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# SIMULTANEOUS STABILIZATION AND SIMULTANEOUS POLE-PLACEMENT

## BY NONSWITCHING DYNAMIC COMPENSATION

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# 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The "simultaneous stabilization problem" - in either discrete or continuous time - consists in answering the following question:

Given an r-tuple $G_1(s),\ldots,G_r(s)$ of $p \times m$ proper transfer functions, does there exist a compensator $K(s)$ such that the closed-loop systems $G_1(s)(I+K(s)G_1(s))^{-1},\ldots,G_r(s)(I+K(s)G_r(s))^{-1}$ are (internally) stable?

As pointed out in [13], this question arises in reliability theory, where $G_2(s),\ldots,G_r(s)$ represents a plant $G_1(s)$ operating in various modes of failure and $K(s)$ is a nonswitching stabilizing compensator. Of course, for the same reason, it is important in the stability analysis and design of a plant which can be switched into various operating modes. The simultaneous stabilization problem can also apply to the stabilization of a nonlinear system which has been linearized at several equilibria. Finally, it has been shown [14], [20] that to solve the case $r = 2$ is to solve the well-known problem considered by Youla et al in [21]: When can a single plant be stabilized by a stable compensator? This correspondence also serves to give some measure of the relative depth of this problem.

In order to describe the results obtained via this correspondence, we need some notation. First, set $n_i = $ McMillan degree of $G_i(s)$. In the scalar input-output setting ($m = p = 1$), we regard each $G_i(s)$ as a point in $\mathbb{R}^{2n_i+1}$, viz. if

$$G_i(s) = p_i(s)/q_i(s), \quad \text{where}$$

$$p_i(s) = a_{0i} + \ldots + a_{n_i i}s^{n_i}, \text{ and } q_i(s) = b_{1i} + \ldots + b_{ni}s^{n_i-1} + s^{n_i}$$

then $G_1(s)$ corresponds to the vector $(a_{o1}, \ldots, a_{n_1 1}, b_{11}, \ldots, b_{n1})$ $\mathbb{R}^{2n_1+1}$.

Moreover, since $p_1$ and $q_1$ are relatively prime, this vector lies in the open dense set $\text{Rat}(n_1) \subset \mathbb{R}^{2n_1+1}$ (see [3] for the strictly proper case). In [14], Saeks and Murray used the techniques of fractional representations [8] and the correspondence mentioned above to give explicit inequalities defining the open set

$$U \subset \text{Rat}(n_1) \times \text{Rat}(n_2)$$

of pairs $(G_1(s), G_2(s))$ which are simultaneously stabilizable. In [20] Vidyasagar and Viswanadham showed, using similar techniques, that provided $\max(m,p) > 1$ the open set $U$ of pairs $(G_1(s), G_2(s))$ which can be stabilized is in fact dense.

This can be made precise by topologizing a point $G_1(s)$ in the set

$$\sum_{m,p}^{n} = \{p \times m \; G_1(s) \; ; \; \text{degree } G_1(s) = n_1\}$$

as a vector in $\mathbb{R}^{(n_1+1)(mp)}$ via its Hankel parameters: If

$$G_1(s) = \sum_{j=0}^{\infty} H_{ij} s^{-j}$$

then $G_1(s)$ corresponds to the $n+1$ $p \times m$ block matrices $\{H_{10}, \ldots, H_{1,n+1}\}$ which determines $G(s)$. It is known that $\sum_{m,p}^{n}$ is an $(n(m+p)+mp)$-manifold (see [7], [12], [5]), although this is not important here. What is important is that $\sum_{m,p}^{n}$ is a topological space.

One of our main results concerns the generic stabilizability problem; that is,

Question 1.1. Fix $m,p,r$, and $n_i$. Is the set $U$ of r-tuples $G_1(s), \ldots, G_r(s)$ which can be simultaneously stabilized open and dense in $\sum_{m,p}^{n_1} \times \ldots \times \sum_{m,p}^{n_r}$ ?

It is also important to ask, for reasons of global robustness of algorithms finding such a compensator, for compensators with a fixed degree of complexity.

