NASA Technical Memorandum 84271

Nonlinear Zero-Sum Differential Game Analysis by Singular Perturbation Methods

J. Shinar and N. Farber

July 1982

National Aeronautics and
Space Administration

NASA Technical Memorandum 84271

Nonlinear Zero-Sum Differential Game Analysis by Singular Perturbation Methods

J. Shinar
N. Farber, Ames Research Center, Moffett Field, California

National Aeronautics and
Space Administration

Ames Research Center
Moffett Field, California 94035

N82-29314#

LIST OF SYMBOLS

D_o	zero lift drag
D_i	lift induced drag in level flight
d	distance of capture
$\vec{e}_{(\ )}$	unit vector in the direction of $(\vec{\ })$
g	acceleration of gravity
$\mathcal{H}$	Hamiltonian functions
$\mathcal{J}$	cost function of the game
$k_1 \ldots k_4$	scalar parameters
n	aerodynamic load factor
R	distance of separation
r	the smallest turning radius at a given speed
T_{max}	maximum thrust force
t	time
V	velocity
W	aircraft weight
X,Y	Cartesian coordinates in figure 1
ε	singular perturbation parameter
λ_R	separation adjoint
λ_V	velocity adjoint
ξ	throttle control parameter
τ	stretched time scale
χ	aircraft flying direction (azimuth)
ψ	line of sight angle defined in figure 1
ω	angular velocity
Ω	angular velocity limit

Subscripts

E evader

f final value

P pursuer

o initial value

Superscripts

c composite zero-order solution

i inner (boundary layer) solution

o outer solution (constant speed game)

r reduced order solution (variable speed game)

* optimal value

$(\dot{\ })$ time derivative

$(\vec{\ })$ three-dimensional vector

NONLINEAR ZERO-SUM DIFFERENTIAL GAME ANALYSIS BY SINGULAR PERTURBATION METHODS

J. Shinar* and N. Farber†

Ames Research Center

SUMMARY

A certain class of nonlinear, zero-sum differential games, exhibiting time-scale separation properties, is suitable for analysis by singular-perturbation techniques. The merits of such an analysis, leading to an approximate game solution, as well as the "well-posedness" of the formulation, are discussed. The singular-perturbation approach is shown to be particularly attractive for investigating pursuit-evasion problems; the original multidimensional differential game is decomposed to a "simple pursuit" (free-stream) game and two independent (boundary-layer) optimal-control problems. Using multiple time-scale boundary-layer models in such games results in a pair of uniformly valid zero-order composite feedback strategies. Though a priori perfect information is assumed, the suboptimal strategies depend only on relative geometry and own-state measurements. This is demonstrated by a three-dimensional, constant-speed example. For game analysis with realistic vehicle dynamics, the technique of forced singular perturbations and a variable modeling approach is proposed. Accuracy of the zero-order singular-perturbation analysis is evaluated by comparison with the exact (numerical) solution of a time-optimal, variable-speed "game of two cars" in the horizontal plane.

I. INTRODUCTION

The notion of differential games, coined by Isaacs (ref. 1), was intended initially to describe a mathematical framework for investigating pursuit-evasion problems. Today, the general theory of differential games has a much larger horizon, but pursuit-evasion games still attract a particular interest. However, the progress in solving nonlinear games, representing realistic problems, has been slow and frustrating. Although it has been proved (refs. 2 and 3) that pursuit-evasion games with separable convex dynamics and properly defined terminal manifolds have a saddle-point solution, only games with very simple structures have been solved (refs. 3-6).

The basic reason for this unsatisfactory progress is the complex nature of the game solution. First, a nonlinear, two-point, boundary-valve problem of high dimension, describing the set of necessary conditions of optimality, has to be solved. Such a solution, generating candidate extremals, is valid only if those trajectories do not reach singular surfaces of the game. And the existence of such surfaces of discontinuity (either in the cost function or in its gradient) is a well-known phenomenon (refs. 7 and 8) in differential game theory. For this reason the numerical

---

*Senior NRC Associate; on sabbatical leave from Technion, Israel Institute of Technology, Haifa, Israel.

†Research Engineer, Department of Aeronautical Engineering, Technion, Israel Institute of Technology, Haifa, Israel.

techniques (refs. 9-12), which can provide a solution of the two-point boundary-value problem, are of limited value. They also require an excessive amount of computation, prohibitive for real-time implementations, and provide only little insight for a systematical analysis. For such purposes a reasonable analytical approximation may be of great value.

In analysis of differential games, where fast and slow variables can be distinguished, the method of singular perturbation — successfully used in optimal control of multiple time-scale systems (refs. 13-17) — can be applied. By this technique the original differential game can be decoupled to a set of lower-dimensional, analytically solvable subgames, enabling the synthesis of an approximation to the exact solution.

There are, however, some conceptual difficulties to be overcome. The first attempts to investigate singularly perturbed linear differential games (refs. 18-21) revealed that because of the differences in the information structure between an open-loop and closed-loop solution, some singularly perturbed differential games may not be "well-posed" (i.e., the solution of the "slow" game does not approximate the solution of the original one, even if the perturbation parameter, characterizing the time-scale separation, tends toward zero). "Well-posedness" was, however, demonstrated (ref. 18) for zero-sum linear games and it was conjectured (ref. 20) for cases in which the "fast" game has identical closed-loop and open-loop solutions.

Other studies (refs. 22 and 23) clarified the significance of an approximate game solution. By defining the outcome of a singularly perturbed, two-person, zero-sum differential game — in which both players follow approximate (synthesized) strategies — as the "extended value," it was demonstrated that

1. The difference between the extended value and the exact saddle-point value is bounded on both sides

2. The extended value satisfies by itself a weak saddle inequality

3. The extended value tends, as a limit, toward the exact value of the singularly perturbed game, as the singular perturbation parameter ε approaches zero

4. The accuracy of the extended value for a zero-order approximation is predicted to be of the order of ε^2.

A class of pursuit-evasion games, analyzed by the technique of singular perturbations, can be characterized by the following properties:

1. There is a time-scale separation between the variables describing the slow relative geometry and the fast variations of vehicle dynamics

2. The dynamics of the relative geometry are separately controlled by the players

3. The dynamics of the fast variables are independent of the slow ones

4. The fast vehicle dynamics are independently controlled by the respective players

5. The terminal surface is defined by the slow variables only.

For this class, the original high-dimensional game can be decomposed to a slow "simple pursuit game" and to two sets of fast, independent, optimal-control problems satisfying the conjecture of reference 20.

The objective of this paper is to summarize the results of singular-perturbation analysis for a set of pursuit-evasion games of gradually increasing complexity. In all of these games, capture is defined as the pursuer approaching the evader within a predetermined distance and is assumed to be guaranteed. The cost to be optimized is the "time of capture."

The scope of the paper is, however, limited to two recently completed investigations. In section II the three-dimensional, constant-speed pursuit-evasion game is analyzed, and it is shown how the singular-perturbation solution (ref. 24) enlightens the apparent controversy raised in previous works (refs. 25-27). The results of a comparison (ref. 28) of the zero-order feedback approximation, obtained by singular-perturbation analysis (ref. 29), and the open-loop optimal solution of variable-speed horizontal pursuit-evasion game (ref. 30) are presented in section III. This comparison has validated the singular-perturbation approach for the analysis of complex pursuit-evasion games for sufficiently large initial distances. In the concluding section, the merits of such game analysis are summarized.

