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Research on Numerical Algorithms for Large Space Structures

Final Report

NASA Grant NSG-1603

March 15, 1981 - October 31, 1982

Department of Electrical Engineering

Cullen College of Engineering

University of Houston

RESEARCH ON NUMERICAL ALGORITHMS FOR LARGE SPACE STRUCTURES

Final Report

NASA Grant NSG-1603

March 16, 1981 - October 31, 1982

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CHAPTER 1

INTRODUCTION

The technical material presented in this report describes the work under NASA Grant NSG-1603, Langley Research Center, Hampton, VA for the period of March 16, 1981 - October 30, 1982. Dr. Eugene D. Denman served as principal investigator for the grant.

The work under this grant was to investigate numerical algorithms for large-space structures with particular emphasis on decoupling method for analysis and design. The investigator and project participants have considered numerous aspects of the analysis of large systems ranging from the algebraic theory to lambda matrices to identification algorithms. Previous reports have described some of the problems considered including the theory of the decoupling procedure. The material presented in this report is a general treatment of the algebraic theory of lambda matrices and application of the theory to second-order lambda matrices. Since the finite-element analysis of the dynamics of a structure can be characterized by a second-order matrix differential equation, the second-order lambda matrix is of importance in analysis and design.

Although published literature on lambda matrices is adequate in some aspects, there are numerous gaps in the general theory and a serious lack of computer software for analysis of large structures. As an example, there is no software package known to this investigator that computes the latent roots and latent vectors with efficiency comparable to the software available for computing eigenvalues and eigenvectors of general matrices. The usual

computational approach for handling lambda matrices is to form a state-variable matrix from the lambda matrix and then use the existing software in computer libraries. Since the state-variable matrix is sparse, the existing software may not be efficient for such problems.

Chapter 2 of this report gives the general theory of lambda matrices. This work was carried out to acquire a better understanding of lambda matrices as well as fill in some of the gaps in the theory. The relationships between eigenvalues, latent roots, eigenvectors, latent vectors, eigenprojectors and latent projectors are presented in this chapter. The decoupling algorithm is based on Riccati matrices and projectors so this chapter plays an important role in the work that follows. The theory of lambda matrices presented in the chapter should also help in developing better software, as well as reference material for other researchers.

A lambda matrix of the general form $A(\lambda)=A_0\lambda^n+A_1\lambda^{n-1}+ \ . \ . \ .+A_n$ is said to be regular if $\det(A_0)\neq 0$. Chapter 2 is restricted to the regular lambda matrix as the dynamic of a structure can generally be defined in the regular form. Lambda matrices which are not regular have been considered in Chapter 3. This type of lambda matrix will sometimes result from the analysis of the dynamics of a system. A fifth order lambda matrix which is not regular was encountered in some related work on the space shuttle. The theory in Chapter 3 complements the development in Chapter 2 and the two chapters cover the general theory of lambda matrice regardless of regularity.

The material in Chapter 4 of the report is not directly related to the large-space structure problem, but is given for completeness as well as for the researcher working with low-order as well as low dimensioned system. The procedure for computing latent projectors or matrix residues given in that chapter would not be preferred to the eigenvector method given in Chapter 2 and 3 unless the worker has the inverse lambda matrix, $[A(\lambda)]^{-1}$.

The finite-element model that this investigator has considered is characterized by a second-order lambda matrix. Chapter 5 presents some theorems on second-order lambda matrices and details on the eigenprojectors and latent projectors of such lambda matrices. The material presented in the chapter can be used as a basis for computing the damping matrix although an in-depth investigation reveals that the computational burden for this method of computing the damping matrix will be high. Work will continue on determining the damping matrix by use of the mathematics of this chapter.

The final chapter of the report describes a decoupled optimal control algorithm tha appears to be reasonably efficient for large space structures. The method discussed a decoupling procedure which can be used with optimal control theory to damp selected modes in a structure. A general software package for determining feedback control for a structure could be developed from the theory. Investigations made to date do not indicate a limitation on the procedure other than the normal ones encountered in large systems. The mode spill-over problem is eliminated by the method.

The material presented in this report required the effort of several individuals. The contributions made by Dr. Graham Goodwin, Jesus Leyva-Ramos and G.I. Jeon have been essential to the investigations and the material presented in this report. Dr. Graham Goodwin was on leave from the University of Newcastle, New South Wales, Australia and was a Visiting Professor for the six month period January-June 1982 at the University of Houston. The contribution to Chapter 3 is hereby acknowledged as well as the time given to research personnel for discussion of various problems.

The Principal Investigator is grateful to NASA Langley Research Center for the support during the grant period as well as the encouragement and suggestions of Dr. Garnett Horner, technical monitor of the grant.

CHAPTER 2

THE ALGEBRAIC THEORY OF LATENT VECTORS AND PROJECTORS IN LAMBDA MATRICES

Multivariable systems and controls are often formulated in terms of n-th order matrix differential equations which give rise to lambda matrices of the form $A(\lambda) = A_o \lambda^n + A_1 \lambda^{n-1} + \ldots\ldots + A_n$. This chapter describes the algebraic theory of latent roots, latent vectors, and latent projectors and gives the relationships to eigenvalues, eigenvectors and eigenprojectors of the companion form matrix. The theory presented in the chapter is general in that distinct as well as repeated eigenvalues or latent roots are treated. The work is limited to regular lambda matrices, $\det(A_o) \neq 0$.

The chapter consists of three sections. Section 1 is introductory in scope and presents the mathematical forms of lambda matrices, and companion matrices as well as the connection between eigenprojectors and latent proctors. Section 2 gives the algebraic theory of latent roots, eigenvalues, latent vectors and eigenvectors. It is shown that the latent roots obtained from the lambda matrices are equivalent to the eigenvalues are computed from the companion matrix. This result is well known and presents no new material. The relationships that exists between the left and right eigenvectors and the left and right latent vectors is then developed and it is shown that the latent vectors are subvectors of the eigenvectors. The algebraic theory of the eigenprojectors and the latent projectors is developed in Section 3. It is shown that the latent projectors are submatrices of the eigenprojectors. Section 3 is a general treatise on eigenprojectors and latent projectors where distinct as well as repeated eigenvalues have been considered.

1. Introduction

Multivariable systems such as a finite-element model of vibrating structures, control systems, and large scale systems can be formulated in terms of second or higher order matrix differential equations. Although such systems can be reformulated in state-variable form, it may be more efficient, from a numerical viewpoint, to analyze the system using the higher-order differential equations.

To illustrate, assume that the dynamics of a system can be characterized by the nonhomogeneous matrix differential equation

$$A_0 \frac{d^n x}{dt^n} + A_1 \frac{d^{n-1} x}{dt^{n-1}} + \ldots \quad \ldots + A_n x = f(t) \tag{1.1}$$

where $A_i \in R^{m \times m}$, $x(t) \in R^m$ and $f(t) \in R^m$, from which it follows that $A(s)$ given by

$$A(s) = A_0 s^n + A_1 s^{n-1} + \ldots \quad \ldots + A_n \tag{1.2}$$

will result when the Laplace transform is taken of (1.1). If $s = \lambda$ then (1.2) becomes a lambda matrix (or matrix polynomial). If $A_0 = I$, then (1.1) can be defined in state-variable form and the companion matrix A_c can be written as

$$A_c = \begin{bmatrix} 0 & I & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots & \vdots \\ -A_n & -A_{n-1} & -A_{n-2} & \ldots & -A_1 \end{bmatrix} \tag{1.3}$$

which has eigenvalues λ_i and eigenvectors Y_{ci}. A_c will be a mn×mn matrix with mn eigenvalues and mn eigenvectors.

When the partial fraction expansion of $[A(\lambda)]^{-1}$ is taken the following general form is obtained

$$[A(\lambda)]^{-1} = \sum_{i=1}^{r} \sum_{\ell=0}^{m_i-q_i} \frac{\hat{P}_{i,\ell}}{(\lambda-\lambda_i)^{\ell+1}} \tag{1.4}$$

where r is the number of distinct latent roots given by det $A(\lambda) = 0$, m_i is the multiplicity of a root and q_i is equal to the nullity of $A(\lambda_i)$ for the latent root λ_i. The matrices $P_{i,\ell} \varepsilon C^{m\times m}$ will be called latent projectors (or matrix residues, [1]).

Similarly, the companion form $A_c(\lambda)$ is given by $[I\lambda-A_c]$ which has eigenvalues λ_i given by $\det[A_c(\lambda)] = 0$. The partial fraction expansion of $A_c^{-1}(\lambda)$ can be written as

$$[A_c(\lambda)]^{-1} = \sum_{i=1}^{r} \sum_{\ell=0}^{m_i-q_i} \frac{P_{i,\ell}}{(\lambda-\lambda_i)^{\ell+1}}$$

with $P_{i,\ell} \varepsilon C^{mn\times mn}$ denoted as eigenprojectors or matrix residues [2].

The purpose of this chapter is to formulate the algebraic theory of lambda matrices and the relationship of latent roots, latent vectors and latent projectors to the eigenvalues, eigenvectors and eigenprojectors of the companion form. The chain rule for latent projectors and eigenprojectors for the repeated latent root or eigenvalues will be given.

This work follows the lines of the earlier work of Lancaster [3], Lancaster and Webber [4], and that of Dennis, Traub and Weber [5] on lambda matrices and matrix polynomials. The reader should refer to Zadeh and Desoer,

[1] and Cullen [6] for material on matrix residues and projectors as well as to the paper by Denman and Leyva-Ramos [2] for some material on eigenprojectors. It is assumed that the reader is familiar with the fundamentals of linear algebra. An excellent source of general material on linear algebra is Gantmacher, [7]. Lambda matrices are discussed by Lancaster in [3], [4] and [8].

2. Lambda Matrices and Companion Forms

A system of differential equations representing a physical model may be given by

$$\bar{A}_0 \frac{d^{(n)}x(t)}{dt^{(n)}} + \bar{A}_1 \frac{d^{(n-1)}x(t)}{dt^{(n-1)}} + \ldots + \bar{A}_n x(t) = f(t) \qquad (2.1)$$

with initial conditions $x(0), \dot{x}(0), \ldots, x^{(n-1)}(0) \in R^m$ where $\bar{A}_i \in R^{m \times m}$, $x(t) \in R^m$ and $f(t) \in R^m$. The existence and uniqueness of the solution to (2.1) is assumed. It is further assumed that (2.1) is Laplace transformable with the Laplace transform taking the general form

$$[\bar{A}_0 s^n + \bar{A}_1 s^{n-1} + \ldots + \bar{A}_n] x(s) = B(s) \qquad (2.2)$$

where B(s) contains the initial condition information as well as the Laplace transform of f(t).

If $s = \lambda$ in (2.2), then the resulting equation can be considered as a lambda matrix equation. The left hand bracketed term in (2.2) is of interest in the remaining parts of this paper. The equation in brackets, $\bar{A}(\lambda)$ or $\bar{A}(s)$, is frequently called a matrix polynomial although Lancaster [3] and Dennis, Traub, and Webber [5] defined $\bar{A}(\lambda)$ as a lambda matrix. This

paper will adhere to the terminology of Lancaster , et. al., and refer to $a(\lambda)$ as a lambda matrix.

2.1 Definition. Given a set of constant matrices $\bar{A}_i \in R^{m\times m}$, and a scalar λ such that $\lambda \in C$, then a lambda matrix will be defined as $\bar{A}(\lambda) = \bar{A}_0\lambda^n + \bar{A}_1\lambda^{n-1} + \ldots + \bar{A}_n$. This matrix will be called regular when $\det \bar{A}_0 \neq 0$. For $\bar{A}_0$ non-singular, then $\bar{A}_0$ can be factored out of $\bar{A}(\lambda)$ such that

$$\bar{A}(\lambda) = \bar{A}_0(I\lambda^n + A_1\lambda^{n-1} + \ldots + A_n) = \bar{A}_0 A(\lambda) \tag{2.3}$$

2.2 Definition. Let $\bar{A}(\lambda)$ be as defined in (2.3) and let $\lambda_i \in C$ be a latent root of $\bar{A}(\lambda)$ where a latent root λ_i is a root of $\det \bar{A}(\lambda) = 0$.

If $\bar{A}(\lambda)$ is regular, then there will be mn latent roots of $\bar{A}(\lambda)$ which are equal to the eigenvalues of A_c.

2.3 Theorem. Let $\bar{A}(\lambda)$ be as defined in (2.1) and let the determinant of $\bar{A}(\lambda)$ be given by the general form

$$\det \bar{A}(\lambda) = c_0\lambda^{mn} + c_1\lambda^{mn-1} + \ldots + c_{mn} = 0 \tag{2.4}$$

The lambda matrix will have at most mn-nullity $\bar{A}_0$ latent roots.

Proof. Let $\bar{A}(\lambda)$ be represented by

$$\bar{A}(\lambda) = \begin{bmatrix} a_{11}(\lambda) & a_{12}(\lambda) & \cdots & a_{1m}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) & \cdots & a_{2m}(\lambda) \\ \vdots & \vdots & \vdots\vdots & \vdots \\ a_{m1}(\lambda) & a_{m2}(\lambda) & \cdots & a_{mm}(\lambda) \end{bmatrix} \tag{2.5}$$

where each $a_{ij}(\lambda)$ is a scalar polynomial of at most order n, i.e.

$$a_{ij}(\lambda) = a_{ij0}\lambda^n + a_{ij1}\lambda^{n-1} + \ldots \ldots + a_{ijn} \tag{2.6}$$

where a_{ijk} are the coefficients of the scalar polynomial with $a_{ijk} = A_k(i,j)$. The determinant of (2.5) has been defined by Franklin [9] as

$$\det \bar{A}(\lambda) = \sum_{(j_1,j_2,\ldots,j_m)} s(j_1,j_2\ldots j_m) a_{1j_1}(\lambda) a_{2j_2}(\lambda) \ldots a_{mj_m}(\lambda). \tag{2.7}$$

The summation extends over m! permutations $j_1,j_2\ldots j_m$ of 1,2...m with

$$s(j_1,j_2\ldots j_m) = \text{sign} \prod_{1 \le p < q \le m} (j_q - j_p) \tag{2.8}$$

The coefficient C_0 is obtained from the leading coefficient of det $\bar{A}(\lambda)$ as given in (2.7) which will be

$$C_0 = \sum_j s(j) a_{1j_1 0}\, a_{2j_2 0} \cdots a_{mj_m 0} = \det \bar{A}_0 \tag{2.9}$$

If det $\bar{A}_0 = 0$ then $\bar{A}(\lambda)$ will have at most mn-1 latent roots. The coefficient C_1 will be given as a linear combination of the determinants of all the submatrices $(m-1)\times(m-1)$ of $\bar{A}_0$, C_2 will be given as a linear combination of the determinants of all submatrices $(m-2)\times(m-2)$ of $\bar{A}_0$ and so on until $\bar{A}_0$ is the zero matrix.

2.3.1 Corollary. There will be at least nullity $\bar{A}_n$ latent roots at the origin, i.e. $\lambda = 0$. In the particular case when det $\bar{A}_n = 0$, there is at least one latent root at $\lambda = 0$.

The proof of this corollary follows directly from (2.7) when the coefficient C_{mm} in the summation is considered.

2.4 Definition. Let $A(\lambda)$ be defined as in (2.3) with matrix coefficients $A_\ell \in \mathbb{R}^{m\times m}$ and latent roots λ_i of multiplicity m_i. The number of primary right or left latent vectors will be q_i = nullity $A(\lambda_i)$. The right latent vectors will be denoted by $y_i^{(j)}$ and the left by $z_i^{(j)}$ where $y_i^{(j)}$ and $z_i^{(j)}$ satisfy the relations

$$A(\lambda_i)y_i^{(j)} = 0_{m\times 1} \qquad j = 1,2,\ldots,q_i \tag{2.10}$$

$$A^T(\lambda_i)z_i^{(j)} = 0_{m\times 1} \qquad j = 1,2,\ldots,q_i \tag{2.11}$$

with $y_i^{(j)} \in C^{m\times 1}$ and $z_i^{(j)} \in C^{m\times 1}$.

The primary right latent vector $y_i^{(j)}$ is a subvector of the linear* independent right eigenvector $y_{ci}^{(j)}$ of the companion matrix of (1.3) with

$$y_{ci}^{(j)} = \begin{bmatrix} y_i^{(j)} \\ \lambda_i y_i^{(j)} \\ \lambda_i^2 y_i^{(j)} \\ \vdots \\ \lambda_i^{n-1} y_i^{(j)} \end{bmatrix} \tag{2.12}$$

where $y_{ci}^{(j)}$ satisfies the usual algebraic equation $(A_c - \lambda_i I)y_{ci}^{(j)} = 0_{mn\times 1}$.

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* Such eigenvectors will be referred to as primary eigenvectors in the rest of this paper.

It follows that the primary right eigenvector $y_{ci}^{(j)}$ is a function of $y_i^{(j)}$ and λ_i. The primary left eigenvector $z_{ci}^{(j)}$ satisfies a similar form as $y_{ci}^{(j)}$ with

$$z_{ci}^{(j)} = \begin{bmatrix} (\lambda_i^{n-1}I+A_1^T\lambda_i^{n-2}+ \ldots +A_{n-1}^T)z_i^{(j)} \\ (\lambda_i^{n-2}I+A_1^T\lambda_i^{n-3}+ \ldots +A_{n-2}^T)z_i^{(j)} \\ \vdots \\ (\lambda_i I+A_1^T)z_i^{(j)} \\ z_i^{(j)} \end{bmatrix} \tag{2.13}$$

where $z_i^{(j)}$ is a primary latent vector.

The two forms of $y_{ci}^{(j)}$ and $z_{ci}^{(j)}$ given in (2.12) and (2.13) hold when $q_i \geq j$ and the maximum number of primary right or left latent vectors will be m. If $m_i > q_i$ then m_i-q_i generalized latent vectors must be constructed to completely define the lambda matrix from its latent roots and vectors. The m_i-q_i generalized latent vectors satisfy a chain rule as given in Lancaster, [4].

2.5 Theorem. Let $a(\lambda)$ be defined as in (2.3), a set of right latent vectors $y_i^{(1)}, y_i^{(2)}, \ldots, y_i^{(h_j)} \in C^{m\times 1}$ form a right Jordan chain associated with the latent root λ_i and the jth primary latent vector. The chain rule is given by

$$A(\lambda)y_i^{(\ell)} + \frac{dA(\lambda_i)}{d\lambda}y_i^{(\ell-1)} + \frac{1}{2!}\frac{d^2A(\lambda_i)}{d\lambda^2}y^{(\ell-2)} + \ldots.$$

$$+ \frac{1}{(\ell-1)!}\frac{d^{(\ell-1)}A(\lambda_i)}{d\lambda^{(\ell-1)}}y_i^{(1)} = 0_{m\times 1} \qquad \ell = 1,2,\ldots,h_j \tag{2.14}$$

where $y_i^{(1)}$ is the jth primary right latent vector and h_j is the length of the Jordan chain. The vectors $y_i^{(k)}$ for $1 < k \leq h_j$ are the generalized right latent vectors of the jth primary latent vector.

Proof. The proof of this theorem is obtained from consideration of the chain rule for generalized eigenvectors. The chain rule is

$$\begin{aligned}
(A_c-\lambda_i I)y_{ci}^{(2)} &= y_{ci}^{(1)} \\
(A_c-\lambda_i I)y_{ci}^{(3)} &= y_{ci}^{(2)} \\
&\vdots \qquad\qquad \vdots \\
(A_c-\lambda_i I)y_{ci}^{(h_j)} &= y_{ci}^{(h_j-1)}
\end{aligned} \tag{2.15}$$

being $y_{ci}^{(1)}$ the jth primary eigenvector and $y_{ci}^{(k)}$ for $1 < k \leq h_j$ its generalized eigenvectors associated with the eigenvalue λ_i of A_c. When (2.15) is expanded the chain rule is obtained where $y_i^{(k)}$ is formed from the first m rows of $y_{ci}^{(k)}$.

The generalized right latent vectors $y_i^{(1)}$ and $y_i^{(2)}$ will be obtained from

$$A(\lambda_i)y_i^{(2)} + \frac{dA(\lambda_i)}{d\lambda} y_i^{(1)} = 0_{m\times 1}$$

$$A(\lambda_i)y_i^{(3)} + \frac{dA(\lambda_i)}{d\lambda} y_i^{(2)} + \frac{1}{2!}\frac{d^2A(\lambda_i)}{d\lambda^2} y_i^{(1)} = 0_{m\times 1}$$

or (2.14) in general.

The chain rule can also be utilized to modify (2.12) for the relation between the generalized eigenvectors and the generalized latent vectors with

$$y_{ci}^{(k)} = \begin{bmatrix} y_i^{(k)} \\ \lambda_i y_i^{(k)} + y_i^{(k-1)} \\ \lambda_i^2 y_i^{(k)} + 2\lambda_i y_i^{(k-1)} + y_i^{(k-2)} \\ \vdots \\ \sum_{\substack{j=0 \\ [k-j\geq 1]}}^{n-1} \binom{n-1}{j} \lambda_i^{n-j-1} y_i^{(k-j)} \end{bmatrix} \tag{2.16}$$

2.6 Theorem. Let $A(\lambda)$ be defined as in (2.3), a set of left latent vectors $z_i^{(1)}, z_i^{(2)}, \ldots, z_i^{(h_j)} \in C^{m\times 1}$ form a left Jordan chain associated with the latent root λ_i and the jth primary latent vector. The chain rule is given by

$$A^T(\lambda_i) z_i^{(\ell)} + \frac{dA^T(\lambda_i)}{d\lambda} z_i^{(\ell-1)} + \frac{1}{2!} \frac{d^2 A^T(\lambda_i)}{d\lambda^2} z_i^{(\ell-2)} + \ldots$$

$$+ \frac{1}{(\ell-1)!} \frac{d^{(\ell-1)} A^T(\lambda_i)}{d\lambda^{(\ell-1)}} z_i^{(1)} = 0_{m\times 1} \qquad \ell = 1,2,\ldots,h_j \tag{2.17}$$

where $z_i^{(1)}$ is the jth primary left latent vector and h_j is the length of the Jordan chain. The vectors $z_i^{(k)}$ for $1 < k \leq h_j$ are the generalized left latent vectors of the jth primary latent vector.

Proof. The proof of this theorem follows directly from the generalized left eigenvectors $z_{ci}^{(k)}$ of the companion form, the generalized left latent vector $z_i^{(k)}$ will be formed from the last m rows of $z_{ci}^{(k)}$.

The generalized left eigenvector $z_{ci}^{(k)}$ can also be defined from the latent vectors $z_i^{(k)}$ of λ_i, the latent roots λ_i and the lambda matrix $A(\lambda)$. Utilizing (2.13) and the chain rule for $z_{ci}^{(k)}$, it follows that the left generalized eigenvectors satisfy the relation

$$z_{ci}^{(k)} = \begin{bmatrix} (\lambda_i^{n-1}I+A_1^T\lambda_i^{n-2}+\ldots+A_{n-1}^T)z_i^{(k)}+((n-1)\lambda^{n-2}I+(n-2)A_1^T\lambda_i^{n-3}+\ldots+A_{n-2}^T)z_i^{(k-1)}+\ldots \\ \vdots \\ (\lambda_i^2I+A_1^T\lambda_i+A_2^T)z_i^{(k)}+(2\lambda_iI+A_1^T)z_i^{(k-1)}+z_i^{(k-2)} \\ (\lambda_iI+A_1^T)z_i^{(k)}+z_i^{(k-1)} \\ z_i^{(k)} \end{bmatrix} \tag{2.18}$$

The latent vector $z_i^{(k)}$ is defined only for $1 < k \leq h_j$.

The number of primary latent vectors associated with the latent root λ_i is q_i and each of these latent vectors could have a chain of generalized latent vectors. The structure of a Jordan block $J_i \in C^{m_i \times m_i}$ could be given by

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}_{h_1} \quad \ddots \quad \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}_{h_{q_i}} \tag{2.19}$$

such that $\sum_{j=1}^{q_i} h_j = m_i$, and every Jordan chain has a very well defined number of generalized latent vectors. The value of h_j is such that the chain rules for (h_j+1) as given in (2.14) and (2.17) are not satisfied.