**Question 1.2.** Fix $m, p, r$, and $n_i$. What is the minimal value of $q$ (if one exists) for which the set $W_q$ of $r$-tuples which can be simultaneously stabilized, by a compensator of degree $\leq q$, is open and dense in $\sum_{m,p}^{n_1} \times \ldots \times \sum_{m,p}^{n_r}$ ?

It should be noted that, in the case $r = 1$, Question 1.2 is an outstanding, unsolved, classical problem. In this paper, we prove:

**Theorem 1.1.** In either discrete or continuous time, a sufficient condition for generic simultaneous stabilizability is

$$\max (m,p) \geq r \qquad (1.1)$$

Indeed, if (1.1) holds, then the generic $r$-tuple can be stabilized by a compensator of degree less than or equal to $q$, where $q$ satisfies:

$$q[\max(m,p)+1-r] \geq \sum_{i=1}^{r} n_i - \max(m,p) \qquad (1.2)$$

In the case $r = 1$, it is unknown whether generic stabilizability implies generic pole-assignability; that is, whether or not these properties of $m, n$, and $p$ are really different (see [4]). Perhaps not surprisingly then, Theorem 1.1 follows from:

**Theorem 1.2.** A sufficient condition for generic simultaneous pole-assignability is (1.1), where the compensator $K(s)$ can be taken to be of degree $q$ satisfying (1.2).

Here, simultaneous pole-assignability means the assignability of $r$ sets of self-conjugate sets of numbers $\{s_{11}, \ldots, s_{n_1+q,1}\} \subset \mathbb{C}$. In fact sharper bounds on $q$ can be obtained (see [18], [11]). Our proof relies on the recent pole-placement techniques derived for $r=1$ by P.K. Stevens in his thesis [18], which contains an improvement on existing results in the literature, see also [9], [17]. We shall prove Theorem 1.2 only in the strictly proper case; the proper case involves more technical arguments from algebraic geometry which can be found in [11]. We shall, however, give an independent proof of Theorem 1.1 in the nonstrictly proper case, based on the equivalence of generic stabilizability and existence of a solution to a generic "deadbeat control" problem, which we can solve if (1.1) is satisfied. This argument extends the argument given in [4] for the case $r=1$ and $q=0$.

Note that if $r=1$, then (1.1) is always satisfied in which case (1.2) implies

**Corollary 1.3.** (Brasch-Pearson [2]). The generic $p \times m$ plant $G(s)$ of degree $n$ can be stabilized by a compensator of order $q$, where $q$ satisfies

$$(q+1)\max(m,p) \geqslant n \tag{1.3}$$

If $r=2$ and $\max(m,p) > 1$, then (1.1) is again satisfied, so we obtain rather easily:

**Corollary 1.4.** (Vidyasagar-Viswanadham [20]). If $r = 2$ and $\max(m,p) > 1$, then the generic pair $(G_1(s), G_2(s))$ is simultaneously stabilizable.

Moreover, in this case we know an upper bound on the order of the required compensator. For example, if $m = p = 2$, $r = 2$, then $q$ can be taken to satisfy

$$q \geqslant n_1 + n_2 - 2$$

On the other hand, in [20] the explicit conditions defining the closed set

$$\Sigma_{m,p}^{n_1} \times \Sigma_{m,p}^{n_2} - U$$

of pairs not simultaneously stabilizable were derived. Such conditions can be derived from our proof, but instead we refer to [10], where Theorem 1.1 (excepting (1.2)) is proved by interpolation methods also yielding a set of explicit conditions in the range $r \leqslant \max(m,p)$.

Finally, we prove that the condition (1.1) is sharp in the following sense.

**Theorem 1.5.** If $\min(m,p) = 1$, then for fixed $m, p, r$ and $n_i$ the following statements are equivalent for proper plants:

(i)  $q \in \mathbb{N}$ satisfies  $q(\max(m,p) + 1 - r) + \geqslant \sum_{i=1}^{r} n_i$;

(ii)  the generic $r$-tuple $G_1(s), \ldots, G_r)$ is simultaneously stabilizable in discrete or continuous time by a compensator of degree $\leqslant q$;

(iii)  the generic $r$-tuple $G_1(s), \ldots, G_r(s)$ is simultaneously stabilizable in discrete or continuous time.

In the strictly proper case it follows that (i)-(iii) is also equivalent to generic simultaneous pole assignability. This holds in the proper case as well, but requires a separate argument [11].