II. THREE-DIMENSIONAL "GAME OF TWO CARS"

Summary of Two-Dimensional Results

As a first example of a singularly perturbed pursuit-evasion game, the two-dimensional, constant-speed game with constrained turning rates (Game of Two Cars) was analyzed in reference 22, since its optimal solution (ref. 31) was available for comparison. This optimal solution, obtained for a sufficiently large initial distance, requires that each player uses his maximum turning rate until the final line of sight is reached. Then the game continues as a straight line pursuit.

The singular-perturbation parameters in this problem are the ratio of the turning radii of the players to the initial distance of separation. The approximate strategy pair for this game, obtained by using the singular-perturbation technique, is given in a feedback form. It directs the players to align their velocity vectors with the current line of sight. If both players can do it simultaneously, the approximate strategies yield a result which is identical to the exact solution. Otherwise, the extended value will shift in favor of the player who reaches the current line of sight later than his opponent. Numerical comparison showed that the payoff accuracy of the approximation is better than the predicted order of ε^2. This successful demonstration of merits of the singular-perturbation approach encouraged investigation of more complex pursuit-evasion game models, such as (1) three-dimensional, constant speed; (2) two-dimensional, variable speed; and (3) three-dimensional, variable speed.

In the following subsection the first game model is formulated, using vector notations.

Problem Formulation

The equations of motion of the constant-speed pursuit-evasion in a three-dimensional space are

$$\frac{d\vec{R}}{dt} = \vec{V}_E - \vec{V}_P \ ; \qquad \vec{R}(t_o) = \vec{R}_o \tag{1}$$

$$\frac{d\vec{V}_P}{dt} = \vec{\omega}_P \times \vec{V}_P \ ; \qquad \vec{V}_P(t_o) = \vec{V}_{P_o} \tag{2}$$

$$\frac{d\vec{V}_E}{dt} = \vec{\omega}_E \times \vec{V}_E \ ; \qquad \vec{V}_E(t_o) = \vec{V}_{E_o} \tag{3}$$

The angular velocity vectors $\vec{\omega}_P, \vec{\omega}_E$ are perpendicular to the respective vectors of linear velocity, that is,

$$\vec{\omega}_P \cdot \vec{V}_P = \vec{\omega}_E \cdot \vec{V}_E = 0 \tag{4}$$

and they are constrained in magnitude,

$$|\vec{\omega}_P| \le \Omega_P \tag{5}$$

$$|\vec{\omega}_E| \le \Omega_E \tag{6}$$

The game terminates at $t = t_f$, when the two following conditions are satisfied for the first time:

$$|\vec{R}(t_f)| = d \tag{7}$$

$$\frac{d}{dt}|\vec{R}(t_f)| < 0 \tag{8}$$

It was shown (ref. 32) that if $|\vec{V}_P| > |\vec{V}_E|$ and $\Omega_P > \Omega_E$ then a finite capture time can be always guaranteed. In the sequel, it is assumed that these inequalities are respected. It permits the definition of a game of degree (ref. 1) in which the cost to be optimized is the time of capture t_f, defined as

$$\mathcal{J} = t_f \triangleq \min_{t > t_o} \ \arg_{|\vec{R}_o| > d} \ |\vec{R}(t)| = d \tag{9}$$

The controls of the players are $\vec{\omega}_P$ and $\vec{\omega}_E$, respectively, and their optimal values have to satisfy the necessary saddle-point conditions of

$$\min_{\vec{\omega}_P} \max_{\vec{\omega}_E} \mathcal{H} = \max_{\vec{\omega}_E} \min_{\vec{\omega}_P} \mathcal{H} = 0 \tag{10}$$

where $\mathcal{H}$ is a scalar function (the Hamiltonian of the game) given by

$$\mathcal{H} = 1 + \vec{\lambda}_R \cdot (\vec{V}_E - \vec{V}_P) + \vec{\lambda}_{V_P} \cdot (\vec{\omega}_P \times \vec{V}_P) + \vec{\lambda}_{V_E} \cdot (\vec{\omega}_E \times \vec{V}_E) \tag{11}$$

The adjoint vectors $\vec{\lambda}_R, \vec{\lambda}_{V_P}, \vec{\lambda}_{V_E}$ are the components of the gradient of the cost function (assuming that such a gradient exists) and they have to satisfy

$$\frac{d\vec{\lambda}_R}{dt} = \frac{\partial \mathcal{H}}{\partial \vec{R}} = 0 \qquad \Rightarrow \vec{\lambda}_R = \text{const} \tag{12}$$

$$\frac{d\vec{\lambda}_{V_P}}{dt} = -\frac{\partial \mathcal{H}}{\partial \vec{V}_P} = \vec{\lambda}_R + \left(\vec{\omega}_P \times \vec{\lambda}_{V_P}\right) \tag{13}$$

$$\frac{d\vec{\lambda}_{V_E}}{dt} = -\frac{\partial \mathcal{H}}{\partial \vec{V}_E} = -\vec{\lambda}_R + \left(\vec{\omega}_E \times \vec{\lambda}_{V_E}\right) \tag{14}$$

The transversality conditions, applied at the terminal surface defined by equation (7), lead to

$$\vec{\lambda}_R(t_f) \parallel \vec{R}(t_f) \tag{15}$$

$$\vec{\lambda}_{V_P}(t_f) = \vec{\lambda}_{V_E}(t_f) = 0 \tag{16}$$

It can be thus concluded from equations (12) and (15) that $\vec{\lambda}_R(t)$ is a constant vector parallel to the final line of sight and that its magnitude can be determined from equation (10) applied at the terminal manifold,

$$\vec{\lambda}_R \cdot [\vec{V}_P(t_f) - \vec{V}_E(t_f)] = 1 \tag{17}$$

The optimal-control strategies are obtained from equation (10), using also equations (5) and (6):

$$\vec{\omega}_P^* = \Omega_P \frac{\vec{\lambda}_{V_P} \times \vec{V}_P}{|\vec{\lambda}_{V_P} \times \vec{V}_P|} \tag{18}$$

$$\vec{\omega}_E^* = -\Omega_E \frac{\vec{\lambda}_{V_E} \times \vec{V}_E}{|\vec{\lambda}_{V_E} \times \vec{V}_E|} \tag{19}$$

These expressions reflect the required perpendicularity expressed in equation (4).

The set of necessary conditions of game optimality and the equations of motion represent a nonlinear, two-point, boundary-layer problem with 6 three-dimensional vector variables. In the next subsection the optimal solution is approximated, using the singular-perturbation technique.

