NASA
Technical
Paper
2029

July 1982

Velocity Gradient Method for Calculating Velocities in an Axisymmetric Annular Duct

Theodore Katsanis

NASA

NASA
Technical
Paper
2029

1982

Velocity Gradient Method for Calculating Velocities in an Axisymmetric Annular Duct

Theodore Katsanis
Lewis Research Center
Cleveland, Ohio

NASA
National Aeronautics
and Space Administration

**Scientific and Technical
Information Branch**

SUMMARY

A method has been developed for calculating the velocity distribution along an arbitrary line between the inner and outer walls of an annular duct with axisymmetric swirling flow. The velocity gradient equation is used with an assumed variation of meridional streamline curvature. Upstream flow conditions can vary between the inner and outer walls, and an assumed total pressure distribution can be specified.

INTRODUCTION

Turbomachinery components are often connected by ducts, which are usually annular. The configurations and aerodynamic characteristics of these ducts are crucial to the optimum performance of the turbomachinery blade rows. One available method of duct-flow analysis is a finite-difference, stream-function analysis, such as the meridional analysis of reference 1. This is a good method of analysis, but it requires a large, complex code to handle arbitrary geometries. Computer storage and execution time are fairly large. A faster and easier method of analyzing the flow through a duct with axisymmetric swirling flow is the velocity gradient method, also known as the stream filament or streamline curvature method. This method has been used extensively for blade passages but has not been used much for ducts, except as the radial equilibrium equation. For the present analysis the momentum equation is used to derive a velocity gradient equation, which is used to determine the velocity variation along an arbitrary straight line between the inner and outer walls of an annular duct. The method works best in a well-guided passage and where the curvatures of the walls are small as compared with the width of the passage. Although other duct-analysis methods are available, this analysis is faster and requires less computer storage.

A computer program, ANDUCT, has been written to solve the equations involved in the analysis. Storage requirements are approximately 18 K words. Computer time is approximately 200 msec per station on an IBM 370/3033 computer.

This report gives a derivation of the equations used and describes the solution procedure and the use of the computer program. The computer code is available from COSMIC, 112 Barrow Hall, The University of Georgia, Athens, Ga. 30602.

SYMBOLS

a	coefficient, eq. (A10)
b	coefficient, eq. (A10)
c	coefficient, eq. (A10)
c_p	specific heat at constant pressure, J/kg K
e	coefficient, eq. (A10)
f	coefficient, eq. (A10)

g coefficient, eq. (A10)

h enthalpy, J/kg

h' total enthalpy, J/kg

m meridional streamline distance, meters

n distance normal to streamline, meters

p pressure, N/meter^2

p' total pressure, N/meter^2

q distance along quasi-orthogonal, meters

R gas constant, J/kg K

r radius from axis of rotation, meters

r_c radius of curvature of meridional streamline, meters

r_{cn} radius of curvature of normal to meridional streamline, meters

s entropy, J/kg K

T temperature, K

T' total temperature, K

t time, sec

V velocity, meters/sec

z axial coordinate, meters

α angle between meridional streamline and axis of rotation, rad; fig. 1

β angle between velocity vector and meridional plane, rad; fig. 1

γ specific heat ratio

θ angular coordinate, rad; fig. 1

λ angular momentum, rV_θ, meter^2/sec

ρ density, kg/meter^3

ρ' total density, kg/meter^3

ψ angle between quasi-orthogonal and radial direction, rad

Subscripts:

cr critical

h hub

m m–component

r r–component

t tip

z z–component

θ θ–component

METHOD OF ANALYSIS

The objective of this analysis method is to calculate the quasi-two-dimensional velocity distribution that satisfies a specified mass flow through an annular duct. The velocity variation along a quasi-orthogonal (ref. 2) between the inner and outer walls is determined by the momentum equation along the quasi-orthogonal. The quasi-orthogonal is a straight line between the walls of the annulus. With suitable assumptions, this leads to a velocity gradient equation. The velocity gradient equation is an ordinary differential equation that can be solved numerically. This determines the velocity distribution along the quasi-orthogonal. The analysis for one quasi-orthgonal is independent of that for other quasi-orthogonals. When the analysis is done for several lines, a velocity distribution is obtained for the entire duct.

The basic simplifying assumptions used to derive the equations and to obtain a solution along any quasi-orthogonal are the following:

(1) The flow in the annulus is steady.

(2) The flow is axisymmetric.

(3) The fluid is a perfect gas with constant specfic heat c_p.

(4) The only forces along a quasi-orthogonal are those due to momentum and pressure gradient.

(5) There is linear variation of meridional streamline curvature along a quasi-orthogonal.

(6) There is linear variation of meridional streamline angle along a quasi-orthogonal.

The flow may be axial, radial, or mixed. Whirl, stagnation pressure, and stagnation temperature must be specified but may vary between the inner and outer walls. Losses and heat transfer are not included in the analysis but may be simulated by specifying appropriate stagnation temperature and pressure distributions. Within the given assumptions, no terms are omitted from the basic velocity gradient equation (A10). Equation (A10), which is derived in appendix A, is an ordinary differential equation with the meridional component of velocity as the unknown. Equation (A10) is solved numerically and iteration is used to satisfy global continuity. Appendix B outlines the solution procedure.

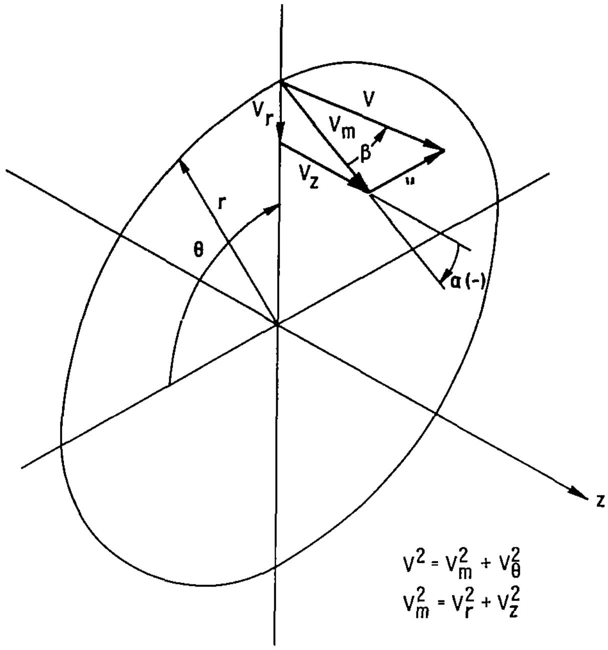


Figure 1. - Cylindrical coordinate system and velocity components.

