frac{d\omega}{dt} = \Omega \left(\frac{dm_1}{dt} \hat{e}_1 + \frac{dm_2}{dt} \hat{e}_2 + \frac{dm_3}{dt} \hat{e}_3 \right) \tag{V-35}$$

since the derivatives $d\hat{e}_1/dt \; d\hat{e}_2/dt \; d\hat{e}_3/dt$ all vanish in the body-fixed frame of a rigid earth.

The dynamical equations governing the Eulerian motion of the rotation axis in the body-fixed frame are given by Equations (V-1), (V-2) and (V-6) as

$$\frac{d\bar{m}}{dt} - i \sigma_r \bar{m} = 0 \tag{V-36}$$

$$\frac{dm_3}{dt} = 0 \tag{V-37}$$

where

$$\bar{m} = m_1 + i m_2$$

and which when substituted into Equation (V-35) yield

$$\frac{d\omega}{dt} = - \sigma_r \Omega \left(m_2 \hat{e}_1 - m_1 \hat{e}_2 + 0 \hat{e}_3 \right) . \tag{V-38}$$

Using the relationships $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3, \hat{e}_2 \times \hat{e}_3 = \hat{e}_1, \hat{e}_3 \times \hat{e}_1 = \hat{e}_2$, between the body-fixed basis vectors we can rewrite Equation (V-38) as

$$\frac{d\omega}{dt} = \sigma_r \Omega \hat{e}_3 \times \left(m_1 \hat{e}_1 + m_2 \hat{e}_2 \right) \tag{V-39}$$

and since $\hat{e}_3 \times \hat{e}_3 = 0$ we can add zero to the RHS of Equation (V-39) to obtain

$$\frac{d\omega}{dt} = \sigma_r \Omega \hat{e}_3 \times [m_1 \hat{e}_1 + m_2 \hat{e}_2 + (1 + m_3) \hat{e}_3] \quad \text{(V-40)}$$

or finally, using Equation (V-34),

$$\frac{d\omega}{dt} = \sigma_r \hat{e}_3 \times \omega \, . \quad \text{(V-41)}$$

The general transformation relating the space-fixed time derivative $\dot{\mathbf{G}}$ to the body-fixed time derivative $d\mathbf{G}/dt$ of an arbitrary vector $\mathbf{G}$ is

$$\dot{\mathbf{G}} = \frac{d\mathbf{G}}{dt} + \omega \times \mathbf{G} \quad \text{(V-42)}$$

and so for the rotation vector ω we have in general

$$\dot{\omega} = \frac{d\omega}{dt} + \omega \times \omega \quad \text{(V-43)}$$

which, since $\omega \times \omega = 0$, reduces to

$$\dot{\omega} = \frac{d\omega}{dt} \quad \text{(V-44)}$$

as was shown in Equation (III-116). Substituting Equations (V-33) and (V-41) into Equation (V-44) gives

$$\frac{\mathbf{L} \times \omega}{|\mathbf{L}|\,|\omega|} |\omega| \, \dot{\lambda} = \sigma_r \hat{e}_3 \times \omega \, . \quad \text{(V-45)}$$

Equating the magnitudes on both sides of the vector Equation (V-45) gives

$$\frac{|\mathbf{L} \times \omega|}{|\mathbf{L}|\,|\omega|} |\omega| \, \dot{\lambda} = \sigma_r |\hat{e}_3 \times \omega| \quad \text{(V-46)}$$

Now from Figure V-3 we have

$$|\hat{e}_3 \times \omega| = |\omega| \sin \beta_e \quad \text{(V-47)}$$

and using Equations (V-31) and (V-47) in Equation (V-46) we have

$$|\omega| \sin \gamma \, \dot{\lambda} = \sigma_r \, |\omega| \sin \beta_e \quad \text{(V-48)}$$

or finally

$$\frac{\dot{\lambda}}{\sigma_r} = \frac{\sin \beta_e}{\sin \gamma} \, . \quad \text{(V-49)}$$

The result of Equation (V-49) expresses algebraically the result of the Poinsot construction. Poinsot (1852) showed that any continuous rotation of a rigid body is geometrically equivalent to the rolling of a cone, *fixed* within the body, on a cone *fixed* within space. The cone fixed within the body is called the polhode cone and the cone fixed within space is called the herpolhode cone. The instantaneous rotation axis of the body relative to inertial space ω lies along the line of contact between the two cones.

The geometry of this arrangement is illustrated in Figure V-5. The motion is a continuous rotation ω around the line of contact between the cones, during which the axis of rotation ω describes successive circuits around the cone of apex angle 2γ *in space* and also around the cone of apex angle $2\beta_e$ *in the earth*.

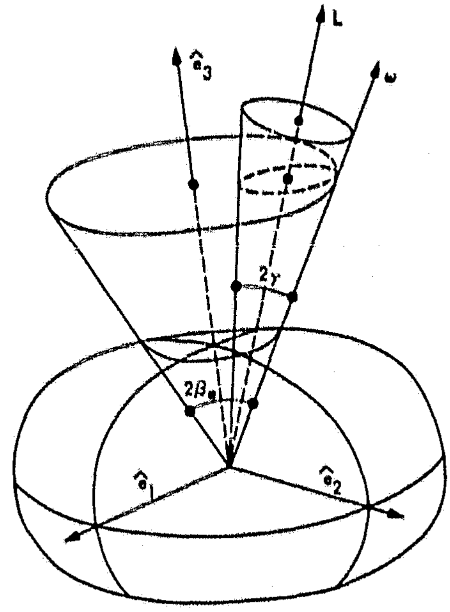

Figure V-5. The resultant Poinsot construction for Eulerian motion on a rigid axis symmetric earth. The large body-fixed cone of apex angle $2\beta_e$ centered on the figure axis $\hat{e}_3$ rolls without slipping on the small space-fixed cone of apex angle 2γ centered on the angular momentum vector L.

During each complete circuit around the cone in space, the axis of rotation ω progresses along the cone in the earth only through a distance equal to the circumference of the small space-fixed cone. Hence as the rotation axis successively returns to the same position in space at the end of each circuit around $\mathbf{L}$ it lies at successively different positions within the earth, with the consequence that the earth lies in different positions in space.

This aspect of the motion can be understood by recalling that σ_r is the angular rate *in the body-fixed frame* of the azimuth of the moving axis of rotation for which $\dot{\lambda}$ is the angular rate *in the space-fixed frame* of the azimuth of the moving axis of rotation. Equation (V-49) can be written

$$\dot{\lambda} = \sigma_r \frac{\sin \beta_e}{\sin \gamma} = \sigma_r \frac{\cos \beta_e}{\cos \gamma} \cdot \frac{\tan \beta_e}{\tan \gamma} \qquad \text{(V-50)}$$

and using Equation (V-27) in Equation (V-50)

$$\dot{\lambda} = \sigma_r \frac{\cos \beta_e}{\cos \gamma} \frac{\omega_3}{\sigma_r} \left(\frac{\sigma_r}{\omega_3} + \sec^2 \beta_e \right). \qquad \text{(V-51)}$$

Since all angles are small $\cos \beta_e \approx 1$, $\cos \gamma \approx 1$, and $\sec^2 \beta_e \approx 1$. Equation (V-51) reduces to its approximate form

$$\dot{\lambda} = \omega_3 + \sigma_r . \qquad \text{(V-52)}$$

Equation (V-52) shows that the angular rate of ω about **L** is slightly *larger* than the diurnal rate, being equal to the diurnal rate plus the polar motion or wobble rate.

C. Non-Eulerian (Forced) Motion of an Axially Symmetric Rigid Earth

In reality the motion of the earth departs from the ideal case of Eulerian motion for two reasons:

(1) The real earth is subjected to rotational excitation of both an internal and external origin.

(2) The real earth is not an infinitely rigid body but a deformable solid with a strength comparable to that of steel $\sim 10^9$ dynes cm^{-2} and in addition possesses fluid portions in the form of a liquid outer core, an atmosphere, and oceans.

We shall examine here the consequences for the earth's rotation of the geophysical forcing functions and leave the investigation of the consequences of the departure of the earth from a rigid body for a later portion of this work.

The geophysical forcing functions can be broadly classified into rotational excitation of external origin and rotational excitation of internal origin. Rotational excitation of external origin would include effects such as the lunisolar gravitational torques, the gravitational effects of the other bodies of the solar system, coupling to the solar wind by fluid or electromagnetic processes, meteors passing through the atmosphere, meteorites striking the earth and so on. Of the effects listed above only the lunisolar gravitational torques can be reliably demonstrated to have any observable effect on the earth's rotation. Even the direct gravitational effect of the other bodies of the solar system can be shown (Woolard, 1953) to be smaller than their indirect effect manifested through the perturbations these bodies produce on the positions of the sun and moon relative to the earth.

External rotational excitations are distinguished by the fact that they, and *only* they, may alter the total angular momentum vector **L** of the earth. Such processes which cause a change of the magnitude and orientation of **L** in a space-fixed frame are studied under the general rubric of the theory of the precession and nutation of the earth.

Rotational excitation of internal origin would include effects such as variations in ocean current systems; variations in atmospheric wind systems; redistribution of ground water; changes in sea level; oceanic, atmospheric, and solid earth tides; fluid motions in the earth's core; electromagnetic effects involving the operation of the geodynamo responsible for the main magnetic field of the earth; long term geologic processes such as post-glacial rebound, erosion and sedimentation, geologic uplift, and continental drift; and so on.

Internal rotational excitations are distinguished by the fact that they may not alter the total angular momentum vector of the earth. This is true even of the electromagnetic processes involving the geodynamo and the earth's magnetic field. The total angular momentum of the earth **L** necessarily includes the angular momentum of all its associated fields and in particular the angular momentum of the geomagnetic field. When this is done, angular momentum is conserved on the earth for all internal processes.

To prescribe the earth's orientation in space it is necessary to specify the orientation of the body-fixed basis vectors $\hat{e}_1\ \hat{e}_2\ \hat{e}_3$ relative to the space-fixed basis vectors $\hat{E}_1\ \hat{E}_2\ \hat{E}_3$. The external rotational excitations of the earth, principally the lunisolar gravitational torques, are related by physical theory to the time derivative of the earth's angular momentum vector $\dot{\mathbf{L}}$. The earth's angular momentum vector and its time derivative are not directly observable and so in order to deduce observational consequences from physical theory it is necessary to invoke some geophysical model for the earth to relate the angular momentum vector to some observable geophysical quantity. If the angular momentum vector **L** can be related to some observable body-fixed vector within the earth, then in observational as well as in theoretical practice the earth's orientation in space can be described by determining the orientation of this observable body-fixed vector with respect to both the set of basis vectors $\hat{e}_1\ \hat{e}_2\ \hat{e}_3$ and the set of basis vectors $\hat{E}_1\ \hat{E}_2\ \hat{E}_3$. In actual practice the "observable" body-fixed vectors chosen for this role have been the earth's figure axis, $\hat{e}_3$ itself, and the earth's instantaneous rotation axis, ω.

Current astronomical theory holds the fundamental reference direction in space to be the mean celestial pole of the ecliptic being defined by the mean orbital angular momentum vector of the earth. The present theory of the precession is based on known astronomical gravitational torques and known moments of inertia of an assumed rigid earth and as such describes the theoretical secular motion of the earth's axis of figure $\hat{e}_3$ relative to the mean pole of the ecliptic. This motion consists of the sum of the secular motion of $\hat{e}_3$ about the instantaneous celestial pole of the ecliptic, known as lunisolar precession and due to the gravitational torques of the sun and the moon on the earth, plus the secular motion of the instantaneous celestial pole of the ecliptic about the mean celestial pole of the ecliptic, known as planetary precession and due to the perturbations imposed on the earth's orbital plane by the other planets of the solar system. Together these two motions combine to make up general precession.

The present theory of nutation is also based on known astronomical gravitational torques and known moments of inertia of an assumed rigid earth and is tabulated in such a way that it describes the periodic motion of the earth's rotation axis ω relative to the mean pole of the ecliptic.

The complete motion is the sum of the secular and periodic components and strictly speaking should be obtained by adding the secular motion of $\hat{e}_3$ to the periodic motion of $\hat{e}_3$ or by adding the secular motion of ω to the periodic motion of ω. However, the secular motion of $\hat{e}_3$ and the secular motion of ω are identical (Goldreich and Toomre 1969) and so the complete motion is described in practice by adding the secular motion of $\hat{e}_3$ to the periodic motion of ω.

We see that the present theory of precession and nutation together describe the orientation of ω relative to the basis vectors $\hat{E}_1\ \hat{E}_2\ \hat{E}_3$ for a rigid earth. In order to orient the earth in space it is also necessary (but insufficient) to describe the orientation of ω relative to $\hat{e}_1\ \hat{e}_2\ \hat{e}_3$. This requires knowledge of the location of the axis of rotation relative to body of the earth or the effects of polar motion. However, that this is an insufficient condition to fix the orientation of $\hat{e}_1\ \hat{e}_2\ \hat{e}_3$ relative to $\hat{E}_1\ \hat{E}_2\ \hat{E}_3$ can be seen from the fact that fixing the orientation of ω in the system $\hat{e}_1\ \hat{e}_2\ \hat{e}_3$ and $\hat{E}_1\ \hat{E}_2\ \hat{E}_3$ still allows both the set $\hat{e}_1\ \hat{e}_2\ \hat{e}_3$ and $\hat{E}_1\ \hat{E}_2\ \hat{E}_3$ to be rotated arbitrarily about ω. The set of basis vectors $\hat{E}_1\ \hat{E}_2\ \hat{E}_3$, being space fixed, are assumed to be not rotating about the direction ω and the rotation of the set of basis vectors $\hat{e}_1\ \hat{e}_2\ \hat{e}_3$ about the direction ω is measured (very nearly) by UT1.

(It should be mentioned that known errors in the present theory of precession indicate that the set of basis vectors $\hat{E}_1\hat{E}_2\hat{E}_3$ is rotating at a rate of roughly 1."1 arc per century. The new theory of the precession is intended to reduce this error to the level of roughly 0."1 arc per century, which, while quite small by the standards of conventional astronomical measurements, is still roughly 1 milli arc second per year and probably observable by long baseline interferometry techniques.)

Although the title of this section, "Non-Eulerian Motion of an Axially Symmetric Rigid Earth," clearly embraces both the changes in the rotation vector relative to the space-fixed frame (precession and nutation) and the changes of the rotation vector relative to the body-fixed frame (polar motion and UT1), it is the latter phenomena with which this document will primarily concern itself and it is the latter phenomena which are described by the equations (IV-22) and (IV-23). Consequently we shall not be concerned with a general development of the theory of precession and nutation but with a general development of the theory of polar motion and UT1.

The superposition of the two processes, precession and nutation occurring simultaneously with polar motion and UT1 variations, can be understood *only approximately* by referring to Figure V-5 and imagining the external lunisolar gravitational torques displacing the previously space-fixed vector **L** around on the surface of a space fixed cone of apex angle about 47° (twice the obliquity of the ecliptic) with a period of roughly 26,000 years. The "space fixed" cone in Figure V-5 is now no longer space fixed but follows the vector **L**. This description is only approximate and is in error for two reasons.

First, the above description neglects the effects of nutation which would be manifested by small amplitude (~9″ arc), high frequency (~2 days – 18.6 years) periodic departures of **L** from the surface of this space fixed cone.

Second, the external gravitational torques perturb the Eulerian (torque-free) motion described so elegantly by the Poinsot construction. This can be seen by the appearance of the torque components $N_1\ N_2\ N_3$ in the RHS of Equations (IV-22) and (IV-23). In particular the lunisolar gravitational torques displace the instantaneous rotation vector in a circuit around its Eulerian position in a retrograde sense. The radius of this circle is about 0."02 arc and the period of the circuit is very nearly one sidereal day. Thus the simple "cone-on-cone" description breaks down.

The problem of the body-fixed rotational perturbations which occur in a rigid axially symmetric earth in response to a prescribed forcing function is of considerable interest in geophysics for its solution allows us to model a variety of geophysical processes and investigate their possible role in exciting polar motion and UT1 fluctuations in the earth. A general solution to the equations governing polar motion and UT1 variations in terms of arbitrary excitation functions and their integrals is easily obtained. If the geophysical excitation functions were sufficiently well known this solution could be used to predict the position of the rotation pole and the value of UT1 in advance. However, such a program is not practical on the basis of our present geophysical knowledge.

1. General solution to the dynamical equations governing polar motion and UT1 fluctuations. The dynamical equations governing the motion of the rotation axis in a body fixed frame are given by Equations (IV-22) and (IV-23). Using Equation (V-6) these can be written as

$$\frac{d\overline{m}}{dt} - i\sigma_r \overline{m} = \overline{e} \qquad \text{(V-53)}$$

$$\frac{dm_3}{dt} = \frac{de_3}{dt} \qquad \text{(V-54)}$$

where $\overline{e}$ is the complex wobble excitation function given by

$$\overline{e} = e_1 + ie_2 \qquad \text{(V-55)}$$

and where Equation (IV-22) gives

$$\overline{e} = \frac{1}{A\Omega}\left[\overline{N} - \Omega\frac{dr}{dt} - \frac{d\overline{h}}{dt} - i\,(\Omega^2 r + \Omega\overline{h})\right]. \qquad \text{(V-56)}$$

In Equation (V-56) the quantity de_3/dt is the UT1 excitation function given by

$$\frac{de_3}{dt} = \frac{1}{C\Omega}\frac{d}{dt}\left(\int_0^t N_3(t')\,dt' - \Omega\, r_{33} - h_3\right) \qquad \text{(V-57)}$$

in Equation (IV-23).

Since $\overline{m}$, m_3 are dimensionless angles the excitation functions $\overline{e}$, de_3/dt, have the units of sec^{-1} or "frequency".

The solution to Equation (V-56) can be obtained by the usual method of variation of parameters. Introducing the dimensionless complex excitation function $\overline{e}'$ where

$$\overline{e}' = e_1' + i\,e_2' \qquad \text{(V-58)}$$

given by

$$\overline{e}' = \frac{\overline{e}}{i\sigma_r} = \frac{1}{i\sigma_r}(e_1 + i\,e_2) \qquad \text{(V-59)}$$

Equation (V-56) can be written

$$\frac{d\overline{m}}{dt} = i\sigma_r(\overline{e}' + \overline{m})\,. \qquad \text{(V-60)}$$

The general solution to the homogeneous ($\overline{e}' = 0$) equation (V-60) is

$$\overline{m}(t) = \overline{m}^o\, e^{i\sigma_r(t-t_0)} \qquad \text{(V-61)}$$

to which must be added a particular integral

$$\overline{m}_p(t) = i\sigma_r\, e^{i\sigma_r t}\int_0^t \overline{e}'(t')\, e^{-i\sigma_r t'}\,dt' \qquad \text{(V-62)}$$

to give the general solution to the inhomogeneous ($e' \neq 0$) Equation (V-60) as

$$\overline{m}(t) = \overline{m}^o e^{i\sigma_r(t-t_0)} + i\sigma_r e^{i\sigma_r t}\int_0^t \overline{e}'(t)\, e^{-i\sigma_r t'}\,dt' \qquad \text{(V-63)}$$

That Equation (V-63) is the most general solution to Equation (V-60) can be verified by direct differentiation. In Equation (V-63) $\overline{m}^o$ is a complex integration constant.

The general solution to Equation (V-57) we have seen in Equation (IV-30) is given by

$$m_3(t) = e_3(t) + m_3^0 \qquad \text{(V-64)}$$

where

$$e_3(t) = \frac{1}{C\Omega}\left[\int_0^t N_3(t')\,dt' - \Omega\, r_{33} - h_3\right] \qquad \text{(V-65)}$$

and where m_3^0 is an integration constant.

2. Some idealized examples of polar motion excitation in an axially symmetric rigid earth. As an illustration of the usefulness of the solutions (V-63) and (V-64) we shall consider the case of a few idealized examples of the excitation of polar motion (Munk and MacDonald, 1960).

It can be seen from the governing equations for polar motion (V-60) that the complex wobble excitation function $\tilde{e}' = e_1' + i e_2'$ and the complex coordinate of the pole of the rotation axis $\tilde{m} = m_1 + i m_2$ have the same "dimensions" of radians. This leads naturally to the concept of a *wobble excitation axis*. It will prove mathematically convenient to define $\tilde{\phi}$ as the complex coordinate of the pole of the wobble excitation axis where

$$\tilde{\phi} = -\tilde{e}'$$

$$\tilde{\phi} = \phi_1 + i\phi_2 = -(e_1' + i e_2'). \qquad \text{(V-66)}$$

If the wobble excitation is small in the sense that $|\tilde{e}'| << 1$ then the excitation axis defining the pole of wobble excitation is associated with the unit vector $\hat{\phi}$ where

$$\hat{\phi} = \phi_1 \hat{e}_1 + \phi_2 \hat{e}_2 + \left[1 - \frac{1}{2}(\phi_1^2 + \phi_2^2)\right]\hat{e}_3. \qquad \text{(V-67)}$$

The excitation pole $\tilde{\phi}$ can be expressed in terms of the geophysical entities such as the components of the external torque $N_1\ N_2$, the perturbations to the inertia tensor $r_{13}\ r_{23}$, and the relative angular momentum components $h_1\ h_2$ as

$$\tilde{\phi} = -\frac{1}{A\Omega\sigma_r}\left[\left(N_2 - \Omega\frac{dr_{23}}{dt} - \frac{dh_2}{dt} - \Omega^2 r_{13} - \Omega h_1\right)\right.$$

$$\left. - i\left(N_1 - \Omega\frac{dr_{13}}{dt} - \frac{dh_1}{dt} + \Omega^2 r_{23} + \Omega h_2\right)\right]. \qquad \text{(V-68)}$$

a. Step function wobble excitation. Step function wobble excitation can be represented mathematically as

$$\tilde{\phi}(t) = \tilde{J}H(t - t_0) \qquad \text{(V-69)}$$

where $\tilde{J}$ is a complex constant given by $\tilde{J} = J_1 + iJ_2$ and where $H(t - t_0)$ is the Heaviside step function defined by

$$H(t - t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t \geq t_0 \end{cases} \qquad \text{(V-70)}$$

Physically we might expect step function wobble excitation to be an approximate model for the effect of earthquakes on polar motion. In such a simple model the fault dislocation produces a discontinuous and permanent change in the earth's inertia tensor and acts as an abrupt generator of the perturbations $r_{13}\ r_{23}$ in Equation (V-68) (Smylie and Mansinha, 1971a; Mansinha, Smylie, and Chapman 1979).

Substituting the wobble excitation function (V-69) into the general solution, Equation (V-63), gives

$$\tilde{m}(t) = \tilde{m}^{\circ} e^{i\sigma_r(t - t_0)} - i\sigma_r e^{i\sigma_r t} \int_0^t \tilde{J}H(t' - t_0)\, e^{-i\sigma_r t'}\, dt'$$

(V-71)

Since we are assuming an absence of wobble excitation for $t < t_0$ we have

$$\tilde{m}^{\circ} = 0$$

and so Equation (V-71) becomes

$$\tilde{m}(t) = -i\sigma_r e^{i\sigma_r t}\, \tilde{J} \int_{t_0}^t e^{-i\sigma_r t'}\, dt'$$

which integrates to give

$$\tilde{m}(t) = \tilde{J}\,[1 - e^{i\sigma_r(t - t_0)}]. \qquad \text{(V-72)}$$

The geographic coordinates $m_1(t)\ m_2(t)$ of the pole of rotation can be obtained from Equation (V-72) by setting $\tilde{J} = J_1 + iJ_2, \tilde{m} = m_1 + im_2$, to obtain

$$m_1(t) = J_1 - J_1 \cos\sigma_r(t - t_0) + J_2 \sin\sigma_r(t - t_0)$$

$$m_2(t) = J_2 - J_2 \cos\sigma_r(t - t_0) + J_1 \sin\sigma_r(t - t_0).$$

(V-73)

In this solution we see that at $t \leq t_0$ the rotation pole coordinates are $m_1 = 0$, $m_2 = 0$. As soon as the step function excitation is imposed and the excitation pole appears at the coordinates

$$\tilde{\phi} = \tilde{J}$$

the rotation axis begins to describe a steady prograde circular path of radius $|\tilde{\phi}|$ about the excitation pole. This is illustrated in Figure V-6. The angular rate of the motion of the pole of rotation is σ_r. The pole completes one circuit in an interval $2\pi/\sigma_r$.

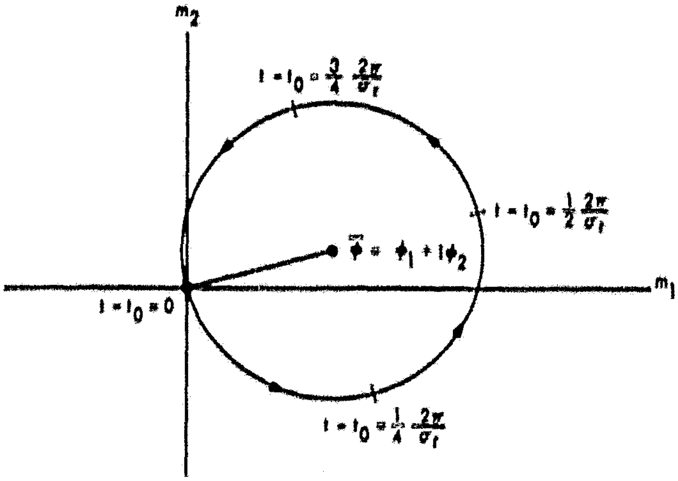

Figure V-6. The polar motion which results from a step function excitation at time $t = t_0$. The rotation axis executes uniform prograde circular motion about the excitation axis.

b. Delta function wobble excitation. Impulsive or δ-function wobble excitation can be represented mathematically as

$$\overline{\phi}(t) = \mathcal{J}\,\delta\,(t - t_0) \tag{V-74}$$

where $\mathcal{J}$ is a complex constant given by $\mathcal{J} = J_1 + iJ_2$ and *has the dimensions of time* and where $\delta(t - t_0)$ is the Dirac δ-function. The dimensions of $\mathcal{J}$ follow from the requirement that

$$\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1 \tag{V-75}$$

hence $\delta(t - t_0)$ has the dimensions of time^{-1}, and that $\overline{\phi}(t)$ be dimensionless.

Physically we might expect impulsive wobble excitation to be an approximate model for the effects of short lived atmospheric storms on polar motion largely as a result of the changes in h_1 and h_2 which might accompany such events.

Substituting this wobble excitation function (V-74) into the general solution, Equation (V-63) gives

$$\overline{m}(t) = \overline{m}^{\circ} e^{i\sigma_r(t-t_0)} - i\sigma_r e^{i\sigma_r t} \int_0^t \mathcal{J}\delta(t' - t_0)\, e^{-i\sigma_r t'}\, dt' \,. \tag{V-76}$$

Since we are assuming the absence of wobble for $t < t_0$ we have

$$\overline{m}^{\circ} = 0$$

and so Equation (V-76) becomes

$$\overline{m}(t) = -i\sigma_r e^{i\sigma_r t} \mathcal{J} \int_0^t \delta(t' - t_0)\, e^{-i\sigma_r t'}\, dt' \tag{V-77}$$

which integrates to give

$$\overline{m}(t) = \begin{cases} 0 & t < t_0 \\ -i\sigma_r \mathcal{J} e^{i\sigma_r(t-t_0)} & t \geq t_0 \end{cases} \,. \tag{V-78}$$

Since $\mathcal{J}$ has the dimensions of time and σ_r has the dimensions of time^{-1} the quantity $\sigma_r\mathcal{J}$ is dimensionless and will serve as a dimensionless wobble amplitude $\overline{K}$

$$\overline{K} = \sigma_r \mathcal{J} \tag{V-79}$$

and the solution, Equation (V-78), can be written

$$\overline{m}(t) = \begin{cases} 0 & t < t_0 \\ -i\overline{K} e^{i\sigma_r(t-t_0)} & t > t_0 \end{cases} \,. \tag{V-80}$$

The geographic coordinates $m_1(t)\, m_2(t)$ of the pole of rotation can be obtained from Equation (V-80) by setting $\overline{K} = K_1 + iK_2$, $\overline{m} = m_1 + i\,m_2$, to obtain

$$\begin{aligned} m_1(t) &= K_1 \sin\sigma_r(t - t_0) + K_2 \cos\sigma_r(t - t_0) \\ m_2(t) &= K_2 \sin\sigma_r(t - t_0) - K_1 \cos\sigma_r(t - t_0) \end{aligned} \,. \tag{V-81}$$

In this solution we see that at time $t = t_0$ the pole of rotation moves discontinuously at the time of the impulse to the complex coordinate $-i\overline{K}$ or to $m_1(t) = K_2$, $m_2(t) = -K_1$. For times $t > t_0$ the pole of rotation moves in a steady prograde circular path of radius $|\overline{K}| = \sigma_r|\mathcal{J}|$ about the excitation pole, which for times $t > t_0$ resides at the origin since for times $t > t_0$ the excitation is zero. This is illustrated in Figure V-7.