Zero-Order Feedback Approximation

The first step in applying singular-perturbation analysis is the transformation of the original problem to exhibit a singularly perturbed structure. The results of the two-dimensional analysis (ref. 22) suggest that if the turning radius of the players, defined by

$$r = |\vec{V}|/|\vec{\omega}| \tag{20}$$

is small relative to the initial distance of separation $|\vec{R}_o|$, there is a time-scale separation between trajectory and vehicle dynamics which can be demonstrated by a scaling transformation. Such a transformation can be avoided by using the technique of "forced" singular perturbations, proposed first by Kelley (ref. 33) and used since in many optimal-control problems (refs. 34-37). This method is based on the insertion of a unit-value, "artificial" perturbation parameter as a multiplier of the respective time derivatives of the fast variables. Judicious ordering of the state variables, representing the actual hierarchy of the time-scale separations, is an essential prerequisite for a meaningful approximation. Recently it was formally demonstrated (ref. 38) that the "forced" singular-perturbation approach and the properly scaled transformation of the variables (using singular-perturbation parameters of physical significance) yield identical zero-order feedback solutions. The example of the constant-speed two-dimensional game solved by this method in reference 23 illustrated this equivalence. By using this technique, equations (1)-(3) can be transformed to

$$\frac{d\vec{R}}{dt} = (\vec{V}_E - \vec{V}_P) \; ; \quad \vec{R}(t_o) = \vec{R}_o \tag{21}$$

$$\varepsilon \frac{d\vec{V}_P}{dt} = (\vec{\omega}_P \times \vec{V}_P) \; ; \quad \vec{V}_P(t_o) = \vec{V}_{P_o} \tag{22}$$

$$\varepsilon \frac{d\vec{V}_E}{dt} = (\vec{\omega}_E \times \vec{V}_E) \; ; \quad \vec{V}_E(t_o) = \vec{V}_{E_o} \tag{23}$$

By setting $\varepsilon = 0$ the reduced-order game is obtained (variables of this game are denoted by the subscript o) yielding

$$\frac{d\vec{R}^\circ}{dt} = (\vec{V}_E^\circ - \vec{V}_P^\circ) \; ; \quad \vec{R}^\circ(t_o) = \vec{R}_o \tag{24}$$

and the constraints

$$(\vec{\omega}_P^\circ \times \vec{V}_P^\circ) = (\vec{\omega}_E^\circ \times \vec{V}_E^\circ) = 0 \tag{25}$$

Combining equations (25) and (4) leads to

$$\vec{\omega}_P^\circ = \vec{\omega}_E^\circ = 0 \tag{26}$$

In this reduced game, the controls are $\vec{V}_P^\circ$ and $\vec{V}_E^\circ$, and they have to satisfy

$$\min_{\vec{V}_P^\circ} \max_{\vec{V}_E^\circ} \mathcal{H}^\circ = \max_{\vec{V}_E^\circ} \min_{\vec{V}_P^\circ} \mathcal{H}^\circ = 0 \tag{27}$$

where

$$\mathcal{H}^\circ = 1 + \vec{\lambda}_R^\circ \cdot (\vec{V}_E^\circ - \vec{V}_P^\circ) \tag{28}$$

The adjoint vector $\vec{\lambda}_R^\circ$ satisfies an equation similar to equation (12), and consequently it is a constant vector, parallel to the final line of sight of the reduced game $\vec{R}^\circ(t_f)$.

Optimization of the reduced-order Hamiltonian $\mathscr{H}^\circ$ leads to

$$\vec{V}_P^\circ = |\vec{V}_P|\vec{e}_R^\circ(t_f) \tag{29}$$

$$\vec{V}_E^\circ = |\vec{V}_E|\vec{e}_R^\circ(t_f) \tag{30}$$

where $\vec{e}_R^\circ$ is a unit vector in the $\vec{R}^\circ$ direction. We also obtain from equation (27) that

$$\vec{\lambda}_R^\circ = \frac{\vec{e}_R^\circ(t_f)}{|\vec{V}_P| - |\vec{V}_E|} \tag{31}$$

The solution of the reduced game is therefore a straight-line pursuit. Substituting equations (29) and (30) into equation (21) indicates that the direction of the line of sight does not change in this subgame, that is, $\vec{R}^\circ(t_f)$ is parallel to $\vec{R}_o$, and consequently

$$\vec{e}_R^\circ(t_f) = \vec{e}_{R_o} \tag{32}$$

Since equations (29) and (30) are generally incompatible with the prescribed initial conditions $\vec{V}_{P_o}$ and $\vec{V}_{E_o}$, a boundary-layer game has to be introduced, using a time-stretching transformation

$$\tau = \frac{t - t_o}{\varepsilon} \tag{33}$$

By substituting equation (33) into equations (21)-(23), the following set of equations (variables are denoted by the superscript i) is obtained:

$$\frac{d\vec{R}^i}{d\tau} = \varepsilon(\vec{V}_E^i - V_P^i) \ ; \quad \vec{R}^i(0) = \vec{R}_o \tag{34}$$

$$\frac{d\vec{V}_P^i}{d\tau} = (\vec{\omega}_P^i \times \vec{V}_P^i) \ ; \quad \vec{V}_P^i(0) = \vec{V}_{P_o} \tag{35}$$

$$\frac{d\vec{V}_E^i}{d\tau} = (\vec{\omega}_E^i \times \vec{V}_E^i) \ ; \quad \vec{V}_E^i(0) = \vec{V}_{E_o} \tag{36}$$

Setting $\varepsilon = 0$ leads to

$$\vec{R}^i(t) = \text{const} = \vec{R}_o \tag{37}$$

The Hamiltonian of this boundary-layer game is similar to equation (11). It is obtained by expressing the adjoints of the singularly perturbed game equations (21)-(23) (denoted by the superscript ε) as

$$\begin{aligned} \vec{\lambda}_R^{\varepsilon} &= \vec{\lambda}_R \\ \vec{\lambda}_{V_P}^{\varepsilon} &= \varepsilon\vec{\lambda}_{V_P} \\ \vec{\lambda}_{V_E} &= \varepsilon\vec{\lambda}_{V_E} \end{aligned} \tag{38}$$

$$\mathscr{H}^i = 1 + \vec{\lambda}_R^i \cdot (\vec{V}_E^i - \vec{V}_P^i) + \vec{\lambda}_{V_P}^i \cdot (\vec{\omega}_P^i \times \vec{V}_P^i) + \vec{\lambda}_{V_E}^i \cdot (\vec{\omega}_E^i \times \vec{V}_E^i) \tag{39}$$

The adjoint equations of the boundary layer are

$$\frac{d\vec{\lambda}_R^i}{d\tau} = -\varepsilon \frac{\partial \mathscr{H}^i}{\partial \vec{R}^i} = 0 \Rightarrow \vec{\lambda}_R^i = \text{const} \tag{40}$$

$$\frac{d\vec{\lambda}_{V_P}^i}{d\tau} = - \frac{\partial \mathscr{H}^i}{\partial \vec{V}_P^i} = \vec{\lambda}_R^i + \left(\vec{\omega}_P^i \times \vec{\lambda}_{V_P}^i\right) \tag{41}$$

$$\frac{d\lambda_{V_E}^i}{d\tau} = - \frac{\partial \mathscr{H}^i}{\partial \vec{V}_E^i} = -\vec{\lambda}_R^i + \left(\vec{\omega}_E^i \times \vec{\lambda}_{V_E}^i\right) \tag{42}$$

The matching condition requires

$$\vec{\lambda}_R^i = \vec{\lambda}_R^{\circ} \tag{43}$$

where $\vec{\lambda}_R^{\circ}$ is already determined by equations (31) and (32). The other adjoint vectors have to satisfy

$$\lim_{\tau\to\infty} \vec{\lambda}_{V_P}^i(\tau) = \vec{\lambda}_{V_P}(t_f) = 0 \tag{44}$$

$$\lim_{\tau\to\infty} \vec{\lambda}_{V_E}^i(\tau) = \vec{\lambda}_{V_E}(t_f) = 0 \tag{45}$$

The optimality conditions, similar to equation (10), lead to

$$(\vec{\omega}_P^i)^* = \Omega_P \frac{\vec{\lambda}_{V_P}^i \times \vec{V}_P^i}{|\vec{\lambda}_{V_P}^i \times \vec{V}_P^i|} \tag{46}$$

$$(\vec{\omega}_E^i)^* = -\Omega_E \frac{\vec{\lambda}^i_{V_E} \times \vec{V}^i_E}{|\vec{\lambda}^i_{V_E} \times \vec{V}^i_E|} \tag{47}$$

It can be already seen that equations (35), (41), (44), and (46) are independent of equations (36), (42), (45), and (47), and consequently each of the equation sets can be solved separately.