1 5	6 10	11 15	16 20	21 30	31 40	41 50	51 60	61 70	71 80
	TITLE (first station of each duct)								
	BELTRAMI FLOW								
	LABEL (every station)								
	FORCED VORTEX								
NHT	LSFR	IPRINT	NEXT						
3	1	1	0						
GAM		AR		ZMSFL					
1.4		287.		317.					
RHUB		RTIP		ZHUB	ZTIP	CURVH	CURVT	ALH	ALT
.5		1.5		0.	0.	0.	0.	0.	0.
	STRFN or QDIST array								
0.		.5		1.					
	ZLAMDA array								
7.5		30.		67.5					
	TIP array								
288.15		288.15		288.15					
	RHOIP array								
1.225		1.225		1.225					

Figure 2. - Input form.

DESCRIPTION OF INPUT AND OUTPUT

Figure 1 shows the cylindrical coordinate system and velocity components. Figure 2 shows the input required for a single station. Sample input is shown with the numerical example.

Input

The input variables are described in terms of a consistent set of the International System of Units (SI). The program, however, will run with input in any consistent set of units.

The first line is the general title for a given geometry consisting of several quasi-orthogonals in a sequence. Succeeding quasi-orthogonals do not require a title, unless a new geometry with a new sequence of quasi-orthogonals is desired. The second line is a label that is required for every quasi-orthogonal. The remaining lines are data:

NHT — number of input points along quasi-orthogonal between inner and outer walls, maximum of 50

LSFR — integer (0 or 1) indicating whether flow conditions are given as a function of stream function (0) or distance from inner wall (1)

IPRINT — integer (0 or 1) indicating whether a detailed solution should be printed (1) or not printed (0) at each station

NEXT — integer (-1, 0, or 1) indicating whether this is the last input station (0). If more input stations follow, it also indicates whether the following station is still for the same duct (1) or whether a new series of input stations for another duct will follow (-1)

GAM — specific heat ratio, γ

AR — gas constant, R, J/kg K

ZMSFL — total mass flow through annulus, kg/sec

RHUB — radius at inner wall, r_h, meters

RTIP — radius at outer wall, r_t, meters

ZHUB — z coordinate at inner wall, meters

ZTIP — z coordinate at outer wall, meters

CURVH — meridional streamline curvature $1/r_c$ at inner wall, 1/meter

CURVT — meridional streamline curvature $1/r_c$ at outer wall, 1/meter

ALH — meridional streamline angle α at inner wall, deg

ALT — meridional streamline angle α at outer wall, deg

STRFN — array of stream function values for input points where flow conditions are specified. STRFN is given when LSFR = 0

QDIST — array of distances from wall along quasi-orthogonal, meters. QDIST is given when LSFR = 1

ZLAMDA — array of values of angular momentum λ corresponding to STRFN or QDIST array, meter^2/sec

TIP — array of total temperatures T' corresponding to STRFN or QDIST array, K

RHOIP array of total densities ρ' corresponding to STRFN or QDIST array, kg/meter^3

Units of Measurement

The International System of Units (ref. 3) is used throughout this report. However, the program does not use constants that depend on the system of units being used. Therefore, any consistent set of units can be used; in particular U.S. customary units can be used.

Output

An example of output is given in table I. This output corresponds to the input given in figure 2. The first output is a listing of input for a given station in format similar to the input sheet. After the input listing, detailed output for each station is printed if IPRINT = 1 is given as input. A summary of the inner and outer wall results for a given geometry is printed separately.

Error Messages

Several error messages have been incorporated into the program. These messages are listed here. Where necessary, suggestions for finding and correcting the error are given.

(1) THE PASSAGE IS CHOKED AT THIS STATION.
THE CHOKING MASS FLOW IS X.XXXX.

This message is self-explanatory.

(2) SUPERSONIC MERIDIONAL VELOCITY COMPONENT AT THIS STATION

If the flow has a supersonic meridional velocity component, without shocks, all the way from the hub to the shroud, a reasonable solution can be obtained. However, this is not the usual situation and caution should be exercised.

(3) SONIC MERIDIONAL VELOCITY COMPONENT AT THIS STATION.
THIS MAY RESULT IN AN INACCURATE SOLUTION

The velocity gradient equation (A10) is singular when the meridional velocity component is sonic. Because of this the solution becomes inaccurate when the meridional velocity is near sonic. This message is printed whenever the meridional velocity component is within 1 percent of the sonic velocity at some point on the quasi-orthogonal.

(4) NO SOLUTION COULD BE FOUND IN 100 ITERATIONS

This message is printed if no solution can be found. Most likely no solution exists for the given input. A common difficulty is an input distribution of whirl, total temperature, and total density that is not possible at the given mass flow.

(5) A FULLY CONVERGED SOLUTION COULD NOT BE OBTAINED IN 1000 ITERATIONS AT THIS STATION
THE STREAM FUNCTION CHANGED BY X.XXX BETWEEN THE LAST TWO ITERATIONS

Even though the inner iteration converges, it may be possible that the corrections due to streamline shift when LSFR = 0 will not converge.

(6) ITERATION PROCEDURE HAD TO BE RESTARTED TO AVOID EITHER A NEGATIVE TEMPERATURE OR A NEGATIVE VELOCITY
RESTART PROCEDURE WAS ABORTED AFTER 1000 TOTAL NUMBER OF ITERATIONS

Most likely no solution exists for the given input. A common difficulty is an input distribution of whirl, total temperature, and total density that is not possible at the given mass flow.

(7) THE MAXIMUM MASS FLOW FOR WHICH A SOLUTION COULD BE OBTAINED WAS X.XXXX
THE MAXIMUM VALUE OF VSUBM AT THE HUB FOR WHICH A SOLUTION COULD BE OBTAINED WAS X.XXXX
THE MINIMUM VALUE OF VSUBM AT THE HUB FOR WHICH A SOLUTION COULD BE OBTAINED WAS X.XXXX
THE TOTAL NUMBER OF ITERATIONS WAS XXX
NSUB = XX
NADD = XX

These messages give debug information when one of the previous error messages is printed.