An example will now be considered to illustrate the computational procedures. Let $A(\lambda)$ be defined as

$$A(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \lambda^2 + \begin{bmatrix} -4.0 & 2.0 \\ -0.2 & -3.0 \end{bmatrix} \lambda + \begin{bmatrix} 3.0 & -1.0 \\ 0.2 & 2.6 \end{bmatrix} \tag{2.20}$$

which has latent roots $\lambda_1 = 1$ with multiplicity $m_1 = 1$ and $\lambda_2 = 2$ with multiplicity $m_2 = 3$. The right latent vectors are

$$Y = [y_1^{(1)} \; y_2^{(1)} \; y_2^{(2)} \; y_2^{(3)}] = \begin{bmatrix} 1 & 3 & -1 & -2 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

and transposed left latent vectors.

$$Z = [z_1^{(1)} \; z_2^{(3)} \; z_2^{(2)} \; z_2^{(1)}] = \begin{bmatrix} 3 & -2 & -1 & -1 \\ -5 & 5 & 0 & 5 \end{bmatrix}$$

The right eigenvectors can be constructed from (2.12) and (2.16) with W_R being the right eigenvector matrix

$$W_R = \begin{bmatrix} y_1^{(1)} & y_2^{(1)} & y_2^{(2)} & y_2^{(3)} \\ \lambda_1 y_1^{(1)} & \lambda_2 y_2^{(1)} & y_2^{(1)}+\lambda_2 y_2^{(2)} & y_2^{(2)}+\lambda_2 y_2^{(3)} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & -1 & -2 \\ 0 & 1 & -1 & -1 \\ 1 & 6 & 1 & -5 \\ 0 & 2 & -1 & -3 \end{bmatrix} = [y_{c1}^{(1)} \quad y_{c2}^{(1)} \quad y_{c2}^{(2)} \quad y_{c2}^{(3)}]$$

The transposed left eigenvectors can be constructed from (2.13) and (2.18) with W_L being the left eigenvector matrix

$$W_L = \begin{bmatrix} (I\lambda_1+A_1^T)z_1^{(1)} & z_2^{(2)}+(I\lambda_2+A_1^T)z_2^{(3)} & z_2^{(1)}+(I\lambda_2+A_1^T)z_2^{(2)} & (I\lambda_2+A_1^T)z_2^{(1)} \\ z_1^{(1)} & z_2^{(3)} & z_2^{(2)} & z_2^{(1)} \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 2 & 1 & 1 \\ 16 & -9 & 3 & -7 \\ 3 & -2 & -1 & -1 \\ -5 & 5 & 0 & 5 \end{bmatrix} = [z_{c1}^{(1)} \quad z_{c2}^{(3)} \quad z_{c2}^{(2)} \quad z_{c2}^{(1)}]$$

It should be noted that the left eigenvectors and latent vectors are reversed in order with respect to the right eigenvectors and latent vectors. The reason for this reverse ordering will be discussed later.

The companion matrix A_c can be generated from the right eigenvector matrix W_R or the left eigenvector matrix W_L

$$A_c = W_R J W_R^{-1} = W_L^{-T} J W_L^T \tag{2.21}$$

where

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

In the next section it will be assumed for simplicity that the (q_i-1), primary latent vectors do not have any generalized latent vectors and the q_ith latent vector has (m_i-q_i) generalized latent vectors although a more general treatment will require the structure of J_i as given in (2.19).

3. Eigenprojectors and Latent Projectors

It was shown in the previous section that the latent vectors of a regular lambda matrix with leading coefficient $A_0 = I$ are given as subvectors of the eigenvectors of the companion matrix A_c constructed from $A(\lambda)$. The structures of A_c and $A(\lambda)$ as given in (1.3) and (2.3) respectively will be assumed in this section. The computation of eigenprojectors and latent projections from the eigenvectors and the latent vectorswill be given in this section and their relationship to partial fraction expansions of $[A_c(\lambda)]^{-1}$ as well as $[A(\lambda)]^{-1}$ will be given. The chain rule for the latent projectors will be derived and an example will be presented.

Zadeh and Desoer [1] have given the general partial fraction expansion of $(\lambda I-A_c)^{-1}$ as

$$[A_c(\lambda)]^{-1} = (\lambda I-A_c)^{-1} = \sum_{i=1}^{r} \sum_{\ell=0}^{m_i-q_i} \frac{P_{i,\ell}}{(\lambda-\lambda_i)^{\ell+1}} \tag{3.1}$$

where $P_{i,\ell} \varepsilon C^{mn \times mn}$, m_i is the multiplicity of the eigenvalue λ_i and q_i is the number of primary eigenvectors. The $P_{i,\ell}$ matrices will be called the eigenprojectors since they can be constructed from the eigenvectors of A_c. The eigenprojectors are also the matrix residues which can be computed from the

usual formula

$$P_{i,m_i-q_i-k} = \lim_{\lambda \to \lambda_i} \left\{ \frac{1}{k!} \frac{d^{(k)}}{d\lambda^{(k)}} (\lambda-\lambda_i)^{m_i-q_i+1} [A_c(\lambda)]^{-1} \right\} \qquad (3.2)$$

$$k = 0,1,\ldots,m_i-q_i$$

or from the proper selection of the right and left eigenvectors as will be seen later.

The eigenprojectors (or matrix residues) satisfy the properties of the resolution of the identity and spectral decomposition

I. $\sum_{i=1}^{r} P_{i,0} = I_{mn \times mn}$

II. $P_{i,0} P_{i,0} = P_{i,0}$

III. $P_{i,0} P_{j,0} = 0_{mn \times mn} \qquad i \neq j$

IV. $P_{i,\ell+1} = P_{i,\ell} (A_c - \lambda_i I) \qquad \ell = 0,1,2,\ldots,m_i-q_i-2$

V. $P_{i,m_i-q_i} (A_c - \lambda_i I) = 0_{mn \times mn}$

VI. $A_c = \sum_{i=1}^{r} \lambda_i P_{i,0} + \sum_{i=1}^{r} P_{i,1} \qquad (3.3)$

Property VI describes the spectral decomposition of $A_c \in R^{mn \times mn}$ and the set of eigenvalues λ_i is called the spectrum of A_c. An additional property can be added for defining functions of a matrix which is obtained directly from VI with

VII. $f(A_c) = \sum_{i=1}^{r} P_{i,0} f(\lambda_i) + \sum_{i=1}^{r} \sum_{\ell=1}^{m_i-q_i} P_{i,\ell} \frac{1}{\ell!} \frac{d^{(\ell)}}{d\lambda^{(\ell)}} f(\lambda_i) \qquad (3.4)$

Properties I-V and VII are given in Zadeh and Desoer [1].

3.1 Definition. Let W_R, $W_L \in C^{mn \times mn}$ be formed from the set of normalized right and left eigenvectors, $y_{ci}^{(j)}$ and $z_{ci}^{(j)} \in C^{mn \times 1}$ respectively with $W_R W_L^T = I$. Let $Y_{ci}^{mn \times m_i}$ and $Z_{ci}^{mn \times m_i}$ be the rectangular matrices (or vectors when $m_i = 1$) for λ_i with

$$Y_{ci}^{mn \times m_i} = [y_{ci}^{(1)} \quad y_{ci}^{(2)} \quad \cdots \quad y_{ci}^{(q_i-1)} \quad y_{ci}^{(q_i)} \quad \cdots \quad y_{ci}^{(m_i)}] \tag{3.5}$$

where it is assumed that the first q_i vectors in (3.5) are the primary eigenvectors and the last $m_i - q_i$ are generalized eigenvectors associated to λ_i. Similarly, assume that

$$Z_{ci}^{mn \times m_i} = [z_{ci}^{(1)} \quad z_{ci}^{(2)} \quad \cdots \quad z_{ci}^{(q_i-1)} \quad z_{ci}^{(m_i)} \quad \cdots \quad z_{ci}^{(q_i)}] \tag{3.6}$$

where $z_{ci}^{(1)}, z_{ci}^{(2)}, \ldots, z_{ci}^{(q_i)}$ in (3.6) are primary eigenvectors and the vectors $z_{ci}^{(q_i+1)}, \ldots, z_{ci}^{(m_i)}$ the generalized eigenvectors for λ_i. The ordering of the left eigenvectors in (3.6) must be as shown because the ones in the Jordan blocks for λ_i are located on the subdiagonal of J_i due to the transpose operation in A_c.

The matrices W_R and W_L are the right and left eigenvector matrices (or modal matrices) with

$$A_c = W_R \, J \, W_R^{-1} = W_R \, J \, W_L^T \tag{3.7}$$

where $W_R^{-1} = W_L^T$ when the eigenvectors have been properly normalized. It follows from (3.7) that

$$(\lambda I - A_c) = W_R(\lambda I - J)W_R^{-1} = W_R(\lambda I - J)W_L^T \tag{3.8}$$

thus $(\lambda I - A_c)^{-1} = W_R(\lambda I - J)^{-1}W_R^{-1}$. See the appendix for the structure of J.

3.2 Theorem. Let $P_{i,0} \in C^{mn \times mn}$ be a primary eigenprojector of (3.2) for a distinct, nonrepeated eigenvalue λ_i of A_c. The primary eigenprojectors are given by

$$P_{i,0} = Y_i Z_i^T = y_{ci}^{(1)} z_{ci}^{(1)T} \tag{3.9}$$

where $y_{ci}^{(1)}$ and $z_{ci}^{(1)}$ are normalized right and left eigenvectors respectively of A_c.

Proof. Let $(\lambda I - A_c)^{-1}$ be as given in (3.8) with $P_{i,\ell}$ defined as in (3.2) then

$$P_{i,0} = \lim_{\lambda \to \lambda_i} (\lambda - \lambda_i)(\lambda I - A_c)^{-1} = \lim_{\lambda \to \lambda_i} (\lambda - \lambda_i) W_R (\lambda I - J)^{-1} W_L^T$$
$$= Y_i Z_i^T = y_{ci}^{(1)} z_{ci}^{(1)T}$$

as all diagonal elements of $(\lambda - \lambda_i)(\lambda I - J)^{-1}$ will be zero in the limit except for the ith diagonal element which will be one.

3.3 Theorem. Let $P_{i,0} \in C^{mn \times mn}$ be the primary eigenprojector of (3.2) for a repeated eigenvalue of A_c with $q_i = m_i$. The primary eigenprojectors for the repeated eigenvalues are given by

$$P_{i,0} = Y_i Z_i^T = \sum_{j=1}^{m_i} y_{ci}^{(j)} z_{ci}^{(j)T} \tag{3.10}$$

where $y_{ci}^{(j)}$ and $z_{ci}^{(j)}$ are the normalized eigenvectors for λ_i.

Proof. The proof follows from (3.8). The diagonal elements of $(\lambda-\lambda_i)(\lambda I-J)^{-1}$ will have ones only on the diagonals of the Jordan block associated with λ_i in the limit and zero elsewhere. This means that each right eigenvector pairs off with a left eigenvector giving (3.10).

3.3.1 Corollary. Given that $\hat{y}_{ci}^{(j)}$ and $\hat{z}_{ci}^{(j)}$ are not normalized, then the primary eigenprojectors are given by

$$P_{i,0} = \sum_{j=1}^{m_i} \frac{\hat{y}_{ci}^{(j)}\hat{z}_{ci}^{(j)T}}{\hat{z}_{ci}^{(j)T}\hat{y}_{ci}^{(j)}} = \sum_{j=1}^{m_i} \frac{\hat{y}_{ci}^{(j)}\hat{z}_{ci}^{(j)T}}{\eta_i^{j,j}} \tag{3.11}$$

where $\eta_i^{j,j} = \hat{z}_{ci}^{(j)T}\hat{y}_{ci}^{(j)}$.

Proof. Assume that the normalized right and left eigenvectors are scaled by α_j and β_j respectively. The numerator of (3.11) then becomes

$$\hat{y}_{ci}^{(j)}\hat{z}_{ci}^{(j)T} = \alpha_j\beta_j y_{ci}^{(j)} z_{ci}^{(j)T}$$

whereas the inner product in the denominator terms are

$$\eta_i^{j,j} = \hat{z}_{ci}^{(j)T}\hat{y}_{ci}^{(j)} = \alpha_j\beta_j z_{ci}^{(j)T} y_{ci}^{(j)} = \alpha_j\beta_j$$

since $z_{ci}^{(j)T} y_{ci}^{(j)} = 1$. The product $\alpha_j\beta_j$ in the numerator and denominator then cancel, giving (3.10). It is important to notice that $z_{ci}^{(j)T} y_{ci}^{(j)} = 1$ because the primary eigenprojectors satisfy Property II in (3.3).

3.4 Theorem. When the eigenvalue λ_i is repeated with multiplicity m_i and

$q_i < m_i$ then A_c will have q_i primary eigenvectors and $m_i - q_i$ generalized eigenvectors. The primary eigenprojector for λ_i is given by

$$P_{i,0} = \sum_{j=1}^{q_i-1} y_{ci}^{(j)} z_{ci}^{(j)T} + \sum_{j=q_i}^{m_i} y_{ci}^{(j)} z_{ci}^{(m_i-j+q_i)T} \tag{3.12}$$

where the first summation is over the set of q_i-1 normalized primary eigenvectors and the second summation is over the qth primary eigenvector and the set of (m_i-q_i) normalized generalized eigenvectors.

Proof. As before, $(\lambda I-A_c)^{-1}$ can be written as in (3.8) with J. Utilizing the material as given in Appendix A, the primary eigenprojector is

$$P_{i,0} = \lim_{\lambda \to \lambda_i} W_R \{ \frac{1}{(m_i-q_i)!} \frac{d^{(m_i-q_i)}}{d\lambda^{(m_i-q_i)}} (\lambda-\lambda_i)^{m_i-q_i} (\lambda I-J)^{-1} \} W_L^T \tag{3.13}$$

which after differentiating (m_i-q_i) times and taking the limit gives

$$P_{i,0} = Y_{ci} Z_{ci}^T \tag{3.14}$$

and substituting for Y_{ci} and Z_{ci} as defined in (3.5) and (3.6) gives (3.12).

3.4.1 Corollary. Let $\hat{y}_{ci}^{(j)}$ and $\hat{z}_{ci}^{(j)}$ be unnormalized right and left eigenvectors for the eigenvalue λ_i which has multiplicity m_i and $q_i < m_i$. The primary eigenprojector for λ_i is given by

$$P_{i,0} = \sum_{j=1}^{q_i-1} \frac{\hat{y}_{ci}^{(j)} \hat{z}_{ci}^{(j)T}}{\hat{z}_{ci}^{(j)T} \hat{y}_{ci}^{(j)}} + \sum_{j=q_i}^{m_i} \frac{\hat{y}_{ci}^{(j)} \hat{z}_{ci}^{(m_i-j+q_i)T}}{\hat{z}_{ci}^{(m_i-j+q_i)T} \hat{y}_{ci}^{(j)}}$$

$$= \sum_{j=1}^{q_i-1} \frac{\hat{y}_{ci}^{(j)} \hat{z}_{ci}^{(j)T}}{\eta_i^{j,j}} + \sum_{j=q_i}^{m_i} \frac{\hat{y}_{ci}^{(j)} \hat{z}_{ci}^{(m_i-j+q_i)T}}{\eta_i^{m_i-j+q_i,j}} \tag{3.15}$$

where $\eta_i^{j,k} = z_{ci}^{(j)^T} y_{ci}^{(k)}$.

Proof. The proof of this corollary follows from the proof in Corollary 3.3.1.

3.4.2 Corollary. The normalization coefficients $\eta_i^{j,k}$ for the summation over the qth primary eigenvector and the generalized eigenvectors is given by

$$\eta_i^{m_i,q_i} = \eta_i^{m_i-1,q_i+1} = \ldots = \eta_i^{q_i,m_i} \tag{3.16}$$

Proof. The proof of this corollary follows immediately from the definition of generalized right eigenvectors as given in (2.18) and from the generalized left eigenvectorsas follows

$$\begin{aligned}
(A_c-\lambda_i I)^T z_{ci}^{(q_i+1)} &= z_{ci}^{(q_i)} \\
(A_c-\lambda_i I)^T z_{ci}^{(q_i+2)} &= z_{ci}^{(q_i+1)} \\
&\vdots \qquad\qquad \vdots \\
(A_c-\lambda_i I)^T z_{ci}^{(m_i)} &= z_{ci}^{(m_i-1)}
\end{aligned}$$

and taking appropriate inner product the following result is obtained

$$\eta_i^{j,k} = z_{ci}^{(j)^T} y_{ci}^{(k)} = z_{ci}^{(m_i)^T} (A-\lambda_i I)^{m_i-q_i} y_{ci}^{(m_i)}$$

for $q_i \leq j \leq m_i$ and $q_i \leq k \leq m_i$ such that $j+k = m_i+q_i$; then (3.16) is proven.

3.5 Theorem. Let $y_{ci}^{(j)}$ and $z_{ci}^{(j)}$ be the right and left eigenvectors for the eigenvalue λ_i of multiplicity m_i and $q_i < m_i$, then there will be m_i-q_i

generalized eigenprojectors associated to λ_i and given by

$$P_{i,\ell} = \sum_{j=q_i}^{m_i-\ell} y_{ci}^{(j)} z_{ci}^{(m_i-j+q_i-\ell)^T} \tag{3.17}$$

where $\ell = 1,2,\ldots,m_i-q_i$.

Proof. This theorem is proven by considering the structure of $(\lambda I-J_i)^{-1}$ as given in Appendix A, and by carrying out the algebraic operations of (3.2).

3.5.1 Corollary. Given that the secondary eigenprojectors are to be constructed from the set of unnormalized eigenvectors, the required normalization factor is $\eta_i^{m_i,q_i}$, i.e.

$$P_{i,\ell} = \frac{1}{\eta_i^{m_i,q_i}} \sum_{j=q_i}^{m_i-\ell} \hat{y}_{ci}^{(j)} \hat{z}_{ci}^{(m_i-j+q_i-\ell)^T} \quad . \quad \ell=1,2,\ldots,m_i-q_i \tag{3.18}$$

Proof. The normalization coefficients $\eta_i^{j,k}$ are constants over the qth primary and generalized eigenvectors of λ_i, therefore, each vector product in (3.17) will be normalized by the same constant.

The algebraic theory of eigenprojectors is useful in developing the algebraic theory of latent projectors (or matrix residues) of a lambda matrix. The partial fraction expansion of $[A(\lambda)]^{-1}$ is

$$[A(\lambda)]^{-1} = \sum_{i=1}^{r} \sum_{\ell=0}^{m_i-q_i} \frac{\hat{P}_{i,\ell}}{(\lambda-\lambda_i)^{\ell+1}} \tag{3.19}$$

where $\hat{P}_{i,\ell}$ is a latent projector. It is known that $\hat{P}_{i,\ell}$ can be computed in the usual manner with

$$\hat{P}_{i,m_i-q_i-k} = \lim_{\lambda \to \lambda_i} \{\frac{1}{k!} \frac{d^{(k)}}{d\lambda^{(k)}} (\lambda-\lambda_i)^{m_i-q_i+1} [A(\lambda)]^{-1}\} \tag{3.20}$$

for $k = 0,1,2,\ldots,m_i-q_i$.

The development on eigenvectors, eigenprojectors and latent vectors will now be used to develop the algebraic theory of latent projectors as defined in (3.19) and (3.20).

3.6 Theorem. The lambda matrix $[A(\lambda)]^{-1}$ will be given in the upper right m×m block of the inverse of $A_c(\lambda)$ as shown

$$[A_c(\lambda)]^{-1} = \begin{bmatrix} . & . & . & [A(\lambda)]^{-1} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{bmatrix} \tag{3.21}$$

provided that $A(\lambda)$ is regular and $A_0 = I$.

Proof. The proof of this theorem follows directly from the inverse of $A_c(\lambda)$ where

$$A_c(\lambda) = \begin{bmatrix} \lambda I & -I & 0 & \ldots & 0 \\ 0 & \lambda I & -I & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots & \vdots \\ A_n & A_{n-1} & A_{n-2} & \ldots & \lambda I+A_1 \end{bmatrix}.$$

3.7 Theorem. Let $\hat{P}_{i,0}$ denote a primary latent projector of $A(\lambda)$ for the latent root λ_i of multiplicity m_i and $q_i = m_i$. The primary latent projector is given by

$$\hat{P}_{i,0} = \sum_{j=1}^{m_i} y_i^{(j)} z_i^{(j)T} \tag{3.22}$$

where $y_i^{(j)}$ and $z_i^{(j)T}$ are normalized right and left latent vectors respectively of $A(\lambda)$.

Proof. The matrix $A_c(\lambda)$ can be defined by

$$[A_c(\lambda)]^{-1} = W_R(\lambda I - J)^{-1} W_L^T$$

and from the previous theorem

$$[A_c(\lambda)]^{-1} = \begin{bmatrix} . & . & . & [A(\lambda)]^{-1} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{bmatrix} = \begin{bmatrix} Y \\ . \\ . \\ . \end{bmatrix} (\lambda I - J)^{-1} [\ldots Z^T]$$

with $Y = [Y_1, Y_2, \ldots, Y_r]$ and $Z = [Z_1, Z_2, \ldots, Z_r]$. The latent vectors in Y_i and Z_i are in sequential order since there are no generalized latent vectors. Using the definition of an eigenprojector from (3.20) and

$$[A(\lambda)]^{-1} = Y[\lambda I - J]^{-1} Z^T$$

gives the desired results when the individual vectors of Y and Z^T are considered.

3.7.1 Corollary. The lambda matrix $[A(\lambda)]^{-1}$ as defined in (2.3) is given by

$$[A(\lambda)]^{-1} = Y[\lambda I - J]^{-1} Z^T \tag{3.23}$$

where $Y \epsilon C^{m \times mn}$ and $Z \epsilon C^{m \times mn}$ are the matrices of right and left latent vectors respectively. Equation (3.23) will be designated as the canonical form for the inverse of a lambda matrix [3].

Proof. The results stated in this corollary were obtained in the previous theorem.

3.8 Theorem. The primary latent projectors $\hat{P}_{i,0}$ of $A(\lambda)$ for the latent root λ_i of multiplicity m_i and $q_i < m_i$ are given by

$$\hat{P}_{i,0} = \sum_{j=1}^{q_i-1} y_i^{(j)} z_i^{(j)T} + \sum_{j=q_i}^{m_i} y_i^{(j)} z_i^{(m_i-j+q_i)T} \tag{3.24}$$

where $y_i^{(j)}$ and $z_i^{(j)}$ are the normalized right and left latent vectors respectively.

Proof. The proof of this follows from a detailed analysis of $[A(\lambda)]^{-1}$ as defined in Corollary 3.7.1.

3.9 Theorem. Let $P_{i,0} \epsilon C^{mn \times mn}$ be a primary eigenprojector associated with an eigenvalue λ_i of multiplicity m_i with $q_i = m_i$. The $m \times m$ block matrices in the last m-columns of $P_{i,0}$ are given by $(\lambda_i)^j \hat{P}_{i,0}$ for $j = 0,1,2,\ldots,n-1$ where $\hat{P}_{i,0}$ is a primary latent projector.

Proof. When the inverse of $A_c(\lambda)$ is taken the m last columns will be given as follows

$$[A_c(\lambda)]^{-1} = \begin{bmatrix} . & . & [A(\lambda)]^{-1} \\ . & . & \lambda[A(\lambda)]^{-1} \\ . & . & \lambda^2[A(\lambda)]^{-1} \\ \vdots & \vdots & \vdots \\ . & . & \lambda^{n-1}[A(\lambda)]^{-1} \end{bmatrix} \tag{3.25}$$

but the eigenprojector $P_{i,0}$ for $q_i = m_i$ is defined as

$$P_{i,0} = \lim_{\lambda \to \lambda_i} \{(\lambda-\lambda_i)[A_c(\lambda)]^{-1}\} \tag{3.26}$$

then applying the last equation into (3.25) gives

$$P_{i,0} = \begin{bmatrix} \cdot & \cdot & \cdot & \hat{P}_{i,0} \\ \cdot & \cdot & \cdot & \lambda_i \hat{P}_{i,0} \\ \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \lambda_i^{n-1} \hat{P}_{i,0} \end{bmatrix} \tag{3.27}$$

which proves the theorem.

3.10 Theorem. Let $P_{i,0} \in C^{mn \times mn}$ be an eigenprojector for λ_i of multiplicity m_i with q_i primary eigenvectors and $m_i - q_i$ generalized eigenvectors. The $m \times m$ block matrices in the last m columns of $P_{i,0}$ are given by

$$P_{i,0} = \begin{bmatrix} \cdot & \cdot & \cdot & \hat{P}_{i,0} \\ \cdot & \cdot & \cdot & \lambda_i \hat{P}_{i,0} + \hat{P}_{i,1} \\ \cdot & \cdot & \cdot & \lambda_i^2 \hat{P}_{i,0} + 2\lambda_i \hat{P}_{i,1} + \hat{P}_{i,2} \\ \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \lambda_i^{n-1} \hat{P}_{i,0} + \binom{n-1}{1} \lambda_i^{n-2} \hat{P}_{i,1} + \binom{n-1}{2} \lambda_i^{n-3} \hat{P}_{i,2} + \ldots \end{bmatrix} \tag{3.28}$$

Proof. This theorem is proven by considering the eigenprojector $P_{i,0}$ defined as

$$P_{i,0} = \lim_{\lambda \to \lambda_i} \left\{ \frac{1}{(m_i-q_i)!} \frac{d^{(m_i-q_i)}}{d\lambda^{(m_i-q_i)}} (\lambda-\lambda_i)^{m_i-q_i+1} [A_c(\lambda)]^{-1} \right\} \qquad (3.29)$$

when this property is applied to (3.25), then the m×m block matrices in the last columns of $P_{i,0}$ will have the structure as given in (3.28).