Substituting equations (43) and (46) into equation (41) yields

$$\frac{d\vec{\lambda}^i_{V_P}}{d\tau} = \vec{\lambda}^\circ_R + \Omega_P \frac{\left(\vec{\lambda}^i_{V_P} \times \vec{V}^i_P\right) \times \vec{\lambda}^i_{V_P}}{|\vec{\lambda}^i_{V_P} \times \vec{V}^i_P|} \tag{48}$$

Using the triple vector product identity

$$(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a}(\vec{b} \cdot \vec{c}) - \vec{b}(\vec{a} \cdot \vec{c}) \tag{49}$$

we can rewrite equation (48) in a simplified form,

$$\frac{d\vec{\lambda}^i_{V_P}}{d\tau} = \vec{\lambda}^\circ_R + k_1\vec{\lambda}^i_{V_P} + k_2\vec{V}^i_P \tag{50}$$

where k_1 and k_2 are scalar multipliers. Integrating equation (50) backwards from $\vec{\lambda}^i_{V_P}(t_f) = 0$, we observe that initially the derivative vector lies in the plane defined by $\vec{\lambda}^\circ_R$ and $\vec{V}^i_P$. Consequently, the vector $\vec{\lambda}^i_{V_P}$ itself will grow (in the retrograde sense) in this plane only and can be expressed by

$$\vec{\lambda}^i_{V_P}(\tau) = k_3(\tau)\vec{\lambda}^\circ_R + k_4(\tau)\vec{V}^i_P \tag{51}$$

Substituting equation (51) into equation (46) reveals that

$$(\vec{\omega}_P^i)^* = \Omega_P \frac{\vec{\lambda}^\circ_R \times \vec{V}^i_P}{|\vec{\lambda}^\circ_R \times \vec{V}^i_P|} \tag{52}$$

Similarly we obtain for the evader

$$(\vec{\omega}_E^i)^* = \Omega_E \frac{\vec{\lambda}^\circ_R \times \vec{V}^i_E}{|\vec{\lambda}^\circ_R \times \vec{V}^i_E|} \tag{53}$$

Since by equations (31) and (32) $\vec{\lambda}^\circ_R$ is parallel to R°, we have the control strategies in the form

$$(\vec{\omega}_P^i)^* = \Omega_P \frac{\vec{R}_o \times \vec{V}_P^i}{|\vec{R}_o \times \vec{V}_P^i|} \tag{54}$$

$$(\vec{\omega}_E^i)^* = \Omega_E \frac{\vec{R}_o \times \vec{V}_E^i}{|\vec{R}_o \times \vec{V}_E^i|} \tag{55}$$

These boundary-layer strategies imply that each vehicle should turn in the plane defined by the initial line-of-sight vector and its own velocity. A uniformly valid composite zero-order feedback solution (denoted by the superscript c) is obtained if the initial condition $\vec{R}_o$ is replaced in equations (54) and (55) by the current value of the line-of-sight vector $\vec{R}$ (and the superscripts i omitted):

$$\vec{\omega}_P^c = \Omega_P \frac{\vec{R} \times \vec{V}_P}{|\vec{R} \times \vec{V}_P|} \tag{56}$$

$$\vec{\omega}_E^c = \Omega_E \frac{\vec{R} \times \vec{V}_E}{|\vec{R} \times \vec{V}_E|} \tag{57}$$

The accuracy of this approximation depends on the time-scale separation of the variables, that is, on the smallness of the perturbation parameters

$$\varepsilon_j = r_j/|\vec{R}_o| \qquad j = P, E \tag{58}$$

where r is defined in equation (20).

The Optimal Solution

To our knowledge, the complete optimal solution of the game has not been published. Moreover, the results that are available have created some controversy. In several investigations (e.g., refs. 26 and 27) it was indicated that the three-dimensional game tends to take place in an inclined plane. In a recent paper (ref. 28) it is shown, however, that the optimal strategies of this game, starting with noncoplanar initial conditions, are essentially three dimensional. The singular-perturbation analysis can resolve this apparent contradiction. As a matter of fact it extends the original two-dimensional solution of Simakova (ref. 31) to a three-dimensional space. Applying the same approach that was taken in the boundary-layer-game equations (48)-(54) to the exact-game equations (1)-(19) shows that the optimal maneuvers of the players take place in two fixed planes, determined by the final (a priori unknown) direction of the line of sight and the respective velocity vectors. This open-loop solution, expressed in a nonrotating coordinate system, embeds the results of references 25-27, showing that they are complementary and do not contradict each other. The optimal strategies obtained in reference 27 are expressed in a coordinate system aligned with the line of sight and, therefore, the existence of the fixed maneuver planes is not transparent. It also agrees with the observation that if the game starts in (or reaches) a plane, the optimal trajectories are two dimensional.

It is also obvious that if the ratios of the turning radii of the respective players to the initial distance of separation (i.e., the singular-perturbation parameters of the game) are small, the rotation of the line-of-sight vector is negligible, and the current direction of this vector, used in the zero-order feedback strategies, is a good approximation to its final orientation.

III. VARIABLE-SPEED HORIZONTAL PURSUIT-EVASION

Singular-Perturbation Analysis

In variable-speed pursuit-evasion games there exists also a time-scale separation between the velocity and turning dynamics of the players. For such cases the method of forced singular perturbations is particularly rewarding since it has the potential to generate a multiple time-scale differential game. Such a singularly perturbed game, consisting of several consecutively faster subgames, is solvable in a closed form. An example of the variable-speed pursuit-evasion game in the horizontal plane was analyzed first in reference 29. Only recently has it become possible to validate this analysis (ref. 28) by comparing its solution with the open-loop optimal solution of the same game (ref. 30).

Because of the limited scope of the present paper, only the main results of the singular-perturbation analysis and the more important aspects of the comparison are summarized in this section. For further details the reader is referred to the original report (ref. 28).