(8) THE LIMIT OF 100 STATIONS PER CASE HAS BEEN EXCEEDED
OUTPUT IS GIVEN ONLY FOR THE FIRST 100 STATIONS

This message is self-explanatory.

NUMERICAL EXAMPLES

Beltrami Flow with Forced Vortex

A rotational flow with the vorticity vector parallel to the velocity vector is called Beltrami flow. An example of this type of flow is an annular duct with both walls of constant radius and the tangential velocity V_θ proportional to the radius, that is, $V_\theta = kr$, where k is an arbitrary constant. The total temperature is constant. This kind of flow, which is discussed in reference 4, illustrates the limitations on possible solutions. In reference 4, the axial component of velocity V_m is shown to vary with radius as follows:

$$V_m^2 = (V_m)_i^2 - 2k^2 (r^2 - r_i^2)$$

where the subscript i refers to any reference radius. It can be seen from this equation that a solution does not exist for large values of r.

```
                 BELTRAMI FLOW
                      FORCED VORTEX
 NHT    LSFR IPRINT    NEXT
   3       1      1       0
 GAM               AR              ZMSFL
 1.400000          287.0000          317.0000
 RHUB              RTIP             ZHUB            ZTIP            CURVH           CURVT           ALH             ALT
0.5000000          1.500000        0.0000000       0.0000000       0.0000000       0.0000000       0.0000000       0.000000
       QDIST ARRAY
0.0000000         0.5000000         1.000000
       ZLAMDA ARRAY
 7.500000          30.00000         67.50000
       TIP ARRAY
 288.1499          288.1499         288.1499
       RHOIP ARRAY
 1.224999          1.224999         1.224999
```

FORCED VORTEX

I	V	V/VCR	VSUBM	BETA	STATIC PRESSURE	STREAM FUNCTION
1	62.97757	0.2027507	61.16515	13.77914	98897.63	0.0000000
3	62.82813	0.2022697	60.83325	14.47685	98908.75	0.1019999E-01
5	62.67238	0.2017685	60.49181	15.15836	98920.44	0.2080000E-01
7	62.51079	0.2012481	60.14095	15.82709	98932.69	0.3179997E-01
9	62.34114	0.2007020	59.77803	16.48668	98945.44	0.4319999E-01
11	62.16457	0.2001335	59.40379	17.13963	98958.69	0.5499995E-01
13	61.98123	0.1995432	59.01801	17.78833	98972.44	0.6719995E-01
15	61.79137	0.1989320	58.62042	18.43498	98986.56	0.7979989E-01
17	61.59505	0.1982999	58.21066	19.08156	99001.19	0.9279996E-01
19	61.39240	0.1976476	57.78831	19.72992	99016.19	0.1061999
21	61.18346	0.1969749	57.35295	20.38173	99031.63	0.1199999
23	60.96829	0.1962822	56.90407	21.03862	99047.44	0.1341999
25	60.74690	0.1955695	56.44109	21.70213	99063.69	0.1487999
27	60.51926	0.1948366	55.96341	22.37373	99080.31	0.1637999
29	60.28534	0.1940835	55.47037	23.05484	99097.38	0.1791999
31	60.04506	0.1933099	54.96126	23.74687	99114.81	0.1949998
33	59.79839	0.1925158	54.43530	24.45123	99132.63	0.2111999
35	59.54523	0.1917008	53.89168	25.16936	99150.88	0.2277998
37	59.28546	0.1908644	53.32951	25.90260	99169.50	0.2447999
39	59.01892	0.1900063	52.74780	26.65247	99188.50	0.2621998
41	58.74554	0.1891263	52.14555	27.42044	99207.94	0.2799999
43	58.46513	0.1882235	51.52162	28.20808	99227.75	0.2981998
45	58.17751	0.1872975	50.87479	29.01707	99248.00	0.3167999
47	57.88251	0.1863478	50.20377	29.84912	99268.69	0.3357998
49	57.57996	0.1853737	49.50713	30.70610	99289.75	0.3551998
51	57.27228	0.1843830	48.78641	31.58839	99311.31	0.3750000
53	56.95122	0.1833495	48.03065	32.50282	99333.25	0.3951998
55	56.62460	0.1822981	47.24805	33.44569	99355.69	0.4157999
57	56.28954	0.1812193	46.43419	34.41994	99378.50	0.4367998
59	55.94582	0.1801127	45.58772	35.42696	99401.81	0.4581999
61	55.59320	0.1789775	44.70708	36.46861	99425.56	0.4799998
63	55.23151	0.1778131	43.79062	37.54681	99449.81	0.5021996
65	54.86053	0.1766188	42.83636	38.66402	99474.50	0.5247993
67	54.48006	0.1753939	41.84213	39.82300	99499.69	0.5477996
69	54.08994	0.1741379	40.80548	41.02698	99525.31	0.5711994
71	53.68993	0.1728501	39.72351	42.27974	99551.38	0.5949998
73	53.27985	0.1715299	38.59286	43.58582	99577.88	0.6191993
75	52.85950	0.1701766	37.40968	44.95035	99604.88	0.6437998
77	52.42868	0.1687896	36.16936	46.37958	99632.31	0.6687994
79	51.98712	0.1673681	34.86632	47.88101	99660.25	0.6941996
81	51.53471	0.1659116	33.49388	49.46376	99688.56	0.7199993
83	51.07115	0.1644192	32.04381	51.13885	99717.38	0.7461996
85	50.59627	0.1628903	30.50572	52.92035	99746.63	0.7727995
87	50.10976	0.1613240	28.86627	54.82602	99776.31	0.7997994
89	49.61142	0.1597197	27.10786	56.87935	99806.44	0.8271995
91	49.10098	0.1580764	25.20651	59.11200	99837.00	0.8549995
93	48.57816	0.1563932	23.12801	61.56902	99867.94	0.8831992
95	48.04266	0.1546692	20.82037	64.31819	99899.31	0.9117994
97	47.49423	0.1529036	18.19789	67.47041	99931.06	0.9407992
99	46.93251	0.1510952	15.09911	71.23303	99963.25	0.9701996
101	46.36078	0.1492544	11.14995	76.08366	99995.81	1.000000

ANDUCT, of course, cannot get a solution where none exists but will obtain a solution reasonably close to the limit.