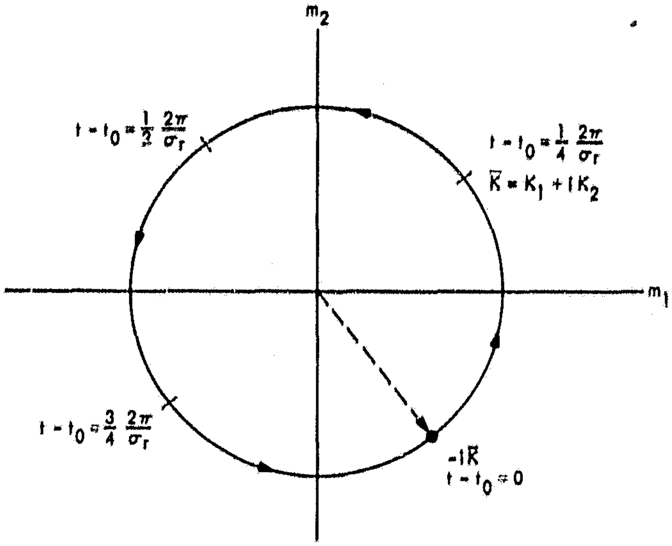

Figure V-7. The polar motion which results from a δ-function excitation at time $t = t_o$. The rotation axis is displaced discontinuously at time $t = t_o$ from its original position and subsequently executes uniform prograde circular motion.

A comparison of step function wobble excitation and δ function wobble excitation leads to the following general conclusions

(1) Step function excitation displaces the excitation pole but not the rotation pole at $t = t_0$.

(2) δ function excitation displaces the rotation pole but not the excitation pole (except for an interval of measure zero) at $t = t_0$.

(3) Both excitations lead to prograde polar motion about the excitation pole.

c. *Harmonic wobble excitation.* Harmonic wobble excitation can be represented mathematically as

$$\bar{\phi}(t) = \mathcal{J}^c \cos \sigma (t - t_0) + \mathcal{J}^s \sin \sigma (t - t_0) \quad \text{(V-82)}$$

where $\mathcal{J}^c$ $\mathcal{J}^s$ are complex constants given by $\mathcal{J}^c = J_1^c + i J_2^c$, $\mathcal{J}^s = J_1^s + i J_2^s$, and where σ is the frequency of the harmonic excitation which *is arbitrary* and not necessarily equal to σ_r, the "resonance" wobble frequency of the axially symmetric rigid earth.

Physically harmonic wobble excitation on the earth occurs as a result of the external lunisolar gravitational torques N_1 N_2 which, when viewed in the body-fixed rotating frame, are harmonically varying. It is in fact this excitation which produces the retrograde motion of the rotation pole about its Eulerian position previously mentioned.

Alternate forms of the harmonic wobble excitation function of Equation (V-82) are

$$\bar{\phi}(t) = \mathcal{J}^+ e^{i\sigma(t-t_0)} + \mathcal{J}^- e^{-i\sigma(t-t_0)} \quad \text{(V-83)}$$

and

$$\bar{\phi}(t) = |\mathcal{J}^+| e^{i[\sigma(t-t_0) + \lambda^+]} + |\mathcal{J}^-| e^{-i[\sigma(t-t_0) + \lambda^-]} \quad \text{(V-84)}$$

where

$$\mathcal{J}^+ = \frac{1}{2}(\mathcal{J}^c - i\mathcal{J}^s) \qquad \mathcal{J}^- = \frac{1}{2}(\mathcal{J}^c + i\mathcal{J}^s) \quad \text{(V-85)}$$

and where

$$\lambda^+ = \tan^{-1}\left(\frac{J^+_{IMAG}}{J^+_{REAL}}\right) \qquad \lambda^- = \tan^{-1}\left(\frac{J^-_{IMAG}}{J^-_{REAL}}\right) \quad \text{(V-86)}$$

In the formulation of Equation (V-84) the quantities $|\mathcal{J}^+|$ and $|\mathcal{J}^-|$ are the modulii of the complex amplitudes $\mathcal{J}^+$ $\mathcal{J}^-$ respectively. $|\mathcal{J}^+|$ represents the amplitude of a prograde rotating excitation function $\mathcal{J}^+$ whose phase angle at the epoch $t = t_0$ is λ^+. $|\mathcal{J}^-|$ represents the amplitude of a retrograde rotating excitation function $\mathcal{J}^-$ whose phase angle at the epoch $t = t_0$ is λ^-.

It is most convenient to proceed with the solution to the problem of harmonic wobble excitation by choosing the form of the excitation function given in Equation (V-83). If we consider the harmonic wobble excitation to have commenced at time $t = t_0$ then substituting the wobble excitation function (V-83) into the general solution equation (V-63), gives

$$\overline{m}(t) = \overline{m}^\circ e^{i\sigma_r(t-t_0)} - i\sigma_r e^{i\sigma_r t}\left[\int_{t_0}^{t} \mathcal{J}^+ e^{i\sigma(t'-t_0)} e^{-i\sigma_r t'} dt' - i\sigma_r \int_{t_0}^{t} \mathcal{J}^- e^{-i\sigma(t'-t_0)} e^{-i\sigma_r t'} dt'\right] \quad \text{(V-87)}$$

which integrates to give

$$\overline{m}(t) = \overline{m}^{\circ} e^{i\sigma_r(t-t_0)} - i\sigma_r e^{i\sigma_r t} \left\{ e^{-i\sigma t_0} \left[\overline{J}^{+} \frac{e^{i(\sigma-\sigma_r)t}}{i(\sigma-\sigma_r)} \Bigg|_{t_0}^{t} \right] \right.$$

$$\left. + e^{i\sigma t_0} \left[\overline{J}^{-} \frac{e^{-i(\sigma+\sigma_r)t}}{-i(\sigma+\sigma_r)} \Bigg|_{t_0}^{t} \right] \right\} \tag{V-88}$$

which becomes

$$\overline{m}(t) = \overline{m}^{\circ} e^{i\sigma_r(t-t_0)} - \frac{\sigma_r}{\sigma-\sigma_r} \overline{J}^{+} \left[e^{i\sigma(t-t_0)} - e^{i\sigma_r(t-t_0)} \right]$$

$$+ \frac{\sigma_r}{\sigma+\sigma_r} \overline{J}^{-} \left[e^{-i\sigma(t-t_0)} - e^{i\sigma_r(t-t_0)} \right]. \tag{V-89}$$

Equation (V-89) can be rewritten as

$$\overline{m}(t) = \left[\overline{m}^{\circ} + \frac{\sigma_r}{\sigma-\sigma_r} \overline{J}^{+} - \frac{\sigma_r}{\sigma+\sigma_r} \overline{J}^{-} \right] e^{i\sigma_r(t-t_0)}$$

$$- \frac{\sigma_r}{\sigma-\sigma_r} \overline{J}^{+} e^{i\sigma(t-t_0)} + \frac{\sigma_r}{\sigma+\sigma_r} \overline{J}^{-} e^{-i\sigma(t-t_0)} \tag{V-90}$$

We see that the solution consists of *two* wobble components: one occurring at the Eulerian frequency σ_r and one occurring at the forcing frequency σ. The Eulerian component is a prograde rotation $\sigma_r > 0$ of the pole of rotation with amplitude $|\overline{M}|$ where

$$|\overline{M}| = \left| \overline{m}^{\circ} + \frac{\sigma_r}{\sigma-\sigma_r} \overline{J}^{+} - \frac{\sigma_r}{\sigma+\sigma_r} \overline{J}^{-} \right| \tag{V-91}$$

The wobble occurring at the forcing frequency consists of a prograde component of amplitude $[\sigma_r/(\sigma - \sigma_r)]\,|\overline{J}^{+}|$ and a retrograde component of amplitude $[\sigma_r/(\sigma + \sigma_r)]\,|\overline{J}^{-}|$.

The fact that harmonic excitation at frequency σ can also excite the Eulerian wobble at frequency σ_r is intimately connected with the presence of both the annual and Chandler frequencies in the spectrum of the earth's wobble.

3. The lunisolar harmonic wobble excitation in an axially symmetric rigid earth. The effect of the lunisolar torques on the position of the instantaneous rotation axis in a body-fixed reference frame has been analyzed in detail by Woolard (1953) for the case of an axially symmetric rigid earth. The lunisolar wobble motion can be expressed as the vector addition of the complex coordinate $\overline{m}_p$ to the Eulerian motion $\overline{m}_e$, where as usual

$$\overline{m}_p = m_{p1} + i\, m_{p2}$$

$$\overline{m}_e = m_{e1} + i\, m_{e2}\ . \tag{V-92}$$

Formulae for the lunisolar wobble motion are obtained by solving Equation (IV-22) written as

$$\frac{d\overline{m}_p}{dt} - i\,\sigma_r \overline{m}_p = \frac{\overline{N}}{A\Omega} \tag{V-93}$$

where $\overline{N} = N_1 + i\,N_2$ is the complex linisolar torque on the earth expressed in the body-fixed frame. The solution to this problem is given in Woolard and Clemence (1966)

$$m_{p1} = +0''\!.0087 \sin\phi - 0''\!.0062 \sin(\phi - 2L_{☾})$$

$$- 0''\!.0029 \sin(\phi - 2L_{\odot}) + \cdots \tag{V-94}$$

$$m_{p2} = +0''\!.0087 \cos\phi - 0''\!.0062 \cos(\phi - 2L_{☾})$$

$$- 0''\!.0029 \cos(\phi - 2L_{\odot}) + \cdots \tag{V-95}$$

where

(1) ϕ is Greenwich Mean Sidereal Time (GMST).

(2) $L_{☾}$ is the mean longitude of the moon.

(3) $L_{\odot}$ is the mean longitude of the sun.

The lunisolar torques superimpose on the slow Eulerian motion of angular rate σ_r a retrograde nearly diurnal circular motion of the rotation axis with radius ranging from zero to $0''\!.02$ arc depending on the positions of the sun and the moon. This motion, which results from the non-vanishing complex torque $\overline{N}$ in Equation (IV-22), is illustrated in Figure V-8.

4. The damping of the wobble and wobble Q. It is apparent from the form of the solution for harmonically forced wobble on the earth, Equation (V-90), that even infinitesimal harmonic excitation at the Eulerian frequency σ_r will produce an infinite wobble amplitude. This unphysical prediction results from the fact that we have considered the earth to be infinitely rigid and hence free of any internal dissipation.

The real earth is not infinitely rigid. The earth's finite rigidity, in addition to greatly altering the character of the

ORIGINAL PAGE IS
OF POOR QUALITY

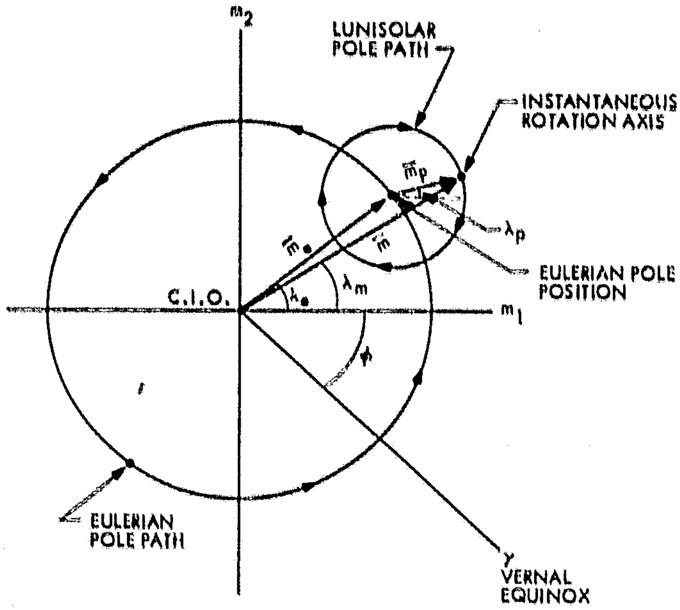

Figure V-8. The combined effect of the Eulerian polar motion $\vec{m}_e$ and the lunisolar polar motion $\vec{m}_p$. The rotation of $\vec{m}_e$ is prograde while the rotation of $\vec{m}_p$ is retrograde. The vector sum of $\vec{m}_e$ and $\vec{m}_p$ constitutes the total polar motion m. The amplitude $|\vec{m}_p|$ is time-dependent.

Eulerian response in a manner to be investigated in a later section of this work, will also result in internal dissipation of wobble energy within the earth. Since the spectrum of naturally occurring wobble excitation is quite complex it will inevitably contain some power in the infinitesimal frequency band containing the Eulerian frequency σ_r. It is the internal dissipation within the earth which results in the observed finite wobble amplitude even in the presence of continual wobble excitation.

Strictly speaking the question of the damping of the earth's wobble does not belong in a theoretical treatment of the rotational dynamics of a rigid earth for its answer necessarily lies in an investigation of the detailed mechanism of the wobble damping and hence an investigation of the general rheological nature of the earth including its fluid portions. However, by introducing the "specific dissipation" or Q it is possible to introduce dissipation into the theory without confronting the question of the detailed mechanism responsible for the dissipation.

The Q of an oscillating system with total energy E and internal dissipation rate dE/dt is defined to be

$$\frac{1}{Q} = \frac{1}{2\pi E} \oint \frac{dE}{dt}\, dt \qquad \text{(V-96)}$$

where the integral is taken over one complete cycle of the oscillation. It can be shown (Munk and MacDonald, 1960) that the Q is related to the sharpness of the resonance peak of the oscillator by

$$\frac{1}{Q} = \frac{2\Delta\sigma}{\sigma_r} \qquad \text{(V-97)}$$

where $\sigma_r \pm \Delta\sigma$ are the frequencies at the half power points of the resonance curve.

The Q of the earth's wobble is generally estimated by this method from spectral analysis of polar motion data. Estimates for Q based on present day data are only precise enough to place it roughly within the bounds $30 \leqslant Q \leqslant 60$ (Pedersen and Rochester, 1972). Improved data which could yield a more precise figure for Q would be of assistance in understanding the mechanism of dissipation within the earth.

VI. Equilibrium Deformation Fields in a Real Deformable Earth

A complete theoretical understanding of the rotational dynamics of the "real" earth which incorporates realistic models for the rheology of the deformable earth and its fluid portions as well as all the forces acting on them is a distant goal for geodynamical theory. Such an achievement is greatly hampered by:

(1) Incomplete knowledge of the properties of the earth's fluid core and its interaction with the rest of the earth including its electromagnetic and viscous effects.

(2) Incomplete knowledge of the long term rheological properties of the inner core, mantle, and crust of the earth.

(3) Incomplete knowledge of the current systems in the oceans and atmosphere and their interaction with the shell and each other, to name but a few major items on a very long list.

Nevertheless a simple technique developed originally by A. E. H. Love (1909) allows reasonably rigorous treatment of the effects of earth deformations for certain classes of deforming force fields. Three important geophysical disturbing forces which are capable of deforming the earth and which can be treated adequately by Love's technique are:

(1) Centrifugal forces and their effects on earth rotation.

(2) Tidal forces and their effects on earth rotation.

(3) Surface loads and their effects on earth rotation.

Since Love's theoretical technique plays a major role in the incorporation of earth deformation into the theory of earth rotation for a wide class of deforming force fields we shall first review Love's theory in some detail before proceeding to discuss the effects of the above three phenomena on earth rotation.

A. Love Numbers and Equilibrium Earth Deformations

The use of Love numbers in geophysics is subjected to a set of restricting assumptions which are:

(1) The earth is assumed to be spherically symmetric in its elastic parameters and overall structure.

(2) The earth is assumed to be in static equilibrium with the system of deforming forces. Strictly speaking this restrictive assumption prevents the use of Love numbers in anything but problems of *geostatics*. However, their use in *geodynamics* is justified in instances where the time scale of the changes in the system of perturbing forces is large compared to the elastic response time of the earth in which case the internal displacement field **u**(**r**) is at all times infinitesimally close to being in equilibrium with the deforming forces. The time scale of the earth's elastic response is of the order of the transit time of a seismic wave across an earth diameter or of the period of the gravest mode in the earth's free oscillation spectrum. Both these intervals are of the order of one hour and so the use of Love numbers to describe the geodynamical response of the earth to perturbing forces whose characteristics are changing significantly on time scales large compared to one hour is possible.

(3) The perturbing force field is assumed to be weak enough that the resulting stresses are small compared to the strength of earth materials, which is typically 10^9 dynes cm^{-2}. In this case the response of the earth will be linearly related to the perturbing stresses.

We begin by considering the earth in equilibrium in its unperturbed state characterized by a gravitational potential $V^0(r)$ and a density profile $\rho^0(r)$, both functions only of radius $r = |\mathbf{r}|$. We then consider the earth subjected to a perturbing force field $\mathbf{f}(\mathbf{r})$ which results in an internal deformation field $\mathbf{u}(\mathbf{r})$. When the perturbing force field $\mathbf{f}(\mathbf{r})$ is derivable from a potential $V^p(\mathbf{r})$,

$$\mathbf{f}(\mathbf{r}) = -\nabla V^p(\mathbf{r}) \tag{VI-1}$$

then it can be shown (Smylie and Mansinha, 1971) that under the restrictions stated above the radial displacement $u_r(\mathbf{r})$ and the dilation $\nabla \cdot \mathbf{u}(\mathbf{r})$ are proportional to the perturbing potential. In general for any radius $r = |\mathbf{r}|$ within the earth we can write

$$u_r(\mathbf{r}) = H'(r)\, V^p(\mathbf{r}) \tag{VI-2}$$

$$\nabla \cdot \mathbf{u}(\mathbf{r}) = F'(r)\, V^p(\mathbf{r}) \tag{VI-3}$$

where $H'(r)$, $F'(r)$ are radial functions which depend on the earth's elastic properties.

Following Love it will be convenient to define functions $H(r)$, $F(r)$ such that

$$H(r) = g^0(r)\, H'(r) \tag{VI-4}$$

$$F(r) = g^0(r)\, F'(r) \tag{VI-5}$$

where $g^0(r)$ is the positive scalar magnitude of gravity at radius r within the undeformed earth. In other words

$$g^0(r) = \frac{Gm^0(r)}{r^2} \tag{VI-6}$$

where $m^0(r)$ is the mass contained in a sphere of radius r given by

$$m^0(r) = 4\pi \int_0^r \rho^0(r)\, r^2 dr\,.$$

It follows that

$$u_r(\mathbf{r}) = \frac{H(r)}{g^0(r)}\, V^p(\mathbf{r}) \tag{VI-7}$$

$$\nabla \cdot \mathbf{u}(\mathbf{r}) = \frac{F(r)}{g^0(r)}\, V^p(\mathbf{r})\,. \tag{VI-8}$$

In general the perturbing potential $V^p(\mathbf{r})$ will have a solid spherical harmonic expansion

$$V^p(\mathbf{r}) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{r}{a}\right)^n P_n^m(\cos\theta)\,(C_n^m \cos m\lambda + S_n^m \sin m\lambda) \tag{VI-9}$$

valid for $r \leqslant a$; where θ is the geocentric colatitude and λ is the geocentric east longitude and P_n^m $(\cos\theta)$ is the associated

Legendre polynomial of degree n and order m. Symbolically we may write

$$V^p(\mathbf{r}) = \sum_n \sum_m V_n^{pm}(\mathbf{r}), \quad r \leq a, \tag{VI-10}$$

where $V_n^{pm}(\mathbf{r})$ is the n, mth element in the double sum.

It follows that

$$u_r(\mathbf{r}) = \frac{1}{g^0(r)} \sum_n \sum_m H_n(\mathbf{r})\, V_n^{pm}(\mathbf{r}) \tag{VI-11}$$

$$\nabla \cdot \mathbf{u}(\mathbf{r}) = \frac{1}{g^0(\mathbf{r})} \sum_n \sum_m F_n(r)\, V_n^{pm}(\mathbf{r}) \tag{VI-12}$$

where the quantities $H_n(r)$, $F_n(r)$ are radially varying functions depending on the earth's elastic properties and depending on the degree n of the spherical harmonic $V_n^{pm}(\mathbf{r})$. That $H_n(r)$, $F_n(r)$ depend only on the degree n and not the order m of the spherical harmonic is a consequence of the fact that spherical harmonics of the same degree but different order all have the same radial dependence.

The deformation of the earth $\mathbf{u}(\mathbf{r})$ produces a perturbation in the equilibrium density profile of the earth. If $\rho(\mathbf{r})$ is the density profile after deformation and $\rho^0(r)$ is the density profile before deformation, then the perturbation to the density $\rho^1(\mathbf{r})$ is defined

$$\rho^1(\mathbf{r}) = \rho(\mathbf{r}) - \rho_0(\mathbf{r}) \tag{VI-13}$$

and is related to the displacement field and the original density field by

$$\rho^1(\mathbf{r}) = -\,\mathbf{u}(\mathbf{r}) \cdot \nabla\rho^0(r) - \rho^0(r)\nabla \cdot \mathbf{u}(\mathbf{r}). \tag{VI-14}$$

Now ρ^0 depends only on radius $r = |\mathbf{r}|$ and so

$$\nabla\rho^0(\mathbf{r}) = \frac{d\rho^0(r)}{dr}\hat{r} \tag{VI-15}$$

which allows Equation (VI-14) to be written

$$\rho^1(\mathbf{r}) = -\frac{d\rho^0(\mathbf{r})}{dr} u_r(\mathbf{r}) - \rho^0(r)\nabla \cdot \mathbf{u}(\mathbf{r}). \tag{VI-16}$$

Using Equations (VI-7)(VI-8) in Equation (VI-16) we see that

$$\rho^1(\mathbf{r}) = -\frac{1}{g^0(r)}\left[\frac{d\rho^0(r)}{dr}H(r) + \rho^0(r)\,F(r)\right] V^p(\mathbf{r}). \tag{VI-17}$$

It follows from Equations (VI-10)(VI-17) that $\rho^1(\mathbf{r})$ can be written

$$\rho^1(\mathbf{r}) = G(r) \sum_n \sum_m V_n^{pm}(\mathbf{r}) \tag{VI-18}$$

where

$$G(r) = -\frac{1}{g^0(r)}\left[\frac{d\rho^0(r)}{dr} H(r) + \rho^0(r)\,F(r)\right]. \tag{VI-19}$$

The deformation of the earth and subsequent redistribution of the mass of the earth produces a perturbation to the equilibrium gravitational potential. If $V(\mathbf{r})$ is the gravitational potential after deformation and $V^0(r)$ is the gravitational potential before deformation then the perturbation to the gravitational potential $V^1(\mathbf{r})$ is defined

$$V^1(\mathbf{r}) = V(\mathbf{r}) - V^0(r). \tag{VI-20}$$

Both $V(\mathbf{r})$ and $V^0(r)$ necessarily satisfy Poisson's equation for the density distributions $\rho(\mathbf{r})$ and $\rho^0(r)$ respectively,

$$\nabla^2 V(\mathbf{r}) = -\,4\pi\, G\rho(\mathbf{r}) \tag{VI-21}$$

$$\nabla^2 V^0(r) = -\,4\pi\, G\rho^0(r) \tag{VI-22}$$

and since the Laplacian is a linear differential operator we can conclude from Equations (VI-13) (VI-20) (VI-21) (VI-22) that

$$\nabla^2 V^1(\mathbf{r}) = -\,4\pi\, G\rho^1(\mathbf{r}). \tag{VI-23}$$

It can be shown (Kaula, 1968, pp. 61-69) that the gravitational potential $V^1(\mathbf{r})$ resulting from a density distribution $\rho^1(\mathbf{r})$ given by Equation (VI-18) will have the form

$$V^1(\mathbf{r}) = \sum_n \sum_m K_n(r)\, V_n^{pm}(\mathbf{r}) \tag{VI-24}$$

where the quantities $K_n(r)$ are radially varying functions depending on radial integrals involving the density distribution $\rho^1(\mathbf{r})$ and depending on the degree n of the spherical harmonic $V_n^{pm}(\mathbf{r})$. That $K_n(r)$ depends *only* on the degree n and not the order m of the spherical harmonic is again a consequence of the fact that spherical harmonics of the same degree but different order all have the same radial dependence.

We now consider a perturbing potential field $V^p(\mathbf{r})$ of *single fixed degree n*. In other words we take

$$V^p(\mathbf{r}) = V_n^p(\mathbf{r}) = \sum_{m=0}^{n} V_n^{pm}(\mathbf{r}) . \qquad \text{(VI-25)}$$

It follows from Equation (VI-11) that the radial displacement $u_r(\mathbf{r})$ within the earth in response to the perturbing potential $V_n^p(\mathbf{r})$ is

$$u_r(\mathbf{r}) = \frac{H_n(r)}{g^0(r)} V_n^p(\mathbf{r}) \qquad \text{(VI-26)}$$

and that the perturbation $V^1(\mathbf{r})$ to the earth's gravitational potential in response to the perturbing potential $V_n^p(\mathbf{r})$ is

$$V^1(\mathbf{r}) = K_n(r)\, V_n^p(\mathbf{r}) . \qquad \text{(VI-27)}$$

The total perturbed potential $V'(\mathbf{r})$ in the region $r \leqslant a$ inside the earth is the sum of

(1) $V^0(r)$ the original unperturbed potential,

(2) $V_n^p(\mathbf{r})$ the perturbing potential of the deforming force field causing the deformation $\mathbf{u}(\mathbf{r})$,

(3) $V^1(\mathbf{r})$ the increment to the gravitational potential resulting from the redistribution of the earth's mass accompanying the displacement $\mathbf{u}(\mathbf{r})$,

$$V'(\mathbf{r}) = V^0(r) + V_n^p(\mathbf{r}) + V^1(\mathbf{r}), \quad r \leqslant a . \qquad \text{(VI-28)}$$

For a perturbing potential $V_n^p(\mathbf{r})$ of single fixed degree n we can use Equation (VI-24) to write

$$V'(\mathbf{r}) = V^0(r) + [1 + K_n(r)]\, V_n^p(\mathbf{r}), \quad r \leqslant a . \qquad \text{(VI-29)}$$

There are a number of aspects to this theoretical development which require emphasis at this point.

First, the potential $V_n^p(\mathbf{r})$ refers to a potential field from which the system of deforming body forces is to be derived and can only be defined in the region $r \leqslant a$. Also the radial function $K_n(r)$ depending on the elastic properties of the earth can only be defined in the region $r \leqslant a$. In general Equation (VI-28) only has meaning in the region $r \leqslant a$.

Second, the Newtonian gravitational potential of the earth $V(\mathbf{r})$ (exclusive of the disturbing potential) depends *only* on the mass distribution. In the undeformed earth we had

$$V(\mathbf{r}) = V^0(r) \qquad \text{(VI-30)}$$

but in the deformed earth $V(\mathbf{r})$ consists of the sum of

$$V(\mathbf{r}) = V^0(r) + V^1(\mathbf{r}) \qquad \text{(VI-31)}$$

and so is given in the region $r \leqslant a$ by

$$V(\mathbf{r}) = V^0(r) + K_n(r)\, V_n^p(\mathbf{r}) . \qquad \text{(VI-32)}$$

Third, the analysis presented thus far is carried out from an Eulerian viewpoint in which the vector $\mathbf{r}$ refers to some fixed position relative to the origin of coordinates (geocenter) and involves the comparison of $V^0(r)$ before deformation with $V'(\mathbf{r})$ and $V(\mathbf{r})$ after deformation (Equations (VI-28) and (VI-32)). For an observer or a particle moving with the deforming earth a Lagrangian description is more appropriate. A Lagrangian description would involve a comparison of $V^0(r)$ and $V'[\mathbf{r} + \mathbf{u}(\mathbf{r})]$ and $V[\mathbf{r} + \mathbf{u}(\mathbf{r})]$.

