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Construction of Solutions for Some Nonlinear Two-Point Boundary Value Problems

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ABSTRACT

Constructive existence and uniqueness results for boundary value problems associated with some simple special cases of the second order equation $y'' = f(x,y,y')$, $0 \le x \le 1$, are sought. The approach we consider is to convert the differential equation and boundary conditions to an integral equation via Green's functions, and then to apply fixed point and contraction map principles to a sequence of successive approximations. The approach is tested on several applied problems. Difficulties in trying to prove general theorems are discussed.

1. INTRODUCTION. (General Problem of Interest.)

The topic of interest is the establishment of constructive existence and uniqueness for nonlinear, two-point boundary value problems

$$y'' = f(x,y,y'), \qquad 0 \le x \le 1, \tag{1.1a}$$

$$a_1y(0) + a_2y'(0) = c_1, \qquad b_1y(1) + b_2y'(1) = c_2. \tag{1.1b}$$

By constructive existence and uniqueness, we mean results whose proofs suggest a method for computing the solution numerically. The type of approach considered involves one of many already known methods for obtaining solutions numerically. The equation and boundary conditions are converted to an integral equation via Green's functions. Then the solution of the integral equation is sought using successive approximations. However, the equation is first parametrized in a special way which depends on conditions assumed to be satisfied by f. Some new existence and uniqueness results can then be obtained.

For the initial value problem associated with (1a), there exists theory that assures a unique local solution for a large class of equations. However, the question of existence and uniqueness or just existence for boundary value problems associated with (1a) is hard to answer unless very strong assumptions are made on f. Even in many recent papers on existence and uniqueness, assumptions made on f usuall include a condition (such as a Lipschitz condition, a boundedness condition, or nondecreasing behavior) in the argument y for all y. Thus, although they may contain more general results with weaker conditions on f than in earlier comprehensive works (such as Keller [1], and Bailey, Shampine and Waltman [2]) they can still be too strict for many applications. Also, proofs are not always constructive in nature, and in many practical applications solutions are computed without establishing existence.

Our basic idea is to seek constructive existence and uniqueness for various cases of (1) based on assumptions on f that hold only for y(x) that satisfy a certain constraint. Although our primary objective is to have results which enable the solution to be computed numerically based on the constructive nature of the proofs, we have a secondary goal. We would like numerical methods to lend themselves to the application of results from research in vector acceleration of sequences and series.

2. MOTIVATION.

In the interest of applied mathematics, we can show problems which arise naturally in the applied sciences and which exemplify the general problem (1.1). Examples of problems (1.1) can be shown to arise for instance from problems in heat transfer, problems in the analysis of chemical reactions, and problems in fluid mechanics. In particular, we mention the following.

In the analysis of the stagnation point shock layer [3], it is shown that the total enthalpy is governed by

$$\text{(2.1a)} \qquad y''(x) = KR(y(x))^n - xRy'(x), \qquad n \geq 1, \qquad 0 \leq x \leq 1,$$

$$\text{(2.1b)} \qquad y(0) = 0, \qquad y(1) = 1.$$

The quantities K and R are positive constants characterized as a radiation loss parameter and a Reynolds number respectively.

In a problem concerning the analysis of heat and mass transfer in a porous catalyst [4], the following boundary value problem is obtained.

$$\text{(2.2a)} \qquad y'' = \alpha\, y \exp\left(\frac{\gamma\beta(1-y)}{1 + \beta(1-y)}\right)$$

(2.2b) $$y'(0) = 0, \qquad y(1) = 1.$$

The quantities γ, β and α are positive constants representative of dimensionless energy of activation, heat evolution and Thiele's modulus respectively.

We also consider vector cases of problem (1.1). In particular, for the 2 by 2 case, $y(x) = (y_1(x), y_2(x))^T$, and $f = (f_1, f_2)^T$. A general form of the linear separated boundary conditions would be

$$A_1 y(0) + A_2 y'(0) = C_1, \qquad B_1 y(1) + B_2 y'(1) = C_2$$

where A_1, A_2, B_1 and B_2 are two by two matrices and $C_1 = (c_{11}, c_{21})^T$ and $C_2 = (c_{12}, c_{22})^T$. Examples can be shown to arise in fluid mechanics. For instance, consider the following.

The unsteady squeezing of a viscous fluid between two parallel plates is discussed in [5]. With the normal velocity prescribed, the unsteady Navier-Stokes equations admit a similarity solution. The similarity equation for the axisymmetric case is

(2.3a) $$S(xf''' + 3f'' - ff''') = f'''',$$

(2.3b) $$f(0) = f''(0) = 0, \qquad f(1) = 1, \qquad f'(1) = 0.$$

which can be written as a vector case of (1.1),

(2.4a) $$\begin{pmatrix} y_1'' \\ y_2'' \end{pmatrix} = \begin{pmatrix} y_2 \\ S(xy_2' + 3y_2 - y_1 y_2') \end{pmatrix}$$

(2.4b) $$y_1(0) = 0,\ y_2(0) = 0, \qquad y_1(1) = 1,\ y_1'(1) = 0.$$

where $y_1 = f$ and $y_2 = f''$.

For more examples, see Appendix A.

3. A BASIC RESULT.

Investigation into the construction of theorems for various subclasses of problem (1.1) is based on a result in [6] established for the problem

(3.1a) $$y'' = f(x,y), \qquad 0 \le x \le 1,$$

(3.1b) $$y(0) = 0, \qquad y(1) = 0.$$

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where the gradient y' does not appear explicitly. This result is an extension of a similar result due to Keller [1]. First, subtract k^2y from both sides of (3.1a) and consider the equivalent problem

(3.2) $$y'' - k^2y = f(x,y) - k^2y, \qquad y(0) = y(1) = 0.$$

Then, for $k^2 \neq 0$, (3.2) can be converted by the Green's function procedure into the equivalent integral equation

(3.3a) $$y(x) = \int_0^1 g_k(x,\xi)\,(k^2\,y(\xi) - f(\xi,y(\xi)))\,d\xi$$

where

(3.3b) $$g_k(x,\xi) = \frac{1}{k \sinh k} \begin{cases} \sinh kx \sinh k(1-\xi), & 0 \le x < \xi, \\ \sinh k(1-x) \sinh k\xi, & \xi < x \le 1. \end{cases}$$

The theorem in [6] that we are refering to is the following.