**Corollary 1.6.** If $\min(m,p) = 1$ and $r \leq \max(m,p)$ then the generic r-tuple is simultaneously stabilizable by a compensator of order precisely given by the least integer $q$ satisfying (1.2).

As a further corollary, we obtain one of the results obtained by Saeks and Murray in [ ], see also [15]:

**Corollary 1.7.** (Saeks-Murray). Suppose $m = p = 1$ and $r = 2$. Simultaneous stabilizability is not a generic property.

We remark that these results hold also over the field $\mathbb{C}$ of complex numbers - in particular, the complex analogue of Corollary 1.7 dispels a folklore conjecture concerning simultaneous stabilization using compensators with complex coefficients.

Finally, over any field, the method of proof of Theorem 1.2 gives linear equations for a compensator simultaneously placing $r(n+q)$ poles when the generic hypothesis is satisfied.

## 2. POLE PLACEMENT AND THE GENERALIZED SYLVESTOR MATRIX: A PROOF OF THEOREM 1.2

In this section we proceed to prove Theorem 1.2. Note that Theorem 1.1 and Corollaries 1.3 and 1.4 follow immediately in the strictly proper case from this theorem. Without any loss of generality we can assume that $m \geq p$, for, if $K(s)$ stabilizes $G_i^t(s)$ then $K^t(s)$

stabilizes $G_i(s)$.

Suppose, first of all, that $p = 1$, so that we are given a set of $r$, $m$ input 1 output plants of McMillan degree $\leq n$ represented as

$$
\left[
\frac{\sum\limits_{i=0}^{n} p_{11}^{k} s^{i}}{\sum\limits_{i=0}^{n} p_{m+p1}^{k} s^{i}}
\quad , \quad
\frac{\sum\limits_{i=0}^{n} p_{21}^{k} s^{i}}{\sum\limits_{i=0}^{n} p_{m+p1}^{k} s^{i}}
\quad , \quad \ldots\ldots\ldots\ldots \quad , \quad
\frac{\sum\limits_{i=0}^{n} p_{m+p-1,1}^{k} s^{i}}{\sum\limits_{i=0}^{n} p_{m+p,1}^{k} s^{i}}
\right]
\tag{2.1}
$$

for $k = 1, 2, \ldots, 4$. A 1 input, m output, compensator of McMillan degree $\leq q$ is represented as

$$
\left[
\frac{\sum\limits_{i=0}^{q} a_{11} s^{i}}{\sum\limits_{i=0}^{q} a_{m+pi} s^{i}}
\quad , \quad
\frac{\sum\limits_{i=0}^{q} a_{21} s^{i}}{\sum\limits_{i=0}^{q} a_{m+pi} s^{i}}
\quad , \quad \ldots\ldots\ldots\ldots \quad , \quad
\frac{\sum\limits_{i=0}^{q} a_{m+p-1,1} s^{i}}{\sum\limits_{i=0}^{q} a_{m+p_1} s^{i}}
\right]
\tag{2.2}
$$

Note that in (2.1) and (2.2) the coefficients $p_{ji}^{k} \; \forall k$ and $a_{ji}$ has been defined up to a nonzero scale factor. Moreover, for a strictly proper plant or compensator, $p_{jn_i}^{k} = 0$, $a_{jq} = 0 \; \forall \; j = 1, \ldots, m+p-1; \; k = 1, \ldots, r$.

The associated return difference equation, $\det(I + K(s)G_k(s) = 0$ is given by

$$
\Pi_k(s) = \sum_{j=1}^{m+p} \left[ \sum_{i=0}^{n} p_{ji}^{k} s^{i} \right] \left[ \sum_{i=0}^{q} a_{ji} s^{i} \right]
\tag{2.3}
$$

$$
\forall \; k = 1, 2, \ldots, r
$$

A generic r-tuple of plants define a mapping $\chi$, via equation (2.3), between the plant parameters and the coefficient of the return difference polynomials given by

$$
\chi : \; \mathbb{R}^{(q+1)(m+p)} \to \mathbb{R}^{r(n+q+1)}
\tag{2.4}
$$

where

$$
\chi \ (A_0, \ldots, A_q) = (A_0, \ldots, A_q)
\begin{bmatrix}
P_0 & \cdots & \cdots & P_n & & \\
 & P_0 & & & P_n & \\
 & & \ddots & & & \ddots \\
 & & & \ddots & & \\
 & & & P_0 & & P_n
\end{bmatrix}
\qquad (2.5)
$$