Table I gives the input for an example with hub radius of 0.5 and tip radius of 1.5. The value of k is 30, and the input is given in SI units at standard atmosphere conditions. With a value of $V_m = 61$ at the hub, V_m equals 11 at the outer wall. This solution is obtained very close to the maximum possible radius (1.522). The calculated distribution of V_m is plotted in figure 3 and is indistinguishable from the theoretical distribution.

Boundary Layer Simulation

Any desired boundary layer profile can be simulated by specifying an appropriate total pressure distribution. The total pressure is specified indirectly by specifying both total temperature and total density. Care

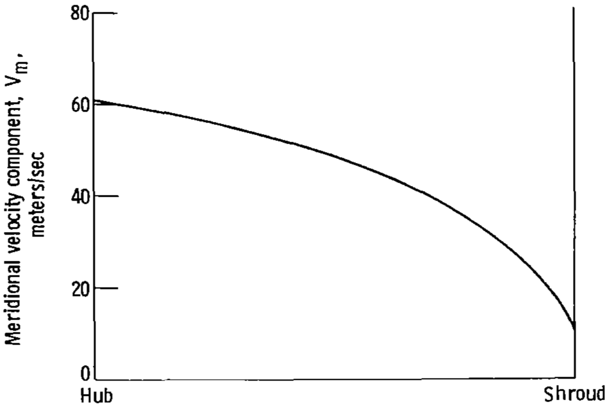


Figure 3. - Beltrami flow solution.

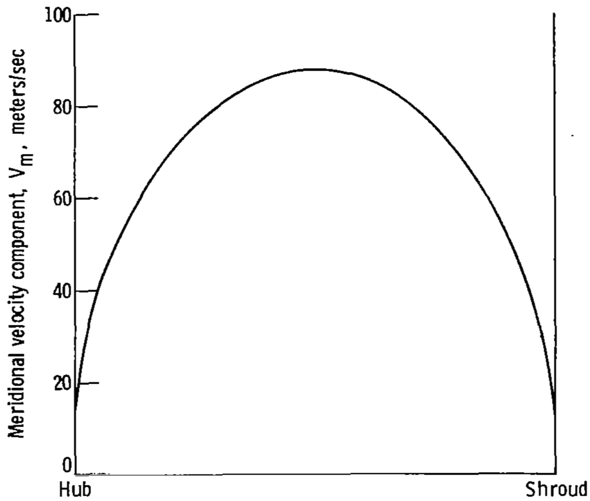


Figure 4. - Fully developed laminar flow.

must be taken that the total pressure variation is not excessive. Wall velocities very close to zero are difficult to approximate. Although this could be improved with more mesh points and double-precision calculations, it is not warranted because of the approximate nature of the entire calculation.

An example case is given for a parabolic velocity distribution corresponding to fully developed laminar flow. The corresponding total pressure is calculated, and from this the total density is calculated, with a uniform total temperature. The resulting total density must be modified (increased) slightly at the walls to obtain a non-zero wall velocity. A reasonable input for approximating fully developed laminar flow is given in table II. The calculated velocity distribution is plotted in figure 4. Turbulent or other boundary layer profiles can be approximated in a similar manner.

Source Flow

Because one of the features of this code is the ability to obtain a reasonable solution for a case where the hub-to-shroud line is not orthogonal to the flow, dV_m/dm is important to the solution. In previous velocity gradient codes several hub-to-shroud lines must be used to estimate dV_m/dm (e.g., ref. 2). This is avoided by using the continuity equation in conjunction with the assumed variation of the meridional flow angle α and the meridional streamline curvature $1/r_c$.

TABLE II. - BOUNDARY LAYER SIMULATION

```
                 FULLY DEVELOPED FLOW
                   LAMINAR CASE
  NHT   LSFR IPRINT   NEXT
   3      1      1      0
 GAM            AR            ZMSFL
 1.400000       287.0530        769.6902
 RHUB           RTIP            ZHUB            ZTIP            CURVH           CURVT           ALH             ALT
 1.000000       2.000000        0.0000000       0.0000000       0.0000000       0.0000000       0.0000000       0.0000000
      QDIST ARRAY
0.0000000       0.5000000       1.000000
      ZLAMDA ARRAY
0.0000000       0.0000000       0.0000000
      TIP ARRAY
 288.1499       288.1499        288.1499
      RHOIP ARRAY
 1.169999       1.224999        1.169999
```

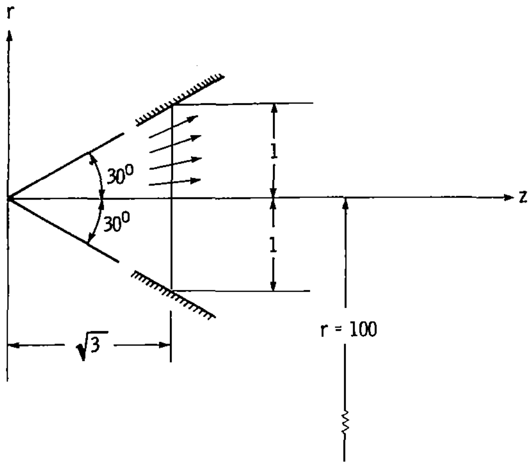


Figure 5. - Two-dimensional source flow.

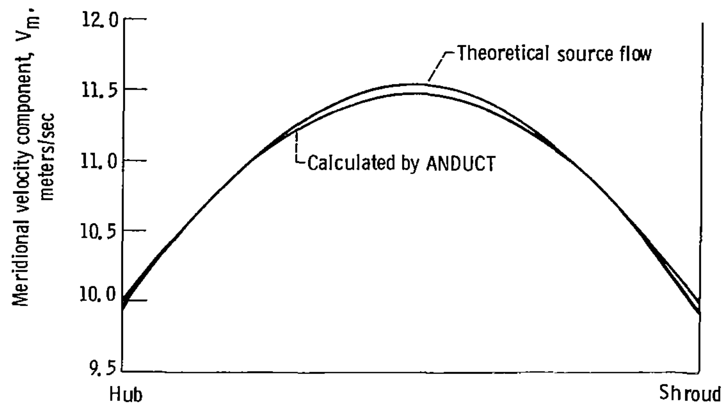


Figure 6. - Source flow distribution.