Theorem 0. In the boundary value problem (3.1), let $\partial f/\partial y$ be continuous for all $x \in [0,1]$ and all y. Suppose that there exists $N > 0$ and $\delta \ge 0$ such that $0 \le \delta \le \partial f/\partial y \le N$ for all $x \in [0,1]$ and all y. Then a unique solution of (3.1) exists. For $k^2 = (1/2)\,(\delta + N)$, it is given by the limit of the convergent sequence of functions

(3.4a) $$y^0(x) = 0,$$

(3.4b) $$y^{m+1}(x) = \int_0^1 g_k(x,\xi)\,[k^2y^m(\xi) - f(\xi,y^m(\xi))]\,d\xi, \qquad m = 0,1,\cdots.$$

Proof. Let

(3.5) $$e^{m+1}(x) = y^{m+1}(x) - y^m(x)$$

and

(3.6) $$\|e^{m+1}\| = \max_{0\le x\le 1} |e^{m+1}(x)|, \qquad m = 0,1,\cdots.$$

Then for $m = 1,2,\cdots$, we can apply the mean-value theorem to $f(x,$ [illegible] $)) -$ $f(x,y^{m-1}(x))$ to obtain

$$(3.7) \qquad e^{m+1}(x) = \int_0^1 g_k(x,\xi)\,[k^2 - \frac{\partial f}{\partial y}(\xi, y^m(\xi) - \theta(\xi)e^m(\xi))]\, e^m(\xi)\, d\xi,$$

where $0 < \theta(\xi) < 1$. Note that $g_k(x,\xi) \geq 0$, and with the choice

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$$(3.8) \qquad k^2 = (1/2)\,(\delta + N)$$

and the bounds on $\partial f/\partial y$, the bracketed term in the integrand of (3.7) satisfies $0 \leq |k^2 - \partial f/\partial y| \leq (1/2)\,(N - \delta)$. Therefore, from (3.7),

$$|e^{m+1}(x)| \leq \frac{1}{2}\,(N - \delta) \int_0^1 g_k(x,\xi)\, d\xi \cdot \|e^m\|,$$

$$(3.9) \qquad = (1/2)(n - \delta)\,\frac{1}{k^2}(1 - \frac{\cosh k((1/2)-x)}{\cosh(k/2)})\,\|e^m\|$$

$$\leq \frac{N - \delta}{N + \delta}\,(1 - \frac{1}{\cosh(k/2)})\,\|e^m\|, \qquad m = 1,2,\cdots.$$

Since this relation holds for all $x \in [0,1]$,

$$(3.10) \qquad \|e^{m+1}\| \leq \mu_k\,\|e^m\|$$

where

$$(3.11) \qquad \mu_k = \frac{N - \delta}{N + \delta}\,(1 - \frac{1}{\cosh(k/2)}).$$

Observe that $\mu_k < 1$, and $\|e^{m+1}\| \leq \mu_k^m\,\|e^1\|$. Thus $\{y^m\}$ is a Cauchy sequence in the space of continuous functions on $[0,1]$ with the norm defined by (3.6). Therefore, a continuous limit $y(x)$ exists, to which $\{y^m(x)\}$ converges uniformly. Since the order of the limit operation and the integration can be interchanged the limit function satisfies the integral equation (3.3). To establish uniqueness, let $y_1(x)$ and $y_2(x)$ be two solutions to (3.1). Then they both satisfy (3.3) for $k^2 = (1/2)\,(\delta + N)$. By the same analysis that leads to (3.10), $\|y_1 - y_2\| \leq \mu_k\,\|y_1 - y_2\|$. Since $\mu_k < 1$, $\|y_1 - y_2\| = 0$, or $y_1 = y_2$.

4. PROGRESS. (Some New Results)

In [7], two theorems were constructed for

(4.1a) $$y'' = f(x,y), \qquad 0 \le x \le 1,$$

(4.1b) $$y(0) = y_0, \qquad y(1) = y_1.$$

One theorem establishes existence and uniqueness among all y for which $|y(x)| \le \max\{|y_0|, |y_1|\}$, $x \in [0,1]$. The other establishes existence and uniqueness among all y for which $0 \le y(x) \le M$ where y_0 and y_1 are assumed nonnegative and $\max\{y_0, y_1\} \le M$. The conditions assumed on f differ slightly. Recently [8], we have obtained results for a more general subclass of (1.1), namely

(4.2a) $$y'' = f_1(x,y) + p(x)y'$$

Theorems have been established for three different sets of boundary conditions,

(4.2b-1) $$y(0) = y_0, \qquad y(1) = y_1,$$

(4.2b-2) $$y'(0) = 0, \qquad y(1) = y_1,$$

(4.2b-3) $$y(0) = y_0, \qquad y'(1) = 0.$$

Since the results for (4.2a) together with (4.2b-1) have as special cases the results reported in [7] for (4.1), we shall show a theorem and proof for (4.2a) together with (4.2b-1).

The approach taken on (4.2a) is as follows. Assuming p(x) has a continuous derivative, multiply both sides of (4.2a) by the integrating factor $e^{-V(x)}$, where

(4.3) $$v(x) = \frac{1}{2}\int_0^x p(\xi)\, d\xi \ .$$

Then write (4.2a) as

(4.4a) $$u'' = F(x,u),$$

where

(4.5a) $$u(x) = e^{-V(x)}\, y(x),$$

(4.5b) $$F(x,u(x)) = e^{-V(x)} f_1(x, e^{V(x)}u(x)) + q(x)u(x),$$

(4.5c) $$q(x) = (p(x)/2)^2 - p'(x)/2 \ .$$

In terms of u, the boundary conditions (4.2b-1) become

(4.4b) $$u(0) = e^{-V(0)}y_0, \qquad u(1) = e^{-V(1)}y_1.$$

Now we can follow an approach similar to the one for (3.1). First replace (4.4a) with the equivalent equation