For the forced singular-perturbation analysis, the variable-speed pursuit-evasion game between two airplanes in the horizontal plane is described by the following multiple time-scale mathematical model (variables are defined in fig. 1):

$$\dot{R} = V_E \cos(\chi_E - \psi) - V_P \cos(\chi_P - \psi) \quad ; \quad R(t_o) = R_o \tag{59}$$

$$\dot{\psi} = [V_E \sin(\chi_E - \psi) - V_P \sin(\chi_P - \psi)]/R \quad ; \quad \psi(t_o) = \psi_o \tag{60}$$

$$\varepsilon\dot{V}_P = g[\xi_P(T_{max})_P - (D_o)_P - n_P^2(D_i)_P]/W_P \quad ; \quad V_P(t_o) = V_{P_o} \tag{61}$$

$$\varepsilon\dot{V}_E = g[\xi_E(T_{max})_E - (D_o)_E - n_E^2(D_i)_E]/W_E \quad ; \quad V_E(t_o) = V_{E_o} \tag{62}$$

$$\varepsilon^2\dot{\chi}_P = g(n_P^2 - 1)^{1/2}/V_P \quad ; \quad \chi_P(t_o) = \chi_{P_o} \tag{63}$$

$$\varepsilon^2\dot{\chi}_E = g(n_E^2 - 1)^{1/2}/V_E \quad ; \quad \chi_E(t_o) = \chi_{E_o} \tag{64}$$

In this model, the maximum thrust of the airplanes "T_{max}," as well as their zero-lift drag "D_o" and the straight-flight-induced drag "D_i," is a known function of the velocity. Weight W is assumed to be constant. The control variables of the players are the throttle setting

$$0 \le \xi \le 1 \tag{65}$$

and the aerodynamic load factor (or lift to weight ratio)

$$n \triangleq (L/W) \tag{66}$$

This last control variable is subject to two different constraints; a structural limit

$$n \leq n_{max} \tag{67}$$

and an aerodynamic (speed-dependent) one:

$$n \leq n_L(V) \tag{68}$$

Singular perturbation analysis (refs. 28 and 29) leads to an approximation of the optimal-control strategies in an explicit feedback form. These strategies are similar for both players. They require full thrust

$$\xi^c = 1 \tag{69}$$

and a load factor given by the equation

$$(n^c)^2 = 1 + \frac{V}{V^r - V} \cdot \frac{T_{max} - (D_o + D_i)}{D_i} [1 - \cos(\psi - \chi)] \tag{70}$$

but subject to the constraints of equations (67) and (68). In this expression, V^r is the velocity in the reduced-order game, which is estimated to be the maximum velocity of the aircraft.

As can be seen, equation (70) expresses the required load factor as an explicit function of the state variables (ψ,χ,V) and the performance parameters (thrust, drag) of the airplanes. Note that no range measurements or estimation of the opponent's parameters are required. Based on equation (70), a simple feedback chart, as shown in figure 2, can be prepared for each airplane.

The very form of equation (70) provides an important insight into the nature of the suboptimal maneuver, which can be summarized as follows:

1. The required load factor can be related to the highest constant speed (sustained) turning performance since

$$\dot{\chi}(\dot{V} = 0) = \frac{g}{V}\left[\frac{T_{max} - (D_o + D_i)}{D_i}\right] \tag{71}$$

indicating the compromise between a fast turn and longitudinal acceleration, both being necessary for a successful pursuit-evasion.

2. The function $[1 - \cos(\psi - \chi)]^{1/2}$ is almost linear for $|\psi - \chi| < 60°$. Therefore, the required turning rate can be considered, for small deviations from the line of sight, as a proportional control with a speed-dependent gain.

3. A part of this gain, given by $V/(V^r - V)$, provides an additional insight. It indicates that if the speed is near its predicted value in the reduced game (i.e.,

when there is a little need for acceleration), a faster turn can be made. However, if $(V^r - V)$ is large, it is better not to lose velocity by making a sharp turn, but to use the excess thrust for acceleration.

4. The control strategies (eqs. (69)-(70)) predict that the trajectory of both airplanes will asymptotically approach the line-of-sight vector and that the engagement will end by an accelerating "tail chase."

Comparison with the Optimal Solution

The zero-order feedback approximation obtained by singular-perturbation analysis was compared recently (ref. 28) with the open-loop optimal solution of the same game (ref. 30). This solution is based on using the direction of the terminal line of sight as a reference. In this particular coordinate system, the necessary conditions of game optimality are decoupled into two one-sided optimization problems. Thus, extremal trajectories can be generated by backward integration from the terminal surface independently. A recent implementation of this approach (ref. 39) makes it possible to compare trajectories and time-histories of variables to suboptimal candidates, such as the zero-order feedback approximation. Initial conditions of an example are depicted in figure 3. Those initial conditions were obtained by simultaneous backward integration of two different trajectories (one for the pursuer and one for the evader) from different end conditions satisfying $V_P(t_f) > V_E(t_f)$, using an identical aircraft model from the open literature (ref. 40). Note that the initial separation in the x-direction can be adjusted by selecting different capture ranges $(R_f = d)$.

The time-histories of the control variables seem to be similar, even for a relatively short initial range (only 4 times larger than the pursuer's turning radius), as can be seen in figures 4 and 5. Inspection of turning time-histories in figures 6 and 7 reveals more differences between the optimal solution and the feedback approximation. The evader, using the suboptimal strategy, reaches the actual line of sight very nearly at the same time as the open-loop optimal trajectory closely approaches the final (reference) direction. His turning strategy can, therefore, be considered as close to optimal. One cannot say the same about the pursuer. Since the initial conditions of the pursuer are more favorable than those of the evader, the pursuer can reach the line of sight in a shorter time. Because the evader has not yet completed his turn, the line of sight continues to rotate. Consequently, the pursuer, using a suboptimal strategy, has made an unnecessary turn (an overshoot) which has to be corrected later. As a result, it can be expected that the capture time of the suboptimal game will be longer than the optimal one. Indeed, for this example, the time of capture predicted by the feedback approximation is about 6.5% higher than the optimal value of 97.1 sec.

The major reason for this level of inaccuracy is the relatively large value of the singular-perturbation parameter $(\varepsilon_P = r_P/R_o = 0.25)$ in the example. By changing the capture range R_f, and accordingly the initial distance of separation R_o, a large set of pursuit-evasion games is generated and the suboptimal outcomes of the games are compared with the optimal time of capture. The results of this comparison, presented in figure 8, demonstrate a very satisfactory accuracy for small values $(\varepsilon_P < 0.125)$ of the perturbation parameter. Since in this example the turning requirement of the pursuer is less than that of the evader, the suboptimal strategy of the pursuer deviates more from the optimal than that of the evader. Consequently the actual time of capture obtained when both players are using zero-order feedback strategies is longer than the optimal value.

In summary, the zero-order, singular-perturbation strategies preserve all of the essential elements of the optimal solution and provide a good payoff accuracy for sufficiently large initial distances.

CONCLUSIONS

It has been shown that singular-perturbation methods are useful mathematical tools for analyzing a class of interesting nonlinear pursuit-evasion games. The explicit analytical form of the zero-order control strategies indicates the simplicity of an eventual implementation. This point is emphasized by the fact that in spite of the assumption of perfect information made in the game formulation, the suboptimal strategies depend only on the relative geometry and on own-state measurements.

The analytical approximation generates an insight, which hardly can be obtained from an open-loop optimal solution. This point has been clearly illustrated in both examples discussed here. The payoff accuracy of the zero-order, singular-perturbation approximation was demonstrated for a variable-speed pursuit-evasion game in the horizontal plane.

Variable-speed pursuit-evasions in the vertical plane and in three-dimensional space were also analyzed recently by using a variable-modeling forced singular-perturbation technique. This last investigation is reported in a contemporary paper (ref. 41).

REFERENCES

1. Isaacs, R.: Differential Games. John Wiley and Sons, New York, 1967.

2. Friedman, A.: Existence of Value and of Saddle Points for Differential Games of Pursuit and Evasion. J. Diff. Eqs., vol. 7, 1970, pp. 92-110.

3. Merz, A. W.: The Homicidal Chauffeur, A Differential Game. SUDAAR Report No. 418, Department of Aeronautics and Astronautics, Stanford U., Stanford, Calif., 1971.

4. Merz, A. W.: The Game of Two Identical Cars. J. Optimization Theory Appl., vol. 9, no. 5, 1972, pp. 324-343.

5. Breakwell, J. V.: Pursuit of a Faster Evader. The Theory and Applications of Differential Games, J. D. Grote, ed., Reidel, 1975.