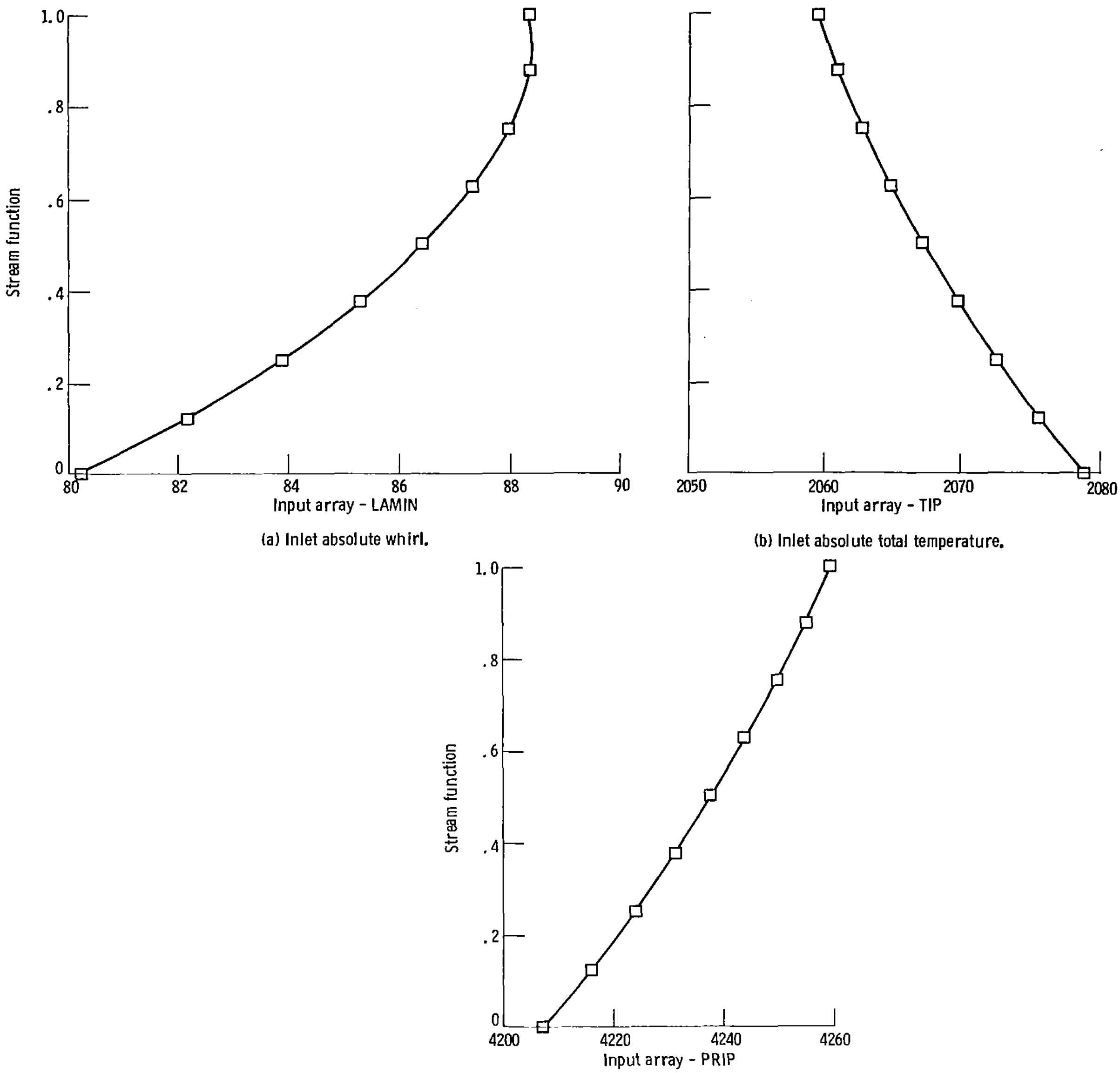
TABLE III. - SOURCE FLOW

```
                    SOURCE FLOW
                     30 DEGREE ANGLE, EACH SIDE
 NHT    LSFR IPRINT    NEXT
  3       1      1       0
GAM              AR             ZMSFL
1.400000         287.0530         16120.35
RHUB             RTIP             ZHUB          ZTIP          CURVH         CURVT         ALH          ALT
99.00000         101.0000        0.0000000     0.0000000     0.0000000     0.0000000     30.00000     -30.00000
      QDIST ARRAY
0.0000000        1.000000         2.000000
      ZLAMDA ARRAY
0.0000000        0.0000000        0.0000000
      TIP ARRAY
288.1499         288.1499         288.1499
      RHOIP ARRAY
1.224999         1.224999         1.224999
```

When compressibility is neglected, the velocity from a source is inversely proportional to the distance from the source. By choosing a large radius of 100, a two-dimensional source is approximated. Since there is no whirl in this example, $V_m = V = k/d$, where k is an arbitrary constant and d is the distance from the source. Figure 5 shows the flow configuration chosen for this example. The value of k was chosen to be 20. This results in values of $V = 10$ at the inner and outer walls and $V = 20/\sqrt{3} = 11.5470$ at the mean radius. The input for this example is given in table III. Figure 6 compares the theoretical source velocity variation with the approximate solution calculated by ANDUCT. The difference in the calculated curve is primarily due to the assumption of linear variation in α between the hub and shroud. It can be seen that the loss in accuracy is modest even with a 60° change in α across the passage.

Transition Duct

This example illustrates a transition duct between turbomachinery components. The flow conditions at the duct entrance are shown in figure 7 and in table IV. A linear loss variation along the length of the duct is included, but the whirl distribution at the inlet to the duct is assumed to be constant along the length of the duct. Figure 8 shows the duct geometry, and figure 9 compares the velocities calculated by ANDUCT with those calculated by MERIDL (ref. 1). MERIDL obtains a finite-difference, stream-function solution and is considered to be reasonably accurate. ANDUCT requires less than 1/3 the computer time required by MERIDL for this solution.



(a) Inlet absolute whirl.

(b) Inlet absolute total temperature.

(c) Inlet absolute total pressure.

Figure 7. - Inlet flow conditions for transition duct.