A Lagrangian analysis begins with the definition

$$V(\mathbf{r}) = V^0(r) + V^1(\mathbf{r}) \qquad \text{(VI-33)}$$

from which it follows

$$V(\mathbf{r} + \mathbf{u}) = V^0(\mathbf{r} + \mathbf{u}) + V^1(\mathbf{r} + \mathbf{u}) . \qquad \text{(VI-34)}$$

A Taylor series expansion of Equation (VI-34) gives

$$V(\mathbf{r} + \mathbf{u}) = V^0(\mathbf{r}) + \nabla V^0(\mathbf{r}) \cdot \mathbf{u}(\mathbf{r}) + V^1(\mathbf{r}) + \nabla V^1(\mathbf{r}) \cdot \mathbf{u}(\mathbf{r}) \qquad \text{(VI-35)}$$

which to first order in small quantities reduces to

$$V(\mathbf{r} + \mathbf{u}) = V^0(r) + \nabla V^0(\mathbf{r}) \cdot \mathbf{u}(\mathbf{r}) + V^1(\mathbf{r}) . \qquad \text{(VI-36)}$$

Now by definition

$$\mathbf{g}^0(\mathbf{r}) = -\nabla V^0(\mathbf{r})$$

and since $\mathbf{g}^0(\mathbf{r})$ has only a radial component owing to the assumed spherical symmetry of the undeformed earth we can write

$$\nabla V^0(\mathbf{r}) \cdot \mathbf{u}(\mathbf{r}) = -\mathbf{g}^0(\mathbf{r}) \cdot \mathbf{u}(\mathbf{r}) = -g^0(r)\, u_r(\mathbf{r}) \quad \text{(VI-37)}$$

which when substituted into Equation (VI-36) gives

$$V(\mathbf{r}+\mathbf{u}) = V^0(r) - g^0(r)\, u_r(\mathbf{r}) + V^1(\mathbf{r}) \,. \quad \text{(VI-38)}$$

Using Equations (VI-11) and (VI-24) in Equation (VI-38) gives

$$V(\mathbf{r}+\mathbf{u}) = V^0(r) - g^0(r) \frac{1}{g^0(r)} \sum_n \sum_m H_n(r)\, V_n^{pm}(\mathbf{r})$$

$$+ \sum_n \sum_m K_n(r)\, V_n^{pm}(\mathbf{r}), \quad r \leqslant a \,, \quad \text{(VI-39)}$$

or

$$V(\mathbf{r}+\mathbf{u}) = V^0(r) + \sum_n \sum_m \left[K_n(r) - H_n(r)\right] V_n^{pm}(\mathbf{r}) \,,$$

$$r \leqslant a \,. \quad \text{(VI-40)}$$

Equation (VI-40) expresses the Lagrangian variation in the gravitational potential for an observer moving with the deforming earth. The total effective potential $V^t(\mathbf{r}+\mathbf{u})$ sensed by a particle would include the perturbing potential $V^p(\mathbf{r})$ responsible for the body force deformation field. Using Equation (VI-60) we can write

$$V^t(\mathbf{r}+\mathbf{u}) = V^0(r) + \sum_n \sum_m \left[1 + K_n(r) - H_n(r)\right] V_n^{pm}(\mathbf{r}) \,,$$

$$r \leqslant a \,. \quad \text{(VI-41)}$$

Considering once again a perturbing potential of single harmonic degree n we can summarize the results:

Eulerian Viewpoint

$$V(\mathbf{r}) = V^0(r) + K_n(r)\, V_n^p(\mathbf{r}) \,, \quad r \leqslant a \,, \quad \text{(VI-42)}$$

$$V^t(\mathbf{r}) = V^0(r) + \left[1 + K_n(r)\right] V_n^p(\mathbf{r}) \,, \quad r \leqslant a \,, \quad \text{(VI-43)}$$

Lagrangian Viewpoint

$$u_r(\mathbf{r}) = \frac{H_n(r)}{g^0(r)}\, V_n^p(\mathbf{r}) \,, \quad r \leqslant a \,, \quad \text{(VI-44)}$$

$$V(\mathbf{r}+\mathbf{u}) = V^0(r) + \left[K_n(r) - H_n(r)\right] V_n^p(\mathbf{r}) \,, \quad r \leqslant a \,, \quad \text{(VI-45)}$$

$$V^t(\mathbf{r}+\mathbf{u}) = V^0(r) + \left[1 + K_n(r) - H_n(r)\right] V_n^p(\mathbf{r}) \,, \quad r \leqslant a \,. \quad \text{(VI-46)}$$

Love numbers of degree n, k_n, h_n are introduced into the theory by specializing the formulae (VI-42)–(VI-46) to the case of an observer or particle at the surface of the earth $r = a$.

Defining

$$k_n = K(r)\big|_{r=a} \quad \text{(VI-47)}$$

$$h_n = H(r)\big|_{r=a} \quad \text{(VI-48)}$$

$$g^0 = g^0(r)\big|_{r=a} \quad \text{(VI-49)}$$

and setting

$$V(\mathbf{r})\big|_{r=a} = V(a) \quad \text{(VI-50)}$$

$$V^t(\mathbf{r})\big|_{r=a} = V^t(a) \quad \text{(VI-51)}$$

$$V_n^p(\mathbf{r})\big|_{r=a} = V_n^p(a) \quad \text{(VI-52)}$$

$$u_r(\mathbf{r})\big|_{r=a} = u_r(a) \quad \text{(VI-53)}$$

where it is understood the argument a indicates that the quantity is still a function of θ, λ, the geocentric colatitude and east longitude respectively, yields the following results:

Eulerian Viewpoint

The potentials along a spherical surface of fixed radius $r = a$.

$$V(a) = V^0(a) + k_n V_n^p(a) \quad \text{(VI-54)}$$

$$V^t(a) = V^0(a) + (1 + k_n)\, V_n^p(a) \quad \text{(VI-55)}$$

Lagrangian Viewpoint

The potentials along the surface of the deformed earth $\mathbf{r} = a + \mathbf{u}$.

$$u_r(a) = \frac{h_n}{g^0} V_n^p(a) \qquad \text{(VI-56)}$$

$$V(a + \mathbf{u}) = V^0(a) + (k_n - h_n) V_n^p(a) \qquad \text{(VI-57)}$$

$$V'(a + \mathbf{u}) = V^0(a) + (1 + k_n - h_n) V_n^p(a) \, . \qquad \text{(VI-58)}$$

B. MacCullagh's Formula and Perturbations to the Earth's Inertia Tensor

The Love numbers allow a calculation of the surface deformations of the real earth in terms of the potential of the perturbing force field and hence are of great utility in geodynamics. Another formula of equal utility is MacCullagh's formula, which when combined with Love numbers allows a calculation of the perturbations r_{ij} to the inertia tensor of the real earth in terms of the potential of the perturbing force field.

We begin by considering the gravitational potential $V(\mathbf{r})$ at a fixed point $\mathbf{r}$ in the region exterior to an extended spherical body of internal density distribution $\rho(\mathbf{R})$. The geometry of the situation is illustrated in Figure VI-1.

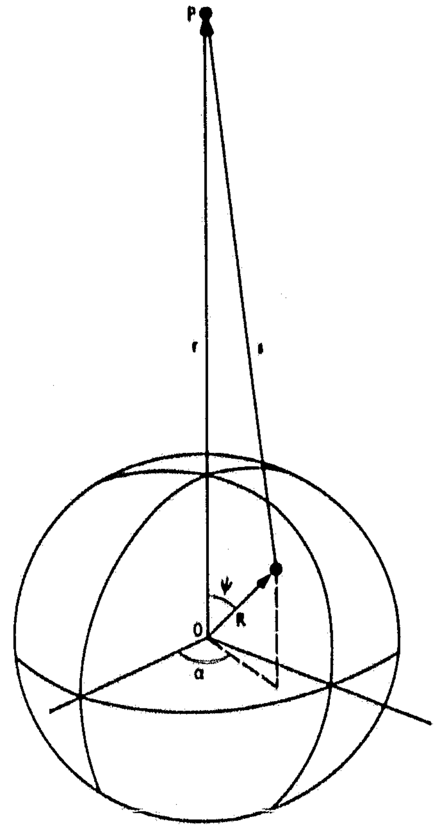

Figure VI-1. The definitions of the geometrical quantities used in the development of the theory of the earth's gravity field.

The contribution $dV(\mathbf{r})$ to the gravitational potential at $\mathbf{r}$ due to the mass element $\rho(\mathbf{R})dV$ at the position $\mathbf{R}$ within the body is

$$dV(\mathbf{r}) = -\frac{G\rho(\mathbf{R})}{|\mathbf{S}|} dV \qquad \text{(VI-59)}$$

where

$$\mathbf{r} = \mathbf{R} + \mathbf{S} \qquad \text{(VI-60)}$$

The total gravitational potential at $\mathbf{r}$ is $V(\mathbf{r})$ given by

$$V(\mathbf{r}) = -G \int_{\text{Vol}} \frac{\rho(\mathbf{R})}{|\mathbf{S}|} dV \qquad \text{(VI-61)}$$

where the integral is to be carried out over the volume of the body. This integral can be expanded in the usual way as

$$V(\mathbf{r}) = -\frac{G}{r} \int_{\text{Vol.}} \sum_{n=0}^{\infty} \rho(\mathbf{R}) \left(\frac{R}{r}\right)^n P_n (\cos\psi)\, dV, \ \ r \geqslant R \qquad \text{(VI-62)}$$

where:

(1) $r = |\mathbf{r}|, R = |\mathbf{R}|$.

(2) ψ is the angle between $\mathbf{r}$ and $\mathbf{R}$.

(3) $P_n(\cos\psi)$ is the Legendre polynominal of degree n.

It can be shown (Mueller 1969, pp. 3-6) that with the origin of coordinates chosen to be at the center of mass of the object

$$\begin{aligned} V(\mathbf{r}) = -\frac{GM}{r} - \frac{G}{2r^3} \Big[& (I_{11} + I_{22} - 2 I_{33}) P_2^0 (\cos\theta) \\ & + 2 I_{23} P_2^1 (\cos\theta) \sin\lambda + 2 I_{13} P_2^1 (\cos\theta) \cos\lambda \\ & + \frac{1}{2}(I_{22} - I_{11}) P_2^2 (\cos\theta) \cos 2\lambda \\ & - I_{12} P_2^2 (\cos\theta) \sin 2\lambda \Big] + 0\left(\frac{1}{r^4}\right) \end{aligned} \qquad \text{(VI-63)}$$

where:

(1) M is the total mass of the extended body.

(2) I_{ij} are the elements of the inertia tensor defined in the body fixed coordinate system for which θ is geocentric colatitude and λ is east longitude.

(3) $P_n^m(\cos\theta)$ is the associated Legendre polynomial of degree n and order m.

Equation (VI-63) is MacCullagh's formula and is valid in this form in the region exterior to the body at fixed (Eulerian) positions $\mathbf{r}$. The usefulness of MacCullagh's formula for the theory of earth deformations is illustrated in the following example.

Suppose the undeformed equilibrium figure of the earth is spherically symmetric with radius $r = a$, in which case the undeformed gravitational potential $V^0(r)$ is given by

$$V^0(r) = \frac{GM_\oplus}{r}, \; r \geq a \; . \qquad \text{(VI-64)}$$

Since, for such an earth, the undeformed inertia tensor $\mathcal{T}^0$ is given by

$$\mathcal{T}^0 = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{bmatrix} \qquad \text{(VI-65)}$$

the undeformed gravitational potential may equally well be written

$$V^0(r) = \frac{GM_\oplus}{r} - \frac{G}{2r^3}\Big[(I_{11}^0 + I_{22}^0 - 2 I_{33}^0) P_2^0(\cos\theta)$$

$$+ 2 I_{23}^0 P_2^1(\cos\theta)\sin\lambda + 2 I_{13}^0 P_2^1(\cos\theta)\cos\lambda$$

$$+ \frac{1}{2}(I_{22}^0 - I_{11}^0) P_2^2(\cos\theta)\cos 2\lambda$$

$$- I_{12}^0 P_2^2(\cos\theta)\sin 2\lambda\Big] \; , r \geq a, \qquad \text{(VI-66)}$$

since the terms inside the brackets vanish.

Now if the earth is subjected to some deforming force field which produces displacements $\mathbf{u}(\mathbf{R})$ internally within the earth the inertia tensor is perturbed from the undeformed value $\mathcal{T}^0$ to the deformed value $\mathcal{T}$ given by

$$\mathcal{T} = \begin{bmatrix} A + r_{11} & r_{12} & r_{13} \\ r_{21} & A + r_{22} & r_{23} \\ r_{31} & r_{32} & A + r_{33} \end{bmatrix} \qquad \text{(VI-67)}$$

and the gravitational potential is perturbed from its undeformed value $V^0(r)$ to its deformed value $V(\mathbf{r})$ where according to Equation (VI-20)

$$V(\mathbf{r}) = V^0(r) + V^1(\mathbf{r}) \qquad \text{(VI-68)}$$

The gravitational potential $V(\mathbf{r})$ of the deformed earth will also be given by MacCullagh's formula Equation (VI-63) and so from Equations (VI-63), (VI-66), (VI-67), and (VI-68) we deduce that

$$V^1(\mathbf{r}) = -\frac{G}{2r^3}\Big[(r_{11} + r_{22} - 2 r_{33}) P_2^0(\cos\theta)$$

$$+ 2 r_{23} P_2^1(\cos\theta)\sin\lambda + 2 r_{13} P_2^1(\cos\theta)\cos\lambda$$

$$+ \frac{1}{2}(r_{22} - r_{11}) P_2^2(\cos\theta)\cos 2\lambda$$

$$- r_{12} P_2^2(\cos\theta)\sin 2\lambda\Big] + 0\left(\frac{1}{r^4}\right) \; , r \geq a \; . \qquad \text{(VI-69)}$$

Combining Equation (VI-24) valid for $r \leq a$ with Equation (VI-69) valid for $r \geq a$ we can obtain, for $r = a$,

$$\sum_n \sum_m K_n(a)\, V_n^{pm}(a) = -\frac{G}{2a^3}\Big[(r_{11} + r_{22} - 2 r_{33})$$

$$P_2^0(\cos\theta) + 2 r_{23} P_2^1(\cos\theta)\sin\lambda + 2 r_{13} P_2^1(\cos\theta)\cos\lambda$$

$$+ \frac{1}{2}(r_{22} - r_{11}) P_2^2(\cos\theta)\cos 2\lambda$$

$$- r_{12} P_2^2(\cos\theta)\sin 2\lambda\Big] + 0\left(\frac{1}{a^4}\right) \qquad \text{(VI-70)}$$

where $V_n^{pm}(\mathbf{r})$ is the potential from which the deforming force field is obtained.

Of particular interest is the special case of a second degree perturbing potential $V_2^p(\mathbf{r})$ which has a general representation valid for $r \leq a$ as

$$V_2^p(\mathbf{r}) = \sum_{m=0}^{2} \left(\frac{r}{a}\right)^2 P_2^m (\cos\theta)\,(C_2^m \cos m\lambda + S_2^m \sin m\lambda) \tag{VI-71}$$

or

$$V_2^p(\mathbf{r}) = \left(\frac{r}{a}\right)^2 \Big[C_2^0 P_2^0 (\cos\theta) + C_2^1 P_2^1 (\cos\theta) \cos\lambda + S_2^1 P_2^1 (\cos\theta) \sin\lambda + C_2^2 P_2^2 (\cos\theta) \cos 2\lambda + S_2^2 P_2^2 (\cos\theta) \sin 2\lambda\Big]. \tag{VI-72}$$

Substituting Equation (VI-72) into Equation (VI-70) and using Equations (VI-47) gives

$$\begin{aligned} &k_2 \Big[C_2^0 P_2^0 (\cos\theta) + C_2^1 P_2^1 (\cos\theta) \cos\lambda + S_2^1 P_2^1 (\cos\theta) \sin\lambda + C_2^2 P_2^2 (\cos\theta) \cos 2\lambda + S_2^2 P_2^2 (\cos\theta) \sin 2\lambda\Big] \\ &= -\frac{G}{2a^3}\Big[(r_{11} + r_{22} - 2r_{33}) P_2^0 (\cos\theta) + 2\, r_{23} P_2^1 (\cos\theta) \sin\lambda + 2\, r_{13} P_2^1 (\cos\theta) \cos\lambda \\ &\quad + \frac{1}{2}(r_{22} - r_{11}) P_2^2 (\cos\theta) \cos 2\lambda - r_{12} P_2^2 (\cos\theta) \sin 2\lambda\Big]. \end{aligned} \tag{VI-73}$$

Since spherical harmonic functions are all linearly independent we can equate the coefficients of harmonics of the same degree n and order m in Equation (VI-73) to obtain the five equations

$$r_{12} = \frac{2k_2 a^3}{G} S_2^2 \tag{VI-74}$$

$$2r_{13} = -\frac{2k_2 a^3}{G} C_2^1 \tag{VI-75}$$

$$2r_{23} = -\frac{2k_2 a^3}{G} S_2^1 \tag{VI-76}$$

$$\frac{1}{2}(r_{22} - r_{11}) = -\frac{2k_2 a^3}{G} C_2^2 \tag{VI-77}$$

$$r_{11} + r_{22} - 2r_{33} = -\frac{2k_2 a^3}{G} C_2^0 \tag{VI-78}$$

These five equations are insufficient to solve for the six independent values of r_{ij}. A full solution for the perturbations to the inertia tensor r_{ij} requires an additional equation to supplement the set (VI-74)–(VI-78). The usual technique for obtaining a "solution" for the r_{ij} is to use as a supplementary equation

$$r_{11} + r_{22} + r_{33} = 0\,. \tag{VI-79}$$

This equation expresses the (assumed) property that the trace of the inertia tensor $T_r\tilde{I}$ is a dynamical invariant whose magnitude is not changed by the deformation of the earth.

Rochester and Smylie (1974) have criticized the use of Equation (VI-79) in geodynamics and point out that, while $T_r\tilde{I}$ is conserved for all earth deformation fields derivable from a potential which is expandable in solid spherical harmonics, there exists a whole class of earth deformation fields for which this is not true and for which $T_r\tilde{I}$ is not a dynamical invariant.

The correct supplementary equation to use to provide a solution to the r_{ij} is

$$r_{11} + r_{22} + r_{33} = \delta(T_r\tilde{I}) \tag{VI-80}$$

where $\delta(T_r\tilde{I})$ is the variation in the trace of the inertia tensor which occurs as a result of the deformation field.

The correct solution for the r_{ij} is then obtained from Equations (VI-74)–(VI-78) and Equation (VI-80) gives

$$r_{12} = \frac{2k_2 a^3}{G} S_2^2 \tag{VI-81}$$

$$r_{13} = -\frac{k_2 a^3}{G} C_2^1 \tag{VI-82}$$

$$r_{23} = -\frac{k_2 a^3}{G} S_2^1 \tag{VI-83}$$

$$r_{11} = \frac{1}{3}\delta(T_r\tilde{I}) + \frac{k_2 a^3}{G}\left(2\, C_2^2 - \frac{1}{3} C_2^0\right) \tag{VI-84}$$

ORIGINAL PAGE IS
OF POOR QUALITY

$$r_{22} = \frac{1}{3}\delta(T_r\tilde{T}) - \frac{k_2 a^3}{G}\left(2\,C_2^2 + \frac{1}{3}C_2^0\right) \qquad \text{(VI-85)}$$

$$r_{33} = \frac{1}{3}\delta(T_r\tilde{T}) + \frac{2}{3}\frac{k_2 a^3}{G}C_2^0\,. \qquad \text{(VI-86)}$$

In addition to r_{33} above, the dynamical equations (IV-22) and (IV-23) governing polar motion and UT1 depend on $\tilde{r} = r_{13} + ir_{23}$. From this analysis we see that

$$\tilde{r} = -\frac{k_2 a^3}{G}(C_2^1 + i\,S_2^1)\,. \qquad \text{(VI-87)}$$

These results, Equations (VI-81)–(VI-86), obtained for the case of a spherically symmetric earth whose undeformed inertia tensor $\tilde{T}^0$ is given by

$$\tilde{T}^0 = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{bmatrix} \qquad \text{(VI-88)}$$

are not altered at all when we consider the flattened real earth whose undeformed inertia tensor $\tilde{T}^0$ is given by

$$\tilde{T}^0 = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix} \qquad \text{(VI-89)}$$

In the case of the flattened earth the equilibrium gravitational potential $V^0(r)$ is given by

$$V^0(\mathbf{r}) = -\frac{GM_*}{r} - \frac{G}{2r^3}\Big[(A + A - 2C)\,P_2^0(\cos\theta)$$
$$+ 2\,I_{23}^0\,P_2^1(\cos\theta)\sin\lambda + 2\,I_{13}^0\,P_2^1(\cos\theta)\cos\lambda$$
$$+\frac{1}{2}(A - A)\,P_2^2(\cos\theta)\cos 2\lambda$$
$$- I_{12}^0\,P_2^2(\cos\theta)\sin 2\lambda\Big] \qquad \text{(VI-90)}$$

for the undeformed earth. The deformed inertia tensor for a flattened earth $\tilde{T}$ is given by

$$\tilde{T} = \begin{bmatrix} A + r_{11} & r_{12} & r_{13} \\ r_{21} & A + r_{22} & r_{23} \\ r_{31} & r_{32} & C + r_{33} \end{bmatrix} \qquad \text{(VI-91)}$$

and the gravitational potential for the deformed earth $V(r)$ is given by

$$V(\mathbf{r}) = V^0(\mathbf{r}) + V^1(\mathbf{r})\,. \qquad \text{(VI-92)}$$

It can readily be seen that substituting Equation (VI-91) into MacCullagh's formula, Equation (VI-63), and using Equations (VI-90) and (VI-92) gives

$$V^1(\mathbf{r}) = -\frac{G}{2r^3}\Big[(r_{11} + r_{22} - 2r_{33})\,P_2^0(\cos\theta)$$
$$+ 2r_{23}\,P_2^1(\cos\theta)\sin\lambda + 2r_{13}\,P_2^1(\cos\theta)\cos\lambda$$
$$+\frac{1}{2}(r_{22} - r_{11})\,P_2^2(\cos\theta)\cos 2\lambda$$
$$- r_{12}\,P_2^2(\cos\theta)\sin 2\lambda\Big] \qquad \text{(VI-93)}$$

which is identical to Equation (VI-69). It follows that degree 2 perturbing potentials will, even in the case of the flattened real earth, result in perturbations to the earth's inertia tensor r_{ij} given by Equations (VI-81)–(VI-86).

VII. The Rotational Dynamics of an Axially Symmetric Deformable Earth

A. The Effect of Rotational Deformations of the Earth on its Eulerian Motion

In the real earth the centrifugal forces of rotation produce deformations which greatly alter the character of its Eulerian motion. It is these deformations which are responsible for the famous lengthening of the Eulerian or Chandler wobble period from 304.6 days predicted on the basis of rigid earth dynamics to the observed period of 435 days. In addition the yielding of the earth to the centrifugal forces of rotation causes an enhancement of the wobble amplitude.

The centrifugal force field $\mathbf{f}^c(r)$ on the rotating earth is given by

$$\mathbf{f}^c(\mathbf{r}) = -\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad \text{(VII-1)}$$

where ω is "the earth's rotation vector" and the superscript c will be used to denote phenomena associated with centrifugal effects.

(Here is an instance where, from the standpoint of strict mathematical rigor, we make an error. It is to be recalled that

ω is *the rotation vector of some rotating geophysical coordinate system* and from the standpoint of formal logic may or may not (depending on its definition) relate to the properties of the rotating earth. In practice, of course, because the geophysical coordinate system is defined to be nearly rigidly attached to the nearly rigid earth the consequences of the "error" are entirely negligible.)

The centrifugal force field can be obtained from the negative gradient of a centrifugal scalar potential field $U^c(\mathbf{r})$.

$$\mathbf{f}^c(\mathbf{r}) = -\nabla U^c(\mathbf{r}) \tag{VII-2}$$

where

$$U^c(\mathbf{r}) = -\frac{1}{2}\left[\omega^2 r^2 - (\boldsymbol{\omega}\cdot\mathbf{r})^2\right] \tag{VII-3}$$

and where

$$\omega^2 = \boldsymbol{\omega}\cdot\boldsymbol{\omega}\,. \tag{VII-4}$$

In terms of the body-fixed basis vectors $\hat{e}_1\ \hat{e}_2\ \hat{e}_3$ we have

$$\begin{aligned}\mathbf{r} &= x\,\hat{e}_1 + y\,\hat{e}_2 + z\,\hat{e}_3\\ \boldsymbol{\omega} &= \omega_1\,\hat{e}_1 + \omega_2\,\hat{e}_2 + \omega_3\,\hat{e}_3\end{aligned} \tag{VII-5}$$

and so

$$\begin{aligned}U^c(\mathbf{r}) = -\frac{1}{2}\,[&(\omega_1^2+\omega_2^2+\omega_3^2)(x^2+y^2+z^2) - \omega_1^2 x^2\\ &-\omega_2^2 y^2 - \omega_3^2 z^2 + 2\,\omega_1\omega_2 xy + 2\,\omega_1\omega_3 xz\\ &+2\,\omega_2\omega_3 yz]\end{aligned} \tag{VII-6}$$

Replacing x, y, z in Equation (VII-6) by their equivalents in spherical polar coordinates

$$x = r\sin\theta\cos\lambda$$

$$y = r\sin\theta\sin\lambda$$

$$z = r\cos\theta$$

the centrifugal potential becomes

$$\begin{aligned}U^c(\mathbf{r}) = -\frac{1}{2}\,r^2\,(&\omega^2 - \omega_1^2\sin^2\theta\cos^2\lambda - \omega_2^2\sin^2\theta\sin^2\lambda\\ &-\omega_3^2\cos^2\theta - 2\,\omega_1\omega_2\sin^2\theta\sin\lambda\cos\lambda\\ &-2\,\omega_1\omega_2\sin\theta\cos\theta\cos\lambda\\ &-2\,\omega_2\omega_3\sin\theta\cos\theta\sin\lambda)\,,\quad r\leqslant a\,.\end{aligned} \tag{VII-7}$$

Equation (VII-7) can be rewritten in terms of the associated Legendre polynomials $P_n^m(\cos\theta)$ of second degree ($n=2$) where

$$\begin{aligned}P_2^0(\cos\theta) &= \frac{1}{2}(3\cos^2\theta - 1)\\ P_2^1(\cos\theta) &= -3\sin\theta\cos\theta\\ P_2^2(\cos\theta) &= 3\sin^2\theta\end{aligned} \tag{VII-8}$$

to become, for $r \leqslant a$,

$$\begin{aligned}U^c(\mathbf{r}) = -\frac{1}{2}\,r^2\Big[&\frac{2}{3}\omega^2 + \Big(\frac{1}{3}\,\omega_1^2 + \frac{1}{3}\,\omega_2^2\\ &-\frac{2}{3}\,\omega_3^2\Big)P_2^0(\cos\theta) + \frac{2}{3}\,\omega_2\omega_3 P_2^1(\cos\theta)\sin\lambda\\ &+\frac{2}{3}\,\omega_1\omega_3 P_2^1(\cos\theta)\cos\lambda\\ &+\frac{1}{6}(\omega_2^2 - \omega_1^2)P_2^2(\cos\theta)\cos 2\lambda\\ &-\frac{1}{3}\,\omega_1\omega_2 P_2^2(\cos\theta)\sin 2\lambda\Big]\end{aligned} \tag{VII-9}$$

This can be written as

$$\begin{aligned}U^c(\mathbf{r}) = -\frac{1}{3}\,r^2\omega^2 + \sum_{m=0}^{2}\left(\frac{r}{a}\right)^2 P_2^m(\cos\theta)\,[&C_2^m\cos m\lambda\\ &+S_2^m\sin m\lambda]\,,\quad r\leqslant a\,,\end{aligned} \tag{VII-10}$$

where the coefficients of the harmonics are given by

$$C_2^0 = -\frac{a^2}{6}(\omega_1^2+\omega_2^2-2\,\omega_3^2) \qquad S_2^0 = 0$$

$$C_2^1 = -\frac{2a^2}{6}\,\omega_1\omega_3 \qquad S_2^1 = -\frac{2a^2}{6}\,\omega_2\omega_3$$

$$C_2^2 = -\frac{a^2}{12}(\omega_2^2-\omega_1^2) \qquad S_2^2 = \frac{a^2}{6}\,\omega_1\omega_2 \;. \qquad \text{(VII-11)}$$

Now the earth's rotation vector ω is given by

$$\omega = \Omega[m_1\,\hat{e}_1 + m_2\,\hat{e}_2 + (1+m_3)\hat{e}_3]$$

and so the coefficients in the centrifugal potential become

$$\omega^2 = \Omega^2 + \Omega^2\,(2m_3 + m_1^2 + m_2^2 + m_3^2)$$

$$C_2^0 = \frac{2a^2}{6}\,\Omega^2 + \frac{2a^2}{6}\,\Omega^2\left(2m_3 - \frac{1}{2}m_1^2 - \frac{1}{2}m_2^2 + m_3^2\right)$$

$$C_2^1 = -\frac{2a^2}{6}\,\Omega^2\,(m_1 + m_1m_3)$$

$$S_2^1 = -\frac{2a^2}{6}\,\Omega^2\,(m_2 + m_2m_3) \qquad \text{(VII-12)}$$

$$C_2^2 = -\frac{a^2}{12}\,\Omega^2\,(m_2^2 - m_1^2)$$

$$S_2^2 = \frac{a^2}{6}\,\Omega^2\,m_1m_2$$

which to first order in $m_1\;m_2\;m_3$ reduce to

$$\omega^2 = \Omega^2 + 2\,\Omega^2 m_3$$

$$C_2^0 = \frac{2}{6}a^2\,\Omega^2 + \frac{4}{6}a^2\,\Omega^2 m_3$$

$$C_2^1 = -\frac{2}{6}a^2\,\Omega^2 m_1$$

$$S_2^1 = -\frac{2}{6}a^2\,\Omega^2 m_2 \qquad \text{(VII-13)}$$

$$C_2^2 = 0$$

$$S_2^2 = 0$$

From Equations (VII-13) we see that the centrifugal potential $U^c(\mathbf{r})$ can be decomposed into a secular part $U_s^c(\mathbf{r})$ due to the steady mean rotation of the earth and independent of m_1, m_2, m_3 and a time-varying perturbation $U_p^c(\mathbf{r})$ due to perturbations in the earth rotation and depending on m_1, m_2, m_3

$$U^c(\mathbf{r}) = U_s^c(\mathbf{r}) + U_p^c(\mathbf{r})\,. \qquad \text{(VII-14)}$$

Selecting those portions of the coefficients which are independent of m_1, m_2, m_3 in Equations (VII-13) and substituting them into Equation (VII-10) to obtain the secular centrifugal potential we find

$$U_s^c(\mathbf{r}) = -\left[\frac{1}{3}r^2\Omega^2 - \frac{2}{6}a^2\Omega^2\left(\frac{r}{a}\right)^2 P_2^0\;(\cos\theta)\right]$$

which can be written as

$$U_s^c(\mathbf{r}) = -\frac{1}{3}r^2\Omega^2\;\left[1 - P_2^0\,(\cos\theta)\right]\,. \qquad \text{(VII-15)}$$

The rotation perturbation potential $U_p^c(\mathbf{r})$ is obtained by selecting those portions of the coefficients which depend on the quantities m_1, m_2, m_3 in Equations (VII-13) and substituting them into Equation (VII-10) to give

$$U_p^c(\mathbf{r}) = -\frac{2}{3}r^2\,\Omega^2 m_3 + \frac{2}{3}a^2\,\Omega^2 m_3\left(\frac{r}{a}\right)^2 P_2^0\,(\cos\theta)$$

$$-\frac{1}{3}a^2\,\Omega^2 m_1\left(\frac{r}{a}\right)^2 P_2^1\,(\cos\theta)\cos\lambda$$

$$-\frac{1}{3}a^2\,\Omega^2 m_2\left(\frac{r}{a}\right)^2 P_2^1\,(\cos\theta)\sin\lambda\,. \qquad \text{(VII-16)}$$

The action of the secular centrifugal potential over geologic time has given rise to the earth's equatorial bulge and the observed polar flattening f where $f^{-1} = 298.256$. The observed value of the flattening together with the value of the earth's mean radius and mean rotation rate when combined with the potential $U_s^c(\mathbf{r})$ allows the computation of a "secular" Love number of degree 2 k_{2s}. It can be shown that k_{2s} is very nearly equal to the fluid Love number k_{2f} which would describe the yielding of the earth to the secular centrifugal potential were the earth a perfect fluid. This is regarded as a demonstration of the fact that for deforming force fields which act over long intervals the global rheology of the earth closely resembles a perfect fluid. We shall not concern ourselves further with the secular centrifugal potential $U_s^c(\mathbf{r})$ but will consider the effects on the earth's Eulerian motion of the rotation perturbation potential $U_p^c(\mathbf{r})$.