6. Lewin, J.; and Breakwell, J. V.: The Surveillance Evasion Game of Degree. J. Optimization Theory Appl., vol. 16, nos. 3/4, 1975.

7. Breakwell, J. V.: Some Differential Games with Interesting Discontinuities. Department of Aeronautics and Astronautics, Stanford U., Stanford, Calif., 1973.

8. Bernhard, P.: Singular Surfaces in Differential Games, An Introduction. Proceedings of the Workshop on Differential Games and Applications, Enschede, Holland, 1977.

9. Roberts, D. A.; and Montgomery, R. C.: Development and Application of a Gradient Method for Solving Differential Games. NASA TN D-6502, 1971.

10. Anderson, G. M.: A Near Optimal Closed-Loop Solution Method for Non-Singular Zero-Sum Differential Games. J. Optimization Theory Appl., vol. 13, no. 3, 1974.

11. Jarmark, B.: Convergence Control in Differential Dynamic Programming Applied to Air-to-Air Combat. AIAA J., vol. 14, no. 1, Jan. 1976.

12. Miller, L. E.: Differential Game Computer Program. AFFDL TM 74-66-FXG, Nov. 1977.

13. Kokotovic, P. V.: A Control Engineers Introduction to Singular Perturbations. Joint Automatic Control Conference, Stanford, Calif., Aug. 1972.

14. Freedman, M. I.; and Granoff, B.: Formal Asymptotic Solution of a Singularly Perturbed Nonlinear Optimal Control Problem. J. Optimization Theory Appl., vol. 19, 1976, pp. 301-325.

15. Kokotovic, P. V.; O'Malley, R. E., Jr.; and Sanutti, P.: Singular Perturbations and Order Reduction in Control Theory — An Overview. Automatica, vol. 12, 1977, pp. 123-132.

16. O'Malley, R. E., Jr.: Singular Perturbations and Optimal Control. Mathematical Control Theory, Springer Verlag Lecture Notes in Mathematics, 1978, p. 680.

17. Mehra, R. K.; Washburn, R. B.; Sajon, S.; and Cornell, J. V.: A Study of the Application of Singular Perturbation Theory. NASA CR-3167, 1979.

18. Gardner, B. F., Jr.: Zero-Sum Nash Strategies for Systems with Fast and Slow Modes. Proceedings 15th Allerton Conference on Communication, Computers, and Controls, U. Illinois, Urbana, Ill., 1977.

19. Gardner, B. F., Jr.; and Cruz, J. B., Jr.: Well Posedness of Singularly Perturbed Nash Games. J. Franklin Inst., vol. 306, 1978, pp. 355-374.

20. Khalil, M. K.: Multimodelling and Multiparameter Singular Perturbation in Control and Game Theory. Ph.D. Dissertation, U. Illinois, Urbana, Ill., 1978.

21. Khalil, H. K.; and Kokotovic, P. V.: Feedback and Well Posedness of Singularly Perturbed Nash Games. IEEE Trans. Auto. Cont. AC-24, vol. 699, 1979.

22. Farber, N.; and Shinar, J.: Approximate Solution of Singularly Perturbed Nonlinear Pursuit-Evasion Games. J. Optimization Theory Appl., vol. 32, no. 1, Sept. 1980, pp. 39-73.

23. Shinar, J.: Solution Techniques for Realistic Pursuit-Evasion Games. Control and Dynamics Systems, C. T. Leondes, ed., vol. 17, Academic Press, N.Y., 1981, pp. 64-125.

24. Farber, N.: Improved Dynamic Models for Air Combat Analysis as a Zero-Sum Differential Game. D.Sc. Dissertation, Technion Israel Institute of Technology, Haifa, Israel, 1981 (in Hebrew).

25. Lynch, U. H. D.: Games Barriers and Their Application in Air-to-Air Combat. Ph.D. Dissertation, Air Force Institute of Technology, Wright-Patterson AFB, 1973.

26. Miller, E. L.: Application of Differential Game to Air-to-Air Combat Problems. AFFDL-TR-7547, 1975.

27. Miloh, T.: A Note on Three Dimensional Pursuit-Evasion Game with Bounded Curvature. To be published in the Journal of Guidance and Control.

28. Shinar, J.: Validation of Zero-Order Feedback Strategies for Medium Range Horizontal Air-to-Air Interception. NASA TM-84237, 1982.

29. Farber, N.; and Shinar, J.: An Approximate Feedback Solution of a Variable Speed Nonlinear Pursuit-Evasion Game Between Two Airplanes in a Horizontal Plane. AIAA Paper 80-1597, Danvers, Mass., Aug. 1980, pp. 337-347.

30. Rajan, N.; Prashad, U. R.; and Rao, N. J.: Pursuit-Evasion of Two Aircraft in a Horizontal Plane. Journal of Guidance and Control, vol. 3, no. 3, May-June 1982, pp. 261-267.

31. Simakova, E. N.: Differential Pursuit Game. Automatika i Telemekhannika, no. 2, Feb. 1967, pp. 5-14.

32. Rublein, G. T.: On Pursuit with Curvature Constraints. SIAM Journal of Control, vol. 10, 1972, pp. 37-43.

33. Kelley, H. J.: Aircraft Maneuver Optimization by Reduced Order Approximation. Control and Dynamic Systems, vol. 10, C. T. Leondes, ed., Academic Press, 1973, pp. 131-178.

34. Ardema, M. K.: Solution of the Minimum Time to a Climb Problem by Matched Asymptotic Expansions. AIAA J., vol. 14, no. 7, 1976, pp. 843-850.

35. Calise, A. J.: Extended Energy Management Methods for Flight Performance Optimization. AIAA J., vol. 15, 1977, pp. 314-321.

36. Sesak, J. R.; and Higgins, T. J.: Sensitivity Constrained Linear Optimal Control via Forced Singular Perturbation Model Reduction. Proceedings IEEE 17th Conference on Decision and Control, Jan. 1979, pp. 1145-1146.

37. Shinar, J.; and Merari, A.: Aircraft Performance Optimization by Forced Singular Perturbation. 12th Congress of ICAS, Munich, Germany, Oct. 1980.

38. Shinar, J.: Remarks on Singular Perturbation Technique Applied in Nonlinear Optimal Control. 2nd IFAC Workshop on Control Applications of Nonlinear Programming and Optimization, Oberpfaffenhofen, Germany, Sept. 1980.

39. Rajan, N.; and Ardema, M. D.: Barrier and Dispersal Surfaces in Minimum Time Interception. NASA TM-84241, 1982.

40. Bryson, A. E., Jr.; Desai, N. M.; and Hoffman, W. L.: The Energy State Approximation in Performance Optimization of Supersonic Aircraft. J. Aircraft, vol. 6, no. 6, Nov.-Dec. 1969, pp. 481-487.

41. Shinar, J.; Farber, N.; and Negrin, M.: A Three-Dimensional Air Combat Game Analysis by Forced Singular Perturbation. AIAA Paper 82-1327, San Diego, Calif., Aug. 1982.