where

$$
A_i = (a_{1i}, a_{2i}, \ldots, a_{m+pi}) \qquad (2.6)
$$

$$
P_i =
\begin{bmatrix}
P_{1i}^1 & P_{1i}^1 & \cdots & P_{1i}^r \\
P_{2i}^1 & P_{2i}^2 & \cdots & P_{2i}^r \\
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
P_{m+pi}^1 & P_{m+pi}^2 & \cdots & P_{m+pi}^r
\end{bmatrix}
\qquad (2.7)
$$

The matrix in the right hand side of (2.5) is classically known as the generalized Sylvestor matrix and is of order $(q+1)$ $(m+p) \times r(n+q+1)$. For $r=1$ its rank has been analyzed by Bitmead, Kailath, Kung in [1]. In particular, for a generic plant, it is known to have full rank. For $r \geqslant 1$, we have the following:

Lemma 2.1. The generalized Sylvestor matrix is of full rank for a generic r-tuple.

Proof: See Appendix I.

**Lemma 2.2.** Assume $\min(m,p) = 1$. A sufficient condition for generic pole assignment, for an $r$ tuple of strictly proper plants by a proper compensator is given by

$$(q+1)(m+p-r) \geqslant \sum_{i=1}^{r} n_i - r + 1 \tag{2.8}$$

**Proof:** We prove this Lemma assuming for notational convariance that $n_i = n \;\forall\; i = 1,\ldots,r$ and analyze the mapping $\chi$ as defined by (2.4), (2.5). Assume

$$a_{m+p,q} = -1 \quad , \quad p_{m+p,n}^k = 1 \;\forall\; k = 1,\ldots,r$$

and that the coefficient of $s^{n+q}$ in all the $r$ return difference polynomials (2.3) has been normalized to 1.

Thus a sufficient condition for generic pole assignment is that $\chi'$ is onto. Here the mapping

$$\chi' : \mathbb{R}^{(q+1)(m+p)-1} \to \mathbb{R}^{(n+q)} \tag{2.9}$$

is given by

$$\chi(A_0,\ldots,A_{q-1}A_q') = (A_0,\ldots,A_{q-1}A_q') \begin{bmatrix} P_0 \cdots\cdots P_n & & & \\ & P_0 \cdots\cdots P_n & & \\ & & \ddots & \\ & & P_0 \cdots\cdots P_n \\ & & P_0' \cdots\cdots P_n' \end{bmatrix} \tag{2.10}$$

where

$$A_q' = (a_{1q}a_{2q},\ldots,a_{m+p-1q})$$

and $p_i'$ is obtained from $p_i$ by deleting its $(m+p)^{th}$ row.

By Lemma 2.1 the matrix in the right-hand side of (2.10) is of full rank for a generic r-tuple of plants, and has the order $(q+1)(m+p-1) \times r(n+q)$. Therefore, a sufficient condition for generic pole placement is given by

$$(q+1)(m+p) - 1 \geqslant r(n+q) \qquad (2.11)$$

which is same as (2.8) for $n_i = n \; \forall \; i = 1, \ldots, r$.

The proof of Theorem 1.2 now proceeds by a reduction to the case $\min(m,p) = 1$, which has been treated in Lemmas 2.1-2.2. This procedure, which is called "vectoring down", is adopted from the case $r = 1$, studied in P.K. Stevens thesis [18].

**Lemma 2.3.** Given an r-tuple of $p \times m$ plants $G_i(s)$ of degrees $n_i$, each with distinct simple poles, there is an open dense set of $1 \times p$ vectors $v \in \mathbb{R}^p$ such that $vG_i(s)$ has degree $n_i$.

**Proof:** If $r = 1$, then we may expand $G(s)$

$$G(s) = \sum_{i=1}^{n} \frac{R_i}{s-\lambda_i}$$

in a partial fraction expansion, where $\lambda_i \in \mathbb{C}$ and each $R_i$ has rank 1. Now, the set $U_1$ of real vectors $v$ such that $vR_1 \neq 0$ is clearly open and dense in $\mathbb{R}^p$. Defining $U_2, \ldots, U_n$ similarly, set $V = \bigcap_{i=1}^{n} U_i$. Thus, $V$ is an open dense set of vectors with the required property.