(4.6) $$u'' - k^2u = F(x,u) - k^2u.$$

Then for $k^2 \neq 0$, (4.6) together with (4.4b) can be converted into an integral equation by the Green's function procedure for the operator $(d^2/dx^2 - k^2)$, i.e.,

(4.7a) $$u(x) = h(x) + \int_0^1 g_k(x,\xi)\,(k^2u(\xi) - F(\xi,u(\xi)))\,d\xi,$$

where

(4.7b-1) $$g_k(x,\xi) = \frac{1}{k \sinh k}\begin{cases} \sinh k(1-\xi)\sinh kx, & 0 \le x < \xi, \\ \sinh k(1-x)\sinh k\xi, & \xi < x \le 1, \end{cases}$$

and

(4.7c-1) $$h(x) = \frac{e^{-V(0)}\,y_0 \sinh k(1-x) + e^{-V(1)}\,y_1 \sinh kx}{\sinh k}.$$

In addition to assuming that p(x) has a continuous derivative on [0,1], we shall also assume q(x) of (4.5c) is nonnegative on [0,1]. Then, if we define

(4.8) $$\delta_2 = \min_{0\le x\le 1} q(x), \qquad N_2 = \max_{0\le x\le 1} q(x),$$

we will have $\delta_2 \ge 0$, and $N_2 \ge 0$. (Note. The assumption $q(x) \ge 0$ can be weakened.)

Theorem 1A. In the boundary value problem which consists of equation (4.2a) together with (4.2b-1), let $\max\{|y_0|,|y_1|\} \le M$. Suppose there exists an $N_1 > 0$ and a $\delta_1 \ge 0$ such that $0 \le \delta_1 \le \partial f_1/\partial y \le N_1$ for all $x \in [0,1]$ and all y such that $|y(x)| \le e^{V(x)}\max\{e^{-V(0)},e^{-V(1)}\}M$, $x \in [0,1]$. Suppose further that $0 \le f_1(x,y) \le (N_1+\delta_1)y$, $y \ge 0$, and $(N_1+\delta_1)y \le f_1(x,y) \le 0$, $y \le 0$, for all $x \in [0,1]$ and all y such that $|y(x)| \le e^{V(x)}\max\{e^{-V(0)},e^{-V(1)}\}M$, $x \in [0,1]$. Then there exists a unique solution of the problem satisfying $|y(x)| \le e^{V(x)}\max\{e^{-V(0)},e^{-V(1)}\}M$, $x \in [0,1]$. Let $\delta = (\delta_1+\delta_2)$ and $N = (N_1+N_2)$ where N_2 and δ_2 are given by (4.8).

For $k^2 = (\delta+N)/2$, the unique solution is given by $y(x) = e^{v(x)}u(x)$, where $v(x)$ is given by (4.3) and $u(x)$ is the limit of the convergent sequence of functions

$$(4.9a) \qquad u^0(x) = h(x) = \frac{e^{-v(0)}\, y_0 \sinh k(1-x) + e^{-v(1)}\, y_1 \sinh kx}{\sinh k},$$

$$(4.9b) \qquad u^{m+1}(x) = h(x) + \int_0^1 g_k(x,\xi)\, [k^2 u^m(\xi) - F(\xi, u^m(\xi))]\, d\xi, \quad m = 0, 1, \cdots,$$

where $g_k(x,\xi)$ is given by (4.7b-i).

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Proof. Let

$$(4.10) \qquad E = \max\{e^{-v(0)}, e^{-v(1)}\}.$$

First, we show that each member of the sequence (4.9) satisfies $|u^m(x)| \le EM$, $m = 0, 1, \cdots$, for all $x \in [0,1]$. Observe that

$$|u^0(x)| \le E\, \frac{|y_0| \sinh k(1-x) + |y_1| \sinh kx}{\sinh k}$$

$$\le E \max\{|y_0|, |y_1|\}\, \frac{\sinh k(1-x) + \sinh kx}{\sinh k}$$

$$(4.11) \qquad \le E\, M\, \frac{\cosh k((1/2)-x)}{\cosh(k/2)}$$

$$\le E\, M, \qquad x \in [0,1].$$

Now assume that $u^i(x)$ satisfies $|u^i(x)| \le EM$ for some $i \ge 0$. Then from (4.9b),

$$(4.12a) \qquad |u^{i+1}(x)| \le |h(x)| + \int_0^1 g_k(x,\xi)\, |k^2 u^i(\xi) - F(\xi, u^i(\xi))|\, d\xi.$$

Now, if $|u^i(x)| \le EM$, then $|e^{v(x)}u^i(x)| = e^{v(x)}|u^i(x)| \le e^{v(x)}EM$. Thus, if $e^{v(x)}u^i(x) \ge 0$, then $0 \le f_1(x, e^{v(x)}u^i(x)) \le (\delta_1+N_1)e^{v(x)}u^i(x)$ by hypothesis. This in turn implies from (4.5) that if $q(x) \ge 0$, $x \in [0,1]$, then $0 \le F(x,u^i(x)) \le e^{-v(x)}(\delta_1+N_1)e^{v(x)}u^i(x) + (\delta_2+N_2)u^i(x) = (\delta+N)u^i(x) = 2k^2u^i(x)$ if $u^i(x) \ge 0$. Similarly, it follows that $2k^2u^i(x) = (\delta+N)u^i(x) \le F(x,u^i(x)) \le 0$, if

$u^i(x) \le 0$. Therefore the term $|k^2 u^i(\xi) - F(\xi, u^i(\xi))|$ in the integrand of (4.12a) will be bounded by $k^2 |u^i(\xi)|$, and it follows from (4.12a) that

$$|u^{i+1}(x)| \le EM \left(\frac{\sinh k(1-x) + \sinh kx}{\sinh k}\right) + \int_0^1 g_k(x,\xi)\, d\xi\, k^2 EM$$

$$(4.12b) \qquad = EM \left(\frac{\sinh k(1-x) + \sinh kx}{\sinh k}\right) + \left(1 - \frac{\sinh k(1-x) + \sinh kx}{\sinh k}\right) EM$$

$$= EM .$$

Thus, by induction, $|u^m(x)| \le EM$, $m = 0, 1, 2, \cdots$. Now define

$$(4.13) \qquad e^{m+1}(x) = u^{m+1}(x) - u^m(x)$$

and

$$(4.14) \qquad \|e^{m+1}\| = \max_{0 \le x \le 1} |e^{m+1}(x)|, \quad m = 0, 1, \cdots.$$