3.11 Theorem. Let $\hat{P}_{i,\ell} \in C^{m\times m}$ be the latent projectors associated to the regular lambda matrix $A(\lambda)$ with $A_0 = I$, the set of latent projectors satisfies the following properties

$$\text{I.} \quad \sum_{i=1}^{r} \hat{P}_{i,0} = 0_{m\times m}$$

$$\text{II.} \quad \sum_{i=1}^{r} \sum_{\ell=0}^{j} \binom{j}{\ell} \lambda_i^{j-\ell} \hat{P}_{i,\ell} = 0_{m\times m} \qquad \text{for } j = 1,\ldots,n-2 \qquad \ell \leq m_i - q_i$$

$$\text{III.} \quad \sum_{i=1}^{r} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} \lambda_i^{n-\ell-1} \hat{P}_{i,\ell} = I_{m\times m} \qquad \ell \leq m_i - q_i \qquad (3.30)$$

If all $q_i = m_i$ then all the generalized latent projectors are zero and the above properties simplify to

$$\text{IV.} \quad \sum_{i=1}^{r} \lambda_i^j \hat{P}_{i,0} = 0_{m\times m} \qquad j = 0,\ldots,n-2$$

$$\text{V.} \quad \sum_{i=1}^{r} \lambda_i^{n-1} \hat{P}_{i,0} = I_{m\times m} \qquad (3.31)$$

Proof. Let $P_{i,0} \in C^{mn\times mn}$ be the primary eigenprojectors of the companion form A_c as given in (3.28), when the resolution of the identity $\sum_{i=1}^{r} P_{i,0} = I$ is applied, the m×m block matrices in the last column satisfy the properties given in (3.30). If all $q_i = m_i$ then $P_{i,0}$ have a simplified structure as given in (3.27) then properties in (3.31) can be easily obtained.

Property I in (3.30) will be called the resolution of the zero matrix. The above properties have been derived when the lambda matrix has the identity as the leading matrix coefficient, but for a general regular lambda matrix the above properties also hold where Properties III and V are modified having A_0^{-1} in their right hand side.

It has been assumed in the theorems on latent projectors that the latent vectors were normalized with normalization $W_R W_L^T = I$. The above normalization is based on complete knowledge of the eigenvectors which may not be available. It is therefore necessary to compute the normalization factors from the latent vectors. Suppose that $\eta_i^{j,k}$ is as given earlier, then for normalized eigenvectors

$$\eta_i^{j,k} = z_{ci}^{(j)T} y_{ci}^{(k)}. \tag{3.32}$$

This normalization factor must be applied to each unnormalized latent vector in the latent projector calculation.

The primary eigenvectors of A_c are given in (2.12) and (2.13) therefore, the normalization factor, is

$$\eta_i^{j,j} = \left[\ldots, z_i^{(j)T}[\lambda_i^2 I + A_1\lambda_i + A_2], z_i^{(j)T}[\lambda_i I + A_1], z_i^{(j)T} \right] \begin{bmatrix} y_i^{(j)} \\ \lambda_i y_i^{(j)} \\ \lambda_i^2 y_i^{(j)} \\ \vdots \\ \lambda_i^{n-1} y_i^{(j)} \end{bmatrix}$$

$$= z_i^{(j)T} \frac{dA(\lambda_i)}{d\lambda} y_i^{(j)} \qquad \text{for } j < q_i \tag{3.33}$$

The normalization factors for the latent vectors in the chain rule are more complex than the normalization factors for the primary latent vectors. A lemma will be given for that case.

3.12 Lemma. Let $J_i \in C^{m_i \times m_i}$ be the Jordan block associated with the latent root λ_i and let $y_i^{(j)}$ and $z_i^{(j)}$ be the latent vectors. The normalization factor is given by

$$\eta_i^{j,k} = \sum_{t+s+p=j+k+q_i} z_i^{(t)T} \frac{1}{s!} \frac{d^{(s)}A(\lambda_i)}{d\lambda^{(s)}} y_i^{(p)} \tag{3.34}$$

where λ_i has $m_i - q_i$ generalized latent vectors and where $q_i \leq t \leq j$ and $q_i \leq p \leq k$.

Proof. The proof of this lemma follows from Corollary 3.3.1 and the corresponding relations for the eigenvectors and latent vectors. As stated Corollary 3.4.2, the normalization factors $\eta_i^{m_i,q_i} = \eta_i^{m_i-1,q_i+1} = \ldots \quad \ldots = \eta_i^{q_i,m_i}$ are equal.

The generalized latent projectors satisfy a chain rule which would be expected since they are constructed from the generalize latant vectors that satisfy a chain rule.

Theorem 3.13. Let $\hat{P}_{i,\ell} \in C^{m \times m}$ be the generalized latent projectors associated with the latent root λ_i. The chain rule for the generalized latent projectors is

$$\sum_{\ell=0}^{m_i-q_i-j} \frac{1}{\ell!} \frac{d^{(\ell)}A(\lambda_i)}{d\lambda^{(\ell)}} \hat{P}_{i,j+\ell} = 0_{m \times m} \qquad j = 0,1,\ldots,m_i-q_i \tag{3.35}$$

where $A(\lambda) \in R^{m \times m}$ is a regular lambda matrix.

Proof. Let $P_{i,\ell} \in C^{mn \times mn}$ be the eigenprojectors of the companion form A_c. Using $P_{i,\ell}$ as defined in (3.2) it is easy to show that the m×m block matrices of the last m columns have the following structure

$$P_{i,\ell} = \begin{bmatrix} \cdot & \cdot & \hat{P}_{i,\ell} \\ \cdot & \cdot & \lambda_i \hat{P}_{i,\ell} + \hat{P}_{i,\ell+1} \\ \vdots & \vdots & \vdots \\ \cdot & \cdot & \lambda_i^{n-1} \hat{P}_{i,\ell} + \binom{n-1}{1} \lambda_i^{n-2} \hat{P}_{i,\ell+1} + \ldots \end{bmatrix} \tag{3.36}$$

when the structure of $P_{i,\ell}$ is used with Properties IV and V in (3.8) then the results in this theorem are obtained.

The results in the last theorem also hold for regular lambda matrix where matrix leading coefficient is different than the identity matrix.

The following example will be given to illustrate some of the computational aspects of the material. Let $A(\lambda)$ be defined as

$$A(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \lambda^2 + \begin{bmatrix} -4.0 & 2.0 \\ -0.2 & -3.0 \end{bmatrix} \lambda + \begin{bmatrix} 3.0 & -1.0 \\ 0.2 & 2.6 \end{bmatrix}$$

with latent roots $\lambda_1 = 1$ of multiplicity 1, and $\lambda_2 = 2$ of multiplicity 3. The latent vectors for the lambda matrix are given by

$$Y = [y_1^{(1)} \; y_2^{(1)} \; y_2^{(2)} \; y_2^{(3)}] = \begin{bmatrix} 1 & 3 & -1 & -2 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

and

$$Z = [z_1^{(1)} \; z_2^{(3)} \; z_2^{(2)} \; z_2^{(1)}] = \begin{bmatrix} 3 & -2 & -1 & -1 \\ -5 & 5 & 0 & 5 \end{bmatrix}$$

Since nullity A(2) = 1, there is only one primary latent projector for $\lambda_2 = 2$.

The latent projector for $\lambda_1 = 1$ can be computed directly from the latent vectors $y_1^{(1)}$ and $z_1^{(1)}$ and $dA(1)/d\lambda$ or

$$\hat{P}_{1,0} = \frac{y_1^{(1)} z_1^{(1)T}}{z_1^{(1)T} \frac{dA(1)}{d\lambda} y_1^{(1)}} = \frac{1}{5} \begin{bmatrix} -3 & 5 \\ 0 & 0 \end{bmatrix}$$

The second primary latent projector $\hat{P}_{2,0}$ is given by

$$\hat{P}_{2,0} = \frac{y_2^{(1)} z_2^{(3)T} + y_2^{(2)} z_2^{(2)T} + y_2^{(3)} z_2^{(1)T}}{\eta_2^{3,1}} = \frac{1}{5} \begin{bmatrix} 3 & -5 \\ 0 & 0 \end{bmatrix}$$

where $\eta_2^{3,1}$ is computed from (3.34) with

$$\eta_2^{3,1} = z_2^{(3)T} \frac{dA(2)}{d\lambda} y_2^{(1)} + \frac{1}{2!} z^{(2)T} \frac{d^2A(2)}{d\lambda^2} y_2^{(1)} + \frac{1}{3!} z^{(1)T} \frac{d^3A(2)}{d\lambda^{(3)}} y_2^{(1)} = -5$$

It can be verified that $\eta_2^{3,1} = \eta_2^{2,2} = \eta_2^{1,3}$. The generalized latent projectors are

$$\hat{P}_{2,1} = \frac{y_2^{(1)} z_2^{(2)T} + y_2^{(2)} z_2^{(1)T}}{\eta_2^{3,1}} = \frac{1}{5} \begin{bmatrix} 2 & 5 \\ 0 & 5 \end{bmatrix}$$

and

$$\hat{P}_{2,2} = \frac{y_2^{(1)} z_2^{(1)T}}{\eta_2^{3,1}} = \frac{1}{5} \begin{bmatrix} 3 & -15 \\ 1 & -5 \end{bmatrix}$$

The eigenprojectors for A_c are

$$P_{1,0} = \begin{bmatrix} 1.6 & -3.2 & -0.6 & 1 \\ 0 & 0 & 0 & 0 \\ 1.6 & -3.2 & -0.6 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_{2,0} = \begin{bmatrix} -0.6 & 3.2 & 0.6 & -1 \\ 0 & 1 & 0 & 0 \\ -1.6 & 3.2 & 1.6 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_{2,1} = \begin{bmatrix} -0.4 & -3.2 & 0.4 & 1 \\ 0 & -2 & 0 & 1 \\ -1.4 & -2.2 & 1.4 & -1 \\ -0.2 & -2.6 & 0.2 & 1 \end{bmatrix}$$

and

$$P_{2,2} = \begin{bmatrix} -0.6 & 4.2 & 0.6 & -3 \\ -0.2 & 1.4 & 0.2 & -1 \\ -1.2 & 8.4 & 1.2 & -6 \\ -0.4 & 2.8 & 0.4 & -2 \end{bmatrix}$$

where it is noted that the 2×2 upper right blocks are the latent projectors. The eigenprojectors were computed by using W_R sh $F_{i,\ell}$ W_R^{-1} with

$$W_R = \begin{bmatrix} 1 & 3 & -1 & -2 \\ 0 & 1 & -1 & -1 \\ 1 & 6 & 1 & -5 \\ 0 & 2 & -1 & -3 \end{bmatrix}$$

as constructed from (3.5) and the shifting matrices are

$$\text{sh } F_{1,0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{sh } F_{2,0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{sh } F_{2,1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{sh } F_{2,2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The structure of the shifting matrices has also appeared in [10].

The partial fraction expansion for $[A(\lambda)]^{-1}$ is

$$[A(\lambda)]^{-1} = \frac{1}{5(\lambda-1)}\begin{bmatrix} -3 & 5 \\ 0 & 0 \end{bmatrix} + \frac{1}{5(\lambda-2)}\begin{bmatrix} 3 & -5 \\ 0 & 0 \end{bmatrix} + \frac{1}{5(\lambda-2)^2}\begin{bmatrix} 2 & 5 \\ 0 & 5 \end{bmatrix} + \frac{1}{5(\lambda-2)^3}\begin{bmatrix} 3 & -15 \\ 1 & -5 \end{bmatrix}$$

The chain rule for the latent projectors $\hat{P}_{2,\ell}$ satisfy the equations

I. $A(2)\hat{P}_{2,2} = 0$

II. $A(2)\hat{P}_{2,1} + \frac{dA(2)}{d\lambda}\hat{P}_{2,2} = 0$

$$\text{III.} \quad A(2)\hat{P}_{2,0} + \frac{dA(2)}{d\lambda}\hat{P}_{2,1} + \frac{1}{2!}\frac{d^2A(2)}{d\lambda^2}\hat{P}_{2,2} = 0$$

as given in (3.35).

The results given in this chapter hold only for lambda matrices that are regular, i.e., $\bar{A}_0$ is nonsingular and a full set of latent vectors exists.

4. Conclusion

The theory of latent projectors have been presented and their use in the computation of the inverse of $A(\lambda)$ and its residues have been described. The latent projectors have specific properties and are related to the eigenprojectors of the companion form A_c obtained from $A(\lambda)$. The concept of generalized latent projectors has been developed which are constructed from the generalized latent vectors of $A(\lambda)$. The chain rule for the generalized latent projectors has been developed and described. The work on eigenprojectors, latent projectors and lambda matrices presented in this chapter is not exhaustive as many other questions must be resolved; for example, the case of singular $\bar{A}_0$ has not been addressed. The next chapter resolves the problem of singular $\bar{A}_0$.

CHAPTER 3

THE EIGENVALUE-EIGENVECTOR STRUCTURE OF OBSERVER STATE SPACE FORMS AND THEIR RELATIONSHIP TO GENERAL LAMBDA MATRICES

This chapter studies the algebraic theory of latent roots and latent vectors of a row-reduced form of a lambda matrix $A(\lambda) = A_0\lambda^n + A_1\lambda^{n-1} + \ldots + A_n$. In the previous chapters $A(\lambda)$ was assumed to be regular, i.e., $\det A_0 \neq 0$; however, this assumption is very restrictive particularly when $A(\lambda)$ arises as the left inverse matrix in a left polynomial matrix description for a dynamical system. In this chapter the regularity assumption is replaced by a weaker assumption, namely that $A_R(\lambda)$ is now reduced. The latter assumption is shown to be without loss of generality in the context of dynamical system modeling. The development utilitizes the duality between lambda matrices and observer state space forms to reveal the algebraic structure of the latent vectors of row-reduced lambda matrices. This, in turn, exposes the algebraic structure of the eigenvectors of observer state forms.

This chapter along with the material presented in Chapter 3 gives the general algebraic theory of lambda matrices and their relationship to state space forms. The mathematical development in the chapter supplements the material in the previous chapter. The computational procedure for the latent vectors, eigenvectors, latent projectors and eigenprojectors for the singular case is given.

1. Introduction

Multivariable systems, such as finite-element models of vibrating systems, control systems, and large scale systems, can be formulated in terms of second-and higher-order matrix differential equations of the form

$$A_0 \frac{d^n x}{dt^n} + A_1 \frac{d^{n-1} x}{dt^{n-1}} + \ldots \quad \ldots + A_n x = B_1 \frac{d^{n-1} u}{dt^{n-1}} + \ldots \quad \ldots + B_n u \tag{1.1}$$

where $A_i \epsilon R^{m \times m}$, $x(t) \epsilon R^m$, $B_i \epsilon R^{m \times r}$ and $u(t) \epsilon R^r$. Such a model is often called a left matrix description for the dynamical system.

If the Laplace transform is taken of Eq. (1.1), the following matrix polynomial appears

$$A(s) = A_0 s^n + A_1 s^{n-1} + \ldots \quad \ldots + A_n \tag{1.2}$$

on the left side of (1.1). If s is replaced by λ, then (1.2) defines a lambda matrix [1], also called a matrix polynomial.

Our objective here is to investigate the latent roots and the latent vectors of the polynomial matrices. Earlier work on this topic has been described by Lancaster [3], Lancaster and Weber [4], Dennis, Traub and Weber [5], Lancaster [9], and Denman, Leyva-Ramos and Jeon [11]. In all of this work it has been assumed that $A(\lambda)$ is <u>regular</u>, i.e., det $A_0 \neq 0$. This assumption permits the model in (1.1) to be expressed in block companion state space form. The assumption that $A(\lambda)$ is regular is too restrictive in the context of models of the form in (1.1). In general, regularity can be achieved by introducing additional roots, i.e., at the origin. However, this negates one of the prime motivations for the work in [3]-[5], namely

that of achieving numerically efficient procedures for handling eigenvalue-eigenvector problems arising from systems of the form given in (1.1). The general case will be treated in this chapter without the need for introducing additional roots.

2. Row-reduced Lambda Matrices

To ensure existence and uniqueness of the solution to (1.1), it is required that $\det A(\lambda) \neq 0$. The following definitions are introduced, [12] [13]

Definition 2.1 A nonsingular polynomial matrix $A_R(\lambda)$ is said to be **row-reduced** if the matrix formed from the coefficients of the highest powers of λ in each row is nonsingular.

It follows that any row-reduced polynomial matrix $A_R(\lambda)$ can be written in the form

$$A_R(\lambda) = S(\lambda)D_0 + L(\lambda) \tag{2.1}$$

where

$$S(\lambda) = \text{diag}[\lambda^{\nu_1}, \lambda^{\nu_2}, \ldots, \lambda^{\nu_m}] \tag{2.2}$$

$$L(\lambda) = \Psi(\lambda)^T \bar{L} \tag{2.3}$$

$$\Psi(\lambda)^T = \begin{bmatrix} \lambda^{\nu_1-1} & \lambda^{\nu_1-2} & \cdots & 1 & & & & & \\ & & & & \lambda^{\nu_2-1} & \lambda^{\nu_2-2} & \cdots & 1 & \\ & & & & & & & & \ddots \\ & & & & & & & & \lambda^{\nu_m-1} \quad \lambda^{\nu_m-2} \quad \cdots \quad 1 \end{bmatrix} \tag{2.4}$$

and D_0, $\bar{L}$ are matrices of reals with

$$\det D_0 \neq 0 \tag{2.5}$$

A key fact from the theory of polynomial matrices is the following:

Lemma 2.1 Any nonsingular lambda matrix $A(\lambda)$ can be transformed to row-reduced form $A_R(\lambda)$ by left multiplication by a unimodular matrix $U(\lambda)$.

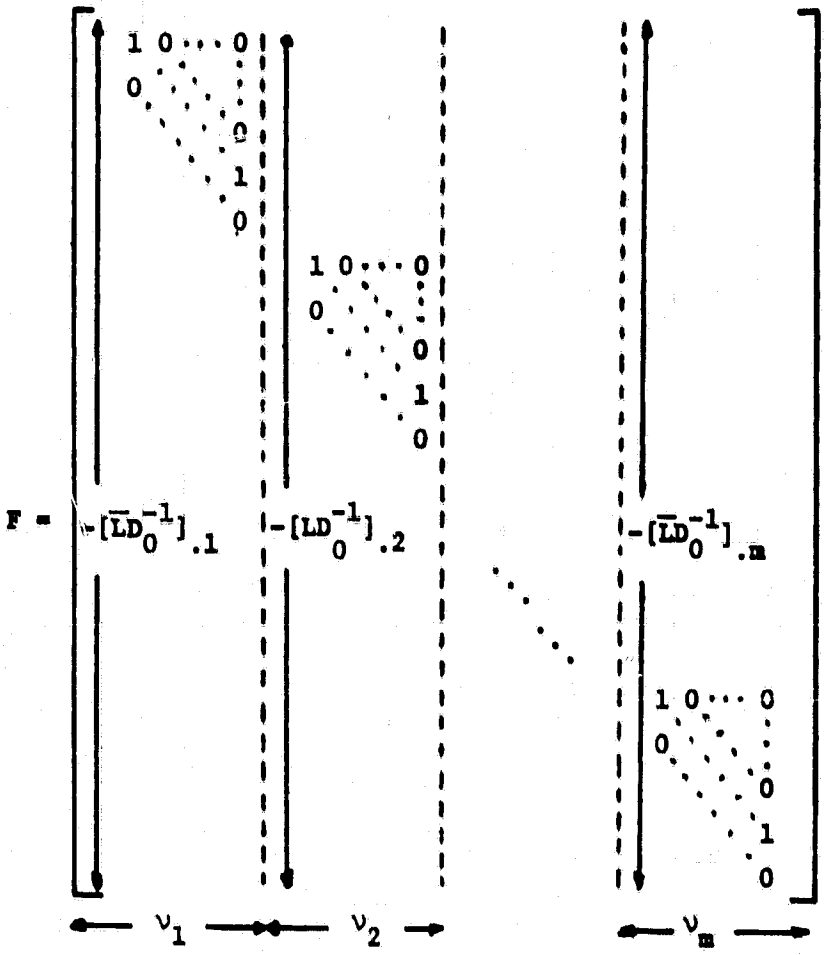
Proof: See [12, p. 27] or [13]. ∇∇∇

Note that operating on Eq. (1.1) by $U(\lambda)$ does not change the number of latent roots since det $U(\lambda)$ is a nonsingular constant matrix independent of λ. Thus, we see that $A(\lambda)$ can be transformed to the row-reduced form without loss of generality. However, it is very restrictive to assume that A_0 in (1.1) is nonsingular since this is tantamount to assuming that the row indices ν_1 to ν_m are all equal. The latter situation is unlikely to occur unless additional roots are artificially introduced. Of course, the case det $A_0 \neq 0$ is simply a special case of row-reducedness in which $\nu_i = \nu_j \; \forall \; i, j \in [1,2,\ldots,m]$.

Theorem 2.1. The number of latent roots of $A(\lambda)$ is precisely $\sum_{i=1}^{m} \nu_i$ where ν_i is the row-degree in the row-reduced form of $A_R(\lambda)$.

Proof: The proof follows directly from (2.1) as det $D_0 \neq 0$. ∇∇∇

By standard arguments, [13], [14], any row-reduced set of differential equations, as given in (2.1), can be associated with a state space model in the observer canonical form, sometimes called the flat canonical form [15]. The system matrix has the following form:



(2.6)

where $[\bar{L}D_0^{-1}]_{.i}$ denotes the ith column of $\bar{L}D_0^{-1}$ and D_0 and $\bar{L}$ are as in (2.1) and (2.3).

The above companion form is the appropriate generalization of the block companion form used elsewhere, (see [3] to [5]). Note that the block companion form requires that det $A_0 \neq 0$, whereas the form in (2.6) needs only the weaker assumption that det $D_0 \neq 0$ and there is no loss of generality in the latter assumption. In the sequel, $D_0 = I$ will be taken since if this is not the case, then one can simply multiply by D_0^{-1} throughout.

3. Eigenvectors and Generalized Eigenvectors

In this section, the definitions for eigenvalues, right and left eigenvectors, and the chain rules for the generalized eigenvectors of a real matrix F will be given.

The following definitions are standard for a real matrix F, (see [16]:

Definition 3.1 A scalar λ_i is an eigenvalue of F if

$$\det[\lambda_i I - F] = 0 \ . \tag{3.1}$$

The eigenvalue λ_i has multiplicity n_i if

$$\det[\lambda I - F] = \prod_{i=1}^{\sigma} (\lambda - \lambda_i)^{n_i} \tag{3.2}$$

where σ is the number of distinct eigenvalues.

Definition 3.2 A vector $y_i^{(1)}$ is said to be an eigenvector of F associated with λ_i iff

$$(F - \lambda_i I) y_i^{(1)} = 0 \tag{3.3}$$

and a set of vectors $\{y_i^{(2)}, y_i^{(3)}, \ldots, y_i^{(h_i)}\}$ are said to be the generalized eigenvectors associated to $y_i^{(1)}$ if

$$(F - \lambda_i I)^k y_i^{(k)} = 0 \qquad k = 2,3,\ldots,h_i \tag{3.4}$$

where h_i is such that $y_i^{(h_i+1)} = 0$ is the only solution in (3.4).

The generalized eigenvectors, as given in (3.4), can be also obtained in the following form

$$(F-\lambda_i I)y_i^{(k)} = y_i^{(k-1)} \qquad k = 2,3,\ldots,h_i \tag{3.5}$$

and the generalized eigenvectors associated to $y_i^{(1)}$ form a chain where h_i is the length of the chain. It is possible that the same eigenvalue produce sets of different chains associated with other eigenvectors but the number of different chains associated to the eigenvalue λ_i is equal to the nullity of $[F-\lambda_i I]$. The length of the longest chain is called the index of λ_i and is such that $\psi(\lambda) \triangleq \prod_{i=1}^{\sigma} (\lambda-\lambda_i)^{\max h_i}$ is the minimal polynomial of F. The expression in (3.5) is called the Jordan chain of generalized right eigenvectors.

From the right eigenvectors the modal matrix can be constructed by the collection of the eigenvectors and their chains where the chained eigenvectors appear in a sequential manner as $W_R = [y_1^{(1)}, y_1^{(2)}, \ldots, y_2^{(1)}, y_2^{(2)}, \ldots, y_\sigma^{(1)}, y_\sigma^{(2)}, \ldots]$, and W_R is called the right eigenvector matrix.

A similar definition of the left eigenvectors of F can be given where $(F-\lambda_i I)$ is replaced by its transpose in definition 3.2. The set of vectors, $\{z_1^{(2)}, z_1^{(3)}, \ldots, z_i^{(h_i)}\}$ form a chain of left generalized eigenvectors associated with the primary eigenvector $z_i^{(1)}$. The chain can be obtained from

$$(F-\lambda_i I)^T z_i^{(k)} = z_i^{(k-1)} \qquad k = 2,\ldots,h_i \tag{3.6}$$

where h_i is the length of the chain and $z_i^{(1)}$ is the primary left eigenvector.