We see from the form of Equation (VII-16) that

$$U_p^c(\mathbf{r}) = -\frac{2}{3} r^2 \Omega^2 m_3 + \sum_{m=0}^{2} \left(\frac{r}{a}\right)^2 P_2^m (\cos \theta)$$

$$(C_2^m \cos m\lambda + S_2^m \sin m\lambda) \qquad \text{(VII-17)}$$

where the harmonic coefficients C_2^m S_2^m are given by

$$C_2^0 = \frac{2}{3} a^2 \Omega^2 m_3$$

$$C_2^1 = -\frac{1}{3} a^2 \Omega^2 m_1$$

$$S_2^1 = -\frac{1}{3} a^2 \Omega^2 m_2 \qquad \text{(VII-18)}$$

$$C_2^2 = 0$$

$$S_2^2 = 0 .$$

From Equations (VI-81)–(VI-86) and Equation (VII-18) we see that the perturbations to the earth's rotation described by the dimensionless parameters m_1, m_2, m_3 cause perturbations to the earth's inertia tensor r_{ij}^c given, to first order in m_1, m_2, m_3 by

$$r_{12}^c = 0 \qquad \text{(VII-19)}$$

$$r_{13}^c = \frac{k_2 a^5 \Omega^2}{3G} m_1 \qquad \text{(VII-20)}$$

$$r_{23}^c = \frac{k_2 a^5 \Omega^2}{3G} m_2 \qquad \text{(VII-21)}$$

$$r_{11}^c = \frac{1}{3} \delta(T_r T) - \frac{2k_2 a^5 \Omega^2}{9G} m_3 \qquad \text{(VII-22)}$$

$$r_{22}^c = \frac{1}{3} \delta(T_r T) - \frac{2k_2 a^5 \Omega^2}{9G} m_3 \qquad \text{(VII-23)}$$

$$r_{33}^c = \frac{1}{3} \delta(T_r T) + \frac{4k_2 a^5 \Omega^2}{9G} m_3 . \qquad \text{(VII-24)}$$

The time derivatives of the perturbations to the inertia tensor reckoned in the rotating frame of the earth are

$$\frac{dr_{12}^c}{dt} = 0 \qquad \text{(VII-25)}$$

$$\frac{dr_{13}^c}{dt} = \frac{k_2 a^5 \Omega^2}{3G} \frac{dm_1}{dt} \qquad \text{(VII-26)}$$

$$\frac{dr_{23}^c}{dt} = \frac{k_2 a^5 \Omega^2}{3G} \frac{dm_2}{dt} \qquad \text{(VII-27)}$$

$$\frac{dr_{11}^c}{dt} = \frac{1}{3} \frac{d}{dt} \delta(T_r T) - \frac{2k_2 a^5 \Omega^2}{9G} \frac{dm_3}{dt} \qquad \text{(VII-28)}$$

$$\frac{dr_{22}^c}{dt} = \frac{1}{3} \frac{d}{dt} \delta(T_r T) - \frac{2k_2 a^5 \Omega^2}{9G} \frac{dm_3}{dt} \qquad \text{(VII-29)}$$

$$\frac{dr_{33}^c}{dt} = \frac{1}{3} \frac{d}{dt} \delta(T_r T) + \frac{4k_2 a^5 \Omega^2}{9G} \frac{dm_3}{dt} . \qquad \text{(VII-30)}$$

Owing to the presence of the term $(2/3)r^2 \Omega^2 m_3$ in the rotation perturbation potential $U_p^c(\mathbf{r})$ (Equation (VII-17)) we see that $U_p^c(\mathbf{r})$ contains a term which cannot be incorporated into a spherical harmonic expansion. The presence of this term in the perturbing potential is sufficient to insure that the deformation field associated with the yielding of the earth to perturbations m_1 m_2 m_3 in its mean rotation will not preserve the value of $T_r T$ (Rochester and Smylie 1974). There does not exist in the literature at this time to my knowledge a solution for the quantities $\delta(T_r T)$ and $(d/dt)\, \delta(T_r T)$ required in order to solve explicitly for the quantities r_{ii}^c and $(dr^c/dt)_{ii}$. As a result the effect of the yielding of the earth to centrifugal forces and its effect on its Eulerian motion is not a completely solved problem in geodynamics today. "Solutions" to the problem which have ignored the terms in $\delta(T_r T)$ such as that presented in Munk and MacDonald (1960, pp. 25) are in error.

Fortunately the effect of $\delta(T_r T)$ is confined to the inertia tensor perturbations r_{ii} $i = 1, 2, 3$, and so only enters into the value of UT1 $[m_3(t)]$ through the term Ωr_{33} in Equation (IV-30). So that while the effect of rotational deformations on UT1 is at present unknown, the effect of rotational deformations on polar motion, Equation (IV-22), can be calculated because these effects are seen to depend only on the quantities $\tilde{r}$ and $d\tilde{r}/dt$ where $\tilde{r} = r_{13} + ir_{23}$.

We can see from Equations (VII-19)–(VII-24) and Equations (VII-25)–(VII-30) that in the event of polar motion on the earth the yielding of the earth to the perturbed centrifugal force field can be accommodated into the dynamical equation governing polar motion, Equation (IV-22), by including the effect in the terms $d\bar{r}/dt$ and $\bar{r}$ appearing in the excitation function on the RHS. This is accomplished in the following way.

The Eulerian (force free) polar motion of a rigid earth is governed by the equations

$$\frac{d\bar{m}}{dt} - i\frac{C-A}{A}\Omega\bar{m} = 0\,, \tag{VII-31}$$

The yielding of the earth in response to its Eulerian motion modifies the rigid body motion by generating a *centrifugal deformation excitation function* $\bar{\epsilon}^c$ given by

$$\bar{\epsilon}^c = -\frac{1}{A\Omega}\left(\Omega\frac{d\bar{r}^c}{dt} + i\,\Omega^2\bar{r}^c\right) \tag{VII-32}$$

where

$$\bar{r}^c = r^c_{13} + ir^c_{23} \tag{VII-33}$$

$$\frac{d\bar{r}^c}{dt} = \frac{dr^c_{13}}{dt} + i\frac{dr^c_{23}}{dt} \tag{VII-34}$$

are given by Equations (VII-20), (VII-21), (VII-26), (VII-27), above. The equation governing Eulerian motion in a deformable earth is then

$$\frac{d\bar{m}}{dt} - i\frac{C-A}{A}\Omega\bar{m} = \bar{\epsilon}^c \tag{VII-35}$$

or

$$\frac{d\bar{m}}{dt} - i\frac{C-A}{A}\Omega\bar{m} = -\frac{1}{A\,\Omega}\left(\Omega\frac{d\bar{r}^c}{dt} + i\,\Omega^2\,\bar{r}^c\right)\,. \tag{VII-36}$$

We see that

$$\bar{r}^c = \frac{k_2 a^5\Omega^2}{3G}\bar{m} \tag{VII-37}$$

$$\frac{d\bar{r}^c}{dt} = \frac{k_2 a^5\Omega^2}{3G}\frac{d\bar{m}}{dt}. \tag{VII-38}$$

Substituting Equations (VII-37) (VII-38) into Equation (VII-36) gives

$$\left[1 + \frac{F}{A}\right]\frac{d\bar{m}}{dt} - i\left(\frac{C-A-F}{A}\right)\Omega\bar{m} = 0 \tag{VII-39}$$

where

$$F = \frac{k_2 a^5\Omega^2}{3G}\,. \tag{VII-40}$$

From Stacey (1977) we have

$$\begin{aligned} k_2 &= 0.29 \\ a &= 6.3708 \times 10^8 \text{ cm} \\ \Omega &= 7.2921 \times 10^{-5} \text{ rad sec}^{-1} \\ G &= 6.6732 \times 10^{-8} \text{ cm}^3 \text{ gm}^{-1} \text{ sec}^{-2} \\ C &= 8.0378 \times 10^{44} \text{ gm cm}^2 \\ A &= 8.0115 \times 10^{44} \text{ gm cm}^2 \end{aligned} \tag{VII-41}$$

from which we can deduce

$$F = 8.082 \times 10^{41} \text{ gm cm}^2 \tag{VII-42}$$

and

$$\frac{F}{A} = 1.008 \times 10^{-3}\,. \tag{VII-43}$$

Equation (VII-39) can be written

$$\left(1 + \frac{F}{A}\right)\frac{d\bar{m}}{dt} - i\frac{C-A}{A}\left(1 - \frac{F}{C-A}\right)\Omega\bar{m} = 0 \tag{VII-44}$$

and since

$$\frac{F}{C-A} = 0.3073 \tag{VII-45}$$

we see that with an accuracy of 1 part in 10^3 we can write Equation (VII-44) as

$$\frac{d\bar{m}}{dt} - i\frac{C-A}{A}\left(1 - \frac{F}{C-A}\right)\Omega\bar{m} = 0\,, \tag{VII-46}$$

The approximation which leads to Equation (VII-46) depends on F/A being much smaller than $F/C\text{-}A$ or on the condition that

$$\frac{F/A}{F/C\text{-}A} = \frac{C-A}{A} << 1 \qquad \text{(VII-47)}$$

which is of course true for the earth.

With the approximation of Equation (VII-46) we see that the dynamical equations governing Eulerian polar motion on the deformable earth can be written to an accuracy of 1 part in 10^3 as

$$\frac{d\overline{m}}{dt} - i\frac{C-A}{A}\Omega\overline{m} = -i\frac{F\Omega}{A}\overline{m} \; . \qquad \text{(VII-48)}$$

Thus we may introduce the *approximate centrifugal deformation excitation function* $\overline{\psi}^c$, to be used instead of the *exact centrifugal deformation excitation function* $\overline{\varepsilon}^c$ (Equation VII-32) because of the simplifications it brings to the theory, and write

$$\frac{d\overline{m}}{dt} - i\frac{C-A}{A}\Omega\overline{m} = \overline{\psi}^c \qquad \text{(VII-49)}$$

for the approximate equation governing Eulerian polar motion on a deformable earth where

$$\overline{\psi}^c = -i\frac{F\Omega}{A}\overline{m} \; . \qquad \text{(VII-50)}$$

Two *equivalent forms* for the equation governing Eulerian polar motion on a deformable earth are

$$\frac{d\overline{m}}{dt} - i\,\sigma_0\,\overline{m} = 0 \qquad \text{(VII-51)}$$

and

$$\frac{d\overline{m}}{dt} - i\,\sigma_r\,\overline{m} = \overline{\psi}^c \qquad \text{(VII-52)}$$

where Equation (VII-51) was obtained from Equation (VII-46) by setting

$$\sigma_0 = \frac{C-A}{A}\left(1 - \frac{F}{C-A}\right)\Omega \qquad \text{(VII-53)}$$

as the angular rate of Eulerian polar motion on a deformable earth and where Equation (VII-52) was obtained by using

$$\sigma_r = \frac{C-A}{A}\Omega \qquad \text{(VII-54)}$$

as the angular rate of Eulerian polar motion on a rigid earth.

We see from Equations (VII-51) and (VII-53) that the principle effect of the deformability of the earth is to reduce the angular rate of Eulerian polar motion by an amount

$$\sigma_r - \sigma_0 = \frac{F}{C-A}\Omega \, . \qquad \text{(VII-55)}$$

This effectively lengthens the period of the Eulerian motion or the "Chandler wobble" to

$$\frac{2\pi}{\sigma_0} = \frac{2\pi A}{\Omega\,(C-A-F)} \, . \qquad \text{(VII-56)}$$

This is a period of about 439 days which is an increase by a factor of 1.44 over the period predicted for an equivalent rigid earth.

In order to understand more fully the effects of centrifugal earth deformation on its Eulerian motion it is instructive to introduce the *dimensionless centrifugal deformation excitation function* in its approximate form, $\overline{\psi}^{c\prime}$, (as opposed to its exact form $\overline{\varepsilon}^{c\prime}$) defined, in accordance with Equation (V-59), by

$$\overline{\psi}^{c\prime} = \frac{\overline{\psi}^c}{i\sigma_r} \qquad \text{(VII-57)}$$

along with the corresponding complex coordinate of the *centrifugal deformation excitation pole* $\overline{\phi}^c$ defined, in accordance with Equations (V-66), by

$$\overline{\phi}^c = -\overline{\psi}^{c\prime} \, . \qquad \text{(VII-58)}$$

The definition of $\overline{\psi}^{c\prime}$ by Equation (VII-57) allows the equation governing Eulerian motion in a deformable earth, Equation (VII-52), to be written in the convenient form

$$\frac{d\overline{m}}{dt} - i\,\sigma_r\,(\overline{m} + \overline{\psi}^{c\prime}) = 0 \, . \qquad \text{(VII-59)}$$

It can be shown that the instantaneous axis of figure of a body with instantaneous moments and small products of inertia given by A, A, C, r_{12}, r_{13}, r_{23} respectively is displaced from the x_3 coordinate axis by angles μ_1 μ_2 parallel to the x_1 x_2 coordinate axes respectively where

$$\mu_1 = \frac{r_{13}}{C-A} \qquad \mu_2 = \frac{r_{23}}{C-A} \tag{VII-60}$$

and that the complex coordinate $\overline{\mu} = \mu_1 + i\mu_2$ is then given in terms of the complex quantity $\overline{r} = r_{13} + ir_{23}$ by

$$\overline{\mu} = \frac{\overline{r}}{C-A} \tag{VII-61}$$

It follows from Equations (VII-50) (VII-54) (VII-57) that

$$\overline{\psi}^{c'} = -\frac{\overline{r}}{C-A} \tag{VII-62}$$

and so we see that the centrifugal deformation excitation pole $\overline{\phi}^c$ given by

$$\overline{\phi}^c = -\overline{\psi}^{c'} = \frac{\overline{r}}{C-A} \tag{VII-63}$$

coincides with the instantaneous figure axis of the rotationally deformed earth.

From Equations (VII-37) (VII-40) we see that

$$\overline{r} = F\overline{m} \tag{VII-64}$$

and so

$$\overline{\phi}^c = \frac{F}{C-A}\,\overline{m} \tag{VII-65}$$

and using Equation (VII-45) we have

$$\overline{\phi}^c = 0.3073\,\overline{m} \tag{VII-66}$$

The result of Equation (VII-66) indicates that the deformability of the earth allows the instantaneous figure axis $\overline{\phi}^c$ to partially adjust itself to the location of the instantaneous Eulerian rotation axis $\overline{m}$, the amount of the adjustment being about 30% of the total displacement of the rotation axis from the mean figure axis.

B. The Effect of the Rotational Deformations of the Earth on Its Non-Eulerian Motion

It is clear from the previous analysis that the elastic yielding of the earth to the changing centrifugal force field which accompanies the changing earth rotation vector ω profoundly alters the Eulerian (force free) motion of the earth. The period of the Eulerian (Chandler) motion is lengthened by approximately 44% and the amplitude of the Eulerian motion is increased by roughly 30%.

The deformability of the earth will also alter the character of its non-Eulerian motion or the forced motions which result from a combination of external torques and internal geophysical excitation. The effects of earth deformations on its non-Eulerian motion can be conveniently broken down into two separate aspects.

The first aspect considers the effect of earth deformations on the geophysically induced polar motion or equivalently the effect of deformations on the changes of ω relative to the basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$.

The second aspect considers the effect of earth deformations on precession and nutation or equivalently the effect of deformations on the motion of ω relative to the basis vectors $\hat{E}_1, \hat{E}_2, \hat{E}_3$.

A complete treatment of the effects of earth deformations on its rotation would include a third aspect; namely, the effects of deformations on the geophysically induced variations in UT1. As pointed out previously in this section (Equations VII-19 ff) this requires a detailed solution, for the case of the real earth, of the variations in $T_r \overline{T}$ which accompany variations in m_3. This shortcoming in present geodynamical theory has been pointed out by Rochester and Smylie (1974) and could be remedied by an extension of the work of Manshina and Shen (1974) or of Saito (1974) but to my knowledge has yet to be done.

1. Non-Eulerian polar motion on a deformable earth. The results of the analysis of Eulerian polar motion on a deformable earth, summarily presented in Equations (VII-49) — (VII-52), indicate that on a deformable earth Eulerian polar displacement $\overline{m}$ gives rise to an additional polar motion excitation $\overline{\psi}^c = -i\,(F\Omega/A)\,\overline{m}$ arising solely from the deformation of the earth in response to the original polar displacement. Thus as far as polar motion is concerned the deformability of the earth acts as positive feedback and enhances the motion. This viewpoint leads naturally to a prescription for the treatment of non-Eulerian polar motion on a deformable earth. This prescription for the general treatment of non-Eulerian (forced) polar motion on a deformable earth is described below.

First, consider the complete ensemble of identifiable geophysical phenomena capable of exciting polar motion on the

ORIGINAL PAGE IS
OF POOR QUALITY

earth and assign an index $k = 1, 2, 3 \ldots m$ to each. Equations (IV-22) and (IV-23) show that according to first-order theory each member of the ensemble of geophysical phenomena only contributes to polar motion and UT1 variations through the effect it has on

(1) The external torques N_i $i = 1, 2, 3$.

(2) The perturbations to the inertia tensor r_{ij}.

(3) The perturbations to the relative angular momentum h_i $i = 1, 2, 3$.

Furthermore, as seen from Equation (IV-22), for consideration of polar motion above it is necessary to be concerned *only* with the elements $N_1, N_2, h_1, h_2, r_{13}, r_{23}$.

The second step of the procedure is to calculate for each member of the ensemble of geophysical processes its individual perturbing contributions $N_1^k, N_2^k, h_1^k, h_2^k, r_{13}^k, r_{23}^k$, $k = 1, 2, 3, \ldots, m$ to the external torque, relative angular momentum, and inertia tensor respectively. Following this it is necessary to form the complex polar motion excitation function $\bar{\epsilon}^k$, $k = 1, 2, 3, \ldots m$ for each member of the ensemble of geophysical processes according to the formula,

$$\bar{\epsilon}^k = \frac{1}{A\Omega}\left[\bar{N}^k - \Omega\,\frac{d\bar{r}^k}{dt} - \frac{d\bar{h}^k}{dt} - i\,(\Omega^2\bar{r}^k + \Omega\bar{h}^k)\right]$$

$$k = 1, 2, 3, \ldots, \qquad \text{(VII-67)}$$

where

$$\begin{aligned} N^k &= N_1^k + iN_2^k & k &= 1, 2, 3, \ldots m \\ h^k &= h_1^k + ih_2^k & k &= 1, 2, 3, \ldots m \\ r^k &= r_{13}^k + ir_{23}^k & k &= 1, 2, 3, \ldots m \end{aligned} \qquad \text{(VII-68)}$$

On a rigid earth each member of the ensemble of geophysical processes would give rise to a component of polar motion $\bar{m}_r^k(t)$ $k = 1, 2, 3, \ldots m$ described by the solution to the equation

$$\frac{d\bar{m}_r^k}{dt} - i\,\frac{C - A}{A}\,\Omega\,\bar{m}_r^k = \bar{\epsilon}^k, \qquad k = 1, 2, 3, \ldots m \qquad \text{(VII-69)}$$

and given by Equation (V-63) as

$$\bar{m}_r^k(t) = e^{i\sigma_r t}\left[\bar{m}_r^k(0) + i\sigma_r \int_0^t \bar{\epsilon}^{k'}(t)\,e^{-i\sigma_r t'}\,dt'\right] \qquad \text{(VII-70)}$$

where

$$\sigma_r = \frac{C - A}{A}\,\Omega \qquad \text{(VII-71)}$$

and

$$\bar{\epsilon}^{k'} = \frac{\bar{\epsilon}^k}{i\sigma_r}, \qquad \text{(VII-72)}$$

Letting $\bar{m}_r(t)$ denote the total polar motion occurring on a rigid earth as a result of the ensemble of geophysical processes, then

$$\bar{m}_r(t) = \sum_{k=1}^{m} \bar{m}_r^k(t). \qquad \text{(VII-73)}$$

It follows from Equation (VII-69) that $\bar{m}_r(t)$ is given by the solution to

$$\frac{d\bar{m}_r}{dt} - i\,\frac{C - A}{A}\,\Omega\bar{m}_r = \sum_{k=1}^{m} \bar{\epsilon}^k \qquad \text{(VII-74)}$$

and is given by

$$\bar{m}_r(t) = e^{i\sigma_r t}\left[\bar{m}_r(0) + i\sigma_r \int_0^t \sum_{k=1}^{m} \bar{\epsilon}^{k'}(t')\,e^{-i\sigma_r t'}dt'\right]. \qquad \text{(VII-75)}$$

On a deformable earth each member of the ensemble of geophysical processes would give rise to a component of polar motion $\bar{m}^k(t)$ $k = 1, 2, 3, \ldots m$. However, on a deformable earth $\bar{m}^k(t)$ must be calculated by recognizing that each member of the ensemble of complex excitation functions $\bar{\epsilon}^k$, $k = 1, 2, 3, \ldots m$ will be affected by the effects of "positive feedback" resulting from the yielding of the earth. The earth deformation will produce, for each member $\bar{\epsilon}^k$, $k = 1, 2, 3 \ldots m$ of the ensemble of complex excitation functions, an additional excitation $\bar{\psi}^{ke}$, $k = 1, 2, 3 \ldots m$, arising solely from the effects of the deformation itself where, according to Equation (VII-50)

$$\bar{\psi}^{ke} = -i\,\frac{F\Omega}{A}\,\bar{m}^k \qquad \text{(VII-76)}$$

On a deformable earth then, each member of the ensemble of geophysical processes will have an effective complex excitation function given by $\overline{\epsilon}^k + \overline{\psi}^{ke}$, $k = 1, 2, 3, \ldots, m$, and will give rise to a component of polar motion $\overline{m}^k(t)$ $k = 1, 2, 3, \ldots m$ described by the solution to the equation

$$\frac{d\overline{m}^k}{dt} - i\,\frac{C-A}{A}\,\Omega\overline{m}^k = \overline{\epsilon}^k + \overline{\psi}^{ke}, \quad k = 1, 2, 3 \cdots m, \tag{VII-77}$$

and given by Equation (V-63) as

$$\overline{m}^k(t) = e^{i\sigma_r t}\left[\overline{m}^k(0) + i\sigma_r \int_0^t (\overline{\epsilon}^{k'} + \overline{\psi}^{k'e})\, e^{-i\sigma_r t'}\, dt'\right] \tag{VII-78}$$

where

$$\overline{\psi}^{k'e} = \frac{\overline{\psi}^{ke}}{i\sigma_r} \qquad k = 1, 2, 3, \cdots m. \tag{VII-79}$$

An alternative and completely equivalent formulation of polar motion on a deformable earth can be obtained by substituting Equation (VII-76) into Equation (VII-77) to obtain

$$\frac{d\overline{m}^k}{dt} - i\left(\frac{C-A-F}{A}\right)\Omega\overline{m}^k = \overline{\epsilon}^k \qquad k = 1, 2, 3, \cdots m \tag{VII-80}$$

which is shown by Equation (V-63) to have the solution

$$\overline{m}^k(t) = e^{i\sigma_0 t}\left[\overline{m}^k(0) + i\sigma_0 \int_0^t \overline{\epsilon}^{k'}\, e^{-i\sigma_0 t'}\, dt'\right]$$
$$k = 1, 2, 3, \cdots m, \tag{VII-81}$$

where we have made use of Equation (VII-53)

$$\sigma_0 = \frac{C-A-F}{A}\,\Omega. \tag{VII-82}$$

Letting $\overline{m}(t)$ denote the total polar motion occurring on a deformable earth as a result of the ensemble of geophysical processes then

$$\overline{m}(t) = \sum_{k=1}^{m} \overline{m}^k(t). \tag{VII-83}$$

It follows from Equations (VII-77) and (VII-80) that $\overline{m}(t)$ is the solution to either of the two equivalent differential equations

$$\frac{d\overline{m}}{dt} - i\,\frac{C-A}{A}\,\Omega\overline{m} = \sum_{k=1}^{m} \overline{\epsilon}^k + \overline{\psi}^{ke} \tag{VII-84}$$

or

$$\frac{d\overline{m}}{dt} - i\left(\frac{C-A-F}{A}\right)\Omega\overline{m} = \sum_{k=1}^{m} \overline{\epsilon}^k. \tag{VII-85}$$

The general solutions to Equations (VII-84) and (VII-85) are given by

$$\overline{m}(t) = e^{i\sigma_r t}\left[\overline{m}(0) + i\sigma_r \int_0^t \sum_{k=1}^{m} (\overline{\epsilon}^{k'} + \overline{\psi}^{k'e})\, e^{-i\sigma_r t'}\, dt'\right] \tag{VII-86}$$

and

$$\overline{m}(t) = e^{i\sigma_0 t}\left[\overline{m}(0) + i\sigma_0 \int_0^t \sum_{k=1}^{m} \overline{\epsilon}^{k'}\, e^{-i\sigma_0 t'}\, dt'\right] \tag{VII-87}$$

respectively.

2. Non-Eulerian precession and nutation on a deformable earth. In principle the elastic yielding of the earth to the system of body forces which give rise to the torques responsible for precession and nutation alters the observed precession and nutation from that which would prevail were the earth a rigid body. It has been shown (Lamb 1945, pp. 724 ff) that in the case of a disturbing force distribution which is fixed in inertial space the precession of a rotating mass of ideal fluid proceeds *exactly* the same as if the mass were solid throughout. Furthermore it has been shown (Lamb 1945, pp. 724 ff) that when the disturbing force distribution varies slowly relative to inertial space with a period $2\pi/n$, then the precession of a mass of ideal fluid rotating with angular velocity ω still proceeds almost exactly the same as if the mass were solid throughout, providing that the ratio of ω/n is small compared to ϵ where ϵ is the ellipticity of the rotating mass. For the earth $\epsilon = 3 \times 10^{-3}$.

a. Precession. Considering the case of precession on a deformable earth where $2\pi/\omega \approx 1$ day and $2\pi/n \approx 26000$ years we see that $\omega/n \approx 10^{-7}$, which is very much less than

$\epsilon \approx 3 \times 10^{-3}$. We conclude that the effect of the yielding of a deformable earth to the system of body forces driving the precession will have a negligible effect on the observed precession. The earth will essentially precess as though it were a rigid body.