Since $\partial f_1/\partial y$ is continuous, $\partial F/\partial u$ is continuous so that we can apply the mean value theorem to $F(x,u^m(x)) - F(x,u^{m-1}(x))$ to obtain

$$(4.15) \quad e^{m+1}(x) = \int_0^1 g_k(x,\xi) \left[k^2 - \frac{\partial F}{\partial u}(\xi, u^m(\xi) - \theta(\xi) e^m(\xi))\right] e^m(\xi)\, d\xi,$$

where $0 < \theta(\xi) < 1$. Note that from (4.5)

$$(4.16) \qquad \partial F/\partial u = e^{-v(x)}\, \partial f_1/\partial y\; \partial y/\partial u + q(x)$$

$$= \partial f_1/\partial y + q(x).$$

Also, if $|e^{v(x)} u(x)| \le e^{v(x)} EM$, then $0 \le \delta_1 \le \partial f_1(x, e^{v(x)} u(x))/\partial y \le N_1$, by hypothesis. This in turn implies from (4.16) that $0 \le \delta_1 + \delta_2 \le \partial F(x,u(x))/\partial u \le N_1 + N_2$, if $q(x) \ge 0$, or $0 \le \delta \le \partial F/\partial u \le N$. Since we have already shown that each member of the sequence (4.9) satisfies $|u^m(x)| \le EM$, $x \in [0,1]$, it follows that

$$(4.17) \qquad 0 \le \delta \le \frac{\partial F}{\partial u}(\xi, u^m(\xi) - \theta(\xi) e^m(\xi)) \le N .$$

Thus with the choice

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$$k^2 = (\delta + N) / 2 \tag{4.18}$$

the bracketed term in the integrand of (4.15) satisfies $|k^2 - \partial F/\partial u| \le (N-\delta)/2$ and

$$|e^{m+1}(x)| \le \frac{1}{2} (N - \delta) \int_0^1 g_k(x,\xi)\, d\xi \cdot \|e^m\| ,$$

$$= (1/2)(n - \delta) \frac{1}{k^2} (1 - \frac{\cosh k((1/2)-x)}{\cosh(k/2)}) \|e^m\| \tag{4.19}$$

$$\le \mu_k \|e^m\| , \qquad m = 0, 1, \cdots,$$

where

$$\mu_k = \frac{N - \delta}{N + \delta} (1 - \frac{1}{\cosh(k/2)}). \tag{4.20}$$

Since (4.19) holds for all $x \in [0,1]$, $\|e^{m+1}\| \le \mu_k \|e^m\|$. Note that $\mu_k < 1$, and $\|e^{m+1}\| \le (\mu_k)^m \|e^1\|$. Thus $\{u^m(x)\}$ given by (4.9) is a Cauchy sequence in the space of continuous functions with the norm defined by (4.14). We can conclude that the sequence (4.9) converges uniformly to a limit $u(x)$ which satisfies the integral equation (4.7a) and is such that $|u(x)| \le EM$, $x \in [0,1]$. Then $y(x) = e^{V(x)}u(x)$ is a solution to (4.2a) with (4.2b-1) satisfying $|y(x)| \le e^{V(x)}EM$, $x \in [0,1]$.

To establish uniqueness, let $y_1(x)$, $y_2(x)$ be two solutions in which $|y_1(x)| \le e^{V(x)}EM$ and $|y_2(x)| \le e^{V(x)}EM$, $x \in [0,1]$. Then $u_1(x) = e^{-V(x)}y_1(x)$ and $u_2(x) = e^{-V(x)}y_2(x)$ both satisfy the integral equation (4.7a) with $g_k(x,\xi)$ given by (4.7b-1), $h(x)$ given by (4.7c-1) and $k^2 = (\delta+N)/2$ as specified in Theorem 1A. By the same analysis that leads to (4.19), we can show that $|u_1(x)-u_2(x)| \le \mu_k \|u_1-u_2\|$, $x \in [0,1]$, or

$$\|u_1 - u_2\| \le \mu_k \|u_1 - u_2\| .$$

Since μ_k is given by (4.20) and $\mu_k < 1$, we must have $\|u_1-u_2\| = 0$, or $u_1(x) = u_2(x)$. Thus $y_1 = y_2$.

The importance of Theorem 1A is that $\partial f_1/\partial y$ and f_1 are required to satisfy conditions only for all $x \in [0,1]$ and all y such that $|y(x)|$ is bounded by an expression dependent on boundary values and $p(x)$. Of course, the existence and uniqueness applies only to functions that satisfy this constraint. This kind of approach is well motivated though for problems that arise naturally in the applied sciences, since $y(x)$ usually represents a physical quantity which may be known to be bounded in absolute value or which may be of one sign. With this in mind the following additional result is obtained. It imposes conditions under the constraint that $y(x)$ be nonnegative and bounded above. In the interest of space we state it without proof.

Theorem 1B. In the boundary value problem (4.2a) with (4.2b-1) let y_0, y_1 be nonnegative and let $\max\{y_0, y_1\} \le M$. Suppose there exists an $N_1 > 0$ and a $\delta_1 \ge 0$ such that $0 \le \delta_1 \le \partial f_1/\partial y \le N_1$ for all $x \in [0,1]$ and all y such that $0 \le y(x) \le e^{v(x)}EM$, $x \in [0,1]$, where E is defined in (4.10). Suppose further that $0 \le f_1(x,y) \le (1/2)(N_1+\delta_1)y$ for all $x \in [0,1]$ and all y such that $0 \le y(x) \le e^{v(x)}EM$, $x \in [0,1]$. Then there exists a unique solution satisfying $0 \le y(x) \le e^{v(x)}EM$, $x \in [0,1]$. Let $\delta = \delta_1+\delta_2$ and $N = N_1+2N_2$ where N_2 and δ_2 are defined in (4.8). For $k^2 = (\delta+N)/2$, the unique solution is given by $y(x) = e^{v(x)}u(x)$ where $v(x)$ is given by (4.3) and $u(x)$ is the limit of the convergent sequence of functions given by (4.9).