TABLE IV. - TRANSITION DUCT

```
                    TRANSITION DUCT
                     STATION 1
  NHT   LSFR IPRINT   NEXT
   9      0      1       1
 GAM              AR              ZMSFL
 1.318999         1716.510          0.4848600E-01
 RHUB             RTIP              ZHUB           ZTIP            CURVH           CURVT          ALH              ALT
0.1496000        0.1928999        -0.4170000E-01  -0.4170000E-01  0.7999998E-01  -0.2000000E-01  -0.1300000        0.4000000E-01
      STRFN ARRAY
0.0000000        0.1250000         0.2500000       0.3750000       0.5000000       0.6250000       0.7500000        0.8750000
 1.000000
      ZLAMDA ARRAY
 80.08499         82.16800          83.89099        85.29500        86.42000        87.33400        88.00400         88.39299
 88.37599
      TIP ARRAY
 2078.900         2075.500          2072.400        2069.600        2067.000        2064.700        2062.600         2060.800
 2059.400
      RHOIP ARRAY
0.1179000E-02   0.1183400E-02     0.1187400E-02   0.1191100E-02   0.1194400E-02   0.1197500E-02   0.1200400E-02   0.1203000E-02
0.1205000E-02
```

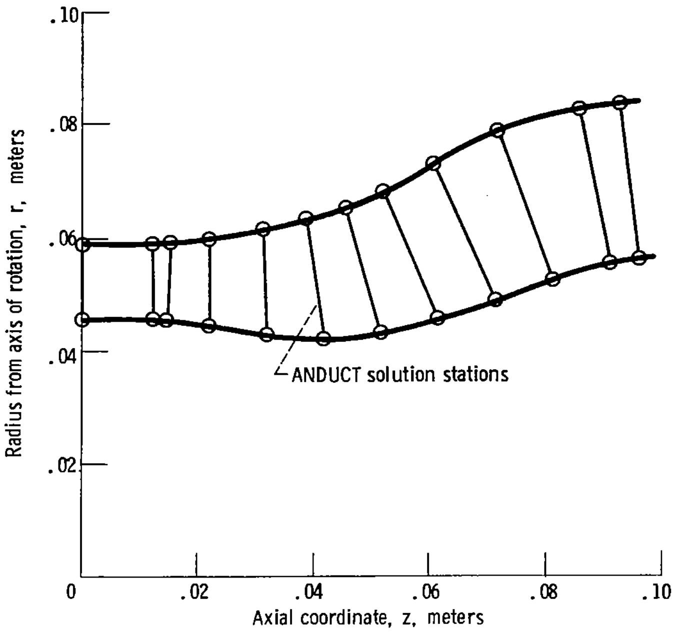
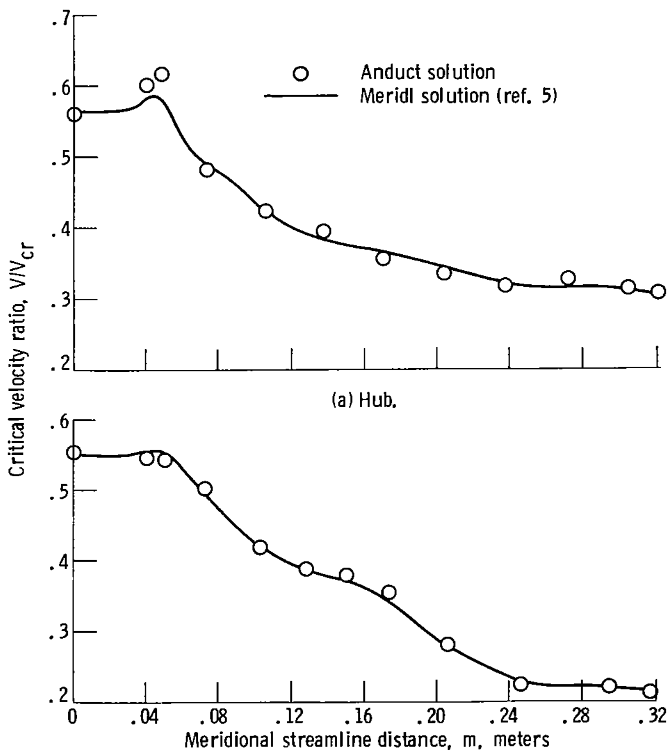


Figure 8. - Transition duct geometry.



(b) Shroud.

Figure 9. - Velocity distribution along walls of transition duct.

CONCLUDING REMARKS

The ANDUCT program calculates the flow field for an arbitrary annular duct with a straight centerline and axisymmetric swirling flow. This flow field could also be calculated by the MERIDL program (ref. 1). However, ANDUCT has the advantages of much less computer time (approximately 1/3 the time for the given numerical example) and very much less storage. The storage required for ANDUCT is 18 K words on the IBM 370/3033 computer with a virtual memory. Since MERIDL is a large, general code for a finite-difference, stream-function solution including a blade row, the storage would be very much larger, even with reduced array sizes. Thus the ANDUCT program is a convenient program to use for analyzing an annular duct with modest computer time on a computer with a small memory.

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, February 11, 1982

APPENDIX A

DERIVATION OF VELOCITY GRADIENT EQUATION

The velocity gradient equation desired is for the meridional velocity component V_m as a function of q, the distance along a quasi-orthogonal. The meridional velocity component V_m is used as the dependent variable since the tangential component is known from the specified whirl λ distribution. It is desired to obtain an equation for dV_m/dq where V_m is the only unknown. All quantities other than V_m are known as a function of either q or the stream function. The velocity gradient equation is based on the momentum equation in the direction of the quasi-orthogonal.

$$-\frac{1}{\rho}\frac{dp}{dq} = \left(\frac{dV_r}{dt} - \frac{V_\theta^2}{r}\right)\frac{dr}{dq} + \frac{dV_z}{dt}\frac{dz}{dq} \tag{A1}$$

Equation (A1) is obtained from equation (B7) of reference 2 with $\omega = 0$. The pressure gradient is related to the velocity gradient by assuming that the entropy variation is known. By combining

$$\frac{dp}{\rho} = dh - T\,ds$$

with

$$h = h' - \frac{V_m^2}{2} - \frac{V_\theta^2}{2}$$

and

$$dh' = c_p\,dT'$$

we get

$$\frac{1}{\rho}\frac{dp}{dq} = c_p\frac{dT'}{dq} - V_m\frac{dV_m}{dq} - V_\theta\frac{dV_\theta}{dq} - T\frac{ds}{dq} \tag{A2}$$