If $r > 1$, one obtains, as above, sets $V_1, \ldots, V_r$ in $\mathbb{R}^p$ having an open dense intersection $\bigcap_{i=1}^{n} V_i$. Q.E.D.

**Lemma 2.4.** Given an r-tuple of $p \times m$ plants $G_i(s)$ there exists a constant gain output feedback $k$ such that the closed loop systems $G_i(s)(I + kG_i(s))^{-1}$ have distinct simple poles.

**Proof:** For $r = 1$, the set $W_1$ of $K$ such that the closed loop system has simple poles is the complement in $\mathbb{R}^{mp}$ of an algebraic set. It is well known [2] that this set is nonempty; therefore, $W_1$ is open and dense. Taking any $K$ in the open dense set $\bigcap_{i=1}^{r} W_i$ gives the desired conclusion. Q.E.D.

Thus, choosing any $(v, K) \in \mathbb{R}^p \times \mathbb{R}^{mp}$ we have a mapping from an an open dense set

$$\Phi_{(v,k)} : \textstyle\sum_{m,p}^n \times \ldots \times \sum_{m,p}^n \to \sum_{m,1}^n \times \ldots \times \sum_{m,1}^n$$

$$\Phi_{(v,k)}(G_i(s))_{i=1}^r = (vG_i(s)(\Phi + KG_i(s))^{-1})_{i=1}^r$$

which is rational in the Hankel parameters $(H_{ij})$ of $(G_i)$. Applying Lemmas 2.1-2.2 to the case $\min(m,p) = 1$, i.e. $\sum_{m,1}^n \times \ldots \times \sum_{m,1}^n$, gives - via composition with $\Phi$ - an open dense set of $\sum_{m,p}^n \times \ldots \times \sum_{m,p}^n$ which can be simultaneously pole-assigned.

Q.E.D.

## 3. GENERIC STABILIZABILITY CONDITION OF AN r-TUPLE OF PROPER PLANTS

In this section we proceed to prove Theorem 1.1 independent of Theorem 1.2. We first show that the following three statements are equivalent.

I.  A generic r-tuple of proper plants is stabilizable with respect
to the open left half plane by a proper compensator of degree $\leq q$.

II.  A generic r-tuple of proper plants is stabilizable with respect to
the interior of the unit disc, by a proper compensator of degree $\leq q$.

III.  A generic r-tuple of proper plants is pole assignable at the
origin by a proper compensator of degree $\leq q$.

Lemma 3.1.            I $\Leftrightarrow$ II

Proof:  Consider the conformal transformation

$$\phi(s) = (s+1)/(s-1) \tag{3.1}$$

which maps the r-tuple of proper plants $g_1, g_2, \ldots, g_r$ onto the r-tuple
of proper plants $g_1', \ldots, g_r'$ where $g_i'(s) = g_i(\phi(s))$ except for the
algebraic set of plants satisfying - "$g_i(s)$ has a pole at $s = 1$ for
some $i = 1, \ldots, r$". The proof now follows from the two facts.

1.  $\phi(s)$ maps the open left half plane onto the interior of the unit
    disc.

2.  The mapping

$$(g_1, \ldots, g_r) \longmapsto (g_1', \ldots, g_r')$$

and its inverse, map the generic r-tuple of proper plants to the
generic r-tuple of proper plants.

<u>Lemma 3.2.</u>         II <=> III

<u>Proof</u>:  Sufficiency is clear and follows by an analagous argument of
Lemma 3.1 with  $\phi(s) = s + a$, $a > 0$, $a \in \mathbb{R}$.