The proof is similar to the proof of Theorem 1A and requires showing that $0 \le u^m(x) \le EM$, $x \in [0,1]$, $m = 0, 1, \cdots$.

Similar theorems can be stated for (4.2a) together with (4.2b-2) and (4.2a) together with (4.2b-3). We make note of one, for example, (4.2a) with (4.2b-3). This problem can also be converted into the integral equation (4.7a) with

$$\text{(4.7b-3)}\quad g_k(x,\xi) = \frac{1}{K}\begin{cases} (k\cosh k(1-\xi) + v'(1)\sinh k(1-\xi))\sinh kx, & 0 \le x < \xi, \\ \sinh k\xi\,(k\cosh k(1-x) + v'(1)\sinh k(1-x)), & \xi < x \le 1, \end{cases}$$

where

$$K = k\,(k\cosh k + v'(1)\sinh k),$$

and

$$\text{(4.7c-3)}\qquad h(x) = e^{-v(0)}\,y_0\,\frac{k\cosh k(1-x) + v'(1)\sinh k(1-x)}{k\cosh k + v'(1)\sinh k}.$$

Theorem 3B. In the problem (4.2a) together with (4.2b-3) assume $0 \le y_0 \le M$. Suppose there exists an $N_1 > 0$ and a $\delta_1 \ge 0$ such that $0 \le \delta_1 \le \partial f_1/\partial y \le N_1$ for all $x \in [0,1]$ and all y such that $0 \le y(x) \le e^{v(x)-v(0)}M$, $x \in [0,1]$. Suppose further that $0 \le f_1(x,y) \le (1/2)(N_1+\delta_1)y$ for all $x \in [0,1]$ and all y such that $0 \le y(x) \le e^{v(x)-v(0)}M$, $x \in [0,1]$. If $v'(1) \ge 0$, then there exists a unique solution satisfying $0 \le y(x) \le e^{v(x)-v(0)}M$, $x \in [0,1]$. define $\delta = \delta_1+\delta_2$ and $N = N_1+2N_2$ where N_2 and δ_2 are given by (4.8). Then for $k^2 = (N+\delta)/2$, the unique solution is given by $y(x) = e^{v(x)}u(x)$ where $v(x)$ is given by (4.3) and $u(x)$ is the limit of the convergent sequence of functions

$$(4.21a) \qquad u^0(x) = h(x) = e^{-v(0)}\, y_0\left\{\frac{k \cosh k(1-x) + v'(1) \sinh k(1-x)}{k \cosh k + v'(1) \sinh k}\right\}$$

$$(4.21b) \qquad u^{m+1}(x) = h(x) + \int_0^1 g_k(x,\xi)\,[k^2u^m(\xi) - F(\xi,u^m(\xi))]\,d\xi, \quad m = 0,1,\cdots,$$

where $g_k(x,\xi)$ is given by (4.7b-3).

5. FUTURE INVESTIGATION

It is of interest to weaken assumptions for some of the results already established. For instance, one could consider removing the assumptions that $q(x)$ of (4.5c) be nonnegative and the assumptions on f_1. They could be replaced with the single assumption that there exists a $\delta \ge 0$ and an $N > 0$ such that

$$(5.1) \qquad 0 \le \delta \le \partial F/\partial u \le N, \qquad 0 \le F(x,u) \le (1/2)(N+\delta)u,$$

for all u as constrained in Theorem 1B, for instance. We would also like to allow $\partial f_1/\partial y$ and/or $\partial F/\partial u$ to be negative, e.g., to consider cases in which

$$(5.2) \qquad -\beta \le \partial f_1/\partial y \le N, \qquad \beta > 0.$$

We can state a simple result for (4.2a) with (4.2b-1) in which we assume that $-\beta \le \partial F/\partial u \le N$ for all u, $0 \le \beta \le N/2$, and take $k^2 = N/2$. It will work if $N < 8(\cosh^{-1}2)^2$. More general boundary conditions can also be considered. However, Green's functions and their analysis will be more complicated.

A generalization to the problem

$$(5.3) \qquad y'' = f(x,y,y'),$$

in which we assume for instance that $0 \le \delta_1 \le \partial f_1/\partial y \le N_1$, and either $0 \le \delta_2 \le \partial f/\partial y' \le N_2$ or $-N_2 \le \partial f/\partial y' \le -\delta_2 \le 0$, will be harder. We propose to consider

(5.4) $$y'' - k_1^2 y \pm 2k_2 y' = f - k_1^2 y \pm 2k_2 y'.$$

In this instance a different norm will have to be used to establish convergence of the sequence of successive approximations and in turn the existence and uniqueness, e.g., max $(k_1^2|y(x)| + 2k_2|y'(x)|$. Preliminary investigation reveals that unlike results reported in Section 3, convergence will depend on the magnitude of N_1 and N_2 and convergence will not always be guaranteed merely if $\partial f/\partial y$ and $\partial f/\partial y'$ are nonnegative and bounded above.

Because of example 3 of Section (2.4), and example (A-4) of Appendix A, it appears that results for

(5.5) $$y'' = \begin{pmatrix} y_1'' \\ y_2'' \end{pmatrix} = \begin{pmatrix} f_1(y_2) \\ f_2(x, y_1, y_2, y_2') \end{pmatrix},$$

$$A_1 y(0) + A_2 y'(0) = C_1, \qquad B_1 y(1) + B_2 y'(1) = C_2,$$

will be useful. Unfortunately, a more complicated integral equation will have to be analized, namely one of the form

(5.6a) $$y(x) = H(x) + \int_0^1 G(x,\xi)\, F(\xi, y(\xi), y'(\xi))\, d\xi$$

where $H = (h_1, h_2)^T$, $F = (f_1, f_2)^T$, and

(5.6b) $$G(x,\xi) = \begin{pmatrix} g_{11}(x,\xi) & g_{12}(x,\xi) \\ g_{21}(x,\xi) & g_{22}(x,\xi) \end{pmatrix}.$$