At the IAU General Assembly in Grenoble in 1976 it was recommended that an improved theory of the precession based on the FK5 star catalogue and referred to a new standard epoch of 2000.0 be adopted. The details of the new theory of earth precession have been published by Lieske et al. (1977). The corrections to the previous theory arose principally from a failure to distinguish between the "catalogue equinox" of the FK4 star catalogue (the zero point of right ascension on the catalogue celestial equator) and the "dynamical equinox" (the intersection at the ascending node of the ecliptic and the terrestrial equator) combined with new values for the masses of the planets of the solar system and some effects due to the galactic rotation of the FK4 catalogue stars. The new theory of the earth's precession is expected to be accurate to ±0.″1 arc per century or roughly ±1.0 milliarc second per year.

b. Nutation. Considering the case of the nutation on a deformable earth where $2\pi/n \approx 18.6$ years we see that $\omega/n \approx 1.4 \times 10^{-4}$, which is sufficiently near to $\epsilon \approx 3 \times 10^{-3}$ to expect that the effect of the yielding of a deformable earth to the system of body forces driving the nutation might produce measurable discrepancies when precise observations of earth nutation are compared against that which is theoretically predicted for a rigid earth. Woolard (1953, pp. 136) was evidently aware of this problem and adopted an "observational" value of 9.″210 arc for the constant of nutation in preference to "theoretical" values of the order of 9.″224 arc derivable from the relationships of the constant of nutation to other astronomical constants which hold in the case of a rigid earth. Recent observational determinations of the nutation constant place it in the range 9.″201 -- 9.″206 arc, with the discrepancy between theory and observation now of the order of 0.″02 arc being attributed to the effects of the deformation of the earth.

c. Woolard's theory of the nutation. The present IAU theory of the earth's nutation is due to Woolard (1953) and describes the theoretical motion of the instantaneous rotation vector ω of an assumed rigid earth with an axially symmetric mass distribution relative to the set of space-fixed basis vectors $\hat{E}_1$, $\hat{E}_2$, $\hat{E}_3$. A description of Woolard's theory requires the introduction of three reference equators: the equator of angular momentum, the equator of figure, and the equator of rotation. Each equator is defined as passing through the earth's center of mass perpendicularly to the angular momentum axis, the figure axis, and the rotation axis respectively. It can be seen from Figure VII-1 that the equator of figure and the equator of rotation intersect each other with an angle β_e where

$$\beta_e \approx |\varpi_e| \qquad \text{(VII-88)}$$

is the amplitude of the Eulerian polar motion. It can also be seen from Figure VII-1 that the equator of angular momentum and the equator of rotation intersect each other with an angle γ where, from Equations (V-6), (V-11), and (V-27), we have

$$\frac{\gamma}{\beta_e} \approx \frac{C-A}{A} \qquad \text{(VII-89)}$$

where

$$\beta_e \approx |\varpi_e|$$

is the amplitude of the Eulerian wobble (of Figure V-7).

It will also be convenient to introduce:

(1) γ_l, the ascending node of the fixed ecliptic of the fundamental epoch on the equator of angular momentum.

(2) γ_r the ascending node of the fixed ecliptic of the fundamental epoch on the equator of rotation.

(3) γ_f the ascending node of the fixed ecliptic of the fundamental epoch on the equator of figure.

In proceeding with his solution Woolard actually integrated a set of differential equations known as Poisson's equations, which approximately describe the motion in inertial space of the earth's angular momentum vector **L** (Kinoshita et al., 1979). The solutions to Poisson's equations we shall denote as $\psi_0\,\theta_0$, where $\psi_0\,\theta_0$ are time-dependent Euler angles defined by Woolard referenced to a set of space-fixed basis vectors $\hat{E}_1$, $\hat{E}_2$, $\hat{E}_3$ defined by the fixed ecliptic and the fixed mean equinox of the fundamental epoch of 1900.0. According to the conventions adopted by Woolard, the angle $2\pi - \psi_0$ approximately represents the combined effect of lunisolar precession (since it is referred to the fixed ecliptic of the epoch) and lunisolar nutation of the angular momentum axis since the epoch and the angle θ_0 approximately represent the obliquity of the fixed ecliptic of the epoch on the angular momentum equator of date. In Woolard's work the corresponding motion of the earth's axis of figure, described by angles $\psi_f\,\theta_f$, and the earth's axis of rotation, described by angles $\psi_r\,\theta_r$, is obtained to the order of the approximations

ORIGINAL PAGE IS
OF POOR QUALITY

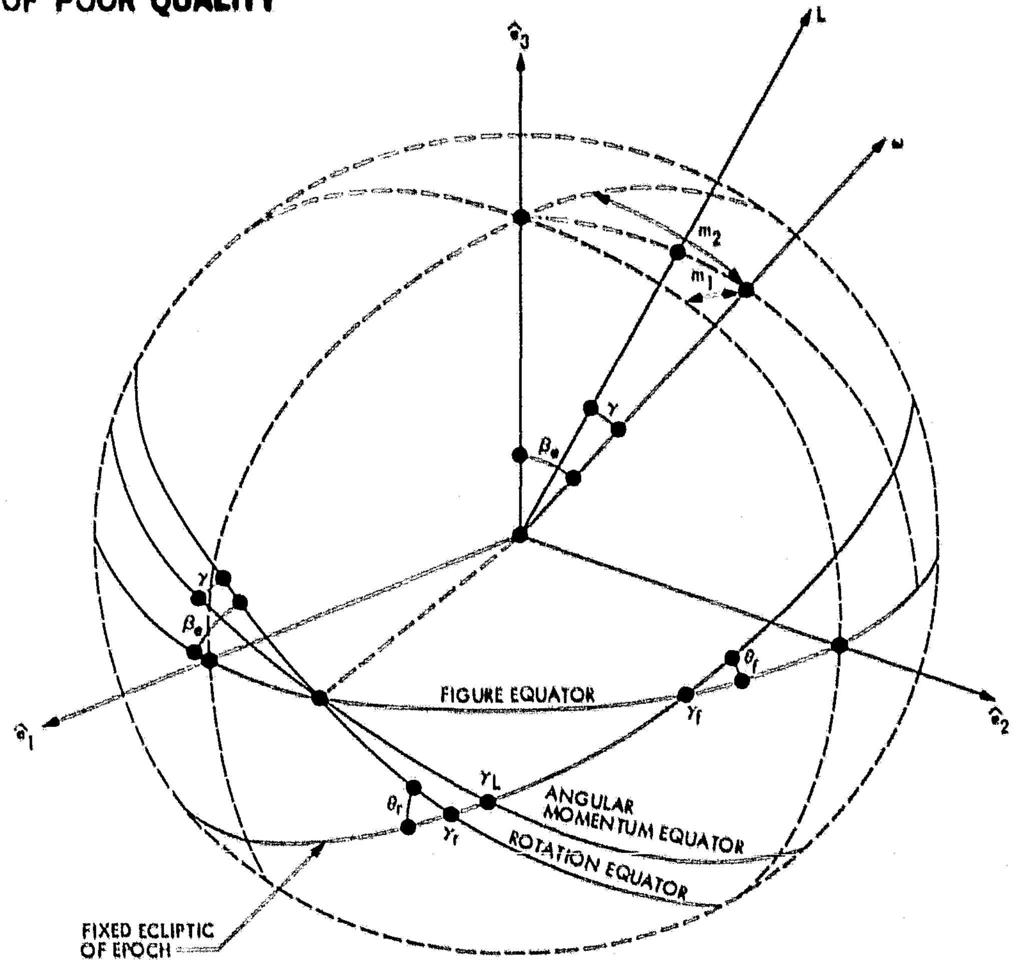

Figure VII-1. The relative location on the celestial sphere of the equator of figure, equator of angular momentum, and equator of rotation. The angles β_e and γ are greatly exaggerated and not drawn to relative scale.

involved in the theory by the additions of correction terms $\delta\psi_f\,\delta\theta_f$ to give

$$\psi_f = \psi_0 + \delta\psi_f$$

$$\theta_f = \theta_0 + \delta\theta_f \qquad \text{(VII-90)}$$

and correction terms $\delta\psi_r\,\delta\theta_r$ to give

$$\psi_r = \psi_0 + \delta\psi_r$$

$$\theta_r = \theta_0 + \delta\theta_r \qquad \text{(VII-91)}$$

respectively. The angles $2\pi - \psi_f$, $2\pi - \psi_r$ represent to the order of the approximations involved the combined effects of lunisolar precession (since they are referred to a fixed ecliptic) and lunisolar nutation in longitude since the fundamental epoch of the axis of figure $\hat{e}_3$ and of the axis of rotation ω respectively. The angles θ_f, θ_r represent, to the order of the approximations involved, the obliquity of the fixed ecliptic of the fundamental epoch on the equator of figure of date and the equator of rotation of date respectively. Woolard then decomposes the angles $\psi_f\theta_f$ and $\psi_r\theta_r$ into secular terms $\psi_f^s\theta_f^s$ and $\psi_r^s\theta_r^s$ and periodic terms $\Delta\psi_f\Delta\theta_f$ and $\Delta\psi_r\Delta\theta_r$ with

$$2\pi - \psi_f = 2\pi - \psi_f^s - \Delta\psi_f$$

$$\theta_f = \theta_f^s + \Delta\theta_f \qquad \text{(VII-92)}$$

and

$$2\pi - \psi_r = 2\pi - \psi_r^s - \Delta\psi_r$$

$$\theta_r = \theta_r^s + \Delta\theta_r \qquad \text{(VII-93)}$$

The secular terms $2\pi - \psi_f^s, \theta_f^s$ and $2\pi - \psi_r^s, \theta_r^s$ are identified by Woolard with the lunisolar precession of the earth's figure axis and rotation axis respectively. The periodic terms $-\Delta\psi_f, \Delta\theta_f$ and $-\Delta\psi_r, \Delta\theta_r$ are identified with the lunisolar nutation of the earth's figure axis and rotation axis respectively.

Note: In his paper Woolard uses ψ_r and ϵ_1 to denote the angles $\psi_r\, \theta_r$ and he uses ψ and θ to denote the angles $\psi_f\, \theta_f$. Furthermore Woolard does not distinguish in his notation between ψ_0, θ_0 and ψ_f, θ_f using ψ, θ for both.

In Woolard's paper, Poisson's equations appear as equations (30) on page 34. The solutions of their descendents, equations (44) and (45) on pages 47 and 48, provide the functions ψ_0, θ_0. Equations (19) on page 24 govern the motion of the figure axis and their solution yields the functions ψ_f, θ_f. Equations (24) on page 26 govern the motion of the rotation axis and their solution yields the functions ψ_r, θ_r. Equations (53) on page 131 govern the correction terms $\delta\psi_f, \delta\theta_f$, which yield the solutions ψ_f, θ_f from the solutions $\psi_0\ \theta_0$. The solutions $\delta\psi_f\delta\theta_f$ are given by the expressions (54) on page 132. The equations governing the correction terms $\delta\psi_r\, \delta\theta_r$ are shown by Woolard to the order of approximations involved to be obtainable directly from equations (53) for the correction terms $\delta\psi_r\, \delta\theta_f$ by multiplying by the factor $-(C - A)/A$. Hence the solutions $\delta\psi_r\, \delta\theta_r$ can be obtained directly from expressions (54) for $\delta\psi_f\delta\theta_f$ by multiplying by $-(C - A)/A$ and are given in expressions (55) on page 133.

$$\delta\psi_r = -\frac{C-A}{A}\,\delta\psi_f \tag{VII-94}$$

$$\delta\theta_r = -\frac{C-A}{A}\,\delta\theta_f. \tag{VII-95}$$

Presumably motivated by the conviction that the motion of the rotation axis in space is that which is observable by astronomic means, Woolard tabulated the results of his solutions for $2\pi - \psi_r^s, \theta_r^s$, and $-\Delta\psi_r, \Delta\theta_r$ in two locations in his work. The first appears in Table 24 p. 138 ff, which gives the precession and nutation in longitude and obliquity of the rotation axis relative to the fixed ecliptic of January 0 1900 Greenwich Mean Noon (JD 2415020.0). The epoch for these expressions is January 0 1900 Greenwich Mean Noon (JD 2415020.0). Since the contents of this table are referenced to a fixed ecliptic, the expressions therein reflect only the motion of the equator since the epoch, and as such represent only the lunisolar contribution to precession and nutation.

The second tabulation appears in Table 26 p. 153, which gives the precession and nutation in longitude and obliquity of the rotation axis relative to the moving ecliptic of date. The epoch for these expressions is also January 0 1900 Greenwich Mean Noon (JD 2415020.0). Since the contents of this table are referenced to a moving ecliptic the expressions therein reflect the combined motion of the ecliptic and equator since the epoch. Table 26 represents the effect of general precession and nutation which is the sum of lunisolar precession and nutation, which perturbs only the equator, plus planetary precession and nutation, which perturbs only the ecliptic. Table 26 is generated from Table 24 by the addition of the contents of Table 25, p. 152, which contains the terms which account for the planetary perturbations to the ecliptic.

At the time of this writing the expressions for the nutation of the rotation axis of Table 26 of Woolard's work constitute the IAU standard series for the nutation and are reproduced on pages 44 and 45 of the *Explanatory Supplement* to the *Astronomical Ephemeris and the American Ephemeris and Nautical Almanac* (*1961*).

The earth's nutation is driven by the gravitational torques of the sun and moon, and it is not surprising that the expressions for the nutation find their most convenient representation in terms of the angles $\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5$ where

$\alpha_1 \equiv \ell$ is the mean anomaly of the moon

$\ell = 296^\circ\ 06'\ 16''\!.59 + 1325^r\ 198^\circ 50'\ 56''\!.79\,T + 33''\!.09T^2 + 0''\!.0518T^3$

$\alpha_2 \equiv \ell'$ is the mean anomaly of the sun

$\ell' = 358^\circ 28'\ 33''\!.00 + 99^r\ 359^\circ 02'\ 59''\!.10\,T - 0''\!.54T^2 - 0''\!.0120T^3$

$\alpha_3 \equiv F$ is the mean argument of the latitude of the moon

$F = 11^\circ\ 15'\ 03''\!.2 - + 1342^r\ 82^\circ 01'\ 30''\!.54T - 11''\!.56T^2 - 0''\!.0012T^3$

$\alpha_4 \equiv D$ is the mean elongation of the moon from the sun

$D = 350^\circ 44'\ 14''\!.95 + 1236^r\ 307^\circ 06'\ 51''\!.18T - 5''\!.17T^2 + 0''\!.0068T^3$

$\alpha_5 \equiv \Omega$ is the mean longitude of the ascending lunar node

$\Omega = 259^\circ 10' 59''\!.79 - 5^r\ 134^\circ 08' 31''\!.23T + 7''\!.4827T^2 + 0''\!.0080T^3$

and where T is measured in Julian centuries of 36525 days from the epoch January 0 1900 Greenwich Mean Noon (JD 2415020.0).

The nutation series is traditionally tabulated in terms of amplitudes $A_j(T)$, $B_j(T)$, $j = 1, 2, 3, \ldots N$, and argument coefficients K_{ji}, $i = 1, 2, 3, \ldots 5$, in such a way that the nutation in longitude $\delta\psi(T)$, reckoned positive to the east along the ecliptic, and the nutation in obliquity, reckoned positive if the obliquity is increased, are given by expressions of the form

$$\delta\psi(T) = \sum_{j=1}^{N} A_j(T) \sin\left[\sum_{i=1}^{5} k_{ji}\,\alpha_i(T)\right]$$

$$+ \text{ diurnal nutation in longitude} \qquad \text{(VII-96)}$$

$$\delta\epsilon(T) = \sum_{j=1}^{N} B_j(T) \cos\left[\sum_{i=1}^{5} k_{ji}\,\alpha_i(T)\right]$$

$$+ \text{ diurnal nutation in obliquity} \qquad \text{(VII-97)}$$

The diurnal motion in the nutation was shown in section V and illustrated in Figure V-3 of this work to arise from the fact that the axis of figure $\hat{e}_3$ and the axis of rotation ω in general depart from the axis of angular momentum **L** by the small angles $\beta_e-\gamma$ and γ respectively. Furthermore the Poinsot construction developed in section V and illustrated in Figure V-5 of the work shows that the figure axis $\hat{e}_3$ and the rotation axis ω make nearly diurnal rotations in a space-fixed frame about the angular momentum axis moving on the surface of the two cones of apex angles $2_{\beta e}$ and 2γ respectively.

d. Woolard's theory for the nutation of the rotation axis. Woolard's series for the nutation of the rotation axis is given in Table VII-1.

Table VII-1. Nutation series for the rotation axis (Woolard, 1953)

Index J	Period, days	α_1	α_2	α_3	α_4	α_5	Amplitude A_j (0″.0001 arc)	Amplitude B_j (0″.0001 arc)
		k_{j1}	k_{j2}	k_{j3}	k_{j4}	k_{j5}		
1	6798	0	0	0	0	1	$-172327 - 173.7T$	$92100 + 9.1T$
2	3399	0	0	0	0	2	$2088 + 0.2T$	$-904 + 0.4T$
3	1305	−2	0	2	0	1	45	−24
4	1095	2	0	−2	0	0	10	0
5	6786	0	−2	2	−2	1	−4	2
6	1616	−2	0	2	0	2	−3	2
7	3233	1	−1	0	−1	0	−2	0
8	183	0	0	2	−2	2	$-12729 - 1.3T$	$5522 - 2.9T$
9	365	0	1	0	0	0	$1261 - 3.1T$	0
10	122	0	1	2	−2	2	$-497 + 1.2T$	$216 - 0.6T$
11	365	0	−1	2	−2	2	$214 - 0.5T$	$-93 + 0.3T$
12	178	0	0	2	−2	1	$124 + 0.1T$	−66
13	206	2	0	0	−2	0	45	0
14	173	0	0	2	−2	0	−21	0
15	183	0	2	0	0	0	$16 - 0.1T$	0
16	386	0	1	0	0	1	−15	8
17	91	0	2	2	−2	2	$-15 + 0.1T$	7
18	347	0	−1	0	0	1	−10	5
19	200	−2	0	0	2	1	−5	3
20	347	0	−1	2	−2	1	−5	3
21	212	2	0	0	−2	1	4	−2
22	120	0	1	2	−2	1	3	−2
23	412	1	0	0	−1	0	−3	0
24	13.7	0	0	2	0	2	$-2037 - 0.2T$	$884 - 0.5T$
25	27.6	1	0	0	0	0	$675 + 0.1T$	0
26	13.6	0	0	2	0	1	$-342 - 0.4T$	183
27	9.1	1	0	2	0	2	−261	$113 - 0.1T$
28	31.8	1	0	0	−2	0	−149	0
29	27.1	−1	0	2	0	2	114	−50
30	14.8	0	0	0	2	0	60	0
31	27.7	1	0	0	0	1	58	−31
32	27.4	−1	0	0	0	1	−57	30
33	9.6	−1	0	2	2	2	−52	22
34	9.1	1	0	2	0	1	−44	23
35	7.1	0	0	2	2	2	32	14
36	13.8	2	0	0	0	0	28	0
37	23.9	1	0	2	−2	2	26	−11
38	6.9	2	0	2	0	2	−26	11

Table VII-1. Nutation series for the rotation axis (Woolard, 1953) (Continued)

Index J	Period, days	α_1 k_{j1}	α_2 k_{j2}	α_3 k_{j3}	α_4 k_{j4}	α_5 k_{j5}	Amplitude A_j (0″.0001 arc)	Amplitude B_j (0″.0001 arc)
39	13.6	0	0	2	0	0	25	0
40	27.0	−1	0	2	0	1	19	−10
41	32.0	−1	0	0	2	1	14	−7
42	31.7	1	0	0	−2	1	−13	7
43	9.5	−1	0	2	2	1	−9	5
44	34.8	1	1	0	−2	0	−7	0
45	13.2	0	1	2	0	2	7	−3
46	9.6	1	0	0	2	0	6	0
47	14.8	0	0	0	2	1	−6	3
48	14.2	0	−1	2	0	2	−6	3
49	5.6	1	0	2	2	2	−6	3
50	12.8	2	0	2	−2	2	6	−2
51	14.7	0	0	0	−2	1	−5	3
52	7.1	0	0	2	2	1	−5	3
53	23.9	1	0	2	−2	1	5	−3
54	29.5	0	0	0	1	0	−4	0
55	15.4	0	1	0	−2	0	−4	0
56	29.8	1	−1	0	0	0	4	0
57	26.9	1	0	−2	0	0	4	0
58	6.9	2	0	2	0	1	−4	0
59	9.1	1	0	2	0	0	3	0
60	25.6	1	1	0	0	0	−3	0
61	9.4	1	−1	2	0	2	−3	0
62	13.7	−2	0	0	0	1	−2	0
63	32.6	−1	0	2	−2	1	−2	0
64	13.8	2	0	0	0	1	2	0
65	9.8	−1	−1	2	2	2	−2	0
66	7.2	0	−1	2	2	2	−2	0
67	27.8	1	0	0	0	2	−2	0
68	8.9	1	1	2	0	2	2	0
69	5.5	3	0	2	0	2	−2	0

c. The diurnal nutation of the rotation axis. The diurnal motion of the rotation axis ω in space can be seen from Figure V-4 to have an amplitude γ where Equation (VII-89) gives

$$\gamma = \frac{C-A}{A}\beta_e \qquad \text{(VII-98)}$$

and where β_e is the amplitude of the Eulerian wobble. The angular rate of the motion of the rotation axis ω about the angular momentum axis **L** in the space-fixed frame is given by Equation (V-52) as $\omega_3 + \sigma_r$; namely, the sum of the earth's spin angular rate (diurnal rate) plus the wobble angular rate. Since the wobble angular rate σ_r is very small (435-day period) we see that this motion of the rotation axis in space has very nearly a diurnal period.

Referring to Figure VII-1 and adopting the angular momentum equator as a slowly moving reference plane relative to inertial space we can see that the diurnal motion in space of the rotation axis ω produces a diurnal nutation in longitude $\delta\psi_{rd}$ with amplitude given by

$$\delta\psi_{rd} = \frac{\gamma}{\sin\theta} = \frac{C-A}{A}\frac{\beta_e}{\sin\theta} \qquad \text{(VII-99)}$$

and a diurnal nutation in obliquity $\delta\epsilon_{rd}$ with amplitude given by

$$\delta\epsilon_{rd} = \gamma = \frac{C-A}{A}\beta_e \qquad \text{(VII-100)}$$

where θ is the obliquity of the ecliptic. To establish the phase of the diurnal nutations we refer to Figure V-7 and see that

ORIGINAL PAGE IS OF POOR QUALITY

the body-fixed plane containing the vectors $\hat{e}_3$ and ω has a geographic east longitude λ_e where

$$\lambda_e = \tan^{-1} \frac{m_{e2}}{m_{e1}} \qquad \text{(VII-101)}$$

and where $\overline{m}_e = m_{e1} + im_{e2}$ specifies the body-fixed position of the Eulerian axis. The right ascension of this plane in the space-fixed frame is $\lambda_e + \phi$, where ϕ is Greenwich Sidereal Time. Referring to Figure VII-1 we see that when

$$\lambda_e + \phi = 0, \pi \qquad \text{(VII-102)}$$

the displacement $\gamma_r - \gamma_L$ or equivalently $\delta\psi_{rd}$ achieves its maximum negative and positive values respectively and that when

$$\lambda_e + \phi = \frac{\pi}{2}, \frac{3\pi}{2} \qquad \text{(VII-103)}$$

the displacement $\gamma_r - \gamma_L$ or $\delta\psi_{rd}$ vanishes. Hence we can set

$$\delta\psi_{rd} = -\frac{C-A}{A}\frac{\beta_e}{\sin\theta}\cos(\phi + \lambda_e) \qquad \text{(VII-104)}$$

and by a similar argument we can set

$$\delta\epsilon_{rd} \approx \frac{C-A}{A}\beta_e \sin(\phi + \lambda_e)\,. \qquad \text{(VII-105)}$$

Now since $\beta_e \leqslant 0''.2$ arc and $(C-A)/A \approx 0.00328$ we see that taking $\theta \approx 23°$ implies that the diurnal nutations in longitude of the rotation axis $\delta\psi_{rd}$ are less than

$$\delta\psi_{rd} \leqslant 0''.0018 \text{ arc} \qquad \text{(VII-106)}$$

and that the diurnal nutations in obliquity of the rotation axis $\delta\epsilon_{rd}$ are less than

$$\delta\epsilon_{rd} \leqslant 0''.0007 \text{ arc}\,. \qquad \text{(VII-107)}$$

f. Woolard's theory for the nutation of the figure axis. Since even on a rigid earth the rotation axis ω does not remain fixed in a body-fixed frame, the motion of the rotation axis in space is not precisely shared by body-fixed axes such as interferometer vector baselines. In fact if the rotation axis is chosen as a reference axis for the nutation, then the motion in space of a body-fixed axis such as an interferometer vector baseline must be obtained from the motion of the reference axis by subtracting from the motion of the reference axis in space the motion of the reference axis relative to the body-fixed frame.

Alternatively on a rigid earth the figure axis $\hat{e}_3$ is a body-fixed axis and its motion in space is shared directly by all other body-fixed axes such as interferometer vector baselines. If the figure axis is chosen as a reference axis for the nutation, then the motion in space of any other body-fixed axis such as an interferometer vector baseline is identical to that of the figure axis. For this reason, in computing the effects of nutation on the orientation in space of an interferometer vector baseline it is useful to have a theory for the nutation of the earth's figure axis $\hat{e}_3$.

In Woolard's theory it is possible to generate a series for the nutation of the figure axis from his series (Table 26) for the nutation of the rotation axis by

(1) Subtracting from the entries of Table 26 the terms of the equations (55) p. 133 of Woolard's work.

(2) Adding (noticing the change in the sign of $\delta\psi$) to the entries of Table 26 the terms of the equations (54) p. 132 of Woolard's work.

The results of this procedure are presented in Table VII-2 below.

g. The diurnal nutation of the figure axis. The diurnal motion of the figure axis $\hat{e}_3$ in space can be seen from Figure V-5 to have an amplitude $\beta_e - \gamma$ where β_e is the amplitude of the wobble and

$$\gamma = \frac{C-A}{A}\beta_e\,. \qquad \text{(VII-108)}$$

Equation (V-18) and Figure V-3 show that the vectors ω **L** and $\hat{e}_3$ are all coplanar and so the rate of the motion of the figure axis $\hat{e}_3$ about the angular momentum vector **L** in the space-fixed frame is identical to the rate of the rotation axis ω about the angular momentum vector **L** in the space-fixed frame and so is also given by Equation (V-52) as $\omega_3 + \sigma_r$. Just as in the case of the diurnal motion of the rotation axis in space this rate is the sum of the earth's spin angular rate (diurnal rate) plus the wobble angular rate, and since the wobble angular rate is very small (435-day period) we see that this motion of the figure axis in space has a nearly diurnal period.

Referring to Figure VII-1 and adopting the angular momentum equator as a slowly moving reference plane relative to

ORIGINAL PAGE IS
OF POOR QUALITY

Table VII-2. Nutation series for the figure axis (Woolard, 1953)

Index J	Period, days	α_1 k_{j1}	α_2 k_{j2}	α_3 k_{j3}	α_4 k_{j4}	α_5 k_{j5}	Amplitude A_j (0".0001 arc)	Amplitude B_j (0".0001 arc)
1	6798	0	0	0	0	1	−172293.1−173.7T	92090+9.1T
2	3399	0	0	0	0	2	2088 + 0.2T	−904 + 0.4T
3	1305	−2	0	2	0	1	45	−24
4	1095	2	0	−2	0	0	10	0
5	6786	0	−2	2	−2	1	−4	2
6	1616	−2	0	2	0	2	−3	2
7	3233	1	−1	0	−1	0	−2	0
8	183	0	0	2	−2	2	−12804.5−1.3T	5549.6−2.9T
9	365	0	1	0	0	0	1261−3.1T	−1.4
10	122	0	1	2	−2	2	−501.4 + 1.2T	217.6−0.6T
11	365	0	−1	2	−2	2	214−0.5T	−93+0.3T
12	178	0	0	2	−2	1	124 + 0.1T	−66
13	206	2	0	0	−2	0	45	0
14	173	0	0	2	−2	0	−21	0
15	183	0	2	0	0	0	16−0.1T	0
16	386	0	1	0	0	1	−15	8
17	91	0	2	2	−2	2	−15 + 0.1T	7
18	347	0	−1	0	0	1	−10	5
19	200	−2	0	0	2	1	−5	3
20	347	0	−1	2	−2	1	−5	3
21	212	2	0	0	−2	1	4	−2
22	120	0	1	2	−2	1	3	−2
23	412	1	0	0	−1	0	−3	0
24	13.7	0	0	2	0	2	2199−0.2T	943.2−0.5T
25	27.6	1	0	0	0	0	675 + 0.1T	−9.7
26	13.6	0	0	2	0	1	−375.5−0.4T	192.9
27	9.1	1	0	2	0	2	−291.8	124.3−0.1T
28	31.8	1	0	0	−2	0	−149	−1.8
29	27.1	−1	0	2	0	2	118.6	−51.7
30	14.8	0	0	0	2	0	60	−1.6
31	27.7	1	0	0	0	1	8.6	−31
32	27.4	−1	0	0	0	1	−54.2	30
33	9.6	−1	0	2	2	2	−57.9	24.1
34	9.1	1	0	2	0	1	−50.4	24.9
35	7.1	0	0	2	2	2	−36.9	15.8
36	13.8	2	0	0	0	0	28	0
37	23.9	1	0	2	−2	2	−27.2	−11
38	6.9	2	0	2	0	2	−30.1	12.5
39	13.6	0	0	2	0	0	25	0
40	27.0	−1	0	2	0	1	19	−10
41	32.0	−1	0	0	2	1	14	−7
42	31.7	1	0	0	−2	1	−13	7
43	9.5	−1	0	2	2	1	−10.2	5
44	34.8	1	1	0	−2	0	−7	0
45	13.2	0	1	2	0	2	7	−3
46	9.6	1	0	0	2	0	6	0
47	14.8	0	0	0	2	1	−6	3
48	14.2	0	−1	2	0	2	−6	3
49	5.6	1	0	2	2	2	−7.2	3
50	12.8	2	0	2	−2	2	6	−2
51	14.7	0	0	0	−2	1	−5	3
52	7.1	0	0	2	2	1	−6	3
53	23.9	1	0	2	−2	1	5	−3
54	29.5	0	0	0	1	0	−4	0
55	15.4	0	1	0	−2	0	−4	0
56	29.8	1	−1	0	0	0	4	0

ORIGINAL PAGE IS
OF POOR QUALITY

Table VII-2. Nutation series for the figure axis (Woolard, 1953) (Continued)

Index J	Period, days	α_1 k_{j1}	α_2 k_{j2}	α_3 k_{j3}	α_4 k_{j4}	α_5 k_{j5}	Amplitude A_j (0".0001 arc)	Amplitude B_j (0".0001 arc)
57	26.9	1	0	−2	0	0	4	0
58	6.9	2	0	2	0	1	−4	0
59	9.1	1	0	2	0	0	3	0
60	25.6	1	1	0	0	0	−3	0
61	9.4	1	−1	2	0	2	−3	0
62	13.7	−2	0	0	0	1	−2	0
63	32.6	−1	0	2	−2	1	−2	0
64	13.8	2	0	0	0	1	2	0
65	9.8	−1	−1	2	2	2	−2	0
66	7.2	0	−1	2	2	2	−2	0
67	27.8	1	0	0	0	2	−2	0
68	8.9	1	1	2	0	2	2	0
69	5.5	3	0	2	0	2	−2	0

inertial space we can see that the diurnal motion in space of the figure axis $\hat{e}_3$ produces a diurnal nutation in longitude $\delta\psi_{fd}$ with amplitude given by

$$\delta\psi_{fd} = \frac{\beta_e - \gamma}{\sin\theta} = \frac{\beta_e}{\sin\theta}\left(2 - \frac{C}{A}\right) \qquad \text{(VII-109)}$$

and a diurnal nutation in obliquity $\delta\epsilon_{fd}$ with amplitude given by

$$\delta\epsilon_{fd} = \beta_e - \gamma = \beta_e\left(2 - \frac{C}{A}\right) \qquad \text{(VII-110)}$$

where θ is the obliquity of the ecliptic.