To prove necessity, we have the following:  For each  $r = 1, 2, \ldots$
(shown easily by assuming statement II and considering  $\phi(s) = as$, $a > 0$,
$a \in \mathbb{R}$).  $\exists$  an open dense set of  $U_r$  of r-tuple of plants for which
there exist a compensator of degree $\leq q$  which places the poles in the
interior of the disc  $D_r$  of radius  $1/r$  centered at the origin.
Consider the set

$$U = \bigcap_{r=1}^{\infty} U_r$$

Clearly,  U  is a dense set by the Baire Category Theorem [13].  Since
the mapping  $\chi$  given by (2.4 is linear, it has a closed image.  Moreover,
every r-tuple of plants in  U  admits a sequence of compensators which
places the poles arbitrary close to the origin.  Since image of  $\chi$  is
closed,  U  is contained in a set  V  of all r-tuple of plants for which
there exists a compensator which places the poles at the origin.  By
the Tarski [19] - Seidenberg [16] theory of elimination over  $\mathbb{R}$,  V  is
indeed defined by union and/or intersection of sets given by polynomial
equations or inequations  $f_\alpha > 0$, $f_\beta = 0$.  Finally, since  U  is dense
in  V,  $f_\beta(U) = 0 \Rightarrow f_\beta \equiv 0$  so that  V  is defined by strict polynomial
inequalities.  Hence  V  is open.  Moreover, since  U  is dense,  V  is
also dense.

<u>Lemma 3.3.</u>  For a generic r-tuple  $(r \leq m + p)$  of  $\min(m, p) = 1$  plants

$$\text{III} \iff (q + 1(m + p - r) \geq r(n - 1) + 1$$

Proof: The only nontrivial part is to prove sufficiency for the case

$$r(n+q) < (q+1)(m+p) < r(n+q+1)$$

(The other cases follow easily from the fact that the associated Sylvestor's matrix is of full rank for a generic r-tuple.)

To prove sufficiency, for the above case we want to show that the vector

$$(0,0,\ldots\ldots,0 \ , \ s_1,s_2,\ldots\ldots s_r)$$

$$\longleftarrow r(n+q) \longrightarrow \qquad \longleftarrow \quad r \longrightarrow$$

indeed belongs to the image of $X$ (defined by (2.5)) for some $s_i \neq 0$, $i = 1,\ldots,r$.

Partition the Sylvestor's matrix in (2.5 as $[S_1 S_2]$ where $S_1$ is of order $(q+1)(m+p) \times r(n+q)$. Clearly we are solving the pair of equations

$$[A_0,\ldots,A_q]S_1 = [0,\ldots,0] \tag{3.2}$$

$$[A_0,\ldots,A_q]S_2 = [s_1,\ldots,s_r] \tag{3.3}$$

We claim that for a generic r-tuple of plants (3.2) has a solution for a nonzero vector $A_q$ for otherwise if $A_q = 0$ we have

$$[A_0,\ldots,A_{q-1}]S_1' = (0,\ldots,0) \tag{3.4}$$

where $S_1'$ is of order $q(m+P) \times r(n+q)$ obtained by deleting the last $m+p$ rows of $S_1$. From (3.4) $(A_0,\ldots,A_{q-1}) = \underline{0}$ since $S_1'$ is of full rank generically. Thus the only solution of (3.2) is the zero vector which is a contradiction since the kernel of $S_1$ is at least of dimension 1. On the other hand, for $A_q \neq 0$, for a generic r-tuple of plants the right-hand side of (3.3) is a vector none of whose entries are zero.

Theorem 1.1 then follows from Lemma 3.1, 3.2, 3.3 and the vectoring down technique used in the proof of Theorem 1.2 in Section 2.


## 4. PROOF OF THEOREM 1.5

To say there exists $q \in \mathbb{N}$ satisfying (1.2) is to $\max(m,p) \geqslant$ Thus, (ii) follows from (i) by Theorem 1.1.


(ii) $\Rightarrow$ (iii) since (iii) is weaker than (ii).

By Lemma 3.1, in order to prove (iii) $\Rightarrow$ (i) it suffices to assume that $G_1(s),\ldots,G_r(s)$ are simultaneously stabilizable in continuous-time.


Proposition 4.1. The generic $(m+1)$-tuple of $1 \times m$ proper continuous time plants of degree $n$ is not simultaneously stabilizable by a proper compensator of finite (but not a priori bounded) degree.