Also, the collection of all the left eigenvectors forms the left eigen-

vector matrix where the chained eigenvectors appear in a sequential manner as $W_L = [y_1^{(1)}, y_1^{(2)}, \ldots, y_2^{(1)}, y_2^{(2)}, \ldots, y_\sigma^{(1)}, y_\sigma^{(2)}, \ldots]$.

4. Latent Roots and Latent Vectors of Lambda Matrices

Analogous definitions to those in Section 3 for a row-reduced lambda matrix $A_R(\lambda)$ will be given. It will be shown that there is an interesting duality between the definitions of Section 3 and those for the corresponding lambda matrix $A_R(\lambda)$ and F, as given in (2.6). Without loss of generality, it is assumed that $D_0 = I$. To simplify the notation in the proofs, it, has been assumed that m=2. The extension to m>2 is along the same lines.

Definition 4.1 Let $A_R(\lambda)$ be as defined in (2.1) and let $\lambda_i \in C$ be a scalar, then λ_i is a latent root of $A_R(\lambda)$ if $\det A_R(\lambda_i) = 0$. Now λ_i has multiplicity n_i if $\det A(\lambda) = \prod_{i=1}^{\sigma} (\lambda-\lambda_i)^{n_i}$, where σ is the number of distinct roots.

Connection 4.1 The determinant of $A_R(\lambda)$ is equal to the determinant of $(\lambda I-F)$, where F is as defined in (2.6) with

$$\det A_R(\lambda) = \det(I\lambda-F) = \prod_{i=1}^{\sigma} (\lambda-\lambda_i)^{n_i}. \tag{4.1}$$

Proof: Let $A_R(\lambda)$ be as defined in (2.1) and represented by

$$A_R(\lambda) = \begin{bmatrix} p^{11}(\lambda) & p^{12}(\lambda) \\ p^{21}(\lambda) & p^{22}(\lambda) \end{bmatrix} = \begin{bmatrix} \lambda^{\nu_1} + a_1^{11}\lambda^{\nu_1-1} + \ldots & \ldots + a_{\nu_1}^{11} \\ a_1^{21}\lambda^{\nu_2-1} + \ldots & \ldots + a_{\nu_2}^{21} \end{bmatrix}$$

$$\cdot \begin{bmatrix} a_1^{12}\lambda^{\nu_1-1} + \ldots & \ldots + a_{\nu_1}^{12} \\ \lambda^{\nu_2} + a_1^{22}\lambda^{\nu_2-1} + \ldots & \ldots + a_{\nu_2}^{22} \end{bmatrix}. \tag{4.2}$$

$\overline{L}$ in (2.3) can be easily obtained as well as the observer canonical form F, then $(\lambda I-F)$ can be represented by

$$(\lambda I-F) = \left[\begin{array}{cccc:cccc} \lambda+a_1^{11} & -1 & \cdots & 0 & a_1^{12} & 0 & \cdots & 0 \\ a_2^{11} & \lambda & \ddots & 0 & & & & \\ \vdots & & \ddots & -1 & \vdots & \vdots & & \vdots \\ a_{\nu_1}^{11} & & & \lambda & a_{\nu_1}^{12} & 0 & \cdots & 0 \\ \hdashline a_1^{21} & 0 & \cdots & 0 & \lambda+a_1^{22} & -1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & a_2^{22} & \lambda & \ddots & 0 \\ & & & & & & \ddots & -1 \\ a_{\nu_2}^{21} & 0 & \cdots & 0 & a_{\nu_2}^{22} & & & \lambda \end{array}\right] . \tag{4.3}$$

If $(\lambda I-F)$ is multiplied by the following unimodular matrix,

$$U(\lambda) = \text{Block Diag}\left\{\left[\begin{array}{ccccc} 1 & 0 & \cdots & \cdots & 0 \\ \lambda & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \lambda^{\nu_1-1} & \cdots & \cdots & \lambda & 1 \end{array}\right], \left[\begin{array}{ccccc} 1 & 0 & \cdots & \cdots & 0 \\ \lambda & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \lambda^{\nu_2-1} & \cdots & \cdots & \lambda & 1 \end{array}\right]\right\} \tag{4.4}$$

then

$$U(\lambda)(\lambda I-F) = \begin{bmatrix} p^{11}_{\nu_1-1}(\lambda) & -1 & 0 & \cdots & 0 & \vdots & p^{12}_{\nu_1-1}(\lambda) & 0 & \cdots & 0 \\ \vdots & & 0 & & 0 & & \vdots & & & \\ p^{11}_1(\lambda) & & & & -1 & & p^{12}_1(\lambda) & & & \\ p^{11}(\lambda) & 0 & & & 0 & & p^{12}(\lambda) & 0 & & 0 \\ \hdots & \hdots & \hdots & \hdots & \hdots & \hdots & \hdots & \hdots & \hdots & \hdots \\ p^{21}_{\nu_{2-1}}(\lambda) & 0 & & & 0 & & p^{22}_{\nu_2-1}(\lambda) & -1 & 0 & 0 \\ \vdots & & & & & & \vdots & 0 & & 0 \\ p^{21}_1(\lambda) & & & & & & p^{22}_1(\lambda) & & & -1 \\ p^{21}(\lambda) & 0 & & & 0 & & p^{22}(\lambda) & 0 & & 0 \end{bmatrix} = G(\lambda) \quad . \quad (4.5)$$

However, by evaluating $|G(\lambda)|$ by minors on the ones, it is clear that

$$\det(\lambda I-F) = \det \begin{bmatrix} p^{11}(\lambda) & p^{12}(\lambda) \\ p^{21}(\lambda) & p^{22}(\lambda) \end{bmatrix} = \det A(\lambda) \quad . \quad (4.6)$$

∇∇∇

<u>Definition 4.2</u> Let $A_R(\lambda)$ have latent roots λ_i then the primary right latent vectors denoted by $y_i^{(1)}$ and the primary left latent vectors denoted by $z_i^{(1)}$ satisfy the relations

$$A_R(\lambda_i)y_i^{(1)} = 0_{m\times 1} \quad (4.7a)$$

and

$$A_R^T(\lambda_i)z_i^{(1)} = 0_{m\times 1} \quad (4.7b)$$

respectively.

We now explore the structure of the eigenvectors of F associated with an eigenvalue and then relate these to the latent vectors associated with $A_R(\lambda)$.

From (3.3) the eigenvector satisfies the relation $(F-\lambda_i I)y_{0i}^{(1)} = 0$ and by operating on this expression by $-U(\lambda)$, as defined in (4.4), the following expression is obtained:

$$G(\lambda_i)y_{0i}^{(1)} = 0 \ , \tag{4.8}$$

where $G(\lambda)$ is as defined in (4.5). Note that the polynomials $p_j^{11}(\lambda)$, $j = 1,\ldots,\nu_1-1$ are the Tschinhausen polynomials [16] associated with $p^{11}(\lambda)$ defined as follows

$$\begin{bmatrix} p_{\nu_1-1}^{11}(\lambda) \\ p_{\nu_1-2}^{11}(\lambda) \\ \vdots \\ p_1^{11}(\lambda) \\ p^{11}(\lambda) \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ \lambda^2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \lambda^{\nu_1-1} & & \ddots & \ddots & 1 \\ \lambda^{\nu_1} & \lambda^{\nu_1-1} & \cdots & \lambda^2 & \lambda \end{bmatrix} \begin{bmatrix} 1 \\ a_1^{11} \\ \vdots \\ a_{\nu_1}^{11} \end{bmatrix} \tag{4.9}$$

and similar expressions can be found for the other polynomials in (4.5).

The eigenvector $y_{0i}^{(1)}$ can now be expressed as a function of the latent vector $y_i^{(1)}$ in the following connection.

Connection 4.2 Let $y_{0i}^{(1)}$ be as defined in (4.8) and let $y_i^{(1)}$ be the corresponding right latent vector associated with the root λ_i, then the eigenvector and latent vector are related by

$$y_{0i}^{(1)} = U(\lambda_i)My_i^{(1)} \tag{4.10}$$

where $U(\lambda_i)$ is as defined in (4.4) and M is given by

$$M = \begin{bmatrix} 1 & 0 \\ a_1^{11} & a_1^{12} \\ \vdots & \vdots \\ a_{\nu_1-1}^{11} & a_{\nu_1-1}^{12} \\ 0 & 1 \\ a_1^{21} & a_1^{22} \\ \vdots & \vdots \\ a_{\nu_2-1}^{21} & a_{\nu_2-1}^{22} \end{bmatrix} . \qquad (4.11)$$

Proof: The expression given in (4.10) is obtained by noting in (4.5) that the first and the (ν_1+1) elements of the eigenvector $y_{0i}^{(1)}$ form the latent vector $y_i^{(1)}$. Then by using simple substitutions to obtain the rest of the elements of the eigenvector as a function of the elements of the latent vector, the expression (4.10) is obtained where the corresponding Tschinhauser polynomials as given in (4.9) have been used. VVV

Also the left eigenvector $z_{0i}^{(1)}$ of the observer form defined as $(\lambda_i I-F)^T z_{0i}^{(1)} = 0$ can be obtained as a function of the left latent vector $z_i^{(1)}$ of the row-reduced form $A_R(\lambda_i)$:

Connection 4.3 Let $z_{0i}^{(1)}$ be the left eigenvector of F and $z_i^{(1)}$ the corresponding left latent vector associated to the root λ_i, then the eigenvector and latent vector are related by

$$z_{0i}^{(1)} = \begin{bmatrix} \lambda_i^{\nu_1-1} & 0 \\ \vdots & \vdots \\ \lambda_i & 0 \\ 1 & 0 \\ 0 & \lambda_i^{\nu_2-1} \\ \vdots & \vdots \\ 0 & \lambda_i \\ 0 & 1 \end{bmatrix} z_i^{(1)} = \Psi(\lambda_i) z_i^{(1)} \qquad (4.12)$$

where $\Psi(\lambda)$ is as given in (2.4).

Proof: Consider $(\lambda I - F)$ as given in (4.3), then the left eigenvector is obtained by $(\lambda_i I - F)^T z_{0i}^{(1)} = 0$ and using the expression for $z_{0i}^{(1)}$, then

$$\begin{bmatrix} \lambda_i + a_1^{11} & a_2^{11} & \cdots & a_{\nu_1}^{11} & a_1^{21} & \cdots & a_{\nu_2}^{21} \\ -1 & \lambda_i & \ddots & 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & & \vdots \\ 0 & \cdots & 0 & -1 & \lambda_i & 0 & \cdots & 0 \\ a_1^{12} & \cdots & & a_{\nu_1}^{12} & \lambda_i + a_1^{22} & a_2^{22} & \cdots & a_{\nu_2}^{22} \\ 0 & \cdots & \cdots & 0 & -1 & \lambda_i & \ddots & 0 \\ \vdots & & & \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & -1 & \lambda_i \end{bmatrix} \begin{bmatrix} \lambda_i^{\nu_1-1} & 0 \\ \lambda_i^{\nu_1-2} & 0 \\ \vdots & \vdots \\ \lambda_i & 0 \\ 1 & 0^{\nu_2-1} \\ 0 & \lambda_i \\ 0 & \lambda_i^{\nu_2-2} \\ \vdots & \vdots \\ 0 & \lambda_i \\ 0 & 1 \end{bmatrix} z_i^{(1)} = 0 . \qquad (4.13)$$

In the above expression, the first and (ν_1+1) rows will contain the lambda matrix $A_R(\lambda)$ as defined in (4.2), therefore $A_R^T(\lambda_i) z_i^{(1)} = 0$ as expected. $\nabla\nabla\nabla$

It is interesting to notice in (4.12) that the elements of the left latent vector appear on the ν_1 and $(\nu_1+\nu_2)$ locations of the left eigenvector.

In the same manner as in section (2), a set of generalized latent vectors associated with the latent vector $y_i^{(1)}$ exist and they satisfy a chain rule, which is given in the next theorem.

Theorem 4.1 Let $A_R(\lambda)$ be defined as in (2.1). A set of right latent vectors $y_i^{(1)}, y_i^{(2)}, \ldots \ldots, y_i^{(\ell_i)} \in C^{m\times 1}$ form a right Jordan chain associated with the latent root λ_i and the primary latent vector $y_i^{(1)}$. The chain is given by

$$A_R(\lambda_i)y_i^{(\ell)} + \frac{dA_R(\lambda_i)}{d\lambda} y_i^{(\ell-1)} + \frac{1}{2!}\frac{d^2A_R(\lambda_i)}{d\lambda^2} y_i^{(\ell-2)} + \ldots$$

$$\ldots + \frac{1}{(\ell-1)!}\frac{d^{(\ell-1)}A_R(\lambda_i)}{d\lambda^{(\ell-1)}} y_i^{(1)} = 0_{m\times 1} \qquad \ell = 2,\ldots,h_j \tag{4.14}$$

where h_j is the length of the Jordan chain. The vectors $y_i^{(\ell)}$ for $1<\ell\leq h_j$ are called the generalized right latent vectors of the latent vector $y_i^{(1)}$.

Proof: The generalized right eigenvector $y_{0i}^{(2)}$ can be obtained from the equation $(F-\lambda_i I)y_{0i}^{(2)} = y_{0i}^{(1)}$ which when multiplied by $-U(\lambda_i)$ as defined in (4.4) gives the following expression:

$$G(\lambda_i)y_{0i}^{(2)} = -U(\lambda_i)y_{0i}^{(1)} \ . \tag{4.15}$$

Using connection 4.2 and considering the ν_1 and $(\nu_1+\nu_2)$ rows of the above expression, the following result is obtained

$$A_R(\lambda_i)y_i^{(2)} = -\Psi^T(\lambda_i)y_{0i}^{(1)} \ , \tag{4.16}$$

where the elements of generalized latent vector $y_i^{(2)}$ are in the first and (ν_1+1) locations of the generalized eigenvector $y_{0i}^{(1)}$. Using the expression $y_{0i}^{(1)}$, as given in (4.10), gives

$$A_R(\lambda_i)y_i^{(2)}+\Psi^T(\lambda_i)U(\lambda_i)My_i^{(1)} = 0 \ . \tag{4.17}$$

Considering the generalized eigenvector $y_{0i}^{(3)}$ as obtained from $(F-\lambda_i I)y_{0i}^{(3)} = y_{0i}^{(2)}$, then a similar expression to that in (4.16) is obtained with

$$A_R(\lambda_i)y_i^{(3)}+\Psi^T(\lambda_i)y_{0i}^{(2)} = 0 \ . \tag{4.16}$$

$y_{0i}^{(2)}$ is needed in the above expression and can be obtained from (4.15) where the elements of $y_i^{(2)}$ are located in the first and (ν_1+1) elements. We then use back substitutions to obtain the rest of the elements of $y_{0i}^{(2)}$ as a function of $y_i^{(1)}$ where the following expression is obtained for $y_{0i}^{(2)}$

$$y_{0i}^{(2)} = U(\lambda_i)My_i^{(2)}+\begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & \ddots & & & & \\ \lambda_i & \ddots & & & & \\ & \ddots & & & & \\ & & & & & \\ \lambda_i^{\nu_1-2} & \cdots & \lambda_i\ 1\ 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ & & & 1 & \ddots & \\ & & & \lambda_i & \ddots & \\ & & & & \ddots & \\ 0 & \cdots & 0 & \lambda_i^{\nu_2-2} & \cdots & \lambda_i\ 1\ 0 \end{bmatrix} y_{0i}^{(1)} \ . \tag{4.19}$$

The second matrix on the second term is not $U(\lambda)$ but is a shifted version of $U(\lambda)$ and will be called sh $U(\lambda)$ where sh stands for shifting. This shifting matrix has the interesting property that

$$\frac{1}{(\ell+1)} \frac{d^{(\ell+1)}U(\lambda)}{d\lambda^{(\ell+1)}} = (\text{sh } U(\lambda)) \frac{d^{(\ell)}}{d\lambda^{(\ell)}} U(\lambda) . \tag{4.20}$$

Now substituting the values of $y_{0i}^{(1)}$ into (4.19) and using the shifting property, the expression in (4.18) can be changed to

$$A_R(\lambda_i)y_i^{(3)}+\Psi^T(\lambda_i)U(\lambda_i)My_i^{(2)}+\Psi^T(\lambda_i) \frac{dU(\lambda_i)}{d\lambda} My_i^{(1)} = 0 . \tag{4.21}$$

Following the same lines as the beginning of the proof and using the shifting property the following chain rule is obtained.

$$A_R(\lambda_i)y_i^{(\ell)}+\Psi^T(\lambda_i)U(\lambda_i)My_i^{(\ell-1)}+\Psi^T(\lambda_i) \frac{dU(\lambda_i)}{d\lambda} My_i^{(\ell-2)}+ \ldots$$

$$\ldots + \Psi^T(\lambda_i) \frac{1}{(\ell-2)!} \frac{d^{(\ell-2)}}{d\lambda^{(\ell-2)}} U(\lambda_i)My_i^{(1)} = 0_{m\times 1} \qquad \ell = 2,3,\ldots,h_i . \tag{4.22}$$

Equation (4.14) of this theorem is obtained by recognizing that

$$\frac{1}{(\ell+1)} \frac{d^{(\ell+1)}}{d\lambda^{(\ell+1)}} A_R(\lambda) = \Psi^T(\lambda) \frac{d^{(\ell)}U(\lambda)}{d\lambda^{(\ell)}} M \quad \text{for } \ell = 0,1,\ldots,h_i-2 \tag{4.23}$$

thus the theorem is proved. ∇∇∇

The left generalized latent vectors associated with $z_i^{(1)}$ satisfies a similar chain rule that is given in the next theorem.

Theorem 4.2 Let $A_R(\lambda)$ be defined as in (2.1), a set of left latent vectors $z_i^{(1)}, z_i^{(2)}, \ldots, z_i^{(h_j)} \in C^{m\times 1}$ form a left Jordan chain associated with the latent root λ_i and the primary latent vector $z_i^{(1)}$. The chain rule is given by

$$A_R^T(\lambda_i) z_i^{(\ell)} + \frac{dA_R^T(\lambda_i)}{d\lambda} z_i^{(\ell-1)} + \frac{1}{2!}\frac{d^2A_R^T(\lambda_i)}{d\lambda^2} z_i^{(\ell-2)} + \ldots$$

$$\ldots + \frac{1}{(\ell-1)!}\frac{d^{(\ell-1)}A_R^T(\lambda_i)}{d\lambda^{(\ell-1)}} z_i^{(1)} = 0_{m\times 1} \qquad \ell = 1,2,\ldots,h_j \tag{4.24}$$

where h_j is the length of the Jordan chain. The vectors $z_i^{(\ell)}$ for $1<\ell\leq h_j$ are the generalized left latent vectors of the primary latent vector $z_i^{(1)}$.

Proof: The proof of this theorem follows along the same lines as theorem 4.1 where the chain of generalized left eigenvectors as given in (3.6) is used with the corresponding values of $z_{0i}^{(\ell)}$. The elements of $z_i^{(\ell)}$ are located in the ν_1 and $(\nu_1+\nu_2)$ entries of $z_{0i}^{(\ell)}$. ∇∇∇

As can be seen in the proof of Theorem 2.1, some interesting relations exist between the generalized eigenvectors of F and the generalized latent vectors of $A_R(\lambda)$.

Connection 4.4 Let $y_{0i}^{(2)}, y_{0i}^{(3)}, \ldots, y_{0i}^{(h_i)}$ be a set of generalized eigenvectors of F and let $y_i^{(2)}, y_i^{(3)}, \ldots, y_i^{(h_i)}$ be the corresponding set of generalized latent vectors of $A_R(\lambda)$ associated with the root λ_i. The eigenvectors and latent vectors are related by

$$y_{0i}^{(\ell)} = U(\lambda_i) My_i^{(\ell)} + \frac{dU(\lambda_i)}{d\lambda} My_i^{(\ell-1)} + \ldots \quad \ldots \frac{1}{(\ell-1)!}\frac{d^{(\ell-1)}U(\lambda_i)}{d\lambda^{(\ell-1)}} My_i^{(1)}$$

$$\ell = 1,2,\ldots,h_i \,. \tag{4.25}$$

Proof: The eigenvector $y_{0i}^{(1)}$ is a function of $y_i^{(1)}$ as was previously stated in connection 4.2. Now using $y_{0i}^{(2)}$ as given in (4.19) and substituting $y_{0i}^{(1)}$

$$y_{0i}^{(2)} = U(\lambda_i)My_i^{(2)}+(sh\ U(\lambda_i))U(\lambda_i)My_i^{(1)} \tag{4.26a}$$

and using (4.20)

$$y_{0i}^{(2)} = U(\lambda_i)My_i^{(2)} + \frac{dU(\lambda_i)}{d\lambda} My_i^{(1)}. \tag{4.26b}$$

The generalized eigenvector $y_{0i}^{(3)}$ can be obtained from the chain rule $(F-\lambda_i I)y_{0i}^{(3)} = y_{0i}^{(2)}$ and following the arguments as in (4.19), with

$$y_{0i}^{(3)} = U(\lambda_i)My_i^{(3)}+(sh\ U(\lambda_i))y_{0i}^{(2)}. \tag{4.27}$$

Substituting the vector $y_{0i}^{(2)}$ as given in (4.26b) in the last expression then

$$y_{0i}^{(3)} = U(\lambda_i)My_i^{(3)} + \frac{dU(\lambda_i)}{d\lambda} My_i^{(2)} + \frac{1}{2!}\frac{d^2U(\lambda_i)}{d\lambda^2} My_i^{(1)}. \tag{4.28}$$

The remaining eigenvectors in the chain are obtained from the extension of (4.28). ∇∇∇

The left generalized eigenvectors of F can be obtained from the left generalized latent vectors of $A_R(\lambda)$ as given in the following connection.

Connection 4.5 Let $z_{0i}^{(2)}, z_{0i}^{(3)}, \ldots \ \ldots, z_{0i}^{(h_i)}$ be a set of generalized left eigenvectors of F and let $z_i^{(2)}, z_i^{(3)}, \ldots \ \ldots, z_i^{(h_i)}$ be the corresponding set of generalized left latent vectors of $A_R(\lambda)$ associated with the root λ_i. The left eigenvectors and left latent vectors are related by

$$z_{0i}^{(\ell)} = \Psi(\lambda_i) z_i^{(\ell)} + \frac{d\Psi(\lambda_i)}{d\lambda} z_i^{(\ell-1)} + \ldots \quad \ldots + \frac{1}{(\ell-1)!} \frac{d^{(\ell-1)}\Psi(\lambda_i)}{d\lambda^{(\ell-1)}} z_i^{(1)}$$

$$\ell = 1,2,\ldots,h_i \,. \tag{4.29}$$

Proof: The left eigenvector $z_{0i}^{(1)}$ as a function of $z_i^{(1)}$ was previously obtained in connection 4.3. The left eigenvector satisfies $(F-\lambda_i I)^T z_{0i}^{(2)} = z_{0i}^{(1)}$ where the elements of the left latent vector $z_i^{(2)}$ are at the ν_1 and $(\nu_1+\nu_2)$ locations of $z_{0i}^{(2)}$ and thus by simple substitution, $z_{0i}^{(2)}$ can be represented by

$$z_{0i}^{(2)} = \Psi(\lambda_i) z_i^{(2)} + (\text{sh } U(\lambda_i))^T z_{0i}^{(1)} \tag{4.30}$$

where $(\text{sh } U(\lambda_i))^T$ is the shifting upwards on the transpose of $U(\lambda)$. This shifting matrix has the property that

$$\frac{1}{(\ell+1)} \frac{d^{(\ell+1)}\Psi(\lambda)}{d\lambda^{(\ell+1)}} = (\text{sh } U(\lambda))^T \frac{d^{(\ell)}}{d\lambda^{(\ell)}} \Psi(\lambda) \,. \tag{4.31}$$

Using $z_{0i}^{(1)}$ as given in (4.12) and the above property with (4.30), then $z_{0i}^{(1)}$ is given by

$$z_{0i}^{(2)} = \Psi(\lambda_i) z_i^{(2)} + \frac{d\Psi(\lambda_i)}{d\lambda} z_i^{(1)} \,. \tag{4.32}$$

Successive application of the chain rule of the left generalized eigenvectors and (4.31) gives equation (4.29) as stated in this connection. ∇∇∇

It has been shown that the eigenvectors of the observer state form can be defined in terms of the latent vectors of the row-reduced form $A_R(\lambda)$.

It therefore follows that the eigenvectors of the observer state form can be obtained directly from the knowledge of the latent roots and latent vectors of $A_R(\lambda)$.