By referring to Figure VII-1 we see that the nutation $\gamma_f - \gamma_L$ or equivalently $\delta\psi_{fd}$ is opposite in phase to the nutation $\gamma_r - \gamma_L$ or $\delta\psi_{rd}$ as a consequence of the fact that the figure axis $\hat{e}_3$ and the rotation axis ω lie on opposite sides of the angular momentum vector **L**. Hence by arguments similar to those used in the case of the rotation axis we can establish the phase of the diurnal nutations of the figure axis as

$$\delta\psi_{fd} = \frac{\beta_e}{\sin\theta}\left(2 - \frac{C}{A}\right)\cos(\phi + \lambda_e) \qquad \text{(VII-111)}$$

and

$$\delta\epsilon_{fd} = -\beta_e\left(2 - \frac{C}{A}\right)\sin(\phi + \lambda_e) \qquad \text{(VII-112)}$$

and where $C/A \approx 1$.

h. Comments on Woolard's theory. It is apparent from Tables VII-1 and VII-2 that the nutations in Woolard's series can be exhaustively grouped into nutations whose period exceeds 91 days, which are normally referred to as the long period nutations, and nutations whose period is less than 35 days, which are normally referred to as the short-period nutations.

Apparent sidereal time is referenced to the nutated equinox. If the rotation axis is chosen as the reference axis for nutation then consistency requires that the rotation equator be used to provide the reference equinox γ_r for apparent sidereal time. From Equations (VII-104) (VII-106) we see that this equinox suffers diurnal nutations in longitude of the order of $\delta\psi_{rd} \approx 0''.0018$ arc which correspond to diurnal inequalities of apparent sidereal time of the order of 0.12 msec. This diurnal inequality in apparent sidereal time is usually ignored in most present-day applications of the theory.

The motion $\overline{m} = m_1 + im_2$ of the rotation axis ω relative to the figure axis $\hat{e}_3$ in the body-fixed frame is composed of the sum of

(1) A geophysical or Eulerian (torque-free) component $\overline{m}_e$ excited by a variety of internal processes on the earth and which is unpredictable on the basis of present-day geophysical knowledge.

(2) A non-Eulerian lunisolar component $\overline{m}_p = m_{p1} + im_{p2}$ excited by the gravitational torques $\overline{N}$ of the sun and the moon acting on the earth, often referred to as the "dynamical variation in latitude," and which is precisely predictable. Expressions for m_{p1} and m_{p2} are given in Equations (V-94) and (V-95) respectively.

The Eulerian motion has an amplitude of the order of 0.″2 arc maximum and the lunisolar motion has an amplitude of 0.″0178 arc maximum. The lunisolar motion or the dynamical variations in latitude is a retrograde motion with a nearly diurnal period. The center of its diurnal circular path corresponds to the Eulerian position of the rotation axis in the body-fixed frame as shown in Figure V-8.

The mean amplitude of the lunisolar motion or dynamical variations in latitude $\mathscr{M}_p$ is roughly $\langle\beta_p\rangle \approx 0.''0089$ arc and so we see that in a body-fixed frame the lunisolar motion of the rotation axis can be described as motion around a body-fixed cone of mean apex angle $2\langle\beta_p\rangle \approx 0.''0178$ arc. In the absence of geophysically induced polar motion or Eulerian motion the precession of the earth's rotation axis ω and figure axis $\hat{e}_3$ in inertial space can be visualized by a Poinsot construction in a manner described by Woolard on p. 31 and illustrated below in Figure VII-2.

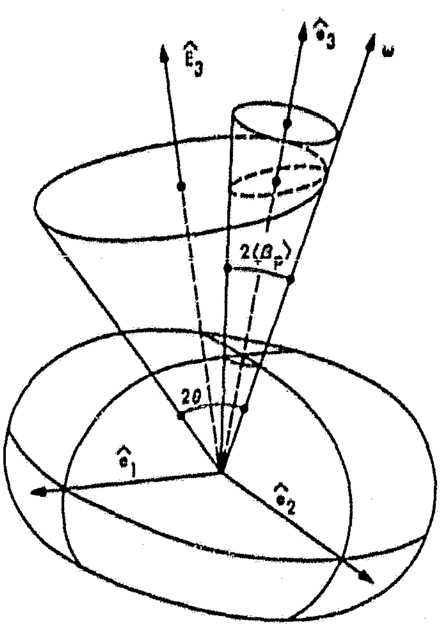

Figure VII-2. The Poinsot construction appropriate for the description of the motion of the earth's figure axis in inertial space or lunisolar precession. The basis vector $\hat{E}_3$ is normal to the ecliptic, θ is the mean obliquity of the ecliptic referenced to the equator of rotation, and $\langle\beta_p\rangle$ is the mean amplitude of the lunisolar polar motion $\mathscr{M}_p$.

The lunisolar body-fixed cone of mean apex angle $2\langle\beta_p\rangle$ rolls without slipping on the interior of a space-fixed cone centered on the pole of the ecliptic $\hat{E}_3$ of apex angle 2θ where θ is the obliquity of the ecliptic and with the rotation vector ω occupying the line of contact between the two cones. The retrograde lunisolar motion of the body-fixed cone drives it in a retrograde sense around the interior of the space-fixed cone. In the absence of geophysically induced polar motion or Eulerian motion the Eulerian position of the rotation axis in the body-fixed frame coincides with the figure axis $\hat{e}_3$. Hence in Figure VII-2 $\hat{e}_3$ occupies the axis of the body-fixed cone. Taking $\theta \approx 23.°473$ (Stacey 1977) the diurnal rotation of the body-fixed cone transports $\hat{e}_3$ and ω around the space fixed cone in about 26,000 years.

Figure VII-2 illustrates the fact that during this motion the angle between $\hat{E}_3$ and $\hat{e}_3$ is systematically smaller than the angle between $\hat{E}_3$ and ω by the amount $\langle\beta_p\rangle \approx 0.''0089$ arc. This is the kinematical reason for the fact that the mean obliquity of the ecliptic for the figure axis $\hat{e}_3$ is less than the mean obliquity of the ecliptic for the rotation axis ω. This kinematical construction accounts for the existence of the constant term of $-0.''00868$ arc appearing in equations (54) p. 132 of Woolard's work involving the transformation from the obliquity of the rotation axis to the obliquity of the figure axis.

The secular term of $-0.''00043T$ appearing in equations (54) p. 132 of Woolard's work involving the transformation from the node of the rotation equator to the node of the figure equation has been shown by Murray (1977) to be spurious. Kinoshita et al. (1977) and Kinoshita (1977) have shown how the spurious secular term identified by Murray and others like it appearing in Woolard's work arise from incorrect mathematical procedures.

Woolard's theory of the nutation is widely regarded as being inadequate for present-day requirements. Firstly the theory treats a rigid earth and there are indications that this restriction is responsible for errors in the predicted nutation of the order of 0.″02 arc. This is insufficient accuracy for the requirements of modern observational techniques, in particular long baseline interferometry.

Furthermore it is apparent that the instantaneous rotation axis ω is not directly observable. This conclusion follows from the fact that any attempt to observe the vector ω necessarily requires observations extending over a finite interval of time, which for the classical methods is typically several hours in duration and often as long as 12 or 24 hours, during which time the vector ω will complete a significant portion of its diurnal circuit in the body-fixed frame. The result of this is that the axis which is observed in practice is not ω but an axis coinciding with some mean position of ω averaged over the observing interval.

i. The revised theory of the nutation. In 1977 at the IAU Symposium No. 78 on Nutation and the Earth's Rotation, a

ORIGINAL PAGE IS
OF POOR QUALITY

working group was convened whose task it was to revise the theory of the earth's nutation and to recommend a new series for the earth's nutation to be adopted by the IAU at its 17th General Assembly in 1979 in Montreal, Canada. At the time of this writing this group has completed its task and its recommendations to the IAU General Assembly are two-fold (J. G. Williams personal communication): (i) a change of reference axis, (ii) the computations of earth nutations for a deformable earth.

i. The change of reference axis. Since all attempts to observe the present nutation reference axis ω invariably yield a mean position for ω, averaged over the finite observing internal in both the body-fixed frame and the space-fixed frame, it is recommended that the revised theory of the nutation make explicit recognition of this fact and replace the instantaneous axis ω with a new reference axis defined explicitly as the mean position of ω when averaged over its predictable (i.e., driven by external gravitational torques $\overline{N}$) diurnal motion $\overline{m}_p$. The new reference axis will be given a name to distinguish it from others with which it might potentially become confused. Suggested names include "celestial reference pole," "mean diurnal axis," "Eulerian pole of rotation" and in Section II of this document it has been referred to as the "spin axis."

The new reference axis is most precisely described mathematically. Before doing so we require the result of a small lemma: we wish to show that if the matrix M is an orthogonal transformation matrix representing a spatial rotation then the matrix $\dot{M}M^T$, where the dot "$\cdot$" denotes differentiation with respect to time, is antisymmetric. The proof of this is as follows:

Since M is orthogonal $M^T = M^{-1}$ and since $MM^{-1} = I$ where I is the identity matrix we have $MM^T = I$. Differentiating with respect to time gives

$$\frac{d}{dt}[MM^T] = \dot{M}M^T + M\dot{M}^T = 0$$

Hence

$$\dot{M}M^T = -M\dot{M}^T = -[\dot{M}M^T]^T$$

Therefore

$$\dot{M}M^T = -[\dot{M}M^T]^T$$

and $\dot{M}M^T$ is antisymmetric. Q.E.D.

The transformation of the components of a body-fixed vector denoted $\underline{r}_b$ to the components denoted $\underline{r}_s$ of the same vector viewed in the space-fixed frame is

$$\underline{r}_s = PNSW\underline{r}_b \tag{VII-113}$$

where the matrix:

P represents the precession of the reference axis of the body-fixed coordinate system.

N represents the nutation of the reference axis of the body-fixed coordinate system

S represents the net rotation (spin) about the reference axis of the body-fixed coordinate system (UT1).

W represents the orientation of the reference axis within the body-fixed coordinate system (polar motion).

In reality, observations are made in a body-fixed frame of the body-fixed components $\underline{r}_b$ of a vector (the direction to a quasar or a star) whose components in the space-fixed space frame $\underline{r}_s$ are considered to be constant.

$$\underline{r}_b = W^T S^T N^T P^T \underline{r}_s . \tag{VII-114}$$

The body-fixed motion of the quasar or star is given by

$$\frac{d\underline{r}_b}{dt} = \left[\frac{dW^T}{dt} S^T N^T P^T + W^T \frac{dS^T}{dt} N^T P^T + W^T S^T \frac{dN^T}{dt} P^T + W^T S^T N^T \frac{dP^T}{dt}\right]\underline{r}_s + W^T S^T N^T P^T \frac{d\underline{r}_s}{dt} \tag{VII-115}$$

Note: Equations (VII-113) and (VII-114) deal explicitly with the components $\underline{r}_s$ of the vector $\mathbf{r}_s$ and the components $\underline{r}_b$ of the vector $\mathbf{r}_b$ and so when differentiating with respect to time there is no need to distinguish between time derivatives taken in the body-fixed frame and time derivatives taken in the space-fixed frame.

If we consider the case where $d\underline{r}_s/dt = 0$ which corresponds to a constant space-fixed direction and is achieved in practice by removing effects of aberration for sources exhibiting no proper motion, then Equations (VII-113) and (VII-115) give

$$\frac{d\underline{r}_b}{dt} = \left[\frac{dW^T}{dt} S^T N^T P^T + W^T \frac{dS^T}{dt} N^T P^T + W^T S^T \frac{dN^T}{dt} P^T + W^T S^T N^T \frac{dP^T}{dt}\right] PNSW\underline{r}_b \tag{VII-116}$$

which can be written as

$$\frac{dr_b}{dt} = \left[\frac{dW^T}{dt} W + W^T \frac{dS^T}{dt} S W + W^T S^T \frac{dN^T}{dt} N S W + W^T S^T N^T \frac{dP^T}{dt} P N S W\right] r_b \qquad \text{(VII-117)}$$

Each term in the square bracket of Equation (VII-117) is either an antisymmetric matrix of the form $\dot{M} M^T$ or is a similarity transformation on such a matrix. Since it is the property of similarity transformations that they preserve antisymmetry of matrices, we conclude that all four terms in the square brackets of Equation (VII-117) are antisymmetric matrices and so Equation (VII-117) can be written in vector form as

$$\frac{dr_b}{dt} = -[\Omega_W + \Omega_S + \Omega_N + \Omega_P] \times \mathbf{r}_b \qquad \text{(VII-118)}$$

where

(1) Ω_W is a polar motion or wobble rate vector given by

$$\Omega_W = 0^W_{23}\,\hat{e}_1 + 0^W_{31}\,\hat{e}_2 + 0^W_{12}\,\hat{e}_3 \qquad \text{(VII-119)}$$

(2) Ω_S is a spin rate vector given by

$$\Omega_S = 0^S_{23}\,\hat{e}_1 + 0^S_{31}\,\hat{e}_2 + 0^S_{12}\,\hat{e}_3 \qquad \text{(VII-120)}$$

(3) Ω_N is a nutation rate vector given by

$$\Omega_N = 0^N_{23}\,\hat{e}_1 + 0^N_{31}\,\hat{e}_2 + 0^N_{12}\,\hat{e}_3 \qquad \text{(VII-121)}$$

(4) Ω_p is a precession rate vector given by

$$\Omega_p = 0^P_{23}\,\hat{e}_1 + 0^P_{31}\,\hat{e}_2 + 0^P_{12}\,\hat{e}_3 \qquad \text{(VII-122)}$$

and where the matrices $0^W\ 0^S\ 0^N\ 0^P$ are given by

$$0^W = \frac{dW^T}{dt} W \qquad \text{(VII-123)}$$

$$0^S = W^T \frac{dS^T}{dt} S W \qquad \text{(VII-124)}$$

$$0^N = W^T S^T \frac{dN^T}{dt} N S W \qquad \text{(VII-125)}$$

$$0^P = W^T S^T N^T \frac{dP^T}{dt} P N S W \qquad \text{(VII-126)}$$

and are all antisymmetric (Goldstein 1950, pp. 124 ff).

The instantaneous rotation vector ω is given by

$$\omega = \Omega_W + \Omega_S + \Omega_N + \Omega_P \qquad \text{(VII-127)}$$

and consists of the vector sum of the separate rotation rates due to polar motion, spin, nutation, and precession. The components of these separate rotation rate vectors *in the body-fixed coordinate frame* $\hat{e}_1\ \hat{e}_2\ \hat{e}_3$ are given by Equations (VII-119) – (VII-126).

The magnitudes of the rotation rates in Equation (VII-127) are of the order of

$$|\Omega_W| \approx 2 \times 10^{-13}\ \text{rad sec}^{-1}\ \text{maximum}$$

$$|\Omega_S| \approx 7.29 \times 10^{-5}\ \text{rad sec}^{-1}$$

$$|\Omega_N|, |\Omega_P| \approx 7.9 \times 10^{-12}\ \text{rad sec}^{-1}$$

and the vector relationship of Equation (VII-113) is illustrated in Figure VII-3.

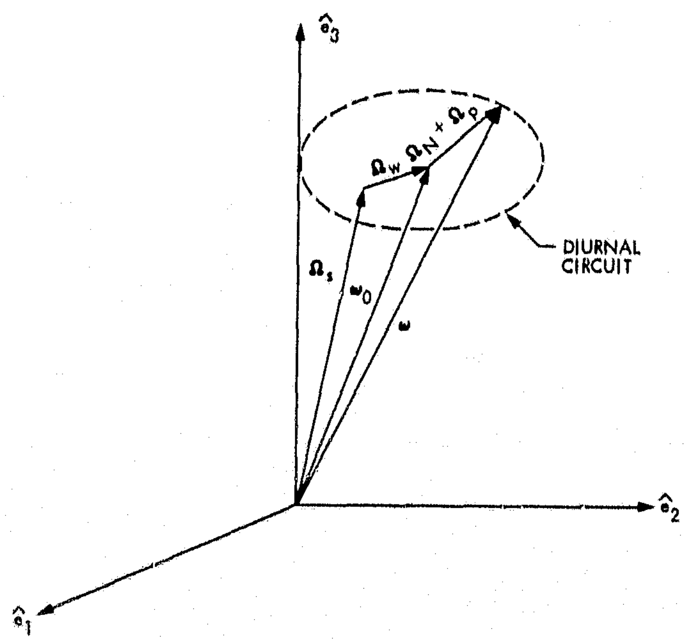

Figure VII-3. The vector relationships pertinent to earth rotation. The instantaneous axis ω executes a diurnal circuit in a body-fixed frame about the Eulerian rotation axis ω_0.

In Woolard's theory of the nutation the vector ω is the reference axis. The small vector $\Omega_N + \Omega_p$ represents the diurnal dynamical variations in latitude.

In the revised theory of the nutation the vector ω_0 is the reference axis where

$$\omega_0 = \omega - \Omega_N - \Omega_p \, . \qquad \text{(VII-128)}$$

From their definition it can be seen that the vectors Ω_N and Ω_p are rotating diurnally in the body-fixed frame and so ω_0 coincides with the mean position of ω when averaged over a diurnal circuit. The angle between ω and ω_0 is of the order of 0."02 arc.

The small vector Ω_W represents the small rotation rate arising as a result of the slowly varying position of ω within the body-fixed frame due to geophysical causes other than external gravitational torques. The angle between the spin vector Ω_S and ω_0 is of the order of $5'' \times 10^{-4}$ arc maximum, and so we see that the reference axis of the revised nutation theory lies very close to the spin vector. This is a convenient choice of reference axis for it coincides with the general conception of the earth's motion as consisting of a rapid spin about an axis which is in turn changing direction *slowly* in both the body-fixed frame and the space-fixed frame.

The new reference axis of the revised nutation theory also has the advantage of removing the diurnal oscillations of the position of the reference equinox on the fixed ecliptic. Since ω_0 coincides with the mean diurnal position of ω it follows that the equator corresponding to ω_0 coincides with the mean diurnal rotation equator or the mean position of the diurnally varying rotation equator of the vector ω (Figure VII-1). If γ_0 is the ascending node on the fixed ecliptic of the fundamental epoch on the mean diurnal rotation equator it is clear that γ_0 coincides with the mean position of γ_r and so can be identified with γ_L. Since the angular momentum vector L and the equinox γ_L are slowly moving in a space-fixed frame, we see that by adopting the new reference axis we remove undesirable diurnal nonuniformities of the order of ±0.12 milliseconds in the definition of apparent sidereal time.

This principal advantage of adopting the new reference axis is that it is an observable axis. In a body-fixed frame it lies at the center of the quasicircular diurnal paths of the stars in the sky. Apart from the slow Eulerian motion the vector ω_0 is a body-fixed vector and its nutation in space will be nearly identical to the figure axis $\hat{e}_3$.

k. The nutation series for a deformable earth. A theory of the nutation for a deformable earth has been developed by Kinoshita et al. (1979) using the Molodenski II earth model (Molodenski 1961) and based on the accurate rigid earth theory of the nutation of Kinoshita (1977) with appropriate modifications to allow for the elastic yielding of the earth. The earth model Molodenski II has a liquid outer core and a solid inner core with radially varying elastic constants.

The modifications to the rigid earth theory are described briefly by Kinoshita et al. (1979) as follows. For each circular component of nutation with angular frequency N_j

$$N_j = \frac{d\Theta_j}{dt} \, , \qquad \text{(VII-129)}$$

where

$$\Theta_j = \sum_{i=1}^{5} K_{ji} \, a_i(t)$$

is the so-called argument of the nutation (Kinoshita 1977), a theoretical ratio, $(a/a_0)_{N_j}$, is computed for the amplitude of the nutation for the deformable earth to the amplitude of the nutation for the rigid earth for each nutation frequency N_j. This ratio is computed by a two-stage formula,

$$\beta = \frac{41.15}{0.2159 - 100 \dfrac{N_j}{N_j - \omega_3}} + 1.7 \qquad \text{(VII-130)}$$

$$\left(\frac{a}{a_0}\right)_{N_j} = 1 + 0.1124 \left[\beta - 4.1\right] \frac{N_j}{\omega_3} \qquad \text{(VII-131)}$$

where ω_3 is the angular rate of earth rotation about the figure axis and the numerical constants in these formulae are derived from the Molodenski II earth model.

Kinoshita et al. (1979) then give the amplitudes of nutations in longitude $\Delta\psi_{N_j}$ and nutations in obliquity $\Delta\theta_{N_j}$ for the Molodenski II "real" earth in terms of the corresponding amplitudes of nutations in longitude $\Delta\psi^R_{N_j}$ and nutations in obliquity $\Delta\theta^R_{N_j}$ for the rigid earth as

$$\Delta\psi_{N_j} \sin\theta = \frac{1}{2}\left\{\left[\left(\frac{a}{a_0}\right)_{N_j} + \left(\frac{a}{a_0}\right)_{-N_j}\right]\Delta\psi^R_{N_j} \sin\theta - \left[\left(\frac{a}{a_0}\right)_{N_j} - \left(\frac{a}{a_0}\right)_{-N_j}\right]\Delta\theta^R_{N_j}\right\} \qquad \text{(VII-132)}$$

$$\Delta\theta_{N_j} = \frac{1}{2}\left\{-\left[\left(\frac{a}{a_0}\right)_{N_j} - \left(\frac{a}{a_0}\right)_{-N_j}\right]\Delta\psi^R_{N_j}\ \sin\theta\right.$$

$$\left. + \left[\left(\frac{a}{a_0}\right)_{N_j} + \left(\frac{a}{a_0}\right)_{-N_j}\right]\Delta\theta^R_{N_j}\right\} \qquad \text{(VII-133)}$$

Using this procedure Kinoshita et al. (1979) have generated the following nutation series for the figure axis $\hat{e}_3$ of a deformable earth whose properties are those of the model "Molodenski II." This series has been recommended to the IAU for adoption at the 17th General Assembly in Montreal in August 1979 and is presented in Table VII-3 below. The fundamental epoch for this series is J2000.0 [JED 2451545.0]. The variable T is measured in Julian centuries from the epoch.

l. Diurnal nutations in the revised theory of the nutation. The diurnal motions of the figure axis in the revised theory of the nutation are essentially identical to the diurnal motions of the figure axis in Woolard's theory of the nutation. This can be seen by combining Figures V-8 and VII-2 and representing the lunisolar motion of Figure VII-2 in the presence of geophysically induced polar motion or Eulerian motion. This is shown below in Figure VII-4.

In this case the body-fixed lunisolar cone is centered on an axis occupying the Eulerian pole position, as can be seen in Figure V-8. The lunisolar body-fixed cone of mean apex angle $2\langle\beta_p\rangle \approx 0''.0178$ arc rolls without slipping on the interior of the space-fixed cone of apex angle $2\,\Theta$, where Θ is the obliquity of the ecliptic. The rotation axis ω occupies the line of contact between the two cones and the motion is retrograde as shown in Figure V-8. The result of this motion is the steady retrograde progression of ω about $\hat{E}_3$ with a period of nearly 26,000 years.

Figure VII-4 indicates clearly that the figure axis $\hat{e}_3$ moves in a prograde sense around the axis ω_0 as a result of retrograde motion of ω around the lunisolar path in the body-fixed frame. The figure axis $\hat{e}_3$ is moving in a prograde sense on a cone of apex angle $2\beta_e$ where $\beta_e = |\bar{m}_e|$.

It follows that the formulae (VII-111) and (VII-112) will serve to describe the diurnal nutation in longitude and obliquity in the revised theory of the nutation as well.

It is also of interest to observe that the reference axis ω_0 is slowly moving in the space-fixed frame as indicated by Figure VII-4. In the absence of Eulerian motion ω_0 coincides with $\hat{e}_3$ in Figure VII-2.

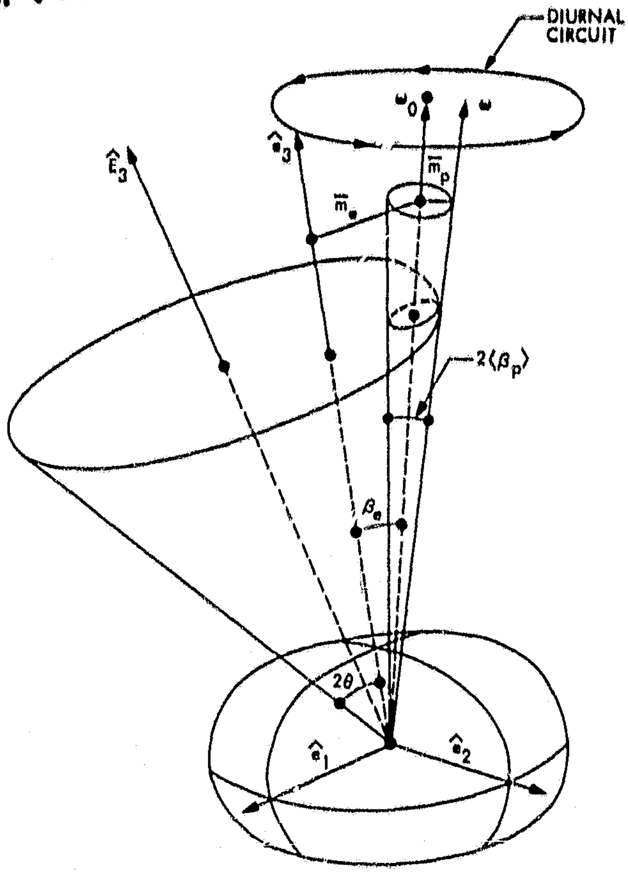

Figure VII-4. The space-fixed diurnal motion of the earth's figure axis $\hat{e}_3$ which occurs as a result of the lunisolar polar motion $\bar{m}_p$ whenever there is a nonvanishing geophysically induced or Eulerian polar motion $\bar{m}_e$.