Proof: Consider the domain of (simultaneous) stability

$$\mathscr{T} = \{(c_{11},\ldots,c_{1n},\ldots,c_{r,n_r}) : \sum_{j=0}^{n_1+q} c_{i,j} s^j \text{ has all roots in } D_1\}$$

and its convex hull $\Omega(\mathscr{T}) \subset \mathbb{R}^{n_1+q} \times \ldots \times \mathbb{R}^{n_r+q}$. Clearly, a necessary condition for generic simultaneous stabilizability is

$$\text{image}(\chi_\eta) \cap \Omega(\mathscr{T}) \neq \emptyset,$$

for an open dense set of $\eta$. Since

$$\Omega(\mathscr{D}) \subset \{(c_{ij}) : c_{ij} > 0\}$$

it will suffice to prove:

**Lemma 4.2.** If $r = m + p$, then there exists an open set of r-tuples $\eta$ such that image ($\chi_\eta$) contains no vector with only positive entries.

We fix the value of $q$ and construct the associated Sylvestor matrix $S$. We claim that the open set $E$ of plants defined by

$$E \overset{\Delta}{=} \{(P_0, P_1, \ldots, P_n) \mid P_0^{-1} P_j \ \forall \ j = 1, \ldots, n \ \text{has all the entries negative}\}$$

cannot be stabilized by a proper compensator of degree $\leq q$.

Suppose the above is not true, then there exist $\eta \in E$, such that

$$\text{image}(\chi_\eta) \ \cap \ \Omega(\mathscr{D}) \neq \emptyset$$

or in other words $\exists \underline{\alpha} \mid \alpha_i > 0 \ \forall \ i = 1, \ r(n + q + 1)$ and

$$\underline{a} \ S = \underline{\alpha} \tag{4.1}$$

has a solution. Writing $S$ as

$$S \overset{\Delta}{=} [S' \quad S'']$$

where

$$S' = \begin{bmatrix} P_0 & P_1 & \cdots & P_q \\ 0 & P_0 & \cdots & P_{q-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 0 & 0 & & P_0 \end{bmatrix} \tag{4.2}$$

and $P_j = 0$ for all $j > n$

Equation (4.1) can be written as

$$\underline{a}' \; [I \mid S'^{-1} \; S''] = \underline{\alpha} \qquad\qquad (4.3)$$

where $S'^{-1}$ is given as follows

$$S'^{-1} = \begin{bmatrix} X_0 & X_1 & \cdots\cdots & X_q \\ & X_0 & \cdots\cdots & X_{q-1} \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \\ & & & X_0 \end{bmatrix}$$

where $X_0 = P_0^{-1}$

$$-(P_1, P_2, \ldots, P_{r+1}) \begin{bmatrix} X_r \\ X_{r-1} \\ \cdot \\ \cdot \\ X_1 \end{bmatrix} = X_{r+1} \;\; \forall \; r = 0, \ldots, q-1$$

$$P_j = P_0^{-1} P_j \;\; ; \;\; j = 1, \ldots, q$$

The identity matrix of order $(q+1)(m+p)$ in (4.3) forces $\underline{a}'$ to have all the entries positive. Moreover, since $\eta \in E, S'^{-1} S''$ has all its entries negative so that $\underline{a}'(S'^{-1} S'')$ has all the entries negative which is a contradiction since $\underline{\alpha}$ is a positive vector.

Finally it is shown that $E$ is not an empty set. For a fixed $P_0 = P_0^*$ choose the vector $\delta$ to be so that $P_0^{*-1} \delta$ has all its entries negative. Let

$$P_j^* = \underbrace{(\delta, \delta, \ldots, \delta)}_{m+p} \qquad j = 1, \ldots, n$$

so that

$$(P_0^*, P_1^*, \ldots, P_n^*) \in E$$

Q.E.D.

**Remark:** If image($\chi_\eta$) is affine hyperplane, then the necessary condition

$$\text{image}(\chi_\eta) \cap \Omega(\mathcal{D}) \neq \emptyset$$

of course is sufficient, i.e. implies

$$\text{image}(\chi_\eta) \cap \mathcal{D} \neq \emptyset$$

This fact was used by Chen, together with

**Lemma 4.3.** (Chen [6]) If $r = 1$, $\Omega(\mathcal{D}) = \{(c_1, \ldots, c_n) : c_1 > 0\}$

to give precise conditions for stabilizability in the case $r = 1$, $q = 0$, $\min(m,p) = 1$, and $\max(m,p) = n-1$. This technique can be adapted in the cases $r \geq 1$ to give explicit conditions - in certain cases - defining the open set of simultaneously stabilizable plants when $r > \max(m,p)$, see [11].