6. SOME APPLICATIONS AND SOME NUMERICAL RESULTS

Lets demonstrate an application of the results reported in Section 3 to, for example, problem (2.1), i.e.,

(6.1a) $$y'' = KRy^n - xRy',$$

(6.1b) $$y(0) = 0, \qquad y(1) = 1.$$

For this specific case of (4.2a) with (4.2b-1), we have

(6.2) $$f_1(x,y) = KRy^n, \qquad p(x) = -xR.$$

In terms of (4.4) and (4.5), (6.1) becomes

(6.3a) $$u''(x) = F(x,u(x)) = e^{-v(x)}KR(e^{v(x)}u(x))^n + q(x)u(x),$$

(6.3b) $$u(0) = 0, \qquad u(1) = e^{-v(1)} = \exp(R/4),$$

where

(6.3c) $$v(x) = -x^2R/4, \qquad q(x) = (-xR/2)^2 + R/2, \qquad u(x) = e^{-v(x)}y(x).$$

For $k^2 \neq 0$, (6.3) can be converted into

(6.4a) $$u(x) = \frac{e^{-v(1)} \sinh kx}{\sinh k} + \int_0^1 g_k(x,\xi)\{k^2u(\xi) - F(\xi,u(\xi))\}\, d\xi,$$

where

(6.4b) $$g_k(x,\xi) = \frac{1}{k \sinh k}\begin{cases} \sinh k(1-\xi) \sinh kx, & 0 \le x < \xi, \\ \sinh k(1-x) \sinh k\xi, & \xi < x \le 1, \end{cases}$$

In this problem, one expects that $0 \le y(x) \le 1$. We shall demonstrate a direct application of Theorem 1B, and show constructively that there exists a unique solution satisfying $0 \le y(x) \le e^{v(x)-v(1)} = \exp(R(1-x^2)/4)$, $x \in [0,1]$. The conditions of Theorem 1B are satisfied as follows.

(a) For this problem, $f_1(x,y) = KRy^n$, so

(6.5a) $$\partial f_1/\partial y = nKRy^{n-1}$$

and

(6.5b) $$0 \le \partial f_1/\partial y \le nKR(\exp(R/4))^{n-1}$$

for all $x \in [0,1]$ and all y such that $0 \le y(x) \le e^{v(x)-v(1)}$, $x \in [0,1]$. Note that

(6.6a) $$\delta_1 = 0, \qquad N_1 = nKR(\exp(R/4))^{n-1}$$

in this case, while

(6.6b) $$\delta_2 = \min_{0\le x\le 1} q(x) = R/2, \qquad N_2 = \max_{0\le x\le 1} q(x) = R^2/4 + R/2.$$

As defined in Theorem 1B,

(6.6c) $$\delta = R/2, \qquad N = nKR(\exp(R/4))^{n-1} + 2(R^2/4 + R/2).$$

(b) If $x \in [0,1]$, and y is such that $0 \le y(x) \le e^{V(x)-V(1)}$, $x \in [0,1]$, then

(6.7) $$0 \le f_1(x,y) \le (n/2)\ KR(\exp(R/4))^{n-1}y = (1/2)(N_1+\delta_1)y, \quad n \ge 2.$$

According to the conclusion of Theorem 1B, for

(6.8) $$k^2 = (1/2)(\delta + N) = (1/2)(nKR(\exp(R/4)^{n-1} + R^2/2 + (3/2)R)$$

the sequence

(6.9a) $$u^0(x) = \exp(R/4)\ \sinh kx\ /\ \sinh k,$$

(6.9b) $$u^{m+1}(x) = \frac{\exp(R/4)\ \sinh kx}{\sinh kx} + \int_0^1 g_k(x,\xi)\{k^2u^m(\xi) - F(\xi,u^m(\xi))\}d\xi,$$

$m = 0, 1, \cdots$, has a limit $u(x)$, and $y(x) = e^{V(x)}u(x) = \exp(-Rx^2/4)u(x)$ is the unique solution of (6.1) satisfying $0 \le y(x) \le \exp(R(1-x^2)/4)$, $x \in [0,1]$. In particular for the sequence (6.9),

$$\|u^{m+1} - u^m\| \le \mu_k\ \|u^m - u^{m-1}\|,$$

where

(6.10) $$\mu_k = \frac{nKR(\exp(R/4))^{n-1} + R^2/2 + R/2}{nKR(\exp(R/4))^{n-1} + R^2/2 + (3/2)R}\left(1 - \frac{1}{\cosh(k/2)}\right).$$

To obtain a numerical solution of the limit $u(x)$ of the sequence (6.9), it can be approximated by the discrete solution $w_0 = u(0)$, $w_1 = u(x_1)$, $\cdots$, $w_{p-1} = u(x_{p-1})$, $w_p = u(1)$ on a uniform grid, $h = 1/p$, $x_1 = h$, $x_2 = 2h$, $\cdots$, $x_{p-1} = (p-1)h$, $x_p = 1$. By using the trapezoidal rule on the integrand, w_i can be computed by the expression

(6.11) $$w_i = h(x_i) + \sum_{j=0}^{p} \alpha_j\ g_k(x_i,x_j)\ [k^2w_j - F(x_j,w_j))],\ i = 0,1,\cdots,\ p,$$

where $\alpha_0 = \alpha_p = h/2$, $\alpha_j = h$, $j = 1,\cdots,p-1$. Keller [1] has shown that this should yield accuracy on the order of h^2.

To compute the approximations (6.11), a sequence of net functions $\{w_i^m\}$, $m = 0, 1, \cdots$, can be defined as

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$$(6.12a) \qquad w_i^0 = h(x_i),$$

$$(6.12b) \quad w_i^{m+1} = h(x_i) + \sum_{j=0}^{p} \alpha_j \, g_K(x_i,x_j) \, [k^2 w_j^m - F(x_j,w_j^m)], \qquad i = 0,1,\cdots, p,$$

where $\alpha_0 = \alpha_p = h/2$, $\alpha_j = h$, $j = 1, \cdots, p-1$. By arguments similar to those given by Keller [1] for (3.1), one can show that the limit of (6.12) exists and is the unique solution of (6.11). Also, as $p \to \infty$, the contractive parameter for the sequence (6.12) will converge to the contractive parameter μ_K for the corresponding continuous sequence.