Solving for dV_m/dq by using equations (A1) and (A2) gives

$$\frac{dV_m}{dq} = \frac{1}{V_m}\left(\frac{dV_r}{dt} - \frac{V_\theta^2}{r}\right)\frac{dr}{dq} + \frac{1}{V_m}\frac{dV_z}{dt}\frac{dz}{dq} - \frac{V_\theta}{V_m}\frac{dV_\theta}{dq} + \frac{c_p}{V_m}\frac{dT'}{dq} - \frac{T}{V_m}\frac{ds}{dq} \tag{A3}$$

It is assumed that the whirl λ and meridional streamline angle α are known functions. Therefore V_r, V_θ, and V_z can be expressed in terms of V_m:

$$V_r = V_m \sin \alpha$$

$$V_\theta = \frac{\lambda}{r}$$

$$V_z = V_m \cos \alpha$$

By differentiating these last two expressions and noting that $d\alpha/dm = 1/r_c$ (where r_c is the radius of curvature of the meridional streamline), we obtain

$$\frac{dV_r}{dt} = \frac{V_m^2 \cos \alpha}{r_c} + V_m \sin \alpha \frac{dV_m}{dm}$$

$$\frac{dV_z}{dt} = - \frac{V_m^2 \sin \alpha}{r_c} + V_m \cos \alpha \frac{dV_m}{dm}$$

The angle between the radial direction and the quasi-orthogonal is denoted by ψ (fig. 10), so $\alpha - \psi$ is the angle between the quasi-orthognal and the true streamline orthogonal. We can use

$$\frac{dr}{dq} = \cos \psi$$

$$\frac{dz}{dq} = - \sin \psi$$

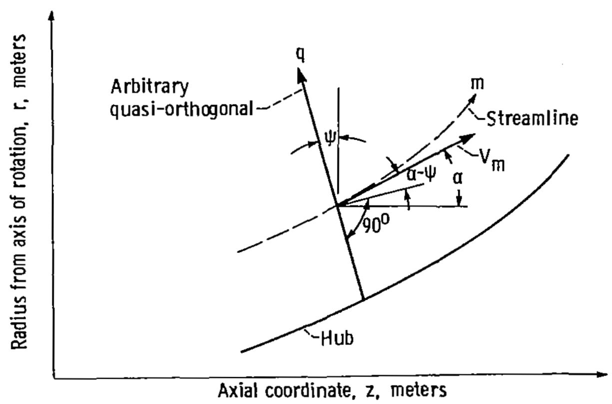


Figure 10. - Streamline and quasi-orthogonal angles.

When these relations are all used in equation (A3) and trigonometric expressions for the difference of angles are used, we obtain

$$\frac{dV_m}{dq} = \frac{V_m \cos(\alpha - \psi)}{r_c} + \sin(\alpha - \psi)\frac{dV_m}{dm} - \frac{\lambda}{r^2 V_m}\frac{d\lambda}{dq} + \frac{1}{V_m}\left(c_p \frac{dT'}{dq} - T\frac{ds}{dq}\right) \tag{A4}$$

Since the entropy variation is usually known as a total temperature and total pressure variation, we use

$$ds = \frac{c_p\ dT'}{T'} - \frac{R\ dp'}{p'}$$

to obtain

$$c_p\ dT' - T\ ds = \left(\frac{\lambda^2}{2r^2T'} + \frac{V_m^2}{2T'}\right) dT' + \frac{RT\ dp'}{p'}$$

This expression can be substituted into equation (A4) to obtain

$$\frac{dV_m}{dq} = \frac{V_m \cos(\alpha - \psi)}{r_c} + \sin(\alpha - \psi)\frac{dV_m}{d_m} - \frac{\lambda}{r^2\ V_m}\frac{d\lambda}{dq}$$

$$+ \frac{\lambda^2\ dT'}{2r^2T'V_m\ dq} + \frac{V_m}{2T'}\frac{dT'}{dq} + \frac{RT}{V_m p'}\frac{dp'}{dq} \tag{A5}$$

All the coefficients of V_m are known, except for dV_m/dm. However, dV_m/dm can be calculated from the continuity equation since the flow angles and the streamline curvature are assumed to be known. In terms of m and θ velocity components, the continuity equation is

$$\frac{1}{r}\frac{\partial(\rho V_\theta)}{\partial\theta} + \frac{\partial(\rho V_m)}{\partial m} + \rho V_m\left(\frac{1}{r}\frac{\partial r}{\partial m} + \frac{1}{r_{cn}}\right) = 0 \tag{A6}$$

(See eq. A3(34) in ref. 4, where $V_3 = V_n = 0.$) The curvature of the normal $1/r_{cn}$ is $\partial\alpha/\partial n$ and can be calculated from the known quantities $\partial\alpha/\partial q$ and $\partial\alpha/\partial m$. We have

$$\frac{\partial\alpha}{\partial q} = \cos(\alpha - \psi)\frac{\partial\alpha}{\partial n} + \sin(\alpha - \psi)\frac{\partial\alpha}{\partial m}$$

or, by solving for $\partial\alpha/dn$

$$\frac{\partial\alpha}{\partial n} = \frac{1}{\cos(\alpha - \psi)}\frac{\partial\alpha}{\partial q} - \tan(\alpha - \psi)\frac{\partial\alpha}{\partial m}$$

Note that $\partial\alpha/\partial m = 1/r_c$, $\partial r/\partial m = \sin\alpha$, and $\partial(\rho V_\theta)/\partial\theta = 0$, substitute in equation (A6), expand the derivatives, and solve for $\partial V_m/\partial m$ to obtain

$$\frac{\partial V_m}{\partial m} = V_m\left[\frac{\tan(\alpha - \psi)}{r_c} - \frac{\sin\alpha}{r} - \frac{\partial\alpha/\partial q}{\cos(\alpha - \psi)} - \frac{1}{\rho}\frac{\partial\rho}{\partial m}\right] \tag{A7}$$

The only quantity that is not immediately known is $\partial\rho/\partial m$. This quantity, however, can be calculated from $\partial V_m/\partial m$:

$$\frac{\rho}{\rho'} = \left(\frac{T}{T'}\right)^{1/(\gamma-1)}$$

where

$$\frac{T}{T'} = 1 - \frac{V_\theta^2 + V_m^2}{2c_pT'}$$

and

$$V_\theta = \frac{\lambda}{r}$$

When these are used and any streamwise variation of p' and T' is neglected, we find that

$$\frac{1}{\rho}\frac{\partial\rho}{\partial m} = \frac{1}{\gamma RT}\left(\frac{\lambda^2\sin\alpha}{r^3} - V_m\frac{\partial V_m}{\partial m}\right) \tag{A8}$$