C. The Effect of Solid Earth Tides on Earth Rotation

1. Tidal perturbations to the earth's inertia tensor. Elementary considerations are sufficient to show (Stacey 1977, pp. 90 ff) that the tidal perturbing force per unit mass $\mathbf{f}_t(\mathbf{r})$ acting throughout the body of the earth due to a celestial body of mass M at a distance R from the earth's center of mass can be expressed as the negative gradient of a tidal potential $U_t(\mathbf{r})$ as

$$\mathbf{f}_t(\mathbf{r}) = -\nabla U_t(\mathbf{r}) \qquad \text{(VII-134)}$$

where

$$U_t(\mathbf{r}) = -\frac{GM}{R}\sum_{n=2}^{\infty}\left(\frac{r}{R}\right)^n P_n(\cos\Theta) \qquad \text{(VII-135)}$$

Table VII-3. Nutation series for the figure axis of the Molodenski II deformable earth model (Kinoshita et al., 1979)

Index J	Period, days	α_1	α_2	α_3	α_4	α_5	Amplitude A_j (0".0001 arc)	Amplitude B_j (0".0001 arc)
		k_{j1}	k_{j2}	k_{j3}	k_{j4}	k_{j5}		
1	6798.4	0	0	0	0	1	−172058−1742.2T	92044+8.9T
2	3399.2	0	0	0	0	2	2063+0.1T	−895+0.5T
3	1305.2	−2	0	2	0	1	46+0.0T	−24+0.0T
4	1095.2	2	0	−2	0	0	11 + 0.0T	0 + 0.0T
5	1615.7	−2	0	2	0	2	−3 + 0.0T	1 + 0.0T
6	3232.9	1	−1	0	−1	0	−3 + 0.0T	0 + 0.0T
7	6786.3	0	−2	2	−2	1	−2 + 0.0T	1 + 0.0T
8	943.2	2	0	−2	0	1	1 + 0.0T	0 + 0.0T
9	182.6	0	0	2	−2	2	−13152 − 1.5T	5719 − 3.1T
10	365.3	0	1	0	0	0	1411 − 3.4T	49 − 0.1T
11	121.7	0	1	2	−2	2	−515 + 1.2T	224 − 0.6T
12	365.2	0	−1	2	−2	2	217 − 0.5T	−95 + 0.3T
13	177.8	0	0	2	−2	1	129 + 0.1T	−70 + 0.0T
14	205.9	2	0	0	−2	0	48 + 0.0T	0 + 0.0T
15	173.3	0	0	2	−2	0	−22 + 0.0T	0 + 0.0T
16	182.6	0	2	0	0	0	17 − 0.1T	0 + 0.0T
17	386.0	0	1	0	0	1	−15 + 0.0T	8 + 0.0T
18	91.3	0	2	2	−2	2	−15 + 0.1T	7 + 0.0T
19	346.6	0	−1	0	0	1	−12 + 0.0T	6 + 0.0T
20	199.8	−2	0	0	2	1	−5 + 0.0T	3 + 0.0T
21	346.6	0	−1	2	−2	1	−5 + 0.0T	3 + 0.0T
22	212.3	2	0	0	−2	1	4 + 0.0T	−2 + 0.0T
23	119.6	0	1	2	−2	1	4 + 0.0T	−2 + 0.0T
24	411.8	1	0	0	−1	0	−4 + 0.0T	0 + 0.0T
25	131.7	2	1	0	−2	0	1 + 0.0T	0 + 0.0T
26	169.0	0	0	−2	2	1	1 + 0.0T	0 + 0.0T
27	329.8	0	1	−2	2	0	−1 + 0.0T	0 + 0.0T
28	409.2	0	1	0	0	2	1 + 0.0T	0 + 0.0T
29	388.3	−1	0	0	1	1	1 + 0.0T	0 + 0.0T
30	117.5	0	1	2	−2	0	−1 + 0.0T	0 + 0.0T
31	13.7	0	0	2	0	2	−2260 − 0.2T	972 − 0.5T
32	27.5	1	0	0	0	0	709 + 0.1T	−7 + 0.0T
33	13.6	0	0	2	0	1	−384 − 0.4T	199 + 0.0T
34	9.1	1	0	2	0	2	−299 + 0.0T	128 − 0.1T
35	31.8	1	0	0	−2	0	−157 + 0.0T	−1 + 0.0T
36	27.1	−1	0	2	0	2	123 + 0.0T	−53 + 0.0T
37	14.8	0	0	0	2	0	63 + 0.0T	−2 + 0.0T
38	27.7	1	0	0	0	1	63 + 0.1T	−33 + 0.0T
39	27.4	−1	0	0	0	1	−58 − 0.1T	32 + 0.0T
40	9.6	−1	0	2	2	2	−59 + 0.0T	25 + 0.0T
41	9.1	1	0	2	0	1	−51 + 0.0T	26 + 0.0T
42	7.1	0	0	2	2	2	−38 + 0.0T	16 + 0.0T
43	13.8	2	0	0	0	0	29 + 0.0T	−1 + 0.0T
44	23.9	1	0	2	−2	2	29 + 0.0T	−12 + 0.0T
45	6.9	2	0	2	0	2	−31 + 0.0T	13 + 0.0T
46	13.6	0	0	2	0	0	26 + 0.0T	−1 + 0.0T
47	27.0	−1	0	2	0	1	21 + 0.0T	−10 + 0.0T
48	32.0	−1	0	0	2	1	15 + 0.0T	−8 + 0.0T
49	31.7	1	0	0	−2	1	−13 + 0.0T	7 + 0.0T
50	9.5	−1	0	2	2	1	−10 + 0.0T	5 + 0.0T
51	34.8	1	1	0	−2	1	−7 + 0.0T	0 + 0.0T
52	13.2	0	1	2	0	2	7 + 0.0T	−3 + 0.0T
53	14.2	0	−1	2	0	2	−7 + 0.0T	3 + 0.0T
54	5.6	1	0	2	2	2	−8 + 0.0T	3 + 0.0T
55	9.6	1	0	0	2	0	6 + 0.0T	0 + 0.0T
56	12.8	2	0	2	−2	2	6 + 0.0T	−3 + 0.0T
57	14.8	0	0	0	2	1	−6 + 0.0T	3 + 0.0T

ORIGINAL PAGE IS
OF POOR QUALITY

Table VII-3. Nutation series for the figure axis of the Kolodenski II deformable earth model (Kinoshita et al., 1979) (Continued)

Index J	Period, days	α_1 k_{j1}	α_2 k_{j2}	α_3 k_{j3}	α_4 k_{j4}	α_5 k_{j5}	Amplitude A_j (0".0001 arc)	Amplitude B_j (0".0001 arc)
58	7.1	0	0	2	2	1	$-7+0.0T$	$3+0.0T$
59	23.9	1	0	2	-2	1	$6+0.0T$	$-3+0.0T$
60	14.7	0	0	0	-2	1	$-5+0.0T$	$3+0.0T$
61	29.8	1	-1	0	0	0	$5+0.0T$	$0+0.0T$
62	6.8	2	0	2	0	1	$-5+0.0T$	$3+0.0T$
63	15.4	0	1	0	-2	0	$-4+0.0T$	$0+0.0T$
64	26.9	1	0	-2	0	0	$4+0.0T$	$0+0.0T$
65	29.5	0	0	0	1	0	$-4+0.0T$	$0+0.0T$
66	25.6	1	1	0	0	0	$-3+0.0T$	$0+0.0T$
67	9.1	1	0	2	0	0	$3+0.0T$	$0+0.0T$
68	9.4	1	-1	2	0	2	$-3+0.0T$	$1+0.0T$
69	9.8	-1	-1	2	2	2	$-3+0.0T$	$1+0.0T$
70	13.8	-2	0	0	0	1	$-2+0.0T$	$1+0.0T$
71	5.5	3	0	2	0	2	$-3+0.0T$	$1+0.0T$
72	7.2	0	-1	2	2	2	$-3+0.0T$	$1+0.0T$
73	8.9	1	1	2	0	2	$2+0.0T$	$-1+0.0T$
74	32.6	-1	0	2	-2	1	$-2+0.0T$	$1+0.0T$
75	13.8	2	0	0	0	1	$2+0.0T$	$-1+0.0T$
76	27.8	1	0	0	0	2	$-2+0.0T$	$1+0.0T$
77	9.2	3	0	0	0	0	$2+0.0T$	$0+0.0T$
78	9.3	0	0	2	1	2	$2+0.0T$	$-1+0.0T$
79	27.3	-1	0	0	0	2	$1+0.0T$	$-1+0.0T$
80	10.1	1	0	0	-4	0	$-1+0.0T$	$0+0.0T$
81	14.6	-2	0	2	2	2	$1+0.0T$	$-1+0.0T$
82	5.8	-1	0	2	4	2	$-2+0.0T$	$1+0.0T$
83	15.9	2	0	0	-4	0	$-1+0.0T$	$0+0.0T$
84	22.5	1	1	2	-2	2	$1+0.0T$	$-1+0.0T$
85	5.6	1	0	2	2	1	$-1+0.0T$	$1+0.0T$
86	7.3	-2	0	2	4	2	$-1+0.0T$	$1+0.0T$
87	9.1	-1	0	4	0	2	$1+0.0T$	$0+0.0T$
88	29.3	1	-1	0	-2	0	$1+0.0T$	$0+0.0T$
89	12.8	2	0	2	-2	1	$1+0.0T$	$-1+0.0T$
90	4.7	2	0	2	2	2	$-1+0.0T$	$0+0.0T$
91	9.6	1	0	0	2	1	$-1+0.0T$	$0+0.0T$
92	12.7	0	0	4	-2	2	$1+0.0T$	$0+0.0T$
93	8.8	3	0	2	-2	2	$1+0.0T$	$0+0.0T$
94	23.8	1	0	2	-2	0	$-1+0.0T$	$0+0.0T$
95	13.1	0	1	2	0	1	$1+0.0T$	$0+0.0T$
96	35.0	-1	-1	0	2	1	$1+0.0T$	$0+0.0T$
97	13.6	0	0	-2	0	1	$-1+0.0T$	$0+0.0T$
98	25.4	0	0	2	-1	2	$-1+0.0T$	$0+0.0T$
99	14.2	0	1	0	2	0	$-1+0.0T$	$0+0.0T$
100	9.5	1	0	-2	-2	0	$-1+0.0T$	$0+0.0T$
101	14.2	0	-1	2	0	1	$-1+0.0T$	$0+0.0T$
102	34.7	1	1	0	-2	1	$-1+0.0T$	$0+0.0T$
103	32.8	1	0	-2	2	0	$-1+0.0T$	$0+0.0T$
104	7.1	2	0	0	2	0	$1+0.0T$	$0+0.0T$
105	4.8	0	0	2	4	2	$-1+0.0T$	$0+0.0T$
106	27.3	0	1	0	1	0	$1+0.0T$	$0+0.0T$

and where

(1) $r = |\mathbf{r}| \leq a$.

(2) Θ is the zenith angle of the celestial body as seen by the observer at **r**.

(3) P_n $(\cos \Theta)$ is the Legendre polynomial of degree n.

(4) G is the gravitational constant.

Figure VII-5 illustrates the geometry of the situation. The geographic coordinates of the observer are θ_0 λ_0 and the

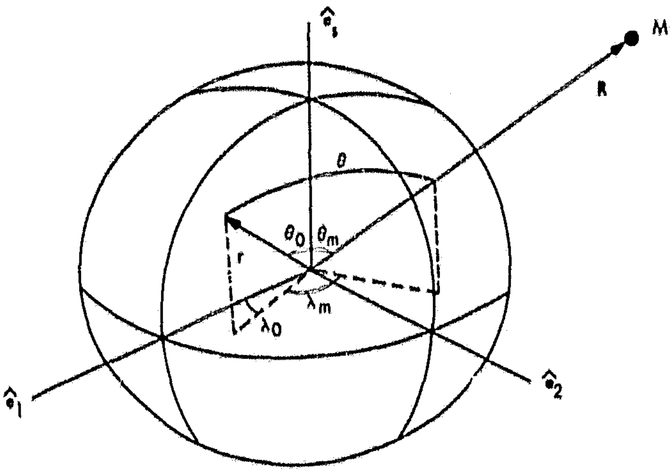

Figure VII-5. The definitions of the geometrical quantities used in the development of the theory of the solid earth tides.

geographic coordinates of a line joining the earth's center of mass and the celestial body of mass M are θ_m λ_m.

The addition theorem for spherical harmonics allows us to write the expression for the tidal potential $U_t(\mathbf{r})$ in terms of the geographic coordinates θ_0 λ_0, θ_n λ_m,

$$P_n(\cos \Theta) = P_n(\cos \Theta_m) P_n(\cos \Theta_0) + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos \Theta_m) \cdot P_n^m(\cos \Theta_0) \cos m(\lambda_m - \lambda_0) \quad , \quad r \leq a \ . \qquad \text{(VII-136)}$$

For the sun and the moon $(r/R_\odot) << 1$ and $(r/R_☾) << 1$ and sufficient accuracy is usually achieved in Equation (VII-135) by truncating the sum at $n = 2$.

$$U_t(\mathbf{r}) = - \frac{GMr^2}{R^3} P_2(\cos \Theta) \ , \ r \leq a \ , \qquad \text{(VII-137)}$$

which in geographic coordinates θ_0 λ_0, θ_m λ_m, becomes

$$U_t(\mathbf{r}) = - \frac{GMr^2}{R^3} \left[P_2(\cos \theta_m) P_2(\cos \theta_0) + 2 \sum_{m=1}^{2} \frac{(2-m)!}{(2+m)!} P_2^m(\cos \theta_m) P_2^m(\cos \theta_0) \cos m(\lambda_m - \lambda_0) \right] , \ r \leq a \ . \qquad \text{(VII-138)}$$

For a celestial body such as the sun or the moon whose celestial coordinates are changing relatively slowly we see that the argument $m(\lambda_m - \lambda_0)$ is periodic in an interval of approximately m^{-1} days. The term $m = 1$ gives rise to the diurnal tidal variations and the term $m = 2$ gives rise to the semidiurnal tidal variations.

The tidal deformations of the earth cause perturbations r_{ij}^{tide} to the elements of the earth's inertia tensor which can be calculated using MacCullagh's formula. Written out in full Equation (VII-138) becomes

$$U_t(\mathbf{r}) = - \frac{GMr^2}{R^3} \left[P_2(\cos \theta_m) P_2(\cos \theta_0) + \frac{2}{6} P_2^1(\cos \theta_m) P_2^1(\cos \theta_0) \cos(\lambda_m - \lambda_0) + \frac{2}{24} P_2^2(\cos \theta_m) P_2^2(\cos \theta_0) \cos 2(\lambda_m - \lambda_0) \right] ,$$

$$r \leq a \ . \qquad \text{(VII-139)}$$

This expression can be rewritten

$$U_t(\mathbf{r}) = - \frac{GMr^2}{R^3} \left[P_2^0(\cos \theta_m) P_2^0(\cos \theta_0) + \frac{1}{3} P_2^1(\cos \theta_m) P_2^1(\cos \theta_0) \cos \lambda_m \cos \lambda_0 + \frac{1}{3} P_2^1(\cos \theta_m) P_2^1(\cos \theta_0) \sin \lambda_m \sin \lambda_0 + \frac{1}{12} P_2^2(\cos \theta_m) P_2^2(\cos \theta_0) \cos 2 \lambda_m \cos 2 \lambda_0 + \frac{1}{12} P_2^2(\cos \theta_m) P_2^2(\cos \theta_0) \sin 2 \lambda_m \sin 2 \lambda_0 \right] ,$$

$$r \leq a \ , \qquad \text{(VII-140)}$$

which in turn can be cast in the form

$$U_t(\mathbf{r}) = \sum_{m=0}^{2} \left(\frac{r}{R} \right)^2 P_2^m(\cos \theta_0) \left[C_2^m \cos m \lambda_0 + S_2^m \sin m \lambda_0 \right] , \ r \leq a \ , \qquad \text{(VII-141)}$$

ORIGINAL PAGE IS
OF POOR QUALITY

where

$$C_2^0 = -\frac{GMr^2}{R^3} P_2^0(\cos\theta_m)$$

$$S_2^0 = 0$$

$$C_2^1 = -\frac{GMr^2}{3R^3} P_2^1(\cos\theta_m)\cos\lambda_m$$

$$S_2^1 = -\frac{GMr^2}{3R^3} P_2^1(\cos\theta_m)\sin\lambda_m$$

$$C_2^2 = -\frac{GMr^2}{12R^3} P_2^2(\cos\theta_m)\cos 2\lambda_m$$

$$S_2^2 = -\frac{GMr^2}{12R^3} P_2^2(\cos\theta_m)\sin 2\lambda_m \ .$$

(VII-142)

It then follows from Section VI-2 and specifically from Equations (VI-74) – (VI-78) that

$$r_{12}^{tide} = -\frac{k_2Ma^5}{6R^3} P_2^2(\cos\theta_m)\sin 2\lambda_m$$

(VII-143)

$$2r_{13}^{tide} = \frac{2k_2Ma^5}{3R^3} P_2^1(\cos\theta_m)\cos\lambda_m$$

(VII-144)

$$2r_{23}^{tide} = \frac{2k_2Ma^5}{3R^3} P_2^1(\cos\theta_m)\sin\lambda_m$$

(VII-145)

$$\frac{1}{2}\left[r_{22}^{tide} - r_{11}^{tide}\right] = \frac{k_2Ma^5}{6R^3} P_2^2(\cos\theta_m)\cos 2\lambda m$$

(VII-146)

$$r_{11}^{tide} + r_{22}^{tide} - 2r_{33}^{tide} = \frac{2k_2Ma^5}{R^3} P_2^0(\cos\theta_m) \ .$$

(VII-147)

The above five equations in six unknowns are supplemental with the additional equation

$$r_{11}^{tide} + r_{22}^{tide} + r_{33}^{tide} = \delta(T_r\widetilde{I})^{tide} \qquad \text{(VII-148)}$$

where $\delta(T_r\widetilde{I})^{tide}$ is the tidally induced variations in the trace of the inertia tensor of the earth.

It has been shown by Darwin (1910) that for an incompressible earth in which all possible deformation fields $\mathbf{u}(\mathbf{r})$ have the property that $\nabla \cdot \mathbf{u} = 0$, the trace of the inertia tensor is preserved under earth deformations. In his analysis of the effects of earth tides on the earth's rotation Woolard (1959) assumed that the earth was incompressible in order to use the property that $\delta(T_r\widetilde{I})^{tide} = 0$ in his solution. Later Rochester and Smylie (1974) showed that the value of $T_r\widetilde{I}$ was preserved under a much wider class of deformation fields than those for which $\nabla \cdot \mathbf{u} = 0$. In particular they show that even on a compressible earth for which $\nabla \cdot \mathbf{u} \neq 0$ the deformation field arising from the effects of tidal perturbations preserves the value of $T_r\widetilde{I}$. Following Rochester and Smylie we can, without any restrictive assumptions, take

$$r_{11}^{tide} + r_{22}^{tide} + r_{33}^{tide} = \delta(T_r\widetilde{I})^{tide} = 0 \ .$$

(VII-149)

Equations (VII-143) – (VII-149) have as their solution

$$r_{12}^{tide} = -\frac{k_2Ma^5}{6R^3} P_2^2(\cos\theta_m)\sin 2\lambda m$$

(VII-150)

$$r_{13}^{tide} = \frac{k_2Ma^5}{3R^3} P_2^1(\cos\theta_m)\cos\lambda m \qquad \text{(VII-151)}$$

$$r_{23}^{tide} = \frac{k_2Ma^5}{3R^3} P_2^1(\cos\theta_m)\sin\lambda m \qquad \text{(VII-152)}$$

$$r_{11}^{tide} = \frac{k_2Ma^5}{6R^3}\left[2P_2^0(\cos\theta_m) + P_2^2(\cos\theta_m)\cos 2\lambda m\right]$$

(VII-153)

$$r_{22}^{tide} = \frac{k_2Ma^5}{6R^3}\left[2P_2^0(\cos\theta_m) - P_2^2(\cos\theta_m)\cos 2\lambda m\right]$$

(VII-154)

ORIGINAL PAGE IS
OF POOR QUALITY

$$r_{33}^{tide} = -\frac{2k_2 M a^5}{3R^3} P_2^0(\cos\theta_m) . \qquad \text{(VII-155)}$$

For the sun and the moon the angles λ_m, namely $\lambda_\odot, \lambda_☾$, are varying diurnally with a period near 1 day. Hence r_{12}^{tide}, r_{13}^{tide}, r_{23}^{tide} are diurnally varying products of inertia which are periodic in about 1 day. The moments of inertia r_{11}^{tide}, r_{22}^{tide}, r_{33}^{tide} have long period components which depend on the angles θ_m, namely $\theta_\odot, \theta_☾$, which are in fact given by

$$\begin{aligned} \theta_\odot &= 90^\circ - \delta_\odot \\ \theta_☾ &= 90^\circ - \delta_☾ \end{aligned} \qquad \text{(VII-156)}$$

where $\delta_\odot$, $\delta_☾$ are the declinations of the sun and moon respectively.

2. The effect of the solid earth tides on UT1. It is relatively easy to show that in the absence of internal dissipation which introduces phase lags in the tidal response of the earth the tidal forces exert no net torque on the earth.

The net torque $\mathbf{N}_t$ on the earth due to the tidal forces $\mathbf{f}_t(\mathbf{r})$ is

$$\mathbf{N}_t = \int_V \mathbf{r} \times \mathbf{f}_t(\mathbf{r})\, dV \qquad \text{(VII-157)}$$

where the integral is taken throughout the volume of the earth. If the vector field $\mathbf{f}_t(\mathbf{r})$ has components

$$\mathbf{f}_t(\mathbf{r}) = f_{tr}(\mathbf{r})\hat{r} + f_{t\theta}(\mathbf{r})\hat{\theta} + f_{t\lambda}(\mathbf{r})\hat{\lambda} \qquad \text{(VII-158)}$$

then

$$\mathbf{N}_t = \int_V [r f_{t\theta}(\mathbf{r})\hat{\lambda} - r f_{t\lambda}(\mathbf{r})\hat{\theta}]\, dV \qquad \text{(VII-159)}$$

Using Equation (VII-134) in Equation (VII-159) this becomes

$$\mathbf{N}_t = \int_0^a dr \int_0^{2\pi} d\lambda \int_0^{\pi} d\theta\, r^2 \sin\theta \left[-\frac{\partial U_t}{\partial \theta}\hat{\lambda} + \frac{1}{\sin\theta}\frac{\partial U_t}{\partial \lambda}\hat{\theta}\right] \qquad \text{(VII-160)}$$

Using Equation (VII-140) in Equation (VII-160) and integrating over the variables r, λ gives

$$\mathbf{N}_t = \frac{2\pi}{5}\frac{GMa^5}{R^3} P_2^0(\cos\theta_m) \int_0^{\pi} \sin\theta \frac{dP_2^0(\cos\theta)}{d\theta}\, d\theta\, \hat{\lambda} \qquad \text{(VII-161)}$$

The integral in the above equation vanishes with the result that

$$\mathbf{N}_t = 0 \qquad \text{(VII-162)}$$

and so in the absence of dissipation the angular momentum $\mathbf{L}$ of the earth is conserved under the action of the tidal forces.

If the unperturbed earth is described by an inertia tensor $\tilde{I}_0$ where

$$\tilde{I}_0 = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix} \qquad \text{(VII-163)}$$

and a rotation vector ω_0 where

$$\omega_0 = \Omega \hat{e}_3 \qquad \text{(VII-164)}$$

then the tidally perturbed earth is described by an inertia tensor $\tilde{I}$ where

$$\tilde{I} = \begin{bmatrix} A + r_{11}^{tide} & r_{12}^{tide} & r_{13}^{tide} \\ r_{21}^{tide} & A + r_{22}^{tide} & r_{23}^{tide} \\ r_{31}^{tide} & r_{32}^{tide} & C + r_{33}^{tide} \end{bmatrix} \qquad \text{(VII-165)}$$

and a rotation vector ω where

$$\omega = \Omega\, [m_1^{tide}\hat{e}_1 + m_2^{tide}\hat{e}_2 + (1 + m_3^{tide})\hat{e}_3] \qquad \text{(VII-166)}$$

Conservation of angular momentum under the influence of the tides gives

$$\tilde{I}_0 \cdot \omega_0 = \tilde{I} \cdot \omega \qquad \text{(VII-167)}$$

which gives rise to the three equations

$$\Omega\,[(A + r_{11}^{tide})\, m_1^{tide} + r_{12}^{tide}\, m_2^{tide} + r_{13}^{tide}\,(1 + m_3^{tide})] = 0 \qquad \text{(VII-168)}$$

$$\Omega\,[r_{21}^{tide}\, m_1^{tide} + (A + r_{22}^{tide})\, m_2^{tide} + r_{23}^{tide}\,(1 + m_3^{tide})] = 0 \qquad \text{(VII-169)}$$

$$\Omega\,[r_{31}^{tide}\, m_1^{tide} + r_{32}^{tide}\, m_2^{tide} + (C + r_{33}^{tide})\,(1 + m_3^{tide})] = \Omega C \qquad \text{(VII-170)}$$

for the three unknowns m_1^{tide}, m_2^{tide}, m_3^{tide}. To first order in small quantities Equations (VII-168) – (VII-170) reduce to

$$\Omega\,(A m_1^{tide} + r_{13}^{tide}) = 0 \qquad \text{(VII-171)}$$

$$\Omega\,(A m_2^{tide} + r_{23}^{tide}) = 0 \qquad \text{(VII-172)}$$

$$\Omega\,(C + C m_3^{tide} + r_{33}^{tide}) = \Omega C\,. \qquad \text{(VII-173)}$$

The first-order solutions for the unknowns m_1^{tide}, m_2^{tide}, m_3^{tide} are then

$$m_1^{tide} = -\frac{r_{13}^{tide}}{A} \qquad \text{(VII-174)}$$

$$m_2^{tide} = -\frac{r_{23}^{tide}}{A} \qquad \text{(VII-175)}$$

$$m_3^{tide} = -\frac{r_{33}^{tide}}{C} \qquad \text{(VII-176)}$$

The tidal variations in UT1 are dependent on the history of the quantity m_3^{tide}. It is customary to express the variations in the earth's rotation rate by parameter $\tau(t)$ where

$$\tau(t) = -\int_{t_0}^{t} m_3(t')\, dt' \ \text{seconds} \qquad \text{(VII-177)}$$

is a measure of the amount by which the earth's rotation *lags behind* that of a hypothetical uniformly rotating earth. The tidal contribution to $\tau(t)$ denoted $\tau^{tide}(t)$ is

$$\tau^{tide}(t) = -\int_{t_0}^{t} m_3^{tide}(t')\, dt' \ \text{seconds}\,. \qquad \text{(VII-178)}$$

From Equations (VII-155), (VII-176) (VII-178) we have

$$\tau^{tide}(t) = -\int_{t_0}^{t} \frac{2\, k_2\, M a^5}{3\, C R^3(t')}\, P_2^0(\cos\theta_m(t'))\, dt' \qquad \text{(VII-179)}$$

Now

$$P_2^0(\cos\theta_m) = \frac{1}{2}\,(3\cos^2\theta_m - 1) \qquad \text{(VII-180)}$$

and

$$\theta_m = 90^\circ - \delta_m \qquad \text{(VII-181)}$$

where δ_m is the declination of the celestial body of mass M. It follows that Equation (VII-182) can be written

$$\tau^{tide}(t) = \int_{t_0}^{t} \frac{k_2\, M a^5}{3\, C R^3(t')}\, [1 - 3\sin^2\delta_m(t')]\, dt' \qquad \text{(VII-182)}$$

Woolard (1959) integrated the above Equation (VII-182) for the combined effects of the sun and the moon.

$$\tau^{tide}(t) = \frac{k_2 a^5}{3C} \int_{t_0}^{t} \left\{ \frac{M_\odot}{R_\odot^3(t')}\,[1 - 3\sin^2\delta_\odot(t')] + \frac{M_☾}{R_☾^3(t')}\,[1 - 3\sin^2\delta_☾(t')] \right\} dt' \qquad \text{(VII-183)}$$

to obtain an expression for $\tau^{tide}(t)$ which includes both the lunar and the solar tides.

Woolard's expression for $\tau^{tide}(t)$ is given in the form

$$\tau^{tide}(t) = k_2 \sum_{j=1}^{20} A_j \sin \phi_j \text{ milliseconds,} \quad \text{(VII-184)}$$

where the epoch t_0 is 1900.0. Equation (VII-184) can be evaluated from Table VII-4, taken from Woolard's work. The tabulated amplitudes are in milliseconds and the value of the second degree Love number k_2 can be taken to be

$$k_2 \cong 0.29 \ .$$

Table VII-4. The elements used in Equation (VII-187) to generate the theoretical contribution to τ^{tide} = (UT1-UTC)^{tide} due to the effects of the solid earth tides. (After Woolard, 1959).

j	A_j, msec	ϕ_j	Period, days
1	0.32	$2L+g$	9.1
2	0.13	$2L+g-\Omega$	9.1
3	2.47	$2L$	13.7
4	1.02	$2L-\Omega$	13.7
5	0.10	$2L-2\Omega$	13.7
6	0.11	$2g$	13.8
7	0.23	$2L-2\odot$	14.8
8	2.63	g	27.6
9	0.17	$g+\Omega$	27.6
10	0.17	$g-\Omega$	27.6
11	0.14	$2L-g$	27.1
12	0.06	$2L-g-\Omega$	27.1
13	0.58	$2L-g-2\odot$	31.8
14	0.60	$2\odot+g'$	122
15	15.29	$2\odot$	183
16	0.37	$2\odot-\Omega$	365
17	4.88	g'	365
18	0.23	$2\odot-g'$	6793.7 (18.6 year)
19	515.0	Ω	3396.9 (9.3 year)
20	2.7	2Ω	

The arguments ϕ_j of the sine function are defined by taking:

L mean celestial longitude of the moon

g mean anomaly of the moon

Ω mean celestial longitude of the moon's ascending node

$\odot$ mean celestial longitude of the sun

g mean anomaly of the sun

3. **The effect of the solid earth tides on polar motion.** The effect of the solid earth tides on polar motion can be deduced directly from Equations (VII-151) (VII-152) and (VII-174) (VII-175), which taken together give

$$m_1^{tide} = -k_2 \frac{Ma^5}{3AR^3} P_2^1 (\cos \theta_m) \cos \lambda_m \quad \text{(VII-185)}$$

$$m_2^{tide} = -k_2 \frac{Ma^5}{3AR^3} P_2^1 (\cos \theta_m) \sin \lambda_m \quad \text{(VII-186)}$$

where M is the mass of the perturbing body and R is its geocentric distance and where θ_m λ_m are the geographic coordinates of the position vector **R**.