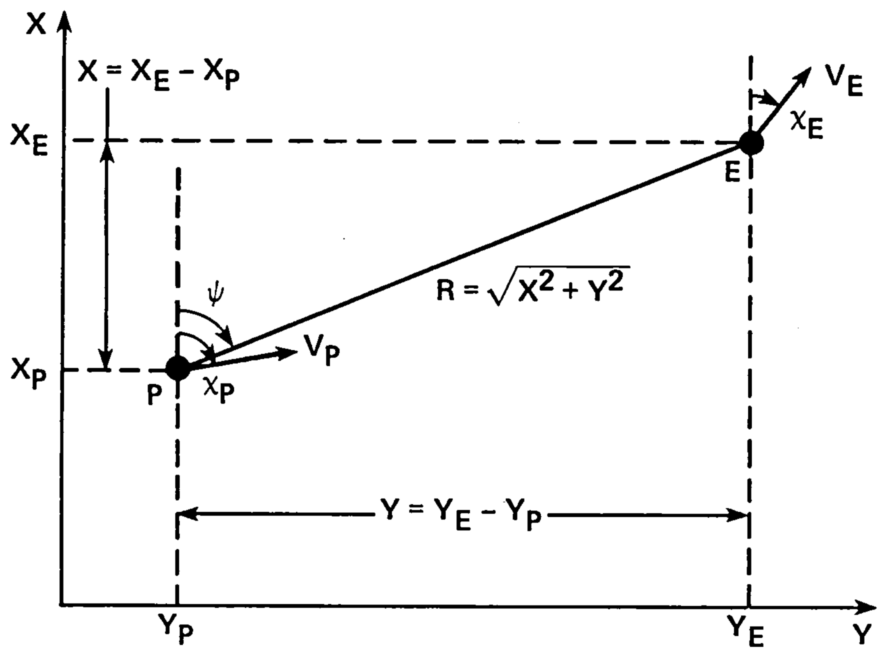

Figure 1. Geometry of horizontal pursuit evasion.

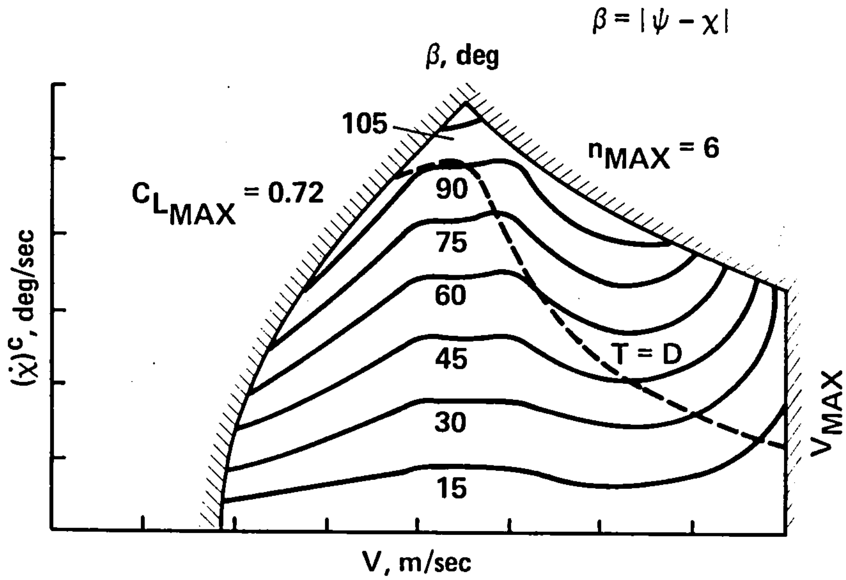

Figure 2. Feedback chart for suboptimal turning.

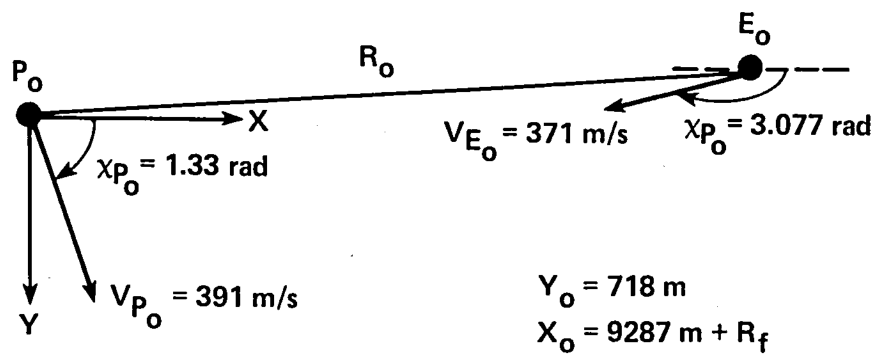

Figure 3. Initial conditions for example.

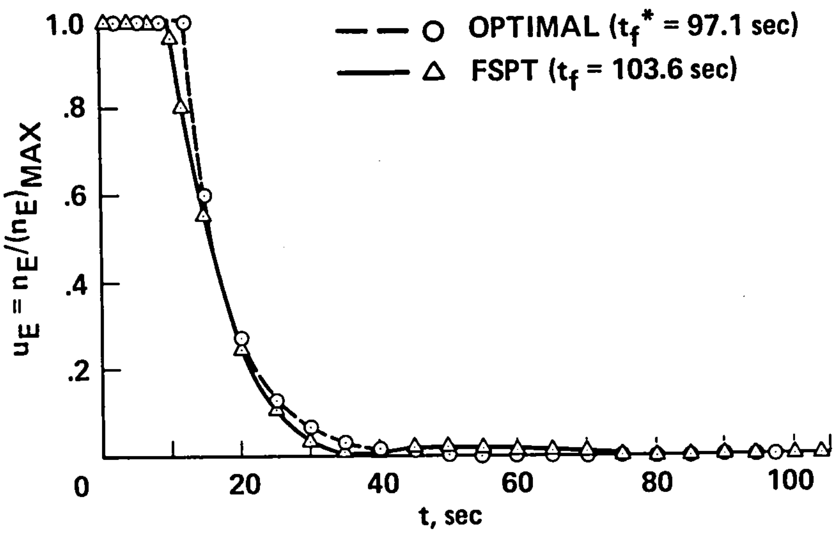

Figure 4. Evader's load factor time-history.

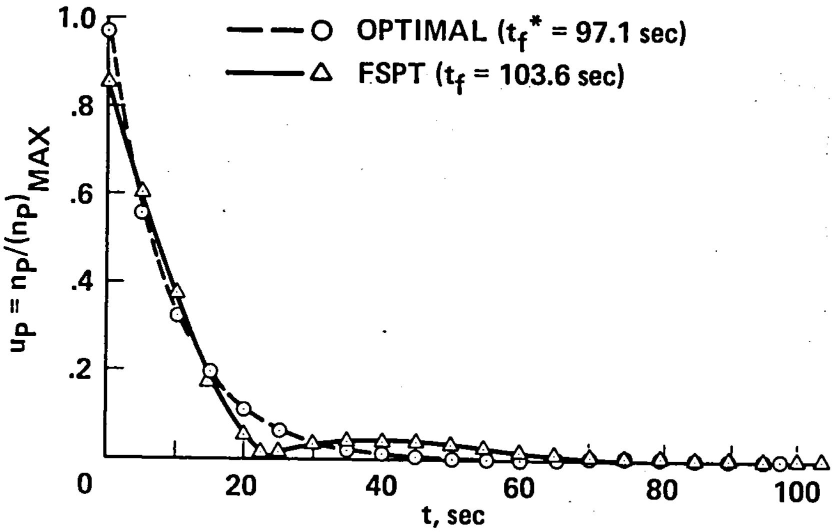

Figure 5. Pursuer's load factor time-history.