5. An Example

An illustrative example is given. Let $A_R(\lambda)$ be the row-reduced form of a lambda matrix that has been subjected to a unimodular matrix

$$A_R(\lambda) = \begin{bmatrix} \lambda^3-\lambda^2+7\lambda+3 & 8\lambda+5 \\ -3\lambda+3 & \lambda^2-6\lambda+5 \end{bmatrix} \tag{5.1}$$

and then expressing the above row-reduced lambda matrix into the form given in (2.1)

$$A_R(\lambda) = \begin{bmatrix} \lambda^3 & 0 \\ 0 & \lambda^2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda^2 & \lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 7 & 8 \\ 3 & 5 \\ -3 & -6 \\ 3 & 5 \end{bmatrix} \tag{5.2}$$

where it can be noted that $\nu_1 = 3$ and $\nu_2 = 2$. The observer canonical form is

$$F = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -7 & 0 & 1 & -8 & 0 \\ -3 & 0 & 0 & -5 & 0 \\ 3 & 0 & 0 & 6 & 1 \\ -3 & 0 & 0 & -5 & 0 \end{bmatrix}. \tag{5.3}$$

The determinant of the lambda matrix is $|A_R(\lambda)| = \lambda(\lambda-1)(\lambda-2)^3$, therefore $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_3 = 2$. The primary right latent vectors can be cal-

culated using (4.7a) with

$$A_R(0)y_1^{(1)} = 0 \qquad A_R(1)y_2^{(1)} = 0 \qquad A_R(2)y_3^{(1)} = 0 \tag{5.4}$$

and the latent vectors are

$$y_1^{(1)} = \begin{bmatrix} -5 \\ 3 \end{bmatrix} \qquad y_2^{(1)} = \begin{bmatrix} 13 \\ -10 \end{bmatrix} \qquad y_3^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$y_3^{(1)}$ has two generalized latent vectors that can be calculated using (4.14)

$$A_R(2)y_3^{(2)} + \frac{dA_R(2)}{d\lambda} y_3^{(1)} = 0 \quad \text{and} \quad A_R(2)y_3^{(3)} + \frac{dA_R(2)}{d\lambda} y_3^{(2)} + \frac{1}{2} \frac{d^2A_R(2)}{d\lambda^2} y_3^{(1)} = 0 \tag{5.5}$$

and the generalized latent vectors are

$$y_3^{(2)} = \begin{bmatrix} -1/3 \\ 0 \end{bmatrix} \qquad y_3^{(3)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The right eigenvectors of (5.3) can be calculated using (4.10) where the unimodular matrix $U(\lambda)$ and M are

$$U(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 & 0 \\ \lambda^2 & \lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \end{bmatrix} \qquad M = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 7 & 8 \\ 0 & 1 \\ -3 & -6 \end{bmatrix}. \tag{5.6}$$

The eigenvectors are then obtained with

$$y_{01}^{(1)} = \begin{bmatrix} -5 \\ 5 \\ -11 \\ 3 \\ -3 \end{bmatrix} \qquad y_{02}^{(1)} = \begin{bmatrix} 13 \\ 0 \\ 11 \\ -10 \\ 11 \end{bmatrix} \qquad y_{03}^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$$

The generalized eigenvectors for $y_{03}^{(1)}$ were obtained from (4.25)

$$y_{03}^{(2)} = U(2)My_3^{(2)} + \frac{dU(2)}{d\lambda} My_3^{(1)}$$

and

$$y_{03}^{(2)} = U(2)My_3^{(3)} + \frac{dU(2)}{d\lambda} My_3^{(2)} + \frac{1}{2}\frac{d^2U(2)}{d\lambda^2} My_3^{(1)} \qquad (5.7)$$

where $y_{03}^{(2)}$ and $y_{03}^{(3)}$ are found to be

$$y_{03}^{(2)} = \begin{bmatrix} -1/3 \\ 2/3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad y_{03}^{(3)} = \begin{bmatrix} 1 \\ 2/3 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$$

The left latent vectors can be calculated using (4.7b) as

$$A_R^T(0)z_1^{(1)} = 0\,, \quad A_R^T(1)z_2^{(1)} = 0 \quad \text{and} \quad A_R^T(2)z_3^{(1)} = 0 \qquad (5.8)$$

then

$$z_1^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad z_2^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad z_3^{(1)} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

and the two generalized left latent vectors $z_3^{(1)}$ are obtained from (4.24)

$$A_R^T(2)z_3^{(2)} + \frac{dA_R(2)}{d\lambda} z_3^{(1)} = 0 \qquad A_R^T(2)z_3^{(3)} + \frac{dA_R^T(2)}{d\lambda} z_3^{(2)}$$

$$+ \frac{1}{2} \frac{d^2A_R^T(2)}{d\lambda^2} z_3^{(1)} = 0 . \tag{5.9}$$

The left generalized latent vectors are

$$z_3^{(2)} = \begin{bmatrix} -1/2 \\ -11/2 \end{bmatrix} \qquad z_3^{(3)} = \begin{bmatrix} -3/4 \\ -7/12 \end{bmatrix} .$$

The left eigenvectors are calculated from (4.29) and they are

$$z_{01}^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \qquad z_{02}^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad z_{03}^{(1)} = \begin{bmatrix} 4 \\ 2 \\ 1 \\ 14 \\ 7 \end{bmatrix} .$$

The left generalized eigenvectors for $z_{03}^{(1)}$ are then obtained from

$$z_{03}^{(2)} = \Psi(2)z_3^{(2)} + \frac{d\Psi(2)}{d\lambda} z_3^{(1)}$$

and

$$z_{03}^{(3)} = \Psi(2)z_3^{(3)} + \frac{d\Psi(2)}{d\lambda} z_3^{(2)} + \frac{1}{2} \frac{d^2\Psi(2)}{d\lambda^2} z_3^{(1)} \tag{5.10}$$

with $z_{03}^{(2)}$ and $z_{03}^{(3)}$ given by

$$z_{03}^{(2)} = \begin{bmatrix} 2 \\ 0 \\ -1/2 \\ -4 \\ -11/12 \end{bmatrix} \qquad z_{03}^{(3)} = \begin{bmatrix} -4 \\ -2 \\ -3/4 \\ -20/3 \\ -7/12 \end{bmatrix}.$$

It is obvious that the eigenvectors of the observer form can be obtained from the latent vectors of the row-reduced form.

6. Conclusions

This chapter has explored the eigenvalue-eigenvector structure of observer canonical form and it has been shown that this structure is related to the simpler underlying structure of the corresponding lambda matrix. This work generalizes the previous results pertaining to a special class of lambda matrices in which the row indices are taken to be equal. The special structure of the eigenvectors has potential for simplifying computations concerned with their evaluation. Dual results also clearly exist for right fraction descriptions, column reduced lambda matrices and the controller canonical forms.

CHAPTER 4

PARTIAL FRACTION EXPANSION METHODS FOR MATRICES OF STRICTLY PROPER RATIONAL FUNCTIONS

Chapters 2 and 3 describes the algebraic theory of latent roots, vectors and projectors. The latent projectors of the lambda matrices are the matrix residues of the partial fraction expansion of $[A(\lambda)]^{-1}$, see Section 1 of Chapter 2. The latent projectors are important in obtaining the time domain solutions since the inverse Laplace transform can be used once the partial fraction expansion is known. The latent projectors or matrix residues can be computed by either using the latent vectors or by the more classical procedure of computing the residues of the partial fraction expansion. The latter method is not recommended for a lambda matrix with high-order matrix coefficients although algorithms for the procedure will be given here for completeness.

This chapter extends several partial fraction expansion methods for matrices of strictly proper rational functions. These methods are 1) eigenprojector, 2) Chen and Leung and 3) escalation which are first given for the scalar problem and then extended to the matrix case by using the Kronecker product. The three methods are competitive for high order functions although the escalation method is the most efficient for scalar functions.

1. Introduction

It is frequently necessary in system analysis to find the partial fraction expansion of a matrix of proper rational functions [1] and [13]. Several methods for the expansion of scalar rational functions have been described in recently published papers. The paper by Karni and Etter [17] was among the first to show that derivatives of the functions were not necessary in the evaluation of the residues of the expansion. Chen and Leung [18], as well as Mahoney [19], also avoid derivatives and describe algorithms that are quite general. The paper by Shahzadi [20] described a procedure that eliminates the necessity for the differentiation of rational functions but derivatives of the numerator and denominator must still be taken separately. The authors have developed an algorithm which is quite similar to the procedure given by Chen and Leung. The algorithm developed by the authors will be described in this paper and will be denoted as the eigenprojector method [11].

The procedure given by Mahoney is the most efficient in a numerical sense for scalar functions but the method does not clearly reveal the algebraic structure of the partial fraction expansion procedure. The Chen and Leung method, as well as the eigenprojector procedure, is based on the algebraic structure of the partial fraction expansion. The Karni and Etter method is algebraic but the referenced version had a limitation in that the order of the numerator polynomial should not exceed the order of the product of the distinct roots of the denominator of the rational function.

This paper will first give a brief mathematical description of the three methods 1) eigenprojector, 2) Chen and Leung and 3) escalation (Mahoney). The eigenprojector method has not been described in the liter-

ature and for that reason, the development will be more extensive than for the other two procedures. The Chen and Leung method will be developed and the Mahoney procedure will be summarized. In Section 2, the description of the methods will be limited to scalar functions.

In Section 3, the methods are extended to matrices of proper rational functions of the form

$$H(s) = \frac{C_1 s^{n-1} + C_2 s^{n-2} + \ldots \quad \ldots + C_n}{s^n + d_1 s^{n-1} + \ldots \quad \ldots + d_n} = \frac{N(s)}{d(s)} \tag{1.1}$$

where $C_i \varepsilon R^{p \times q}$ and the d_i coefficients are scalars. The partial fraction expansion will have the form

$$H(s) = \sum_{i=1}^{r} \sum_{\ell=0}^{m_i - 1} \frac{K_{i,\ell}}{(s-\lambda_i)^{\ell+1}} \tag{1.2}$$

where $K_{i,\ell} \varepsilon C^{p \times q}$, m_i is the multiplicity of the root λ_i and r is the number of distinct roots of d(s). Zadeh and Desoer [1] discuss the expansion of functions of the type given above, but the determination of the residues $K_{i,\ell}$ in their work follows the derivative procedure.

An example will be given in Section 3 to illustrate the three methods as well as the number of operations required where the usual assumptions that the inversion of an n×n matrix requires n^3 operations, the solution of n equations requires $n^3/3$ operations [21] and multiplication of two n×n matrices requires n^3 operations where an operation is defined to be one scalar multiplication together with one scalar addition.

2. Mathematical Development for Scalar Case

a) Eigenprojector Method

A strictly proper rational scalar function H(s) can always be characterized by the triplet (A,B,C). If H(s) is given by

$$H(s) = \frac{c_1 s^{n-1} + c_2 s^{n-2} + \ldots \quad \ldots + c_n}{s^n + d_1 s^{n-1} + \ldots \quad \ldots + d_n} = \frac{n(s)}{d(s)} \tag{2.1}$$

then H(s) can also be defined by

$$H(s) = C[sI-A]^{-1}B \tag{2.2}$$

where $A \epsilon R^{n\times n}$, $B \epsilon R^{n\times 1}$ and $C \epsilon R^{1\times n}$. The structures of A, B and C are given by [9],

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -d_n & -d_{n-1} & \cdots & -d_2 & -d_1 \end{bmatrix} \tag{2.3}$$

$$B^T = [0, 0, \ldots \quad \ldots \quad 0, 1] \tag{2.4}$$

and

$$C = [c_n, c_{n-1}, \ldots \quad \ldots, c_2, c_1] \tag{2.5}$$

where the poles of H(s) are equal to the eigenvalues of A.

Zadeh and Desoer have shown that H(s) as given in (2.2) can be repre-

sented by the expansion

$$H(s) = C\{\sum_{i=1}^{r}\sum_{\ell=0}^{m_i-1}\frac{P_{i,\ell}}{(s-\lambda_i)^{\ell+1}}\}B = \sum_{i=1}^{r}\sum_{\ell=0}^{m_i-1}\frac{k_{i,\ell}}{(s-\lambda_i)^{\ell+1}} \tag{2.6}$$

where the n×n matrices $P_{i,\ell}$ are the matrix residues or eigenprojectors, and $k_{i,\ell}$ are the scalar residues. It is assumed that H(s) has r distinct poles and each pole is repeated m_i times such that

$$\sum_{i=1}^{r} m_i = n \ . \tag{2.7}$$

The eigenprojectors, $P_{i,\ell}$, can be determined from $[sI-A]^{-1}$ which can be represented by the following similarity transformation

$$[sI-A]^{-1} = M[sI-J]^{-1}M^{-1} \tag{2.8}$$

where M is the eigenvector matrix of A, and J is the Jordan canonical form. The eigenprojectors can be defined in the usual sense as

$$P_{i,m_i-j-1} = \lim_{s\to s_i}\{\frac{d^j}{d\lambda^j}[(s-\lambda_i)^{m_i}M(sI-J)^{-1}M^{-1}]\} \quad j = 0,1,\ \ ,m_i-1 \tag{2.9}$$

where the structure of $[sI-J]^{-1}$ is , [10] ,

$$[sI-J]^{-1} = \text{Block Diag}\{[sI-J_1]^{-1},[sI-J_2]^{-1},.....,[sI-J_r]^{-1}\} \tag{2.10}$$

It is not difficult to show that the inverse of $sI-J_i$ is given by

$$[sI-J_i]^{-1} = \begin{bmatrix} (s-\lambda_i)^{-1} & (s-\lambda_i)^{-2} & \cdots & (s-\lambda_i)^{-m_i} \\ 0 & (s-\lambda_i)^{-1} & \ddots & \vdots \\ \vdots & \ddots & (s-\lambda_i)^{-1} & (s-\lambda_i)^{-2} \\ 0 & \cdots & 0 & (s-\lambda_i)^{-1} \end{bmatrix} \quad (2.11)$$

It follows from (2.9) that the eigenprojector P_{i,m_i-1} will have the following form:

$$P_{i,m_i-1} = M \begin{bmatrix} 0 & & \\ & \begin{matrix} 0 & \cdots & 0 & 1 \\ & \ddots & & 0 \\ & & & \vdots \\ & & & 0 \end{matrix} & \begin{matrix} \\ \\ \\ m_i \end{matrix} \\ & \begin{matrix} \\ 0 \end{matrix} & \end{bmatrix} M^{-1} = M[\text{sh } F_{i,m_i-1}]M^{-1} \quad (2.12)$$

$\leftarrow m_i \rightarrow$

where the $m_i \times m_i$ interior block has the same location as the ith Jordan block in J. The matrices [sh F_{i,m_i-1}] will be defined as the shifting matrices. The next eigenprojector, P_{i,m_i-2} for the repeated eigenvalue λ_i is obtained by shifting the ones to the next lower diagonal of the Jordan block. The remaining eigenprojectors for the repeated eigenvalue λ_i are generated by shifting the ones along the remaining lower diagonals until the main diagonal is reached. If the eigenvalue is not repeated, then a one is placed on the diagonal position corresponding to the Jordan block for that eigenvalue.

The eigenvector matrix M of A is constructed from the roots of d(s) as

$$M = \left[\begin{array}{c|c|cccc|c} 1 & 1 & 0 & \cdots & 0 & 1 \\ \lambda_1 & \lambda_i & 1 & \ddots & \vdots & \lambda_r \\ \lambda_1^2 & \lambda_i^2 & 2\lambda_i & \ddots & 1 & \lambda_r^2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^{n-1} & \lambda_i^{n-1} & \binom{n-1}{1}\lambda_i^{n-2} & \cdots & \binom{n-1}{m_i-1}\lambda_i^{n-m_i} & \lambda_r^{n-1} \end{array}\right] \qquad (2.13)$$

$\underbrace{\qquad\qquad\qquad\qquad}_{m_i}$

where the eigenvector for a non-repeated root is as given on the first column of (2.13). Since λ_i is a pole of H(s), the eigenvector matrix M can be constructed directly from the poles of H(s).

The eigenprojectors given in (2.12) are related to the scalar residues of the partial fraction expansion in (2.6) and the scalar residues are obtained by premultiplying the eigenprojectors by vector C and postmultiplying by the vector B as given in (2.4) and (2.5). The residues are then given by

$$k_{i,\ell} = CP_{i,\ell}B = CM[sh\ F_{i,\ell}]M^{-1}B \qquad (2.14)$$

and the only difference between residues for different eigenvalues is in the shifting matrix $[sh\ F_{i,\ell}]$ which is a matrix of ones and zeros which selects the proper columns of CM and the proper rows of $M^{-1}B$.

The computational procedure for the eigenprojector method is determining the residues of the partial fraction expansion of H(s) is as follows:

(i) Set up the eigenvector matrix M, from knowledge of the roots of d(s) as given in (2.13).

(ii) Let $CM = x_{1\times n}$ and $M^{-1}B = y_{n\times 1}$ and compute x and y from C and B as given in (2.4) and (2.5).

(iii) Set flags regarding the multiplicity of each root to carry out the shifting process on $[sh\ F_{i,\ell}]$.

(iv) Compute $k_{i,\ell}$ from $k_{i,\ell} = x[sh\ F_{i,\ell}]y$.

The algorithm will require approximately $\frac{n^3}{3} + 2n^2 - n$ operations for distinct roots provided that the flags set in (iv) are used. For a single root with multiplicity n, the count goes to approximately $\frac{n^3}{3} + 3n^2$. If n is reasonably large, the $n^3/3$ term will dominate, thus this value is an approximate count.

The following example is given to show the procedure of the algorithm. Let H(s) be given by

$$H(s) = \frac{3s^2+8s+9}{s^3+7s^2+15s+9} = \frac{k_{10}}{s+1} + \frac{k_{20}}{s+3} + \frac{k_{21}}{(s+3)^3} \qquad (2.15)$$

for which the eigenvector matrix M can be reconstructed using (2.13) with

$$M = \begin{bmatrix} 1 & 1 & 0 \\ \lambda_1 & \lambda_2 & 1 \\ \lambda_1^2 & \lambda_2^2 & 2\lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -3 & 1 \\ 1 & 9 & -6 \end{bmatrix}.$$

The x row vector and the y column vector are now computed where $C = [9\ 8\ 3]$ and $B^T = [0\ 0\ 1]$ with

$$x = CM = [9\ 8\ 3] \begin{bmatrix} 1 & 1 & 0 \\ -1 & -3 & 1 \\ 1 & 9 & -6 \end{bmatrix} = [4\ 12\ -10]$$

and

$$y = M^{-1}B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -3 & 1 \\ 1 & 9 & -6 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/4 \\ -1/4 \\ -1/2 \end{bmatrix}.$$

and for this example the sh $F_{1,\ell}$ matrices are

$$\text{sh } F_{1,0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{sh } F_{2,0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{sh } F_{2,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

In programming the algorithm for a digital computer, the flags would be set to indicate which combination of elements in x and y are necessary to calculate the residue. The partial fraction expansion of (2.15) is

$$H(s) = \frac{1}{s+1} + \frac{2}{s+3} - \frac{6}{(s+3)^2} \tag{2.16}$$

which is the desired expansion.

b) Chen and Leung Method

The Chen and Leung algorithm, although similar to the eigenprojector method, differs in that all the residues of the partial fraction expansion are computed simultaneously.

Utilizing (2.1) and (2.6), it follows that

$$c_1 s^{n-1}+c_2 s^{n-2}+ \ldots +c_n = \sum_{i=1}^{r} \sum_{\ell=0}^{m_i-1} \frac{k_{i,\ell}}{(s-\lambda_i)^{\ell+1}} \quad (s^n+d_1 s^{n-1}+ \ldots +d_n) \qquad (2.17)$$

The ratio of the two polynomials on the right side of (2.17) can be written in the general form

$$s^{n-\ell-1}+[d_1+ \frac{\ell+1}{1!} \lambda_i] \, s^{n-\ell-2}+[d_2+d_1 \frac{(\ell+1)}{1!} \lambda_i+\frac{(\ell+1)(\ell+2)}{2!} \lambda_i^2] \, s^{n-\ell-3}$$

$$+ \ldots +[d_n+d_{n-1} \frac{(\ell+1)}{1!} \lambda_i+ \ldots +\lambda_i^n] \, s^{-\ell-1} \qquad (2.18)$$

Substituting into (2.17) and equating terms in s^j gives

$$c_1 = \sum_{i=1}^{r} k_{i,0} \qquad (2.19)$$

$$c_2 = \sum_{i=1}^{r} k_{i,0}(d_1+\lambda_i) + \sum_{i=1}^{r} k_{i,1} \qquad (2.20)$$

$$c_3 = \sum_{i=1}^{r} k_{i,0}(d_2+d_1\lambda_i+\lambda_i^2) + \sum_{i=1}^{r} k_{i,1}(d_1+2\lambda_i) + \sum_{i=1}^{r} k_{i,2} \qquad (2.21)$$

or in general

$$c_k = \sum_{i=1}^{r} k_{i,0} \, d_{k-1}(\lambda_i) + \sum_{i=1}^{r} k_{i,1} \frac{d}{d\lambda}[d_{k-1}(\lambda_i)] + \ldots$$

$$+ \frac{1}{k-1!} \sum_{i=1}^{r} k_{i,k-1} \frac{d^{k-1}}{d\lambda^{k-1}} [d_{k-1}(\lambda_i)] \qquad (2.22)$$

where

$$d_{k-1}(\lambda) = \lambda^{k-1}+d_1 \lambda^{k-2}+ \ldots +d_{k-1}$$

If a pole is not repeated, then $k_{i,\ell} = 0$ for all $\ell \geq 1$.

Assuming that all roots of $d(s) = 0$ are distinct, then (2.19)-(2.22) gives the matrix equation.

$$\begin{bmatrix} k_{1,0} \\ k_{2,0} \\ \vdots \\ \vdots \\ k_{n,0} \end{bmatrix} = \begin{bmatrix} 1 & 1 & | & | & | & 1 \\ \lambda_1 & \lambda_2 & | & | & | & \lambda_n \\ \vdots & \vdots & | & | & | & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & | & | & | & \lambda_n^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ d_1 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ d_{n-1} & \cdots & \cdots & d_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_n \end{bmatrix} \tag{2.23}$$

The matrix equation of (2.23) is the desired expression for computing the residues. If the roots of d(s) are repeated, the eigenvector matrix M, as shown on the right side of (2.23) will take the general form given in (2.13) with the residue vector on the left side having the residues $k_{i,\ell}$ in sequential order with $\ell=0,1,\ldots,\ell-1$.

The computational procedure for the Chen and Leung algorithm is quite simple and straightforward, the algorithm is:

i) Set up the eigenvector matrix M and the coefficient matrix D.

ii) Find the solution of the set of equations from D and the coefficients vector.

iii) Find the solution for the set of equations between M and the vector found in (ii).

The previous example will be considered using this method. The eigenvector matrix M is the same as before therefore

$$\begin{bmatrix} k_{10} \\ k_{20} \\ k_{21} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ \lambda_1 & \lambda_2 & 1 \\ \lambda_1^2 & \lambda_2^2 & 2\lambda_2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ d_1 & 1 & 0 \\ d_2 & d_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

and upon subsituting values into the equation, the residues are

$$\begin{bmatrix} k_{10} \\ k_{20} \\ k_{21} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -3 & 1 \\ 1 & 9 & -6 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 15 & 7 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix}$$

which are correct.

The number of operations required for computing the residues using this method is approximately $\frac{2n^3}{3} + n^2 - 2n$ for all distinct roots and $\frac{2n^3}{3} + \frac{3}{2}n^2$ approximately for a single root with multiplicity n, so for large values of n the term $\frac{2n^3}{3}$ will dominate, thus this value is an approximate count.

c) The Escalation Method

Mahoney has recently published a paper on partial fraction expansions by the escalation method. The method is quite efficient although the algebraic structure of the process is not as apparent as in the previous two methods. The mathematical theory will not be given in this paper and only the algorithm will be presented with the extension to the residues of a matrix of proper rational functions in the next section.