The following table indicates some values of the contraction parameter μ_K

n	R	K	μ_K (6.10)
2	10	.01	.76
2	10	.1	.82
5	10	.01	.999
5	10	.1	.9999

Table 1.

It is important to realize that the value of μ_K is merely a bound on the contraction of the sequence. In practice, the actual number of iterations required to achieve a given error tolerance may be significantly less than the number expected by the value of μ_K. To illustrate this we programmed (6.12) for problem (6.1). Programming was done in standard Fortran on and IBM 3033. For example consider the case n=2, R=10, and K=0.01 in Table 1. Assume that p is sufficiently large that the contractive parameter for (6.12) is nearly μ_K of (6.10). Then if $\|u^1 - u^0\| \le 1$, the relation $\|u^{m+1} - u^m\| \le (\mu_K)^m \, \|u^1 - u^0\|$ implies it should take at most 33 iterations to get $\|u^{m+1} - u^m\| \le 10^{-4}$. It actually took m+1 = 14 with h = (1/50). Consider the case n=5, R=10, and K=0.01. Note that μ_K is very close to 1 in this case. In such a case, the rate of convergence of the sequence may be very slow, thus causing the number of iterations required to achieve a given error tolerance in the programming of (6.12) to be large. Keller [1] suggests Newton's method as one alternative to a more rapidly convergent scheme. However, analysis such as the following can also be applied.

First, we observe that the initial approximation $u^0(x)$ of (6.9a) is one suggested by the integral equation itself. If the initial approximation is chosen closer to the limit of the sequence, the number of iterations required to achieve a given error tolerance will be reduced since $||u^1 - u^0||$ will be smaller. To this end we can appeal to comparison theorems [2]. For instance, if $y_2(x)$ is the solution for n=2, R=10, K=.01, and $0 \le y_2(x) \le 1$, then $y_2(x) \le y_3(x)$ where y_3 is the solution for n=3, R=10, K=.01. Thus in the case n=3, R=10, K=.01, the initial approximation in the sequence (6.9) may be taken as $u^0(x) = \exp^{-V(x)} y_2(x)$ instead of $u^0(x)$ given by (6.9a).

Secondly, we suspect that the solution $y(x) = e^{V(x)}u(x) = \exp(-x^2R/4)\, u(x)$ satisfies $0 \le y(x) \le 1$ for problem (6.1). If we can show that $0 \le \exp(-x^2R/4)u_m(x) \le 1$ for each m, then

$$(6.13) \qquad R/2 \le \frac{\partial F}{\partial u}(x,u^m) \le nRK + R^2/4 + R/2$$

for each m. Thus if we choose

$$(6.14) \qquad k^2 = (1/2)\,(nRK + R^2/2 + R)$$

the sequence (3.33) will contract according to

$$||u^{m+1} - u^m|| \le \mu_K \; ||u^m - u^{m-1}||$$

where

$$(6.15) \qquad \mu_K = \frac{nRK + R^2/2}{nRK + R^2/2 + R}\left(1 - \frac{1}{\cosh(k/2)}\right)$$

and k is given by (6.14). Compare the following Table 2 to Table 1.

n	R	K	μ_K (6.15)
5	10	.01	.73
5	10	.1	.75

Table 2.

When we used k^2 given by (6.14) and the solution for the case n=2, R=10, K=.01 as the initial approximation in the sequence (6.12), it only took 4 iterations with $h = 1/50$ to get $\|u^{m+1} - u^m\| \le 10^{-4}$ for the case $n=3$, $R=10$, in problem (6.1). For the case $n=4$, $R=10$, $K=.01$, we used k^2 given by (6.14) and the solution for the case $n=3$, $R=10$, $K=.01$ as the initial approximation and it only took 3 iterations. Finally, for the case $n=5$, $R=10$, $K=.01$, we used k^2 given by (6.14) and the solution for the case $n=4$, $R=10$, $K=.01$ as the initial approximation, and it only took 3 iterations.

Consider the vector problem (2.4) shown in Section 2,

$$\begin{pmatrix} y_1'' \\ y_2'' \end{pmatrix} = \begin{pmatrix} y_1 \\ S(xy_2' + 3y_2 - y_1y_2') \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{6.16a}$$

$$y_1(0) = 0,\ y_2(0) = 0, \qquad y_1(1) = 1,\ y_1'(1) = 0. \tag{6.16b}$$

There are several ways to extend the approach to systems. For instance, we note in this case that

$$\partial f_1/\partial y_1 = 0,\ \partial f_1/\partial y_2 = 1,\ \partial f_1/\partial y_1' = 0,\ \partial f_1/\partial y_2' = 0, \tag{6.17a}$$

$$\partial f_2/\partial y_1 = -Sy_2',\ \partial f_2/\partial y_2 = 3S,\ \partial f_2/\partial y_1' = 0,\ \partial f_2/\partial y_2' = S(x-y_1), \tag{6.17b}$$

Then, write (6.16) in terms of $u_1 = y_1$ and $u_2 = e^{n_2 x}y_2$,

$$D^2u_1 \qquad - l_1^2u_2 = -F_1 \tag{6.18a}$$

$$-m_1^2u_1 + (D^2 - n^2)u_2 = -F_2\,, \qquad n^2 = n_1^2+n_2^2 \tag{6.18b}$$

where $D^2 = d^2/dx^2$ and

$$F_1 = l_1^2u_2 - e^{-n_2x}u_2, \tag{6.18c}$$

$$F_2 = m_1^2u_1 + (n_1^2u_2 - 2n_2(u_2'-n_2u_2) - e^{n_2x}f_2). \tag{6.18d}$$

The boundary conditions are

$$u_1(0) = 0,\ u_2(0) = 0, \qquad u_1(1) = 1,\ u_1'(1) = 0. \tag{6.18e}$$

Problem (6.18) can be converted into an integral equation of the form

$$\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix} + \int_0^1 G(x,\xi) \begin{pmatrix} F_1(\xi,u_2(\xi)) \\ F_2(\xi,u_1(\xi),u_2(\xi),u_2'(\xi)) \end{pmatrix} d\xi. \tag{6.19}$$

For explicit representation of h_1, h_2, F_1, F_2, and G see [8].