Substitute equation (A8) in equation (A7) and solve for dV_m to obtain

$$\frac{\partial V_m}{\partial m} = \frac{\gamma RTV_m}{\gamma RT - V_m^2} \left[\frac{\tan(\alpha - \psi)}{r_c} - \frac{\sin \alpha}{r} - \frac{\partial\alpha/\partial q}{\cos(\alpha - \psi)} - \frac{\lambda^2 \sin \alpha}{r^3 \gamma RT} \right] \tag{A9}$$

When equation (A9) is substituted in equation (A5), we get

$$dV_m = V_m (a\, dq + b\, d\alpha + c\, dT') + \frac{e\, d\lambda + f\, dT' + g\, dp'}{V_m} \tag{A10}$$

where

$$a = \frac{\cos(\alpha - \psi)}{r_c} + \frac{\sin(\alpha - \psi)\ \gamma RT}{\gamma RT - V_m^2} \left[\frac{\tan(\alpha - \psi)}{r_c} - \frac{\sin \alpha}{r} \left(1 + \frac{\lambda^2}{r^3 \gamma RT} \right) \right]$$

$$b = - \frac{\tan(\alpha - \psi)\ \gamma RT}{(\gamma RT - V_m^2)}$$

$$c = \frac{1}{2T'}$$

$$e = - \frac{\lambda}{r^2}$$

$$f = \frac{\lambda^2}{2r^2 T'}$$

$$g = \frac{RT}{p'}$$

$$T = T' - \frac{\lambda^2}{2r^2 c_p} - \frac{V_m^2}{2c_p}$$

APPENDIX B

SOLUTION PROCEDURE

The velocity gradient equation (A10) is an ordinary differential equation that can be readily solved by numerical methods for a given initial value of V_m at the hub. As a solution to equation (A10) is being computed, a corresponding mass flow is computed from

$$w = \int \rho V_m \, 2\pi r \cos (\alpha - \psi) \, dq \tag{B1}$$

where

$$\rho = \rho' \left(1 - \frac{V_m^2 + V_\theta^2}{2c_p T'} \right)^{1/(\gamma-1)}$$

The desired solution is obtained by varying $(V_m)_h$ until a solution to equation (A10) is found that will satisfy equation (B1). This requires an iterative procedure, which is described below.

For the initial solution to equation (A10), $(V_m)_h$ is estimated on the basis of one-dimensional incompressible flow. The numerical solution is calculated by the Heun method (ref. 5) for 100 mesh spaces from inner wall to outer wall. If ZLAMDA, TIP, and RHOIP are all given as a function of position (LSFR = 1), all the coefficients in equation (A10) can be calculated with the solution. However, if ZLAMDA, TIP, and RHOIP are given as a function of the stream function (LSFR = 0), the coefficients can only be approximated until a solution is computed. Thus an outer iteration must be added to correct the coefficients. Usually only one or two outer iterations are required. Within the inner iteration, estimates for $(V_m)_h$ are made by subroutine CONTIN, on the basis of previous calculations. After three estimates are made, CONTIN will fit a parabola through the three points to make the next estimate. This quickly leads to a solution for subsonic flow. If the mass flow specified (ZMSFL) is too large, a solution does not exist. However, CONTIN will make estimates to calculate the largest possible mass flow (which is the choking mass flow for that station). Subroutine CONTIN is more completely described in reference 6.

After the correct mass flow solution has been obtained with the aid of CONTIN, the inner iteration has converged. If LSFR = 0 for input, an outer iteration must be done to correct the coefficients that involve ZLAMDA, TIP, or RHOIP, as mentioned previously.

If difficulty is encountered so that a valid solution cannot be obtained, an appropriate message is printed, as discussed in the main-text section on Error Messages.

REFERENCES

1. Katsanis, Theodore; and McNally, William D.: Revised FORTRAN Program for Calculating Velocities and Streamlines on the Hub-Shroud Midchannel Stream Surface of an Axial-, Radial-, or Mixed-Flow Turbomachine or Annular Duct. I - Users Manual. NASA TN D-8430, 1977.

2. Katsanis, Theodore: Use of Arbitrary Quasi-Orthogonals for Calculating Flow Distribution in the Meridional Plane of a Turbomachine. NASA TN D-2546, 1964.

3. Mechtly, E. A.: The International System of Units, Physical Constants and Conversion Factors. NASA SP-7012, 2nd rev., 1973.

4. Vavra, Michael H.: Aero-Thermodynamics and Flow in Turbomachines. Robert E. Krieger Publ. Co., 1974.

5. Carnahan, Brice; Luther, H. A.; and Wilkes, James O.: Applied Numerical Methods. John Wiley & Sons, Inc., 1969.

6. Katsanis, Theodore; and McNally, William D.: Revised FORTRAN Program for Calculating Velocities and Streamlines on the Hub-Shroud Midchannel Stream Surface of an Axial-, Radial-, or Mixed-Flow Turbomachine or Annular Duct. II - Programmers Manual. NASA TN D-8431, 1977.

1. Report No. NASA TP-2029	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle VELOCITY GRADIENT METHOD FOR CALCULATING VELOCITIES IN AN AXISYMMETRIC ANNULAR DUCT		5. Report Date July 1982
		6. Performing Organization Code 505-32-52
7. Author(s) Theodore Katsanis		8. Performing Organization Report No. E-1104
		10. Work Unit No.
9. Performing Organization Name and Address National Aeronautics and Space Administration Lewis Research Center Cleveland, Ohio 44135		11. Contract or Grant No.
		13. Type of Report and Period Covered Technical Paper
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		14. Sponsoring Agency Code
15. Supplementary Notes		

16. Abstract

A method has been developed for calculating the velocity distribution along an arbitrary line between the inner and outer walls of an annular duct with axisymmetric swirling flow. The velocity gradient equation is used with an assumed variation of meridional streamline curvature. Upstream flow conditions can vary between the inner and outer walls, and an assumed total pressure distribution can be specified.

17. Key Words (Suggested by Author(s)) Duct; Annular duct; Velocity gradient; Flow analysis; Axisymmetric flow		18. Distribution Statement Unclassified - unlimited STAR Category 02	
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 22	22. Price* A02

* For sale by the National Technical Information Service, Springfield, Virginia 22161

NASA-Langley, 1982