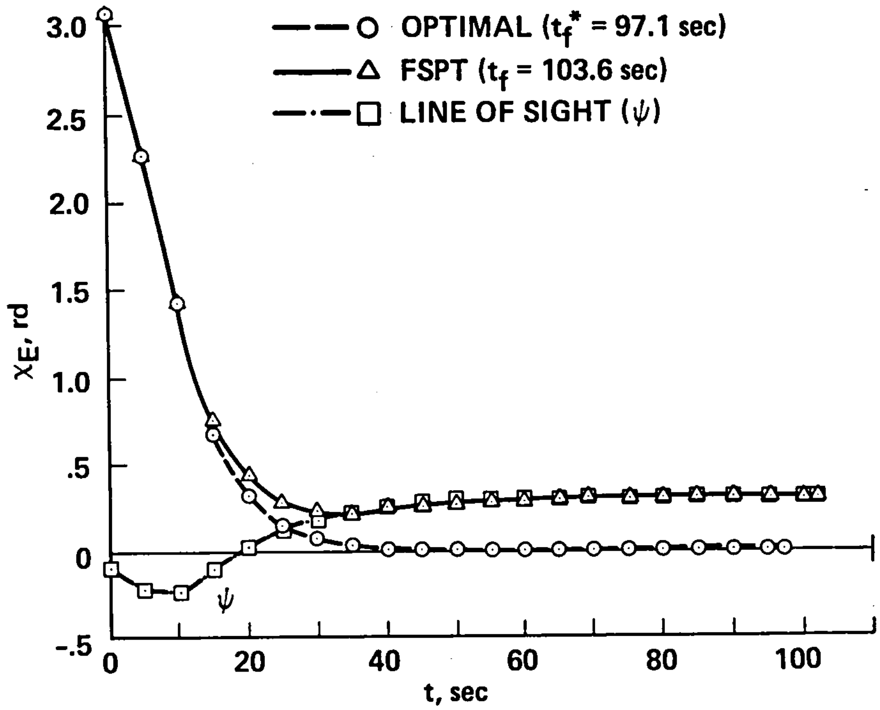

Figure 6. Evader's turning time-history.

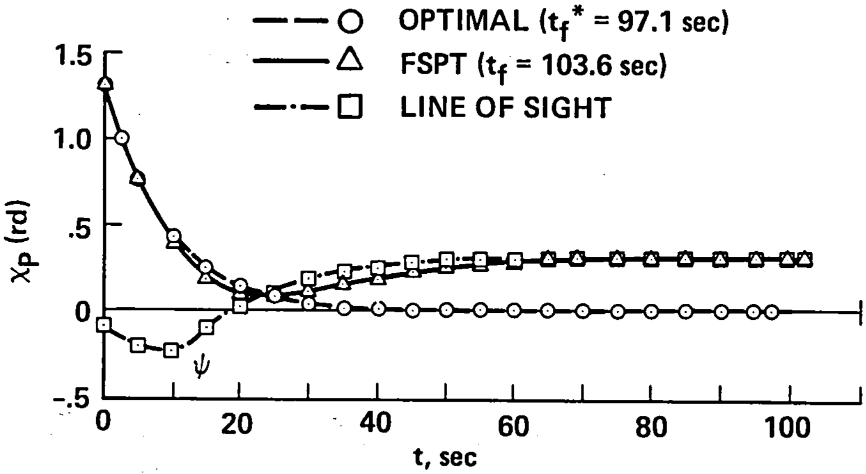

Figure 7. Pursuer's turning time-history.

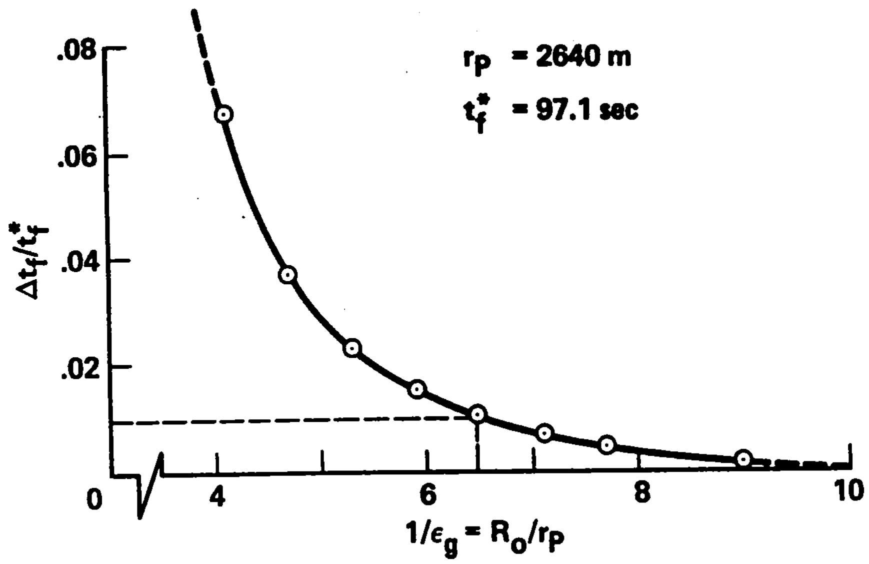

Figure 8. Payoff accuracy of the zero-order feedback solution.

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA TM 84271		

4. Title and Subtitle	
NONLINEAR ZERO-SUM DIFFERENTIAL GAME ANALYSIS BY SINGULAR PERTURBATION METHODS	5. Report Date: July 1982
	6. Performing Organization Code
7. Author(s): J. Shinar and N. Farber	8. Performing Organization Report No.: A-8998
9. Performing Organization Name and Address: NASA Ames Research Center, Moffett Field, California 94035	10. Work Unit No.: T-33094
	11. Contract or Grant No.
12. Sponsoring Agency Name and Address: National Aeronautics and Space Administration, Washington, D.C. 20546	13. Type of Report and Period Covered: Technical Memorandum
	14. Sponsoring Agency Code: 505-34-11-11-00

15. Supplementary Notes

Point of contact: Dr. Heinz Erzberger, M/S 210-9, Ames Research Center, Moffett Field, CA 94035, (415) 965-5450, FTS 448-5450

16. Abstract

A certain class of nonlinear, zero-sum differential games, exhibiting time-scale separation properties, is suitable for analysis by singular-perturbation techniques. The merits of such an analysis, leading to an approximate game solution, as well as the "well-posedness" of the formulation, are discussed. The singular perturbation approach is shown to be particularly attractive for investigating pursuit-evasion problems; the original multidimensional differential game is decomposed to a "simple pursuit" (free-stream) game and two independent (boundary-layer) optimal-control problems. Using multiple time-scale boundary-layer models in such games results in a pair of uniformly valid zero-order composite feedback strategies. Though a priori perfect information is assumed, the suboptimal strategies depend only on relative geometry and own-state measurements. This is demonstrated by a three-dimensional, constant-speed example. For game analysis with realistic vehicle dynamics, the technique of forced singular perturbations and a variable modeling approach is proposed. Accuracy of the zero-order singular-perturbation analysis is evaluated by comparison with the exact (numerical) solution of a time-optimal, variable-speed "game of two cars" in the horizontal plane.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Nonlinear game analysis Pursuit-evasion games Singular perturbations Feedback strategies	Unlimited Subject category 05

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	29	A03

*For sale by the National Technical Information Service, Springfield, Virginia 22161