Note that Corollary 1.6 now follows from our previous results on the generic rank of the generalized Sylvestor matrix, while Corollary 1.7 follows either from Theorem 1.5 or Proposition 4.1.

## APPENDIX I: PROOF OF LEMMA 2.1

The generalized Sylvestor matrix is co-ordinatized by $r(n+1)(m+p)$ parameters, and it is sufficient to show the existence of one principal minor with nonvanishing determinant.

By reordering the rows and columns, the generalized Sylvestor matrix can be written as

$$\bar{S} = [Q_1, Q_2, \ldots, Q_{m+p}]^T \tag{1}$$

where

$$Q_i = [P_{i1}, P_{i2}, \ldots, P_{ir}] \tag{2}$$

$$P_{jk} = \begin{bmatrix} P_{j0}^k & P_{j1}^k & \cdots & P_{jn}^k & & 0 \\ & \ddots & \ddots & & \ddots & \\ 0 & & & P_{j0}^k & P_{j1}^k & \cdots & P_{jn}^k \end{bmatrix} \begin{matrix} \Big\updownarrow \\ (q+1) \end{matrix} \tag{3}$$

$$\longleftarrow (n+q+1) \longrightarrow$$

in the notation of (2.7). Moreover, each $P_{jk}$ is referred to as a 'block' of $S$.

Define a set $M$ of matrices as follows: "$m$ belongs to $M$ provided $m$ is obtainable from one of the matrices $P_{jk}$ in (3) either by deleting the first $\alpha_1$ columns or the last $\alpha_2$ rows $\alpha_1, \alpha_2 \geqslant 0$."

Proposition A.1. Every element $m$ of $M$ has the property that there exists a principal minor $m_p \in M$ of $m$, a coordinate $p_m^*$ and an integer $j_m^*$ such that $p_m^{*j_m^*}$ is a summand in $\det m_p$ where $j_m$ is the order of the minor.

**Proof:** Clear from the structure of $p_{jk}$.

The following is an algorithm to construct a principal minor
with nonidentically-vanishing determinant.

Algorithm:

$$\text{Set} \quad S = \bar{S} \quad , \quad \text{Initialize} \quad \xi = 0$$

1. Set $\xi = \xi + 1$.

2. Look at $P_{11}$. Obtain the principal minor $m$ of $P_{11}$, satisfying
   Proposition 1. If there is more than one possible choice, choose
   the one containing the first column. Define $\alpha_\xi = p_m^*$ and $j_\xi = j_m^*$.

3. Delete the rows and columns corresponding to the coordinate $p_m^*$ from
   S. Renumber the blocks of the resulting matrix and call it S.
   (Every block of S is to be identified as a minor of the correspond-
   ing block in $\bar{S}$ obtained by row or column deletion.)

4. Do the same "delete" operation as in step 3 in $\bar{S}$.

5. If S is empty, terminate. Otherwise go to 6.

6. Set $k = \xi$.
   Construct the principal minor $\bar{m}_p$ of $\bar{S}$ by choosing those elements
   of $\bar{S}$ whose corresponding row and column has been deleted in Step 6.

**Proposition A.2.** During the execution of the above algorithm, S can
always be decomposed into blocks belonging to M.

**Proof:** Clearly $\bar{S}$ satisfies the above proposition, since each block
$p_{jk}$ belongs to M. Each iteration of the algorithm deletes either the

first $\alpha_1$ columns of the first block column of S or the last $\alpha_2$ rows of the first block row of S. The proposition thus follows from the definition of M.

**Proposition A.3.** $\overline{m}_p$ constructed in Step 6 of the algorithm has a nonidentically-vanishing determinant.

**Proof:** We prove the proposition by showing that det $\overline{m}_p$ has a summand given by $\prod\limits_{i=1}^{k} \alpha_i^{j_i}$, in the notation of the algorithm. This is clear, however, by observing that in the $\xi^{th}$ iteration the matrix S has a principal minor, the determinant of which has the summand $\prod\limits_{i=\xi}^{k} \alpha_i^{j_i}$, where k is defined in Step 6 of the algorithm.

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