To get a numerical solution to (6.19), one can try programming the sequence

(6.20a) $\quad u_1^0(x) = h_1(x), \quad u_1^{0\prime}(x) = h_1'(x),$

(6.20b) $\quad u_2^0(x) = h_2(x), \quad u_2^{0\prime}(x) = h_2'(x),$

$$(6.20c)\begin{pmatrix} u_1^{l+1}(x) \\ u_2^{l+1}(x) \end{pmatrix} = \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix} + \int_0^1 G(x,\xi)\begin{pmatrix} F_1(\xi,u_2^l(\xi)) \\ F_2(\xi,u_1^l(\xi),u_2^l(\xi),u_2^{l\prime}(\xi)) \end{pmatrix} d\xi,$$

$$(6.20d)\begin{pmatrix} u_1^{l+1\prime}(x) \\ u_2^{l+1\prime}(x) \end{pmatrix} = \begin{pmatrix} h_1'(x) \\ h_2'(x) \end{pmatrix} + \int_0^1 G'(x,\xi)\begin{pmatrix} F_1(\xi,u_2^l(\xi)) \\ F_2(\xi,u_1^l(\xi),u_2^l(\xi),u_2^{l\prime}(\xi)) \end{pmatrix} d\xi,$$

$$l = 0,1,\cdots.$$

Suppose we take $S > 0$, and try to find a unique solution for which

(6.21) $\quad 0 \le y_1 \le 1, \; 0 \le y_1' \le P, \; -Q \le y_2 \le 0, \; -M \le y_2' \le 0,$

for all $x \in [0,1]$. We propose choosing values

(6.22a) $\quad n_1^2 = 3S/2, \quad 2n_2 = \max|(x-y_1)S|/ 2,$

(6.22b) $\quad m_1^2 = \max(-Sy_2')/2, \quad l_1^2 = (1 + e^{-n_2})/2$

Then, if each member of the sequence (6.20) satisfies the bounds (6.21), and if the sequences converge uniformly with respect to some norm, we can conclude that a unique solution to (6.16) satisfying the bounds (6.21) exists. Its the limit of the sequences (6.20). This has not been done rigorously yet. However, we shall report some numerical results which are encouraging.

The sequence (6.20) was replaced by an approximation in which we evaluated u_1^0, u_2^0, and $u_2^{0\prime}$ discretely at p points $x_j = (j-1)h$, $j = 1,\cdots,p$, $h = 1/(p-1)$. Then we obtained the values of the remaining members of the sequences at each discrete point by evaluating the integral using the trapezoidal rule. The interation was stoped if and when

$$\max_j \{|e_1^m(x_j)|, |(e_1^m(x_j))'|, |e_2^m(x_j)|, |(e_2^m(x_j))'|\} < 10^{-4}$$

where

$$e_i^m(x_j) = y_i^m(x_j)-y_i^{m-1}(x_j), \quad (e_i^m(x_j))' = (y_i^m(x_j)-y_i^{m-1}(x_j))', \quad i = 1,2.$$

The following results were computed using standard Fortran on an IBM 370 3033.

	case 1	case 2	case 3
S	.11576	2.081	6.868
n_1	.4167	1.767	3.210
n_2	.008682	.2601	.5151
m_1	.4167	2.04	6.934
l_1	.9978	.9410	.8937
h	1/50	1/50	1/40
iterations	4	6	11
CPU time	46sec	77sec	93sec
$y_2(1)$ (computed)	-3.0622	-3.968	-5.700
$y_2(1)$ value reported in [5]	-3.0622	-3.961	-5.503

CONCLUDING REMARKS

Analytical Advantages and Disadvantages.

Constructive existence and uniqueness is given for a large class of non-linear problems. However, explicit Green's functions can be difficult to construct and analize. Also, analysis for systems can be complicated.

Numerical Advantages and Disadvantages.

One advantage is that boundary values are built right into the integral equation. In numerical procedures, one does not necessarily need to approximate the derivative at one end if only the unknown is given as in initial value methods. No derivatives need to be approximated and therefore no difference approximations are required. We know before a numerical method of approximating the sequence of successive approximations is applied that the sequence converges to the unique solution of the problem.

One disadvantage is that in practice a large number of mesh points may be required to approximate the function sequences accurately. Another is that slow convergence may be a problem.

Appendix A

In a problem concerning radiation heat transfer for annular fins [9], the temperature distribution is shown to be governed by the energy equation written as

(A-1a) $$y''(x) = q(x)\,(y(x))^4 + p(x)\,y'(x), \qquad 0 \le x \le 1,$$

(A-1b) $$y(0) = 1.0, \qquad y'(1) = 0,$$

where

(A-1c) $$q(x) = \frac{\beta}{(1-x)\tan\alpha + \theta},$$

and

(A-1d) $$p(x) = \frac{\tan\alpha}{(1-x)\tan\alpha + \theta} - \frac{1}{x+\rho}.$$

The quantities α, θ, and ρ are constants related to angle of taper, fin thickness at tip, and radius of fin base and fin tip.

A flat plate model of a catalytic converter [10] leads to the nonlinear two point problem

(A-2a) $$y'' = -\frac{(y')^2}{(2-y)},$$

(A-2b) $$y(0) = y_0, \qquad y(1) = 0.$$

The unknown y represents mole fraction of a gas.

A problem involving fluid flow in a two-dimensional channel [11] leads to the following boundary value problem governing the velocity distribution

(A-3a) $$f'''' = R(ff''' - f'f'')$$

(A-3b) $$f(0) = 0,\ f''(0) = 0, \qquad f(1) = 1,\ f'(1) = 0.$$

We can write this problem in vector form with $y_1 = f$ and $y_2 = f''$,

(A-4a) $$\begin{pmatrix} y_1'' \\ y_2'' \end{pmatrix} = \begin{pmatrix} y_2 \\ R(y_1 y_2' - y_1' y_2) \end{pmatrix}$$

(A-4b) $$y_1(0) = 0,\ y_2(0) = 0, \qquad y_1(1) = 1,\ y_1'(1) = 0.$$

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