Consider a scalar function as given in (2.1) where c_i and d_i are scalars. The basis of the method proceeds as follows; use Horner's algorithm [23] to expand the numerator of H(s) into the form

$$N(s) = \beta_0 + \sum_{i=1}^{n-1} \beta_i \prod_{j=1}^{i} (s-\lambda_j) \qquad (2.24)$$

and, therefore, H(s) can be expressed as

$$H(s) = H_n(s) = \frac{\beta_0 + \sum_{i=1}^{n-1} \beta_i \prod_{j=1}^{i} (s-\lambda_j)}{\prod_{j=1}^{n} (s-\lambda_j)} \tag{2.25}$$

where the above expression can be decomposed into

$$H_\ell(s) = \frac{H_{\ell-1}(s)}{(s-\lambda_\ell)} + \frac{\beta_{\ell-1}}{(s-\lambda_\ell)} \qquad \ell = 1,\ldots,n \; . \tag{2.26}$$

The function $H_{n-1}(s)$ will have a similar structure as (2.25). Letting ν be an index on the roots of d(s), then the partial fraction expansion (2.26) can be written as

$$H_\ell(s) = \sum_{j=1}^{L_\nu^{(\ell)}} \frac{B_{\nu j}^{(\ell)}}{(s-\lambda_\nu)^j} + \sum_{\substack{i=1 \\ i\neq\nu}}^{r} \sum_{j=1}^{L_i^{(\nu)}} \frac{B_{ij}^{(\ell)}}{(s-\lambda_i)^j} \qquad 1 \leq \ell \leq n \tag{2.27}$$

where $1 \leq L_i^{(\ell)} \leq m_i$ and $L_i^{(\ell)}$ is the multiplicity of the root λ_i in $H_\ell(s)$. As can be seen in (2.26) when $(s-\lambda_\ell)$ in H_ℓ is eliminated the multiplicity of that root in $H_{\ell-1}(s)$ is reduced by one, so the scalation method is based on that principle. If λ_ν is taken out in (2.26) then the multiplicities of the roots will be as

$$L_\nu^{(\ell-1)} + 1 = L_\nu^{(\ell)} \qquad s = \lambda_\nu \qquad 1 \leq \nu \leq r \tag{2.28}$$

and

$$L_i^{(\ell-1)} = L_i^{(\ell)} \qquad i = 1,2, \quad ,m_i \qquad i \neq \nu \; .$$

The process terminates when $\ell=n$ and $L_i^{(n)} = m_i$, $i = 1,2,\ldots,r$. The coefficients $B_{ij}^{(\ell)}$ and $B_{\nu j}^{(\ell)}$ are computed from

$$B_{ij}^{(\ell)} = \frac{B_{ij}^{(\ell-1)}}{\lambda_i - \lambda_\nu} \qquad j = L_i^{(\ell)} \geq 1 \tag{2.29}$$

and

$$B_{ij}^{(\ell)} = \frac{B_{ij}^{(\ell-1)} - B_{i,j+1}^{(\ell)}}{\lambda_i - \lambda_\nu} \qquad j = L_i^{(\ell)}-1,\ldots,2,1 \tag{2.30}$$

which Mahoney calls the alien coefficients. The coefficients given by

$$B_{\nu 1}^{(\ell)} = \beta_{\ell-1} - \sum_{\substack{i=1 \\ i\neq r}}^{r} B_{i1}^{(\ell)} \tag{2.31}$$

$$B_{\nu j}^{(\ell)} = B_{\nu,j-1}^{(\ell-1)} \qquad j = 2,3,\ldots,L_\nu^{(\ell)} \tag{2.32}$$

are denoted as the native coefficients.

The escalation procedure is efficient but the programming effort to implement the algorithm is greater than either of the two previous algorithms. The computational procedure is as follows:

i) Use the numerator coefficients and the Horner's algorithm to compute the coefficients β_i in (2.24).

ii) A tableau is then formed from Equations (2.29)-(2.32) and the residues will be obtained in the last row of the n×n tableau. ($k_{i,j} = B_{i,j+1}^{(n)}$)

The example given before will now be considered using this method. The Horner's algorithm is used to express the numerator in the form shown in (2.24). For $n(s) = 3s^2+8s+9$, the Horner's scheme gives

s^3	3	3	9	
s^2	3	5	4	$\lambda_1 = -1$
s^1	3	-4		$\lambda_2 = -3$
s^0	3			

thus, $n(s) = 4-4(s+1)+3(s+1)(s+3)$. The tableau is now constructed using these values and Eqs. (2.29)-(2.32) with the tableau given as

	$B_{11}^{(r)}$	$B_{21}^{(r)}$	$B_{22}^{(r)}$	
r=1	4			$\beta_0 = 4$
r=2	2	-6		$\beta_1 = -4$
r=3	1	2	[illegible]	$\beta_2 = 3$
	k_{10}	k_{20}	k_{21}	

where the residues of the expansion are found in the last row with agreement to the values given earlier.

Mahoney has estimated the number of operations for the procedure as approximately $n^2+n/2$. This method is therefore more efficient when execution time is the only factor considered for digital implementation. The procedure is clearly superior to the other methods when manual calculations are made.

3. Extension of the Methods to Matrices of Proper Rational Functions

The three procedures which have been classified as, 1) the eigenprojector, 2) the Chen and Leung and 3) the escalation method will be extended to strictly proper matrix function defined as

$$\lim_{s\to\infty} H(s) = 0 \ . \tag{3.1}$$

It will be assumed that the function has the form

$$H(s) = \frac{C_1 s^{n-1}+C_2 s^{n-2}+\ldots \quad \ldots+C_n}{s^n+d_1 s^{n-1}+\ldots \quad \ldots+d_n} \tag{3.2}$$

where $C_i \in R^{p\times q}$ with scalar coefficients in the denominator. The Kronecker product will be used to extend the methods as well as to minimize the arithmetic operations required for each method.

a) Eigenprojector Method

It can be shown [24] that a rational matrix functions as given in (3.2) can be expressed in the form

$$H(s) = \bar{C}(Is-\bar{A})^{-1}\bar{B} \tag{3.3}$$

$$\bar{A} = \begin{bmatrix} 0_q & I_q & 0_q & \cdots & 0_q \\ \vdots & & \ddots & & \vdots \\ 0_q & \cdots & 0_q & & I_q \\ -d_n I_q & -d_{n-1} I_q & -d_{n-2} I_q & \cdots & -d_1 I_q \end{bmatrix} = A \otimes I_q \tag{3.4}$$

$$\overline{B}^T = [0_q, 0_q, \ldots \quad \ldots, 0_q, I_q] = B \otimes I_q \tag{3.5}$$

and

$$\overline{C} = [C_n, C_{n-1}, \ldots \quad \ldots, C_2, C_1] \, . \tag{3.6}$$

Using the definition of Kronecker products [25] and [26], the nq×nq matrix in (3.4) can be defined as

$$\overline{A} = (M \otimes I_q)(J \otimes I_q)(M^{-1} \otimes I_q) \tag{3.7}$$

where J and M are as previously defined

It follows directly from (3.3)-(3.7) that the matrix of rational functions H(s) can then be expressed in the form

$$H(s) = \overline{C}(M \otimes I_q)((sI-J)^{-1} \otimes I_q)(M^{-1} \otimes I_q)(B \otimes I_q) \tag{3.8}$$

or

$$H(s) = \overline{C}(M \otimes I_q)((sI-J)^{-1} \otimes I_q)(M^{-1}B \otimes I_q) \, . \tag{3.9}$$

The matrix residues of the expansion in (3.2) can be easily obtained from (3.9) as follows

$$K_{i,\ell} = \overline{C}(M \otimes I_q) \cdot (\text{sh } F_{i,\ell} \otimes I_q) \cdot (M^{-1}B \otimes I_q) \tag{3.10}$$

where [sh $F_{i,\ell}$] is as defined in (2.12). The computational procedure is then as given in the scalar except $\overline{x} = \overline{C}(M \otimes I_q)$ and $\overline{y} = M^{-1}B \otimes I_q$ where $\overline{x}$ is a p×nq matrix and $\overline{y}$ is an nq×q matrix. Note also that in this case the last term in (3.10) can be combined with the shift operation and thus

$$K_{i,\ell} = \bar{C}(M \otimes I_q)(sh\ F_{i,\ell} M^{-1} B) \otimes I_q \tag{3.11}$$

To illustrate, consider the example

$$H(s) = \frac{\begin{bmatrix} 3 & 7 & 2 \\ -6 & 5 & 10 \end{bmatrix} s^2 + \begin{bmatrix} 8 & 24 & 15 \\ -31 & 82 & 45 \end{bmatrix} s + \begin{bmatrix} 9 & 17 & 9 \\ -45 & 51 & 47 \end{bmatrix}}{s^3+7s^2+15s+9} \tag{3.12}$$

where the roots of the denominator are $\lambda_1 = -1$ with $m_1 = 1$ and $\lambda_2 = -3$ with $m_2 = 2$. The eigenvector matrix M is given by

$$M = \begin{bmatrix} 1 & 1 & 0 \\ \lambda_1 & \lambda_2 & 1 \\ \lambda_1^2 & \lambda_1^2 & 2\lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -3 & 1 \\ 1 & 9 & -6 \end{bmatrix}$$

and $\bar{C}(M \otimes I_3)$ is

$$\bar{C}(M \otimes I_3) = \left[\begin{array}{ccc|ccc|ccc} 4 & 0 & -4 & 12 & 8 & -18 & -10 & -18 & 3 \\ -20 & 24 & 12 & -6 & 0 & 2 & 5 & 2 & -15 \end{array}\right] = \bar{x}\ .$$

B will be given by $B^T = [0 \quad 0 \quad 1]$, therefore

$$(M^{-1}B) \otimes I_3 = \begin{bmatrix} 1/4 \\ -1/4 \\ -1/2 \end{bmatrix} \otimes I_3 = \bar{y}\ .$$

Now using the expression for the residues it follows that

$$K_{10} = \bar{x}[\text{sh } F_{10} \otimes I_3]\bar{y} = \begin{bmatrix} 1 & 0 & -1 \\ -5 & 6 & 3 \end{bmatrix}$$

$$K_{20} = \bar{x}[\text{sh } F_{20} \otimes I_3]\bar{y} = \begin{bmatrix} 2 & 7 & 3 \\ -1 & -1 & 7 \end{bmatrix}$$

$$K_{2,1} = \bar{x}[\text{sh } F_{12} \otimes I_3]\bar{y} \begin{bmatrix} -6 & -4 & 9 \\ 3 & 0 & -1 \end{bmatrix}$$

Again, there is no need to construct the [sh $F_{i,\ell}$] matrix as the desired operations are carried out by using the flags. The matrix partial fraction expansion of (3.12) is then

$$H(s) = \frac{\begin{bmatrix} 1 & 0 & -1 \\ -5 & 6 & 3 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 2 & 7 & 3 \\ -1 & -1 & 7 \end{bmatrix}}{s+3} + \frac{\begin{bmatrix} -6 & -4 & 9 \\ 3 & 0 & -1 \end{bmatrix}}{(s+3)^2} \qquad (3.13)$$

which is correct.

The number of arithmetic multiplies and divisions is approximately $\frac{n^3}{3} + 2pqn^2 + n^2$ for a low count and $\frac{n^3}{2} + 2pqn^2 + \frac{3}{2}n^2$ for the high count.

b) Chen and Leung Method

The derivation for the scalar case carries over to the matrix of proper rational functions in an analogous manner to the eigenprojector case. Equation (2.23) is modified by performing a Kronecker product of M and D with I_p, thus for the distinct case

$$\begin{bmatrix} K_{1,0} \\ K_{2,0} \\ \vdots \\ K_{n,0} \end{bmatrix} = \begin{bmatrix} I_p & I_p & \cdots & I_p \\ \lambda_1 I_p & \lambda_2 I_p & \cdots & \lambda_n I_p \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} I_p & \lambda_2^{n-1} I_p & \cdots & \lambda_n^{n-1} I_p \end{bmatrix}^{-1} \begin{bmatrix} I_p & 0_p & \cdots & 0_p \\ d_1 I_p & I_p & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0_p \\ d_{n-1} I_p & \cdots & d_1 I_p & I_p \end{bmatrix}^{-1} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} \tag{3.14}$$

where $C_i \in R^{p \times q}$ and I_p and 0_p are $p \times p$. Using the properties for Kronecker products as given in the Appendix, then (3.14) can be rewritten as

$$\begin{bmatrix} K_{1,0} \\ K_{2,0} \\ \vdots \\ K_{n,0} \end{bmatrix} = (M^{-1}D^{-1} \otimes I_p) \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} \tag{3.15}$$

where M and D are identical to the matrices for the scalar case.

Using the same example as before, the matrices M and D are for the example

$$M^{-1}D^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -3 & 1 \\ 1 & 9 & -6 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 15 & 7 & 0 \end{bmatrix}^{-1} = \frac{1}{4} \begin{bmatrix} 1 & -1 & 1 \\ 3 & 1 & -1 \\ -18 & 6 & -2 \end{bmatrix}$$

and using (3.15) the following expansion is obtained

$$\begin{bmatrix} K_{1,0} \\ K_{2,0} \\ K_{2,1} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 3 & 0 & 1 & 0 & -1 & 0 \\ 0 & 3 & 0 & 1 & 0 & -1 \\ -18 & 0 & 6 & 0 & -2 & 0 \\ 0 & -18 & 0 & 6 & 0 & -2 \end{bmatrix} \begin{bmatrix} 3 & 7 & 2 \\ -6 & 5 & 10 \\ 8 & 24 & 15 \\ -31 & 32 & 45 \\ 9 & 17 & 9 \\ -45 & 51 & 47 \end{bmatrix}$$

where the residues found from the operations are the same as given (3.13).

The method requires about $2n^3+pqn^2+n^2$ for a low count and about $2n^3+pqn^2+\frac{3}{2}n^2$ which are almost the same for large matrices.

c) Escalation Method

The last method to be considered for extension to the expansion of a matrix of proper rational functions is the escalation method. The usual procedure would be to extend the scalar case but where matrices are used in the procedure as follows. The Horner's algorithm is used to calculate the β_i matrices of

$$N(s) = \beta_0 + \sum_{i=1}^{n-1} \beta_i \prod_{j=1}^{i} (s-\lambda_j) \tag{3.16}$$

where $\beta_0,\ldots,\beta_{n-1}\epsilon R^{p\times q}$. Equations (2.29)-(2.32) would then be used in a matrix form of the tableau.

For the example presented in (3.12), the Horner's algorithm gives

$$N(s) = \begin{bmatrix} 4 & 0 & -4 \\ -20 & 24 & 12 \end{bmatrix} + \begin{bmatrix} -4 & -4 & 7 \\ -7 & 12 & 5 \end{bmatrix}(s+1) + \begin{bmatrix} 3 & 7 & 2 \\ -6 & 5 & 10 \end{bmatrix}(s+1)(s+3).$$

Construct the tableau in the same manner as the scalar case using matrices given

	$B_{11}^{(r)}$	$B_{21}^{(r)}$	$B_{22}^{(r)}$	
r=1	$\begin{bmatrix} 4 & 0 & -4 \\ -20 & 24 & 12 \end{bmatrix}$			$\beta_0 = \begin{bmatrix} 4 & 0 & -2 \\ -20 & 24 & 12 \end{bmatrix}$
r=2	$\begin{bmatrix} 2 & 0 & -2 \\ -10 & 12 & 6 \end{bmatrix}$	$\begin{bmatrix} -6 & -4 & 9 \\ 3 & 0 & -1 \end{bmatrix}$		$\beta_1 = \begin{bmatrix} -4 & -4 & 7 \\ -7 & 12 & 5 \end{bmatrix}$
r=3	$\begin{bmatrix} 1 & 0 & -1 \\ -5 & 6 & 3 \end{bmatrix}$	$\begin{bmatrix} 2 & 7 & 3 \\ -1 & -1 & 7 \end{bmatrix}$	$\begin{bmatrix} -6 & -4 & 9 \\ 3 & 0 & -1 \end{bmatrix}$	$\beta_2 = \begin{bmatrix} 3 & 7 & 2 \\ -6 & 5 & 10 \end{bmatrix}$

the last matrix row of the tableau gives the residues and they agree with (3.13).

The arithmetic count, considering only multiplies and divides, is approximately $pq(n^2+n/2)$. The method is more efficient than the other two but the efficiency tends to be less drastic than that for the scalar case particularly for p and q approximately equal to n for large n.

The Mahoney procedure can be modified so that fewer operations are required when compared to the matrix formulation given above. Consider the matrix of rational functions given in (3.2) which can be expressed in the form

$$H(s) = \frac{[C_1, C_2, \ldots\ldots, C_n]}{s^n+d_1 s^{n-1}+\ldots\ldots+d_n} \begin{bmatrix} s^{n-1} \\ s^{n-2} \\ \vdots \\ 1 \end{bmatrix} \otimes I_q \qquad (3.17)$$

The functions

$$h_j(s) = \frac{s^j}{s^n+d_1 s^{n-1}+\ldots\ldots+d_n} \qquad j = 0,1,\ldots\ldots,n-1 \qquad (3.18)$$

appear in (3.17) thus knowledge of the n-expansions of (3.18) will provide the basis for the expansion of (3.17). The Mahoney procedure can be used to expand $h_{n-1}(s)$ and the tableau can be continued to find the expansions of $h_{n-2}(s),\ldots\ldots,h_0(s)$.

If $h_{n-1}(s)$ can be represented by

$$h_{n-1}(s) = \sum_{i=1}^{r} \sum_{\ell=0}^{m_i-1} \frac{k_{i,\ell}^{(1)}}{(s-\lambda_i)^{\ell+1}} \qquad (3.19)$$

it follows that the expansion

$$h_{n-j}(s) = \sum_{i=1}^{r} \sum_{\ell=0}^{m_i-1} \frac{k_{i,\ell}^{(j)}}{(s-\lambda_i)^{\ell+1}} \qquad j = 1,2,\ldots,n \tag{3.20}$$

will contain coefficients given by

$$k_{i,m_i-1}^{(j+1)} = \frac{k_{i,m_i-1}^{(j)}}{\lambda_i} \qquad m_i \geq 1 \tag{3.21}$$

and

$$k_{i,\ell}^{(j+1)} = \frac{k_{i,\ell}^{(j)} - k_{i,\ell+1}^{(j+1)}}{\lambda_i} \qquad \ell = m_i-2,\ldots,1 \tag{3.22}$$

which are the same equations given in (2.29) and (2.30) except $\lambda_\nu = 0$. The latter two equations are then used to continue the tableau such that the n expansions of (3.18) are given in the last n rows of the tableau.

Now it can be seen from (3.17) that the matrix residues $K_{i,\ell} \in C^{p \times q}$ are then obtained

$$K_{i,\ell} = [C_1, C_2, \ldots, C_n] \begin{bmatrix} k_{i,\ell}^{(1)} \\ k_{i,\ell}^{(2)} \\ \vdots \\ k_{i,\ell}^{(n)} \end{bmatrix} \otimes I_q \tag{3.23}$$

where the vectors $[k_{i,\ell}]$ are given in the corresponding columns of the last n-rows of the tableau.

Considering the example in (3.12), the functions given by

$$h_j(s) = \frac{s^j}{(s+1)(s+3)^2} \qquad j = 0,1,2$$

must be expanded. Starting with j=2, the Horner's algorithm gives the table

s^3	1	0	0	
s^2	1	-1	1	$\lambda_1 = -1$
s^1	1	-4		$\lambda_2 = -3$
s^0	1			

and the usual Mahoney tableau for $h_2(s)$ is given by the first three rows of the tableau with the last two rows giving $h_1(s)$ and $h_0(s)$.

	$B_{11}^{(r)}$	$B_{21}^{(r)}$	$B_{22}^{(r)}$		
r=1	1				$\beta_0 = 1$
r=2	-1/2	-9/2			$\beta_1 = -4$
r=3	1/4	3/4	-9/2	s^2	$\beta_2 = 1$
	-1/4	1/4	-3/2	s^1	
	1/4	-1/4	-1/2	s^0	
	$k_{10}^{(j)}$	$k_{20}^{(j)}$	$k_{21}^{(j)}$		

The corresponding vectors of the expansion of (3.18) are

$$\begin{bmatrix} k_{1,0}^{(1)} \\ k_{1,0}^{(2)} \\ k(3)_{1,0} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} k_{2,0}^{(1)} \\ k_{2,0}^{(2)} \\ k_{2,0}^{(3)} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} k_{2,1}^{(1)} \\ k_{2,1}^{(2)} \\ k_{2,1}^{(3)} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -9 \\ -3 \\ -1 \end{bmatrix}$$

and using these vectors in (3.23) will give the correct matrix residues.

The operation count for the above modified procedure is approximately $pqn^2+2n^2-n/2$ which is less than the count for the Mahoney matrix formulation as well as the other procedures.

4. Conclusion

Three methods for computing residues of proper rational functions have been presented and they all have the property that derivatives for the case of repeated roots are not necessary. The methods have been compared in operation counts and the escalation method is the most efficient for the scalar case.

The eigenprojector and the Chen and Leung methods are based on the algebraic structure of the expansions which is elegant and easy to follow and the method can easily be extended to the matrix case by continuing the tableau and then by applying the Kronecker products. This method is shown to be the most efficient in operation counts. When the algorithms are implemented in a digital computer, programming attention should be given to the Kronecker products to minimize the number of arithmetic operations. The Kronecker products have been used in the development as a mathematical tool but that does not mean that the equations are to be implemented as given in the paper.

CHAPTER 5

ALGEBRAIC THEORY OF $A(\lambda) = M\lambda^2 + C\lambda + K$

Chapters 2 and 3 of this report gave the mathematical development of the algebraic theory of lambda matrices. Lambda matrices of general order were considered and the properties of the latent roots, vectors and projectors were considered. The finite-element model of a structure will generally be defined by a second-order lambda matrix $M\lambda^2 + C\lambda + K$ where M is the mass matrix, C is the damping matrix and K is the stiffness matrix. The latent roots of such a lambda matrix will generally be distinct except for the free-free modes which will be repeated and located at the origin of the eigenvalue space of the associated state-variable matrix.

This chapter discusses the properties associated with the specific lambda matrix that arises in the finite-element model, i.e., the second-order lambda matrix. The displacement, damping and identification of the structure is of interest in the development. The objective in this chapter is to consider the second-order lambda matrix utilizing the algebraic theory developed in the previous chapters. The spectral factorization of the second-order lambda matrix will be described as well as properties associated with the spectral matrices.

1. Introduction

The previous chapter of this report gave the general algebraic theory of latent roots, vectors and projectors of lambda matrices. The work did not consider the special case of a second-order lambda matrix such as the one occurring in the finite-element model for structural analysis. The dynamics of a vibrating structure can be characterized by the second-order differential equation

$$M \frac{d^2x}{dt^2} + \bar{C} \frac{dx}{dt} + \bar{K}x = F(t) \tag{1.1}$$

where $M \epsilon R^{m \times m}$ is the mass matrix, $\bar{C} \epsilon R^{m \times m}$ is the damping matrix, $\bar{K} \epsilon R^{m \times m}$ is the stiffness matrix, $x(t) \epsilon R^{m \times 1}$ is the displacement vector and $F(t) \epsilon R^{m \times 1}$ is the force acting on the structure. It will be assumed that M, $\bar{C}$, and $\bar{K}$ are symmetric positive definite matrices.

The homogeneous equation

$$M \frac{d^2x}{dt^2} + \bar{C} \frac{dx}{dt} + \bar{K}x = 0 \tag{1.2}$$

can be transformed to a lambda matrix by assuming that $x(t) = x_0 \exp(\lambda t)$ or by taking the Laplace transform of (1.2) with zero initial conditions. The lambda matrix of interest in the following work will be of the form

$$\bar{A}(\lambda) = M\lambda^2 + \bar{C}\lambda + \bar{K} . \tag{1.3}$$

This lambda matrix will have 2m latent roots that occur in complex conjugate

pairs, 2m right latent vectors in conjugate pairs as well as 2m latent projectors that are also in conjugate pairs. The normalized lambda matrix

$$A(\lambda) = I\lambda^2 + C\lambda + K \tag{1.4}$$

will be taken as the canonic form since the associated block companion matrix

$$A_c = \begin{bmatrix} 0 & I \\ -K & -C \end{bmatrix} \tag{1.5}$$

is considered to be in canonic form. As given previously, the eigenvalues of A_c are equal to the latent roots of $A(\lambda)$ and the eigenvectors of A_c contain the latent vectors of $A(\lambda)$ as subvectors. The latent projectors of $A(\lambda)$ are submatrices of the eigenprojectors of A_c.

The algebraic development in this chapter will be directed toward establishing the theory and structure of the latent roots, vectors and projectors of $\overline{A}(\lambda)$ as given in (1.3) and the structure of $\overline{C}$ for either specified mode dampings, or the location of the latent roots of $\overline{A}(\lambda)$. The first step in the development will be that of transforming $\overline{A}(\lambda)$ into the canonic form and using the canonic form for the algebraic theory of lambda matrices. The damped and undamped lambda matrices will then be studied and the algebraic structure of each form determined.

2. General Theory of Lambda Matrices for Undamped Vibrating Systems

The general theory of vibrating systems and lambda matrices has been covered by Lancaster, [3]. Some of that material is presented here to provide the proper setting for the development that follows.

The undamped system that has no viscous or hysteretic damping and the free vibrating system can be characterized by the n second-order differential equations

$$M\frac{d^2x}{dt^2} + \bar{K}x = 0 \tag{2.1}$$

where $x(t)\in R^{n\times 1}$. The system is undamped and the displacements $x_i(t)$ will be sinusoidal therefore if it is assumed that

$$x(t) = qe^{j\omega t} \tag{2.2}$$

and equation (2.1) takes the form

$$(-M\omega^2+\bar{K})q = 0 \tag{2.3}$$

If $\mu = -\omega^2$, (2.3) can be rewritten as

$$(M\mu+\bar{K})q = 0 \tag{2.4}$$

where $M\mu+\bar{K}$ is a real, regular and symmetric matrix pencil, [3]. A pencil of matrices is regular if a) M and $\bar{K}$ are square matrices and b) M is nonsingular.