Due to earth rotation the argument λ_m decreases by 2π in slightly more than one sidereal day, allowing for the eastward progression of the tide-inducing body, be it the sun or the moon.

The complex quantity $\tilde{m}^{tide}$ given by

$$\tilde{m}^{tide} = m_1^{tide} + i\, m_2^{tide} \quad \text{(VII-187)}$$

defines the angular motion of the tidal perturbations to the rotation axis in the body-fixed frame. Equations (VII-185) (VII-186) (VII-187) together gives

$$\tilde{m}^{tide} = -k_2 \frac{Ma^5}{3AR^3} P_2^1 (\cos \theta_m)\, e^{i\lambda_m} \quad \text{(VII-188)}$$

and since λ_m continuously decreases (moves continuously westward) we see that $\tilde{m}^{tide}$ is a retrograde motion of the rotation axis as shown in Figure VII-6.

Using the formula P_2^1 $(\cos \theta_m) = \cos \theta_m \sin \theta_m$ and recognizing that the coordinate θ_m is related to the declination of the celestial body δ_m by

$$\theta_m = 90° - \delta_m$$

we can rewrite Equation (VII-191) as

$$\tilde{m}^{tide} = -k_2 \frac{Ma^5}{3AR^3} \cos \delta_m \sin \delta_m\, e^{i\lambda_m} \ . \quad \text{(VII-189)}$$

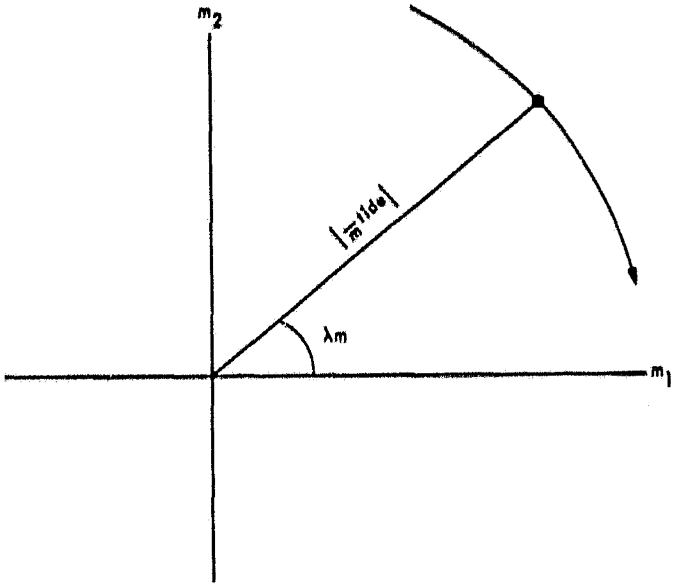

Figure VII-8. The retrograde polar motion $\bar{m}^{tide}$ resulting from tidal perturbations to the earth's inertia tensor.

A general formula which includes the tidal perturbations to the rotation axis of both the sun and the moon can be written directly from Equation (VII-189) as

$$\bar{m}^{tide} = -\frac{k_2 a^5}{A}\left\{\frac{M_\odot}{R_\odot^3}\cos\delta_\odot \sin\delta_\odot\, e^{i\lambda_\odot} + \frac{M_{\leftmoon}}{R_{\leftmoon}^3}\cos\delta_{\leftmoon}\sin\delta_{\leftmoon}\, e^{i\lambda_{\leftmoon}}\right\} \qquad \text{(VII-190)}$$

where:

(1) $M_\odot$ $M_{\leftmoon}$ are the masses of the sun and moon respectively.

(2) $R_\odot$ $R_{\leftmoon}$ are the geocentric distances to the sun and moon respectively.

(3) $\delta_\odot$ $\delta_{\leftmoon}$ are the declinations of the sun and moon respectively.

(4) $\lambda_\odot = \alpha_\odot -$ G.A.S.T. and $\lambda_{\leftmoon} = \alpha_{\leftmoon} -$ G.A.S.T. where $\alpha_\odot$ $\alpha_{\leftmoon}$ are the right ascensions of the sun and moon respectively and GAST refers to Greenwich Apparent Sidereal Time.

The maximum combined amplitude of polar motion due to the lunisolar solid earth tides is of the order of $6'' \times 10^{-3}$ arc or 18 cm of motion. The motion is retrograde with a nearly diurnal period.

D. The Geophysical Excitation Functions for Polar Motion and UT1 Variations

As pointed out previously in this work one of the important objectives of obtaining precise measurements of variations in UT1 and polar motion on the earth is to learn more about the global geophysical processes which are responsible for the variations. In order to carry out such a program it will be necessary to use existing geophysical knowledge to generate realistic theoretical rotation excitation functions whose predicted consequences for UT1 and polar motion variations can be compared against observed data. Such a program will require as inputs, in addition to precise polar motion and UT1 observations, considerable global synoptic data concerning the state of the earth's oceans and atmosphere as well as information concerning the internal state of the earth.

Our ability to generate models for the oceans and atmosphere of the earth has advanced dramatically in the last decade with the development of earth satellites capable of monitoring the global state of the earth's atmosphere and oceans. In addition the ability to theoretically model a number of important internal processes in the earth such as earthquake faulting and to accurately deduce their effect on the earth's inertia tensor for the case of a realistic earth has advanced considerably in the past decade. These trends will no doubt continue into the future and the forthcoming data will provide the basis for generating a priori rotational excitation functions which could be refined by the precise UT1 and polar motion measurements.

Our analysis of the geophysical excitation functions begins by decomposing the density field of the earth $\rho(\mathbf{r},t)$ into a mean density $\rho^0(\mathbf{r})$ and a geophysical perturbation to the mean $\Delta\rho(\mathbf{r},t)$. The position vector $\mathbf{r}$ refers to a fixed position in the rotating geophysical coordinate frame

$$\rho(\mathbf{r},t) = \rho^0(\mathbf{r}) + \Delta\rho(\mathbf{r},t) \qquad \text{(VII-191)}$$

The inertia tensor of the earth $\tilde{I}$ is similarly decomposed according to Equation (IV-5) into a mean inertia tensor $\tilde{I}^0$ and a geophysical perturbation $\tilde{r}$

$$\tilde{I} = \tilde{I}^0 + \tilde{r} \qquad \text{(VII-192)}$$

where

$$\tilde{I} = \int_V \rho(\mathbf{r},t)\,[r^2\,\tilde{I} - \mathbf{r}\,\mathbf{r}]\,dV = \begin{bmatrix} A + r_{11} & r_{12} & r_{13} \\ r_{21} & A + r_{22} & r_{23} \\ r_{31} & r_{32} & C + r_{33} \end{bmatrix} \qquad \text{(VII-193)}$$

ORIGINAL PAGE IS
OF POOR QUALITY

$$\mathcal{T}^0 = \int_V \rho^0(\mathbf{r})\,[r^2\,\mathcal{T} - \mathbf{r}\,\mathbf{r}]\,dV = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix} \tag{VII-194}$$

$$\widetilde{\mathcal{T}} = \int_V \Delta\rho(\mathbf{r},t)\,[r^2\,\mathcal{T} - \mathbf{r}\,\mathbf{r}]\,dV = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}. \tag{VII-195}$$

From Equation (IV-22) we see that the geophysical excitation function for polar motion $\tilde{\epsilon}(t)$ is given by

$$\tilde{\epsilon}(t) = \frac{1}{A\Omega}\left[\widetilde{N} - \Omega\frac{d\tilde{r}}{dt} - \frac{d\tilde{h}}{dt} - i\,(\Omega^2\,\tilde{r} + \Omega\,\tilde{h})\right] \tag{VII-196}$$

where

$$\begin{aligned} \tilde{r} &= r_{13} + i\,r_{23} \\ \tilde{h} &= h_1 + i\,h_2 \\ \widetilde{N} &= N_1 + i\,N_2 \end{aligned} \tag{VII-197}$$

and where

$$\begin{aligned} \mathbf{N} &= N_1\hat{e}_1 + N_2\hat{e}_2 + N_3\hat{e}_3 \\ \mathbf{h} &= h_1\hat{e}_1 + h_2\hat{e}_2 + h_3\hat{e}_3 \end{aligned} \tag{VII-198}$$

are the external torque and relative angular momentum respectively measured in the rotating geophysical coordinate frame. The relative angular momentum $\mathbf{h}$ is given by Equation (III-52) as

$$\mathbf{h}(\mathbf{r},t) = \int_V \rho(\mathbf{r},t)\,[\mathbf{r} \times \mathbf{v}(\mathbf{r},t)]\,dV \tag{VII-199}$$

where $\mathbf{v}(\mathbf{r},t)$ is the velocity of the material of density $\rho(\mathbf{r},t)$ relative to the rotating geophysical coordinate frame.

From Equation (IV-30) we see that the geophysical excitation function for UT1 is

$$\epsilon_3(t) = \frac{1}{C\Omega}\left\{\int_0^t N_3(t')\,dt' - \Omega\,r_{33} - h_3\right\} \tag{VII-200}$$

and so, apart from the question of the externally applied torques $\mathbf{N}$, the excitation of polar motion and UT1 fluctuations depend entirely on the tensor $\widetilde{\mathcal{T}}$ and the vector $\mathbf{h}$ and their time derivatives taken in the rotating frame.

In order to simplify our dynamical theory (Equation (III-10)) we have chosen to define "the earth" to include its oceans and atmosphere and according to this formulation of its rotational dynamics the motion of the oceanic currents and atmospheric circulation by being part of "the earth" are incapable of exerting an "external" torque on the earth through some sort of viscous boundary layer interaction with the solid surface. The effect of the oceanic and atmospheric circulation on the earth's rotational dynamics is entirely included in the relative angular momentum term $\mathbf{h}$.

At the cost of complicating the dynamical description of its rotation we could have defined "the earth" to exclude the oceans and atmosphere. In this case the motion of the oceans and atmosphere do exert an external torque on "the earth" through viscous boundary layer interaction. In addition to being dynamically disadvantageous this formulation of the problem of the earth's rotational dynamics involves the poorly understood phenomenon of the boundary layer interaction of the oceans and atmosphere with the solid earth and with each other by requiring that we model this process in order to express the oceanic and atmospheric torques on "the earth" in terms of their respective velocity fields. While these two approaches to the problem are formally equivalent, the definition of "the earth" to include the oceans and atmosphere is dynamically simpler and has been shown (Lambeck and Cazenave, 1973) to be capable of a more accurate treatment of the effects of oceanic and atmospheric circulation. Similar remarks could presumably be made for the fluid motions of the earth's fluid core and its effect on the earth's rotational dynamics is complicated by the fact that no direct measurements of the fluid velocity of the core are presently possible.

It is our present objective to obtain explicit formulae for the contribution to the rotational excitation arising from internal geophysical processes and will not be concerned at this point with developing detailed expressions for the lunisolar gravitational torques $\mathbf{N}$. It is our objective to obtain expressions for the terms r_{13} r_{23} r_{33} h_1 h_2 h_3 dr_{13}/dt dr_{23}/dt dh_1/dt dh_2/dt which appear as the internal geophysical contributions to the rotational excitation functions in Equations (VII-196) and (VII-200) in terms of observable global fields such as mass density, velocity, mass displacement etc.

If the geocentric position vector $\mathbf{r}$ in the rotating geophysical reference frame is

$$\mathbf{r} = r_1\hat{e}_1 + r_2\hat{e}_2 + r_3\hat{e}_3 \qquad \text{(VII-201)}$$

then we have from Equation (VII-186)

$$r_{ij} = \int_V \Delta\rho(\mathbf{r}, t)\,[r^2\,\delta_{ij} - r_i r_j]\,dV \qquad \text{(VII-202)}$$

and from Equation (VII-199)

$$h_i = \int_V \rho(\mathbf{r}, t)\,\epsilon_{ijk}\,r_j\,v_k\,(\mathbf{r}, t)\,dV \qquad \text{(VII-203)}$$

where δ_{ij} is the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

and where ϵ_{ijk} is the alternating tensor defined by

$$\epsilon_{ijk} = \begin{cases} +1 & ijk \quad \text{cyclic 123} \\ 0 & ijk \quad \text{not all distinct} \\ -1 & ijk \quad \text{noncyclic 123} \end{cases}$$

The time derivative $d\mathbf{h}/dt$, reckoned in the rotating frame, is given by Equation (III-84) as

$$\frac{d\mathbf{h}}{dt} = \int_V \left\{\frac{d\mathcal{h}}{dt} + \nabla\cdot\tilde{\mathcal{H}}_{rot}\right\} dV$$

where Equation (III-81) gives

$$\mathcal{h}(\mathbf{r}, t) = \rho(\mathbf{r}, t)\,[\mathbf{r} \times \mathbf{v}(\mathbf{r}, t)]$$

and Equation (III-82) gives

$$\tilde{\mathcal{H}}_{rot}\,(\mathbf{r}, t) = \rho\,(\mathbf{r}, t)\,\mathbf{v}\,(\mathbf{r}, t)\,[\mathbf{r} \times \mathbf{v}\,(\mathbf{r}, t)]$$

It follows that

$$\frac{d\mathbf{h}}{dt} = \int_V \left[\frac{\partial\rho}{\partial t}(\mathbf{r}\times\mathbf{v}) + \rho\left(\mathbf{r}\times\frac{\partial\mathbf{v}}{\partial t}\right) + \nabla\cdot(\rho\mathbf{v})(\mathbf{r}\times\mathbf{v}) + \rho\mathbf{v}\cdot\nabla(\mathbf{r}\times\mathbf{v})\right] dV$$

which can be written

$$\frac{d\mathbf{h}}{dt} = \int_V \left\{\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right][\mathbf{r}\times\mathbf{v}] + \rho\left[\mathbf{r}\times\frac{\partial\mathbf{v}}{\partial t}\right] + \rho\mathbf{v}\cdot\nabla(\mathbf{r}\times\mathbf{v})\right\} dV \qquad \text{(VII-204)}$$

which by virtue of the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \qquad \text{(VII-205)}$$

reduces to

$$\frac{d\mathbf{h}}{dt} = \int_V \left\{\rho\left[\mathbf{r}\times\frac{\partial\mathbf{v}}{\partial t}\right] + \rho\mathbf{v}\cdot\nabla(\mathbf{r}\times\mathbf{v})\right\} dV\,. \qquad \text{(VII-206)}$$

To obtain the components of $d\mathbf{h}/dt$ we write

$$\frac{dh_i}{dt} = \int_V \left[\rho\,\epsilon_{ijk}\,r_j\frac{\partial v_k}{\partial t} + \rho v_k\frac{\partial}{\partial x_k}(\epsilon_{ijm}\,r_j\,v_m)\right] dV$$

$$\frac{dh_i}{dt} = \int_V \left[\rho\,\epsilon_{ijk}\,r_j\frac{\partial v_k}{\partial t} + \rho v_k\,\epsilon_{ijm}\left(\frac{\partial r_j}{\partial x_k}v_m + r_j\frac{\partial v_m}{\partial x_k}\right)\right] dV$$

$$\frac{dh_i}{dt} = \int_V \left[\rho\,\epsilon_{ijk}\,r_j\frac{\partial v_k}{\partial t} + \rho\,v_k\,\epsilon_{ijm}\left(\delta_{jk}\,v_m + r_j\frac{\partial v_m}{\partial x_k}\right)\right] dV$$

$$\frac{dh_i}{dt} = \int_V \left[\rho\,\epsilon_{ijk}\,r_j\frac{\partial v_k}{\partial t} + \rho\,\epsilon_{ijk}\left(v_j\,v_k + r_j\,v_m\frac{\partial v_k}{\partial x_m}\right)\right] dV\,.$$

Since

$$\epsilon_{ijk}\,v_j\,v_k = 0$$

this reduces to

$$\frac{dh_i}{dt} = \int_V \left[\rho\,\epsilon_{ijk}\,r_j\left(\frac{\partial v_k}{\partial t} + v_m\frac{\partial v_k}{\partial x_m}\right)\right] dV$$

Recognizing

$$\frac{\partial v_k}{\partial t} + v_m \frac{\partial v_k}{\partial x_m} = \frac{D v_k}{Dt}$$

as the Lagrangian time derivative we conclude that

$$\frac{dh_i}{dt} = \int_V \rho\, \epsilon_{ijk}\, r_j \frac{D v_k}{Dt} dV \qquad \text{(VII-207)}$$

or in coordinate free notation,

$$\frac{d\mathbf{h}}{dt} = \int_V \rho \left(\mathbf{r} \times \frac{D\mathbf{v}}{Dt} \right) dV. \qquad \text{(VII-208)}$$

In addition to $d\mathbf{h}/dt$ we also require expressions for $d\tilde{r}/dt$. From Equation (IV-5) we have

$$\frac{d\tilde{I}}{dt} = \frac{d\tilde{I}^0}{dt} + \frac{d\tilde{r}}{dt} \qquad \text{(VII-209)}$$

and since

$$\frac{d\tilde{I}^0}{dt} = 0 \qquad \text{(VII-210)}$$

we have

$$\frac{d\tilde{I}}{dt} = \frac{d\tilde{r}}{dt}. \qquad \text{(VII-211)}$$

From Equation (III-85) and (VII-211) we have

$$\frac{d\tilde{r}}{dt} = \frac{d\tilde{I}}{dt} = \int_V \left[\frac{\partial \rho}{\partial t} (r^2\, \tilde{I} - \mathbf{r}\,\mathbf{r}) + \nabla \cdot \tilde{\mathscr{I}}_{rot} \right] dV \qquad \text{(VII-212)}$$

where $\tilde{\mathscr{I}}_{rot}$ is given by Equation (III-83) as

$$\tilde{\mathscr{I}}_{rot}(\mathbf{r}, t) = \rho(\mathbf{r}, t)\, \mathbf{v}(\mathbf{r}, t)\, (r^2\, \tilde{I} - \mathbf{r}\,\mathbf{r}). \qquad \text{(VII-213)}$$

It follows that

$$\frac{d\tilde{r}}{dt} = \int_V \left\{ \frac{\partial \rho}{\partial t} [r^2\, \tilde{I} - \mathbf{r}\,\mathbf{r}] + \nabla \cdot (\rho \mathbf{v})\, [r^2\, \tilde{I} - \mathbf{r}\,\mathbf{r}] + \rho \mathbf{v} \cdot \nabla\, [r^2\, \tilde{I} - \mathbf{r}\,\mathbf{r}] \right\} dV$$

which can be written as

$$\frac{d\tilde{r}}{dt} = \int_V \left\{ \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right] [r^2\, \tilde{I} - \mathbf{r}\,\mathbf{r}] + \rho \mathbf{v} \cdot \nabla\, [r^2\, \tilde{I} - \mathbf{r}\,\mathbf{r}] \right\} dV$$

which by virtue of the continuity equation, Equation (III-46), reduces to

$$\frac{d\tilde{r}}{dt} = \int_V \rho \mathbf{v} \cdot \nabla\, [r^2 \tilde{I} - \mathbf{r}\,\mathbf{r}]\, dV. \qquad \text{(VII-214)}$$

In component form Equation (VII-214) gives

$$\frac{dr_{ij}}{dt} = \int_V \rho\, v_k \frac{\partial}{\partial x_k} [r_m\, r_m\, \delta_{ij} - r_i r_j]\, dV$$

$$\frac{dr_{ij}}{dt} = \int_V \rho\, v_k\, [2\, r_m\, \delta_{mk}\, \delta_{ij} - r_i\, \delta_{jk} - r_j\, \delta_{ik}]\, dV$$

which finally gives

$$\frac{dr_{ij}}{dt} = \int_V \rho\, \{2\, r_k\, v_k\, \delta_{ij} - r_i\, v_j - r_j\, v_i\}\, dV. \qquad \text{(VII-215)}$$

From Equations (VII-195) (VII-199) (VII-207) (VII-215) we see that the relevant quantities appearing in the geophysical excitation functions are

$$r_{13} = - \int_V \Delta\rho\, (\mathbf{r}, t)\, r_1 r_3\, dV \qquad \text{(VII-216)}$$

$$r_{23} = - \int_V \Delta\rho(\mathbf{r}, t)\, r_2 r_3\, dV \qquad \text{(VII-217)}$$

$$r_{33} = \int_V \Delta\rho\, (\mathbf{r}, t)\, [r_1^2 + r_2^2]\, dV \qquad \text{(VII-218)}$$

$$h_1 = \int_V \rho(\mathbf{r}, t)\, [r_2 v_3(\mathbf{r}, t) - r_3 v_2(\mathbf{r}, t)]\, dV \qquad \text{(VII-219)}$$

$$h_2 = \int_V \rho(\mathbf{r}, t)\,[r_3 v_1(\mathbf{r}, t) - r_1 v_3(\mathbf{r}, t)]\,dV \tag{VII-220}$$

$$h_3 = \int_V \rho(\mathbf{r}, t)\,[r_1 v_2(\mathbf{r}, t) - r_2 v_1(\mathbf{r}, t)]\,dV \tag{VII-221}$$

$$\frac{dr_{13}}{dt} = -\int_V \rho(\mathbf{r}, t)\,[r_1 v_3(\mathbf{r}, t) + r_3 v_1(\mathbf{r}, t)]\,dV \tag{VII-222}$$

$$\frac{dr_{23}}{dt} = -\int_V \rho(\mathbf{r}, t)\,[r_2 v_3(\mathbf{r}, t) + r_3 v_2(\mathbf{r}, t)]\,dV \tag{VII-223}$$

$$\frac{dh_1}{dt} = \int_V \rho(\mathbf{r}, t)\left[r_2\left(\frac{\partial v_3}{\partial t} + \mathbf{v}\cdot\nabla v_3\right) - r_3\left(\frac{\partial v_2}{\partial t} + \mathbf{v}\cdot\nabla v_2\right)\right] dV. \tag{VII-224}$$

$$\frac{dh_2}{dt} = \int_V \rho(\mathbf{r}, t)\left[r_3\left(\frac{\partial v_1}{\partial t} + \mathbf{v}\cdot\nabla v_1\right) - r_1\left(\frac{\partial v_3}{\partial t} + \mathbf{v}\cdot\nabla v_3\right)\right] dV. \tag{VII-225}$$

Substituting Equations (VII-216) – (VII-225) into Equations (VII-196) (VII-197) (VII-200) gives expressions for the geophysical excitation functions for

(1) Polar Motion

$$\tilde{e}(t) = \frac{1}{A\Omega}\Bigg(N_1 + iN_2 - 2\,\Omega^2 \int_V \Delta\rho(r_2 r_3 - i\,r_1 r_3)\,dV + \int_V \Bigg\{\rho\left[2\Omega\, r_3 v_1 + r_3\left(\frac{\partial v_2}{\partial t} + \mathbf{v}\cdot\nabla v_2\right) - r_2\left(\frac{\partial v_3}{\partial t} + \mathbf{v}\cdot\nabla v_3\right)\right] + i\,\rho\left[2\Omega\, r_3 v_2 + r_1\left(\frac{\partial v_3}{\partial t} + \mathbf{v}\cdot\nabla v_3\right) - r_3\left(\frac{\partial v_1}{\partial t} + \mathbf{v}\cdot\nabla v_1\right)\right]\Bigg\} dV\Bigg) \tag{VII-226}$$

(2) UT1

$$\epsilon_3(t) = \frac{1}{C\Omega}\left[\int_0^t N_3(t')\,dt' - \Omega\int_V \Delta\rho\,(r_1^2 + r_2^2)\,dV - \int_V \rho(r_1 v_2 - r_2 v_1)\,dV\right]. \tag{VII-227}$$

Acknowledgments

I would like to acknowledge the use of unpublished research notes of Dr. M. G. Rochester of Memorial University of Newfoundland, Canada, which were of assistance to me in preparing portions of Sections III and V of this document. I would like also to acknowledge the assistance of the Department of Physics and the Faculty of Science of York University, Toronto, Canada, in providing a sabbatical leave for me to undertake this study.

References

Brosche, P., and Sündermann, J., "Effects of Oceanic Tides on the Rotation of the Earth," pp. 133-141. *Scientific Applications of Lunar Laser Ranging*, J. D. Mulholland, ed., D. Reidel 1977.

Darwin, G., "*Scientific Papers*," Vol. 3. Cambridge University Press, London, 1910.

Explanatory Supplement to the Astronomical Ephemeris and American Ephemeris and Nautical Almanac, prepared jointly by the Nautical Almanac Offices of the United Kingdom and the United States of America, Her Majesty's Stationery Office, London, 1961.

Goldreich, P., and Toomre, A., "Some Remarks on Polar Wandering," *J.G.R.*, *74*, 2555-2567, 1969.

Goldstein, H., "*Classical Mechanics*," Addison Wesley, 1950.

Kaula, W. M., "*An Introduction to Planetary Physics*," John Wiley & Sons, 1968.

Kinoshita, H., "Theory of the Rotation of the Rigid Earth," *Celest. Mech.*, *15*, 277-326, 1977.

Kinoshita, H., Nakajima, K., Kubo, Y., Nakagawa, I., Sasao, T., and Yokoyama, K., "Note on Nutation in Ephemerides," Publication of the International Latitude Observatory of Mizusawa, in press March 1979.

Kolaczek, B., and Weiffenbach, G., eds. "*On Reference Coordinate Systems for Earth Dynamics*," Proc. IAU Colloquim No. 26, Warsaw, Poland.

Lamb, H., "*Hydrodynamics*," 6th ed. Dover, 1945.

Lambeck, K., and Cazenave, A., "The Earth's Rotation and Atmospheric Circulation – I Season Variations," Geophys. J. R. astro. Soc. *32*, 79-93, 1973.

Lieske, J. H., Lederle, T., Fricke, W., and Morando, B., "Expressions for the Precession Quantities Based Upon the IAU (1976) System of Astronomical Constants," *Astron. Astrophys.*, *58*, 1-16, 1977.

Love, A. E. H., "The Yielding of the Earth to Disturbing Forces," Proc. R. Soc. London (Ser A.), *82*, 73-88, 1909.

Mansinha, L., and Shen, P., "Rotational Deformation of the Earth," *Phys. Earth Planet. Inter.*, *11*, 200-206, 1974.

Mansinha, L., Smylie, D. E., and Chapman, C. H., "Seismic Excitation of the Chandler Wobble Revisited," Geophys. J. R. astr. Soc. *59*, 1-17, 1979.

Molodenski, M. S., Quatrieme Symposium Internationale sur les Marees Terrestres. Communications de l'observatoire Royale de Belgique. No. 188, 25, 1961.

Mueller, I. I., "*Spherical and Practical Astronomy as Applied to Geodesy*," Frederick Ungar, 1969.

Munk, W., and MacDonald, G. J. F., "*The Rotation of the Earth – A Geophysical Discussion*," Cambridge University Press, 1960.

Murray, C. A., "On the Precession and Nutation of the Earth's Axis of Figure," Mon. Not. R. Astr. Soc. *183*, 677-685, 1978.

Pedersen, G., and Rochester, M. G., "Spectral Analysis of the Chandler Wobble," in *Rotation of the Earth*, 33-38, Melchoir and Yumi, eds., IUA Publication, 1972.

Poinsot, L., *Theorie Nouvelle de la Rotation des Corps*, Paris, 1852.

Prager, W., "*Introduction to Mechanics of Continua*," Dover Publications, 1973.

Rochester, M. G., and Smylie, D. E., "On Changes in the Trace of the Earth's Inertia Tensor," *J.G.R.*, *79*, 4948-4951, 1974.

Saito, M., "Some Problems of the Static Deformation of the Earth," *J. Phys. Earth*, *22*, 123-140, 1974.

Smylie, D. E., and Mansinha, L., "The Elasticity Theory of Dislocations in Real Earth Models and Changes in the Rotation of the Earth," Geophys. J. R. astr Soc., *23*, 329-354, 1971a.

Smylie, D. E., and Mansinha, L., "The Rotation of the Earth," *Scientific American*, *225*, 80-88, 1971b.

Stacey, F. D., "*Physics of the Earth*," 2nd ed., John Wiley and Sons, 1977.

Stolz, A., "Changes in the Position of the Geocenter Due to Seasonal Variations in Air Mass and Ground Water," Geophys. J. R. astr. Soc., *44*, 19-26, 1976.

Williams, J. G., personal communication 1979.

Woolard, E., "Theory of the Rotation of the Earth About Its Center of Mass," Astronomical Papers Prepared for the Use of the American Ephemeris and Nautical Almanac, Nautical Almanac Office U.S.N.O., Washington, D. C., 1953.

Woolard, E., "Inequalities in Mean Solar Time from Tidal Variations in the Rotation of the Earth," *Astr. J.*, *64*, 14-142, 1959.

Woolard, E., and Clemence, G., "*Spherical Astronomy*," Academic Press, 1966.

NASA—JPL—Coml., L.A., Calif.