A simple matrix pencil of order n has the following properties [3].

a) It is a regular pencil

b) It has n linearly independent right and left latent vectors, $\bar{y}_i$ and $\bar{z}_i$ respectively.

Several theorems and corollaries from Lancaster [3] will now be given without proofs.

Theorem 2.1 Latent vectors corresponding to distinct latent roots of a regular matrix pencil are linearly independent.

Corollary 2.2 A matrix pencil of order n having n distinct latent roots is a simple pencil.

Theorem 2.3 A regular matrix pencil $M\mu+\bar{K}$ is simple if and only if for every latent root μ_i the matrix $M\mu+\bar{K}$ has degeneracy equal to the multiplicity of μ_i.

Theorem 2.4 If M and $\bar{K}$ are real symmetric matrices and M is positive definite, then $M\mu+\bar{K}$ is a simple matrix pencil.

Corollary 2.5 Under the assumptions of Theorem (2.4) all latent roots and latent vectors of the pencil $M\mu+\bar{K}$ are real.

Theorem 2.5 The regular pencil $M\mu+\bar{K}$ is defective if and only if there exists a latent root μ_i with a right latent vector y_i such that $z_i^T M y_i = 0$ for all latent vectors y_i of μ_i.

Corollary 2.6 As a result of the theorems, the matrix pencil $M\mu+\bar{K}$ is therefore a simple matrix pencil and has m latent roots with m linearly inde-

dependent latent vectors.

The lambda matrix

$$\bar{A}(\lambda) = \bar{M}\lambda^2 + \bar{K}. \tag{2.5}$$

will therefore have 2m latent roots $\lambda_i = \sqrt{\mu_i} = \pm j\omega_{i0}$. The latent vectors for (2.5) will be real since $(M\lambda_i^2+\bar{K}) \epsilon R^{m\times m}$ and the latent vector for $+j\omega_{i0}$ will be equal to the latent vector of $-j\omega_{i0}$. All latent roots appear in imaginary pairs, $\pm j\omega_{i0}$, and there will be only m linearly independent right and left latent vectors. Furthermore, since $A(\lambda)$ is symmetric, the right and the left latent vectors must be equal for a latent root λ_i, i.e.

$$\bar{z}_i = \bar{y}_i \tag{2.6}$$

where $\bar{A}(\lambda_i)\bar{r}_i = 0$ and $\bar{A}^T(\lambda_i)\bar{z}_i = 0$.

The latent projectors for an undamped vibrating system for a latent root λ_i are given by

$$\hat{\bar{P}}_{i,0} = \frac{\bar{y}_i \bar{z}_i^T}{\bar{z}_i^T \frac{dA(\lambda_i)}{d\lambda} \bar{y}_i} = \frac{\bar{y}_i \bar{y}_i^T}{\bar{y}_i^T \frac{dA(\lambda_i)}{d\lambda} \bar{y}_i} \tag{2.7}$$

but since $d\bar{A}(\lambda_i)/d\lambda = 2M\lambda_i$, (2.7) becomes

$$\hat{\bar{P}}_{i,0} = \frac{1}{2\lambda_i} \left(\frac{\bar{y}_i \bar{y}_i^T}{\bar{y}_i^T M \bar{y}_i}\right). \tag{2.8}$$

Theorem 2.7 The latent projectors for an imaginary pair of latent roots of the lambda matrix for an undamped vibrating system occur in imaginary pairs

C-2

$$\hat{\overline{P}}_{2i-1,0} = \hat{\overline{P}}^{*}_{2i,0} \tag{2.9}$$

where it has been assumed that the latent roots are arranged in the order $\lambda_{2i-1} = j\omega_{i0}$ and, $\lambda_{2i} = -j\omega_{i0}$. The latent projectors for each pair are linearly dependent.

The proof of Theorem 2.6 follows directly from (2.8). If $\lambda_{2i-1} = j\omega_{i0}$ and $\lambda_{2i} = -j\omega_{i0}$ then

$$\hat{\overline{P}}_{2i,0} = \frac{1}{2(-j\omega_{i0})} \begin{pmatrix} \overline{y}_i \overline{y}_i^T \\ \overline{y}_i^T \overline{M} \overline{y}_i \end{pmatrix} = \frac{-1}{2j\omega_{i0}} \begin{pmatrix} \overline{y}_i \overline{y}_i^T \\ \overline{y}_i^T \overline{M} \overline{y}_i \end{pmatrix}$$

$$= \hat{\overline{P}}^{*}_{2i-1,0} = -\hat{\overline{P}}_{2i-1,0} \tag{2.10}$$

<u>Theorem 2.8</u> The latent projectors for different pairs of latent roots, $\mu_i \neq \mu_j$ for an undamped vibrating system must be orthogonal,

$$\hat{\overline{P}}_{i,0} \hat{\overline{P}}_{j,0} = 0 \tag{2.11}$$

The latent projectors for the simple matrix pencil $M\mu+K = 0$ are orthogonal since this latent-problem can be related to the eigen-problem. If $\mu = \omega_{i0}^2$ then

$$\overline{K}-M\mu = M^{1/2}(M^{-1/2}\overline{K}M^{-1/2}-I\mu)M^{1/2} \tag{2.12}$$

where it is assumed that M is positive definite. It then follows that the determination of the eigenvalues and eigenvectors of $K-I\mu$ is a standard eigen-problem. The eigenprojectors of this algebraic system must satisfy

the usual orthogonal property with

$$P_{Ki,0}\, P_{Kj,0} = 0 \tag{2.12}$$

The eigenprojectors of $K-I\mu$ and the latent projectors of $K+\lambda^2 I$ are related with

$$P_{Ki,0} = \hat{P}_{2i-1,0}\lambda_{2i-1} + \hat{P}_{2i,0}\lambda_{2i} = \hat{P}^{*}_{2i-1}\lambda_{2i-1} + \hat{P}^{*}_{2i-1}\lambda^{*}_{2i-1} \tag{2.13}$$

The latent vectors of λ_{2i-1} and λ_{2i} are real and equal and since there are only m linearly independent latent vectors for $K-I\mu$, the latent vectors in $\hat{\overline{P}}_{i,0}$ and $\hat{\overline{P}}_{j,0}$ must be orthogonal in order for (2.12) to hold with $j \neq i$.

3. The Cholesky Decomposition and the Canonic Form

Any symmetric positive matrix $B = B^T$ can be decomposed into a product of two triangular matrices, Q and Q^T by the Cholesky algorithm. The canonic lambda matrix form can be obtained by factoring M out of (1.2), to the right or to the left, but this process would destroy the symmetry of the resulting lambda matrix. Rather than carry out this operation, the mass matrix can first be decomposed by the Cholesky algorithm [26] with

$$M = QQ^T \tag{3.1}$$

where $Q \epsilon R^{m \times m}$ is a lower triangular matrix and Q^T will be an upper triangular matrix. If Q is now factored to the left and Q^T to the right, (1.3) takes the form

$$\overline{A}(\lambda) = Q[I\lambda^2 + C\lambda + K]Q^T = QA(\lambda)Q^T \tag{3.2}$$

where[1] $C = Q^{-1}\overline{C}Q^{-T}$ and $K = Q^{-1}\overline{K}Q^{-T}$. Since $\overline{C}$ and $\overline{K}$ have been assumed to be symmetric, C and K will also be symmetric.

The latent roots of $\overline{A}(\lambda)$ will be equal to those of $A(\lambda)$ which can be easily shown. Defining $A(\lambda)$ as,

$$[A(\lambda)]^{-1} = Y[\lambda I - J]^{-1} Z^T \tag{3.3}$$

where $Z \epsilon C^{m \times 2m}$ is the left latent vector matrix and $Y \epsilon C^{2m \times n}$ is the right latent vector matrix, then

$$[\overline{A}(\lambda)]^{-1} = \overline{Y}[\lambda I - J]^{-1} \overline{Z}^T \tag{3.4}$$

---

(1) The superscript (-T) will be used to denote the transpose of the inverse of the matrix.

The normalization process given in (3.2) will therefore map the latent vectors $\bar{y}_i$ of $\bar{A}(\lambda)$ into y_i with

$$\bar{y}_i = Q^T y_i \qquad y_i \varepsilon C^{m\times 1} \tag{3.5}$$

In addition, $A(\lambda)$ has symmetry thus

$$\bar{z}_i = \bar{y}_i = Q^T y_i \qquad z_i \varepsilon C^{m\times 1} \tag{3.6}$$

The Cholesky decomposition and factorization will leave the latent roots invariant but the latent vectors will be modified as in (3.5) and (3.6).

The block companion matrix associated with $A(\lambda)$ will be as given in (1.5) and its eigenvalues will be equal to the latent roots of $A(\lambda)$. The right eigenvector of A_c are given by

$$y_{ci} = \begin{bmatrix} y_i \\ \lambda_i y_i \end{bmatrix} \tag{3.7}$$

and the left eigenvector of A_c is

$$z_{ci} = \begin{bmatrix} z_i(\lambda_i I + C) \\ z_i \end{bmatrix} = \begin{bmatrix} y_i(\lambda_i I + C) \\ y_i \end{bmatrix} \tag{3.8}$$

for λ_i with $i = 1,3,5,\ldots,m-1$. The eigenvectors for the eigenvalue λ_i with index $i = 2,4,6,\ldots,m$ are equal to the complex conjugate of (3.7) and (3.8).

It will be assumed throughout the development that the latent roots of

$A(\lambda)$ are distinct except for those at the origin, i.e. $\lambda=0$. All latent roots will appear in conjugate pairs including those at zero which must be present in an even number and all latent vectors will occur in conjugate pairs.

The companion form matrix A_c has the spectral resolution

$$A_c = \sum_{i=1}^{2m} P_{i,0}\lambda_i = \sum_{i=1}^{m} (P_{i,0}\lambda_i + P^*_{i,0}\lambda^*_i) \tag{3.9}$$

(E.4)

and the partial fraction expansion of $(\lambda I - A)^{-1}$ is given by

$$[A_c(\lambda)]^{-1} = [\lambda I - A_c]^{-1} = \sum_{i=1}^{2m} \frac{P_{i,0}}{(\lambda+\lambda_i)} \tag{3.10}$$

which can also be defined as

$$[A_c(\lambda)]^{-1} = 2 \sum_{i=1}^{m} \frac{P_{Ri,0}\lambda + (P_{Ri,0}\sigma_i + P_{Ii,0}\omega_i)}{\lambda^2 + 2\sigma_i\lambda + (\sigma_i^2+\omega_i^2)} \tag{3.11}$$

where $P_{Ri,0} = Re(P_{i,0})$ and $P_{Ii,0} = Im(P_{i,0})$. If (3.9) is the spectral resolution of A_c, then

$$A_c(\lambda) = \lambda I - A_c = \sum_{i=1}^{2m} P_{i,0}(\lambda-\lambda_i) =$$

$$= \sum_{i=1}^{m} [P_{i,0}(\lambda-\lambda_i) + P^*_{i,0}(\lambda-\lambda^*_i)] \tag{3.12}$$

which can also be given as

$$A_c(\lambda) = \sum_{i=1}^{m} [(P_{i,0}+P^*_{i,0})\lambda - (P_{i,0}\lambda_i + P^*_{i,0}\lambda^*_i)] \tag{3.13}$$

It follows from (3.13) that

$$\sum_{i=1}^{m} (P_{i,0}+P^{*}_{i,0}) = I \tag{3.14}$$

(E.1)

and

$$\sum_{i=1}^{m} (P_{i,0}\lambda_i+P^{*}_{i,0}\lambda^{*}_i) = A_c = \begin{bmatrix} 0 & I \\ -K & -C \end{bmatrix} \tag{3.15}$$

Recalling that an eigenprojector $P_{i,0}$ is defined as

$$P_{i,0} = y_{ci}\, y^{T}_{ci} = \begin{bmatrix} \tilde{y}_i \\ \lambda_i\tilde{y}_i \end{bmatrix} \begin{bmatrix} \tilde{y}^{T}_i(\lambda_i I+C) & \tilde{y}^{T}_i \end{bmatrix} \tag{3.16}$$

for normalized eigenvectors, then $\tilde{y}_i\, \tilde{y}^{T}_i$ will be properly normalized and

$$P_{i,0} = \begin{bmatrix} \tilde{y}_i\tilde{y}^{T}_i(\lambda_i I+C) & \tilde{y}_i\tilde{y}^{T}_i \\ \tilde{y}_i\tilde{y}^{T}_i(\lambda^2_i I+C\lambda_i) & \lambda_i\tilde{y}_i\tilde{y}^{T}_i \end{bmatrix} = \begin{bmatrix} \hat{P}_{i,0}(\lambda_i I+C) & \hat{P}_{i,0} \\ -\hat{P}_{i,0}K & \hat{P}_{i,0}\lambda_i \end{bmatrix} \tag{3.17}$$

where $\hat{P}_{i,0}$ is a latent projector of the canonic lambda matrix. Equation (3.17) can also be derived from $[A_c(\lambda)]^{-1}$ from which it follows that

$$[A_c(\lambda)]^{-1} = [A(\lambda)]^{-1} \begin{bmatrix} \lambda I+C & I \\ -K & \lambda I \end{bmatrix} \tag{3.18}$$

thus

$$P_{i,0} = (\lambda+\lambda_i)[A_c(\lambda_i)]^{-1} = (\lambda+\lambda_i)[A(\lambda_i)]^{-1}\begin{bmatrix} \lambda_i I+C & I \\ -K & \lambda_i I \end{bmatrix}$$

$$= \begin{bmatrix} \hat{P}_{i,0}(\lambda_i I+C) & \hat{P}_{i,0} \\ -\hat{P}_{i,0}K & \hat{P}_{i,0}\lambda_i \end{bmatrix} \qquad (3.19)$$

Equation (3.14) can be used with (3.17) or (3.19) to derive the result

(L.1) $$\sum_{i=1}^{m} (\hat{P}_{i,0}+\hat{P}^{*}_{i,0}) = 0 \qquad (3.20)$$

and from (2.15)

(L.2) $$\sum_{i=1}^{m} (\hat{P}_{i,0}\lambda_i+\hat{P}^{*}_{i,0}\lambda_i^{*}) = I \qquad (3.21)$$

(L.3) $$\sum_{i=1}^{m} (\hat{P}_{i,0}\lambda_i^2+\hat{P}^{*}_{i,0}\lambda_i^{2*}) = -C \qquad (3.22)$$

Equations (3.20) and (3.22) expresses the invariances for the assignment of damping in the system and are key equations. The assignment of damping will be discussed later.

Two of the properties of the eigenprojectors can be used to derive properties of the latent projectors. The eigenprojector are idempotent matrices thus

(E.2) $$P_{i,0}\,P_{i,0} = P_{i,0}$$

Substituting for $P_{i,0}$ into the above equation gives

$$\text{(L.4)} \qquad \hat{P}_{i,0} = \hat{P}_{i,0}(2\lambda_i I+C)\hat{P}_{i,0} = \hat{P}_{i,0}\frac{d\Delta(\lambda_i)}{d\lambda}\hat{P}_{i,0} \tag{3.23}$$

and

$$\text{(L.5)} \qquad \hat{P}_{i,0}\lambda_i = \hat{P}_{i,0}(\lambda_i^2 I-K)\hat{P}_{i,0} \tag{3.24}$$

The companion matrix A_c will have 2m linearly independent eigenvectors in which case the eigenprojectors are orthogonal with

$$\text{(E.3)} \qquad P_{i,0}\,P_{j,0} = 0 \qquad\qquad i \neq j$$

Substituting for $P_{i,0}$ and $P_{j,0}$ and using (3.17) gives

$$\text{(L.6)} \qquad \hat{P}_{i,0}(\lambda_i I+\lambda_j I+C)\hat{P}_{j,0} = 0 \tag{3.25}$$

and

$$\text{(L.7)} \qquad \hat{P}_{i,0}(\lambda_i\lambda_j I-K)\hat{P}_{j,0} = 0 \tag{3.26}$$

Properties of the eigenprojectors for the companion matrix and the canonic lambda matrix are summarized in Table 5.1. These properties form the basis for the development of the assignment of damping. Each of these properties have been verified with an example and a computer run.

Table 5.1 Summary of the Properties of the Eigen/latent Projectors (r distinct roots of multiplicity m_{i-1})

A_c = Block Companion Form

$P_{i,\ell}$ = eigenprojectors, $\hat{P}_{i,\ell}$ = latent projector

E.1 $$\sum_{i=1}^{r} P_{i,0} = I$$

E.2 $$P_{i,0}\, P_{i,0} = P_{i,0} \qquad i = 1,2,\ldots,r$$

E.3 $$P_{i,0}\, P_{j,0} = 0 \qquad i \neq j$$

E.4 $$P_{i,j} = (A_c - \lambda_i I) P_{i,j-1} \qquad j = 1,2,\ldots m_i - 1$$

E.5 $$A_c = \begin{bmatrix} 0 & I \\ -K & -C \end{bmatrix} = \sum_{i=1}^{r} [P_{i,0}\,\lambda_i + P_{i,1}]$$

E.6 $$[A_c(\lambda_1)]^{-1} = [I\lambda^2 + C\lambda + K]^{-1} \begin{bmatrix} \lambda I + C & I \\ -K & \lambda I \end{bmatrix}$$

E.7 $$P_{i,0} = \{(\lambda - \lambda_i)\,[A(\lambda - \lambda_i)\,[A(\lambda)]^{-1}\}\big|_{\lambda=\lambda_i} \qquad (\lambda_i \text{ distinct root})$$

E.8 $$P_{i,j} = \frac{1}{j!}\,\frac{d^{j-1}}{d\lambda^{j-1}}\{(\lambda - \lambda_i)^{m_i-1}[A(\lambda)]^{-1}\}\big|_{\lambda=\lambda_i} \qquad (\lambda_i \text{ repeated root})$$

E.9 $$P_{i,0} = \begin{bmatrix} \hat{P}_{i,0}\,(\lambda_i I + C) & \hat{P}_{i,0} \\ -\hat{P}_{i,0} K & \lambda_i \hat{P}_{i,0} \end{bmatrix}$$

E.10 $$[A_c(\lambda)]^{-1} = \sum_{i=1}^{r} \sum_{\ell=0}^{m_i-1} \frac{P_{i,\ell}}{(\lambda - \lambda_i)^{\ell+1}}$$

E.11 $$A_{ci} = \begin{cases} P_{i,0}\lambda_i & \cdot\ \lambda_i \text{ distinct root} \\ P_{i,0}\lambda_c + P_{i,1} & \cdot\ \lambda_i \text{ repeated root} \end{cases}$$

E.12 $$A_{ci}A_{cj} = 0 \qquad i \neq j$$

E.13 $$A_c = \sum_{i=1}^{r} A_{ci}$$

E.14 $$A_{ci} = A_c P_{i,0}$$

$A(\lambda) = I\lambda^2+C\lambda+K$ Canonic Form

L.1 $$\sum_{i=1}^{2m} \hat{P}_{i,0} = \sum_{i=1}^{m} (\hat{P}_{2i-1,0}+\hat{P}^{*}_{2i-1,0}) = 0$$

L.2 $$\sum_{i=1}^{2m} \hat{P}_{i,0}\lambda_i = \sum_{i=1}^{m} (\hat{P}_{2i-1,0}\lambda_{2i-1}+\hat{P}^{*}_{2i-1,0}\lambda^{*}_{2i-1}) = I$$

L.3 $$\sum_{i=1}^{2m} (\hat{P}_{i,0}\lambda_i^2+\hat{P}^{*}_{i,0}\lambda_i^{*2}) = -C$$

L.4 $$\hat{P}_{i,0} = \hat{P}_{i,0}(2\lambda_i I+C)\hat{P}_{i,0}$$

L.5 $$\hat{P}_{i,0}\lambda_i = \hat{P}_{i,0}(\lambda_i^2 I-K)\hat{P}_{i,0}$$

L.6 $$P_{i,0}(\lambda_i I+\lambda_j I+C)\hat{P}_{j,0} = 0 \qquad i \neq j$$

L.7 $$\hat{P}_{i,0}(\lambda_i \lambda_j I-K)\hat{P}_{j,0} = 0 \qquad i \neq j$$

L.8 $$\hat{P}_{i,0} = y_i y_i^T/(y_i^T \frac{dA(\lambda_i)}{d\lambda} y_i) = y_i y_i^T$$

4. Spectral Factorization of $A(\lambda)$

Section 3 of this chapter considers the restrictions on the eigenprojectors and the latent projectors of the companion form of A_c and the canonic form of $A(\lambda)$. In addition to these restrictions, studies of $A(\lambda)$ will reveal other restrictions.

Consider the undamped lambda matrix for the structure with

$$A(\lambda) = I\lambda^2+K \tag{4.1}$$

The matrix coefficients of $A(\lambda)$ are m×m and $A(\lambda)$ will have 2m latent roots λ_i. Let K^+ denote a m×m matrix with eigenvalues $\lambda_i = j\omega_i$ where $\omega_i \geq 0$ and similarly let K^- have eigenvalues $\lambda_i = -j\omega_i$ with $\omega_i \geq 0$. The free-free latent roots at $\lambda_i = 0$ will be equally split into the spectrum of K^+ and that of K^-. The lambda matrix admits the spectral factorization

$$A(\lambda) = (I\lambda+K^+)(I\lambda+K^-) \tag{4.2}$$

where for the undamped case of (4.1)

$$K = K^+K^- \tag{4.3}$$

$$0 = K^++K^- \tag{4.4}$$

The stiffness matrix has eigenprojectors $P_{Ki,0}$ thus K can be constructed by the sum

$$K = \sum_{i=1}^{m} P_{Ki,0}\lambda_{Ki} \tag{4.5}$$

where $\lambda_{Ki} = \omega_i^2$. Now K can be factored† as given in (4.3) with

$$K^+ = \sum_{i=1}^{m} P_{Ki,0}\, \lambda_{Ki}^+ \qquad \lambda_{Ki}^+ = j\omega_i \tag{4.6}$$

and

$$K^- = \sum_{i=1}^{m} P_{Ki,0}\, \lambda_{Ki}^- \qquad \lambda_{Ki}^- = -j\omega_i \tag{4.7}$$

with $\lambda_{Ki}^+ \lambda_{Ki}^- = \lambda_{Ki} = \omega_i^2$. Note that (4.4) is satisfied since

$$K^+ + K^- = \sum_{i=1}^{m} (P_{Ki,0} j\omega_i - P_{Ki,0} j\omega_i) = 0 \tag{4.8}$$

as required. An example can be given to illustrate the factorization

K(I,J) MATRIX

9	-5	0
-5	11	-6
0	-6	13

EIGENVALUES

3.11262	10.6609	19.2264

EIGENVECTOR MATRIX

.587523	.737925	.332087
.691794	-.24513	-.679214
.419805	-.62879	.654513

EIGENPROJECTOR FOR LAMBDA = 3.11262

.345184	.406446	.246645
.406446	.47858	.290418
.246645	.290418	.176236

EIGENPROJECTOR FOR LAMBDA = 10.6609

.544534	-.180888	-.464
-.180888	.0600886	.395377
-.464	.154135	.395377

† Note that $-jK^+$ and jK^- are square roots of K, i.e. $K^{1/2}$.

EIGENPROJECTOR FOR LAMBDA = 19.2264

.110282	-.225558	.217355
-.225558	.461331	-.444554
.217355	-.444554	.428387

Using (4.6), K^+ is given by

$$K^+ = j \begin{bmatrix} 2.87052 & -0.862567 & -0.126807 \\ -0.862567 & 3.06338 & -0.933635 \\ -0.126807 & -0.933635 & 3.48026 \end{bmatrix}$$

and $K^- = -K^+$.

When damping is to be added, K is invariant under damping assignment thus the lambda matrix becomes

$$A_D(\lambda) = I\lambda^2+C\lambda+K = (I\lambda+K_D^+)(I\lambda+K_D^-) \tag{4.9}$$

where

$$K = K_D^+K_D^- \tag{4.10}$$

$$C = K_D^++K_D^- \tag{4.11}$$

The invariance of K can be satisfied by postmultiplying K^+ by a unitary matrix and premultiplying K^- by the transpose of the conjugate of the unitary matrix, i.e.

$$K = K^+UU^*K^- = K_D^+ K_D^- \tag{4.12}$$

from which it follows that

$$C = K^{+}U+U^{*}K^{-} = K^{+}U+U^{*}(K^{+})^{*} \tag{4.13}$$

The most general form of U is complex with $U = U_R - jU_I$ and since K^{+} is imaginary with $K^{-} = -K^{+}$

$$C = (K_R^{+}+j\ K_I^{+})(U_R-j\ U_I)+(U_R^{T}+j\ U_I^{T})(K_R^{+}-j\ K_I^{+}) \tag{4.14}$$

where $K_R = 0$ for the undamped case therefore

$$Re(C) = C = K_I^{+}U_I+U_I^{T}K_I^{+} \tag{4.15}$$

$$Im(C) = 0 = K_I U_R - U_R^{T}K_I \tag{4.16}$$

Recalling that $K = K_D^{+}K_D^{-}$, then

$$K = K_D^{+}K_D^{-} = K^{+}(U_R-j\ U_I)(U_R+j\ U_I)^{T}K^{-} \tag{4.17}$$

and that $K = (jK^{1/2})(-jK^{1/2}) = K^{+}K^{-}$, (4.17) takes the form

$$K = K^{1/2}(U_I+j\ U_R)(U_I-j\ U_R)^{T}K^{1/2} = K_D^{+}\ K_D^{-} \tag{4.18}$$

The upper-half spectral matrix K_D^{+} must have eigenvalues equal to the latent roots of $A(\lambda)$ for the upper-half plane. Similarly, K_D^{-} must contain eigenvalues only in the lower-half plane. Figure 4.1 below shows the eigenvalue locations for the undamped and damped matrices, K^{+} and K_D^{+}.

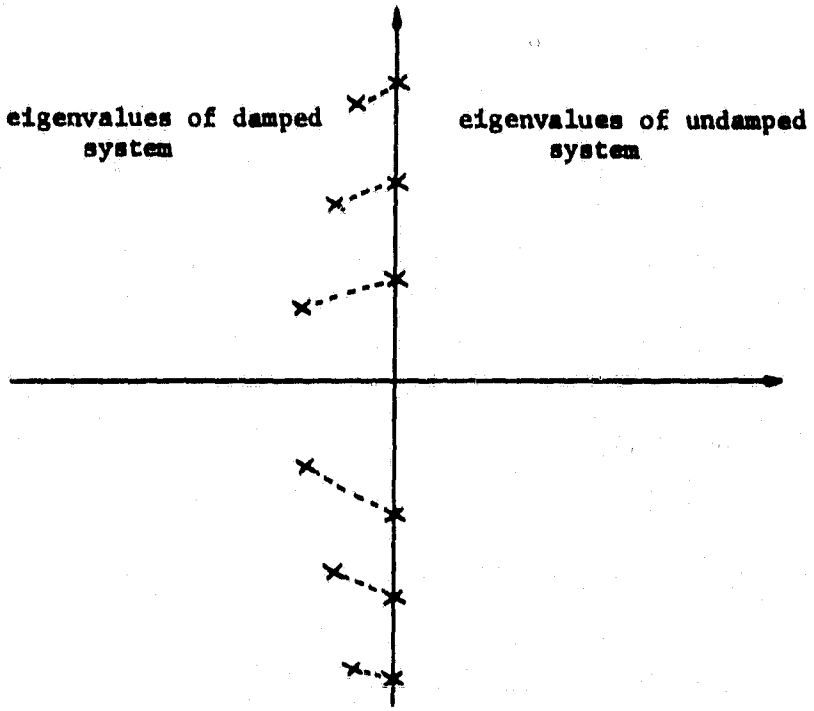


Figure 4.1

The damping problem requires that a unitary matrix be found such that

$$U = U_R + j\, U_I \tag{4.19}$$

$$U(U^*) = I \tag{4.20}$$

$$U_R U_R^T + U_I U_I^T = I \tag{4.21}$$

$$U_I U_R^T - U_R U_I^T = 0 \tag{4.22}$$

where U^* is the complex conjugate transpose of U. The damping matrix is given by

$$C = K^{1/2}\, U_I + U_I^T\, K^{1/2} \tag{4.23}$$

with the additional constraint that C be triadiagonal and that

$$0 = K^{1/2} U_R - U_R^T K^{1/2} \tag{4.24}$$

It follows from the above analysis that the lambda matrix $A(\lambda)$ admits a spectral factorization.

$$A(\lambda) = (I\lambda+K^+)(I\lambda+K^-) \tag{4.25}$$

for the undamped case with $K^- = K^+$ and

$$A_D(\lambda) = [I\lambda+K^+(U_R-j\ U_I)][I\lambda+(U_R^T+j\ U_I^T)K^-] \tag{4.26}$$

for the damped case. The complex unitary matrix must have the property that C as defined in (4.23) properly accounts for the damping implemented in the structure. If C is tridiagonal then $U \epsilon C^{n\times n}$ must satisfy the tridiagonal property as well as satisfy (4.24).

The matrix function

$$C = f(U) = (K^+)U + U^*(K^+)^*$$

given in (4.13) has the following properties: Taussky and Wielandt [28]

a) The $n\times n$ complex matrices K^+ and U are nonsingular if f(U) is a positive definite matrix

b) f(U) is linear over the subfield R of real numbers.

Property a) can be shown as follows; let z denote a vector such that Uz = 0 which implies that U is singular. It follows that

$$K^{+}Uz+z^{*}U^{*}(K^{+})^{*} = z^{*}f(U)z = 0 \qquad (4.27)$$

Thus f(U) is not positive definite. Similarly, if K^{+} is singular then $zK^{+} = 0$ and

$$zK^{+}U+U^{*}(K^{+})^{*}z^{*} = z\ f(U)\ z^{*} = 0$$

and this is a contradiction of a).

Property a) then implies that if K^{+} is singular, e.g. eigenvalues at the origin then C will be singular. Since K^{+} can be positive semi-definite the C will be positive semi-definite whenever K^{+} is semi-definite. The matrix K^{+} is the spectral factor of K thus K will also be positive semi-definite when K^{+} is positive semi-definite.

CHAPTER 6

OPTIMAL CONTROL OF SELECTED MODES

The usual approach taken in applying optimal control theory is to determine state or output feedback for all of the modes of the system. This procedure works quite well when the number of modes in the system is not large but the computational load for several hundred modes makes this type of control impractical if time varying gain is used. Even when a constant gain is used, the computation of the gain is not a trivial task.

There have been numerous papers, see [29]-[30], published in applying optimal control theory to structure with the development based on reduced-order models. The computational load can be reduced significantly by this approach but the reduced-order model must be carefully chosen if mode spill over is to be avoided.

The work in this section will take an entirely different direction. The computational load for the procedure is reasonable and the mode spill over problem can be eliminated. The spectral factorization algorithm will be used to decouple the selected modes from other modes of the structure. The optimal control theory will then be used to construct the feedback for the selected modes. The uncontrolled modes are uncoupled from the control modes and the possibility of mode spill over is eliminated.

The method presented in this chapter is similar to that of Oz and Meirovitch [31] where the modes are decoupled and the optimal control is determined for each mode. The major difference is in the method of decoupling and the computational procedure.

1. Introduction

Consider the state equation for a plant with

$$\dot{X}(t) = AX(t) + BU(t) \tag{1.1}$$

where $A \in R^{n\times n}$ is the system matrix, $B \in R^{n\times m}$ is the input force matrix, $X(t) \in R^{n\times 1}$ is the state vector and $U(t) \in R^{m\times 1}$ is the force vector. Let J be the associated scalar cost function with

$$J(X,U,t) = \frac{1}{2} X^T(0)HX(0) + \frac{1}{2}\int_0^\infty [X^T(t)Q_1X(t)+U^T(t)Q_2U(t)]dt \ . \tag{1.2}$$

The Hamiltonian for the system is

$$H(X,U,P,t) = \frac{1}{2} X^T(t)Q_1X(t) + \frac{1}{2} U^T(t)Q_2u(t)+P^T(t)[AX(t)+BU(t)] \tag{1.3}$$

from which it follows that X(t), P(t) and U(t) must satisfy the equations

$$\dot{X}(t) = AX(t) + BU(t) \tag{1.4}$$

$$\dot{P}(t) = -Q_1X(t)-A^TP(t) \tag{1.5}$$

$$0 = Q_2U(t)+B^TP(t) \ . \tag{1.6}$$

The desired control for minimizing the cost function is

$$U(t) = -Q_2^{-1}B^TP(t) \tag{1.7}$$

where it will be assumed that

$$P(t) = RX(t) \ . \tag{1.8}$$

Differentiating (1.8) with respect to t and using (1.4) and (1.5) gives the algebraic Riccati equation

$$Q_1 + A^TR + RA - RBQ_2^{-1}B^TR = 0 \tag{1.9}$$

for R. Substituting R into (1.8) and the resulting equation into (1.7) gives the control

$$U(t) = -Q_2^{-1}B^T RX(t) \tag{1.10}$$

This control will give the closed loop matrix

$$\overline{A} = A - BQ_2^{-1}B^T R \tag{1.11}$$

It is usually assumed that H, and Q_1 are symmetric positive semi-definite matrices and Q_2 is symmetric but positive definite. The Riccati matrix obtained from (1.9) will also be symmetric and positive definite. The matrices H, Q_1 and Q_2 are weighting matrix chosen to fix the cost penalty for the initial conditions, the displacements and the control effort.

An example will now be given to illustrate the computational procedure. Let A be

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9 & 5 & -1 & 0 \\ 5 & -11 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -K & -C \end{bmatrix}$$

$$B^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H = 0$$

$$Q_1 = I_{4\times 4}$$

$$Q_2 = I_{2\times 2}$$

Substituting into (1.9) gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -9 & 5 \\ 0 & 0 & 5 & -11 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} R+R \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9 & 5 & -1 & 0 \\ 5 & -11 & 0 & -1 \end{bmatrix}$$

$$-R \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R = 0$$

The algebraic solution to this equation is

$$R = \begin{bmatrix} 4.1817 & -2.02521 & 0.0736876 & 0.033291 \\ -2.02521 & 4.99179 & 0.033291 & 0.060371 \\ 0.0736876 & 0.033291 & 0.465215 & 0.022792 \\ 0.033291 & 0.060371 & 0.022792 & 0.456098 \end{bmatrix}$$

and the closed loop matrix is

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -9.07369 & 4.96671 & -1.46522 & -0.022792 \\ 4.96671 & -11.0604 & -0.0227921 & -1.4561 \end{bmatrix}$$

The eigenvalues for the open loop system are $\lambda = -0.5 \pm j2.15661$ and $\lambda = -0.5 \pm j3.85344$ whereas the closed loop eigenvalues are $\lambda = -0.741952 \pm j2.10985$ and $\lambda = -0.718708 \pm j3.82303$.

It should be noted that if the closed loop system matrix represents a model of the form

$$A = \begin{bmatrix} 0 & I \\ -K & -C \end{bmatrix}$$

it is necessary to modify the stiffness matrix. It should also be pointed out that C does not represent a model with passive damping as C does not have the proper structure. If the closed loop system has the feedback defined in (1.10), there are no restrictions since this control law is not for a passive system.

The control vector for optimal control should always be constructed from the velocity elements of the structure if the matrix form given above is to result from the control. Basically this means that the B matrix should have zero elements in the upper half, i.e. in the first $n \times m$ block of B.

The algebraic Riccati solution for the example given above required that a $2n \times 2n$ matrix be used where A is $n \times n$. It is obvious that a system with a large number of modes is not suitable for the above type of analysis. The optimal control procedure above will also add damping to all modes where it may only be necessary to damp a few modes.

The analysis presented above is more difficult when an undamped system is considered. The reason for this is due to the presence of multiple eigenvalues of the matrix considered in solving the algebraic Riccati equation. The mode decoupling method presented in the next section will consider an undamped system and the computational problems with the algebraic Riccati equation will be eliminated.

2. Mode Decoupling

A method will be given in this section that decouples some of the modes from the remaining ones so that the optimal control strategy can be carried out on a lower-order system.

Consider the undamped system matrix with C=0 such that A is

$$A = \begin{bmatrix} 0 & I \\ -K & 0 \end{bmatrix} \tag{2.1}$$

where K is positive definite. The eigenvalues of A are along the $j\omega$ axis and occur in complex conjugate pairs. The eigenvalues of K are given by ω_i^2 where $j\omega_i$ is an eigenvalue of A. This suggests that the spectral decomposition of A can be obtained from considering K rather than A.

Suppose that there exists a similarity transformation matrix T_K such that

$$T_K K T_K^{-1} = \begin{bmatrix} K_{B1} & 0 \\ 0 & K_{B2} \end{bmatrix} \tag{2.2}$$

where K_{B1} has eigenvalues $\lambda_i \leq |\rho|$ and K_{B2} has eigenvalues $\lambda_i > |\rho|$. If such a matrix exists, then K_B gives the spectral decomposition of K. To find T_K, the eigenvectors of K must be found or the sign algorithm can be used to generate T. The eigenvector procedure will probably be the most efficient for large systems so the procedure will be based on that method.

Let Φ_K denote the eigenvector of K from which it follows that the inverse of T is constructed from the columns of Φ_K. Consider the stiffness matrix

$$K = \begin{bmatrix} 9 & -5 & 0 \\ -5 & 11 & -6 \\ 0 & -6 & 13 \end{bmatrix}$$

for which the eigenvalues are

$$\lambda_1 = 3.11262$$

$$\lambda_2 = 10.6609$$

$$\lambda_3 = 19.2264$$

and the eigenvector matrix is

$$\Phi_K = \begin{bmatrix} 0.587523 & 0.737925 & 0.332087 \\ 0.691794 & -0.24513 & -0.679214 \\ 0.419805 & -0.62879 & 0.654513 \end{bmatrix}$$

The eigenvectors are ordered in the same order as the eigenvalues.

Assume now that T_K is to be constructed such that K_{B1} contains the eigenvalue λ_1 and K_{B2} has eigenvalues λ_2 and λ_3. The eigenvector matrix is partitioned with

$$\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} = \begin{bmatrix} 0.587523 & 0.737925 & 0.332087 \\ 0.691794 & -0.24513 & -0.679214 \\ 0.419805 & -0.62879 & 0.654513 \end{bmatrix}$$

and T^{-1} is then given by

$$T_K^{-1} = \frac{1}{2} \begin{bmatrix} -I & \phi_{12}\phi_{22}^{-1} \\ -\phi_{21}\phi_{11}^{-1} & I \end{bmatrix} \qquad (2.3)$$

or numerically

$$T_K^{-1} = \begin{bmatrix} -0.5 & -0.58874 & -0.357266 \\ -0.58874 & 0.5 & 0 \\ -0.357266 & 0 & 0.5 \end{bmatrix}$$

The inverse of T_K is

$$T_K = \begin{bmatrix} -0.690368 & -0.812892 & -0.49329 \\ -0.812892 & 1.04284 & -0.580837 \\ -0.49329 & -0.580837 & 1.64753 \end{bmatrix}$$

The similarity transform as expressed in (2.2) gives

$$K_B = \begin{bmatrix} 3.11262 & 0 & 0 \\ 0 & 16.8874 & -2.42734 \\ 0 & -6 & 13 \end{bmatrix} = \begin{bmatrix} K_{B1} & 0 \\ 0 & K_{B2} \end{bmatrix}$$

which has the correct eigenvalues.

If K_{B1} is to be 2×2 with eigenvalues λ_1 and λ_2, the partitioning of Φ_K would be changed so that ϕ_{11} is 2×2, ϕ_{22} is 1×1, etc. The transformation matrix is then constructed and the similarity transformation applied.

Thus far, the spectral decomposition of K has been carried out but the system matrix given in (2.1) must be considered as this is the matrix that is of concern. Let T be a new transformation matrix with

$$T = \begin{bmatrix} T_K & 0 \\ 0 & T_K \end{bmatrix} \tag{2.4}$$

Using T as given in (2.4) then

$$TAT^{-1} = \begin{bmatrix} T_K & 0 \\ 0 & T_K \end{bmatrix} \begin{bmatrix} 0 & I \\ -K & 0 \end{bmatrix} \begin{bmatrix} T_K & 0 \\ 0 & T_K \end{bmatrix}^{-1} = \begin{bmatrix} 0 & I \\ -T_K K T_K^{-1} & 0 \end{bmatrix} \tag{2.5}$$

gives a new system matrix with K_B in the lower left corner of the matrix. Substituting the value of K_B from the example, the new matrix is

$$TAT^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -3.11262 & 0 & 0 & 0 & 0 & 0 \\ 0 & -16.8874 & 2.42734 & 0 & 0 & 0 \\ 0 & 6 & -13 & 0 & 0 & 0 \end{bmatrix}$$

which is not block diagonalized. To block diagonalize TAT^{-1}, construct a row-column interchange matrix E where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The block diagonal form can then be found with

$$A_E = ETAT^{-1}E = T_A A T_A^{-1}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -3.11262 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -16.8874 & 2.42734 & 0 & 0 \\ 0 & 0 & 6 & -13 & 0 & 0 \end{bmatrix} \qquad (2.6)$$

The spectral decomposition process will modify the state vector X(t) as will now be shown. Let V(t) be defined as the transformed vector

$$V(t) = T_A X(t) \qquad (2.7)$$

therefore

$$\dot{V}(t) = T_A\dot{X}(t) = T_AAX(t) \tag{2.8}$$

but $X(t) = T_A^{-1}V(t)$ thus

$$\dot{V}(t) = T_AAT_A^{-1}V(t) \tag{2.9}$$

The similarity transformation on A to block diagonalize A will map $X(t)$ into a new vector $V(t)$ where

$$V(t) = E\begin{bmatrix} T_K & 0 \\ 0 & T_K \end{bmatrix} X(t) \tag{2.10}$$

or

$$X(t) = \begin{bmatrix} T_K^{-1} & 0 \\ 0 & T_K^{-1} \end{bmatrix} E\ V(t)\ . \tag{2.11}$$

For the example given in this section $T = ET_A$ is

$$T = ET_A = \begin{bmatrix} -0.690368 & -0.812892 & -0.49329 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.690368 & -0.812892 & -0.49329 \\ -0.812892 & 1.04284 & -0.580837 & 0 & 0 & 0 \\ -0.49329 & -0.580837 & 1.64753 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.812892 & 1.04284 & -0.580837 \\ 0 & 0 & 0 & -0.49329 & -0.580837 & 1.64753 \end{bmatrix}$$

thus the vector $V(t)$ has components $v_i(t)$ with

$$v_1(t) = -0.690368x_1(t)-0.812892x_2(t)-0.49329x_3(t)$$
$$v_2(t) = -0.690368\dot{x}_1(t)-0.812892\dot{x}_2(t)-0.49329\dot{x}_3(t)$$
$$\vdots$$
$$v_6(t) = -0.49329\ \dot{x}_1(t)-0.580837\dot{x}_2(t)+1.64753\dot{x}_3(t)$$

where $X^T(t) = [x_1(t)\ x_2(t)\ x_3(t)\ \dot{x}_1(t)\ \dot{x}_2(t)\ \dot{x}_3(t)]$ for the undamped model.

All of the computations for the decomposition given in this section are carried out by considering the K matrix which is m×m. It is not necessary to find the eigenvectors of the A matrix since the necessary information is contained in K.

3. Optimal Control of the Undamped Decoupled State Matrix

It was shown in the previous section that the state matrix could be block diagonalized with selected eigenvalues of A placed in one of the selected block matrices. Let the block matrix for the undamped system have the general form

$$A_B = TAT^{-1} = \begin{bmatrix} A_{B1} & 0 \\ 0 & A_{B2} \end{bmatrix} \tag{3.1}$$

where A_{B1} has eigenvalues $|\lambda_i| < \rho$ and A_{B2} has eigenvalues $|\lambda_i| > \rho$ with ρ a scalar variable and A is the undamped matrix. The value of ρ will be chosen to include the desired modes in A_{B1}.

Consider now the algebraic Riccati equation for $\bar{R}$ and let A_B be the decoupled matrix, thus $\bar{R}$ must satisfy

$$\bar{Q}_1 + A_B^T\bar{R} + \bar{R}A_B - \bar{R}\,\bar{B}\,Q_2^{-1}\,\bar{B}^T\,\bar{R} = 0 \tag{3.2}$$

where $\bar{Q}_1$ and Q_2 are weighting matrices for V(t) and U(t). The matrix $\bar{B}$ represents the control input matrix where

$$\dot{V}(t) = A_B V(t) + \bar{B}U \tag{3.3}$$

with A_B defined in (3.1) and $\bar{B} = TB$. It will be assumed that the algebraic Riccati equation is completely decoupled such that

$$\bar{Q}_{11} + A_{B1}^T\bar{R}_1 + \bar{R}_1A_{B1} - \bar{R}_1(\bar{B}\,Q_2^{-1}\,\bar{B}^T)_1\;\bar{R}_1 = 0 \tag{3.4}$$

$$\bar{Q}_{12} + A_{B2}^T\bar{R}_2 + \bar{R}_2A_{B2} - \bar{R}_2(\bar{B}\,Q_2^{-1}\,\bar{B}^T)_2\;\bar{R}_1 = 0 \tag{3.5}$$

The Riccati matrices $\bar{R}_1$ and $\bar{R}_2$ can be found independently since the equations are decoupled.

Substituting for A_B in (3.2),

$$\bar{Q}_1 + T^{-T}A^TT^T\bar{R} + \bar{R}TAT^{-1} - R\,\bar{B}\,Q_2^{-1}\,\bar{B}^T\,\bar{R} = 0 \tag{3.6}$$

and rearranging gives

$$T^T\bar{Q}_1T + A^TT^T\bar{R}T + T^T\bar{R}TA - T^TRTBQ_2^{-1}B^TT^TRT = 0 \tag{3.7}$$

Defining $R = T^T\bar{R}T$ and $Q_1 = T^T\bar{Q}_1T$ gives

$$Q_1 + A^TR + RA - RBQ_2^{-1}B^TR = 0 \tag{3.8}$$

which is the usual algebraic Riccati equation for the general optimal control problem.

Denoting equation (3.4) as system 1 and (3.5) as system 2, it follows that system 1 has the system equation

$$\dot{V}_1(t) = A_{B1}V_1(t) + \bar{B}_1u_1(t) \tag{3.9}$$

with cost

$$J_1(V_1,u_1,t) = \int_0^\infty [V_1^T(t)\bar{Q}_{11}V_1(t) + u_1^T(t)\bar{Q}_{21}u_1(t)]dt\ . \tag{3.10}$$

The other system has the state equation

$$\dot{V}_2(t) = A_{B2}V_2(t) + \bar{B}_2u_2(t) \tag{3.11}$$

and cost

$$J_2(V_2,u_2,t) = \int_0^\infty [V_2^T(t)\bar{Q}_{12}V_2(t) + u_2(t)\bar{Q}_{22}u_2(t)]dt \tag{3.12}$$

where the initial state cost has been neglected. Assuming that the first system is the desired system for damping, then $\bar{B}_2 = 0$ will leave system 2

undamped $\bar{R}_2 = 0$. It then follows that the uncoupled system Riccati equation is

$$R = T^T \bar{R} T = \begin{bmatrix} T_{11}^T \bar{R}_1 T_{11} & T_{11}^T \bar{R}_1 T_{12} \\ T_{12}^T \bar{R}_1 T_{11} & T_{12}^T \bar{R}_1 T_{12} \end{bmatrix}. \tag{3.13}$$

Since $\bar{R}_1$ will be symmetric then R will be symmetric as desired. The control input vector $\bar{B}$ will have the form

$$\bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} \tag{3.14}$$

therefore

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = T^{-1}\bar{B} = \begin{bmatrix} (T_{11}-T_{12}T_{22}^{-1}T_{21}) & (T_{21}-T_{22}T_{12}^{-1}T_{11})^{-1} \\ (T_{12}-T_{11}T_{21}^{-1}T_{22})^{-1} & (T_{22}-T_{21}T_{11}^{-1}T_{12})^{-1} \end{bmatrix} \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}. \tag{3.15}$$

The numerical value of $\bar{B}_1$ can be chosen such that B_1 is zero and $B_2 \neq 0$.

The closed loop system matrix is then given by

$$\bar{A} = A - BQ_2^{-1}B^T R = \begin{bmatrix} 0 & I \\ -K & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ B_2 B_2^T T_{12}^T \bar{R}_1 T_{11} & B_2 B_2^T T_{12}^T \bar{R}_1 T_{12} \end{bmatrix} \tag{3.16}$$

where it may be possible to make $B_2 B_2^T T_{12}^T \bar{R}_1 T_{11} = 0$ by properly selecting B_2 and the weighting matrices. In general, this matrix will not be zero and the stiffness of the structure will be changed.

The example used in Section 2 will be used to illustrate the computational procedure. The stiffness matrix was

$$K = \begin{bmatrix} 9 & -5 & 0 \\ -5 & 11 & -6 \\ 0 & -6 & 13 \end{bmatrix}$$

where

$$A_{B1} = \begin{bmatrix} 0 & 1 \\ -3.11262 & 0 \end{bmatrix}$$

and

$$A_{B2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -16.8874 & 2.42734 & 0 & 0 \\ 6 & -13 & 0 & 0 \end{bmatrix}$$

The first step is to select B such that $\overline{B}_1 \neq 0$ and $\overline{B}_2 = 0$ which can be obtained from $\overline{B} = TB$ where

$$\begin{bmatrix} -0.690368 & -0.812892 & -0.49329 & 0 & 0 & 0 \\ 0 & 0 & 0 & -.690368 & -.812892 & -0.49329 \\ -0.812892 & 1.04284 & -0.580837 & 0 & 0 & 0 \\ -0.49329 & -0.580837 & 1.64753 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.812892 & 1.04284 & -0.580837 \\ 0 & 0 & 0 & -0.49329 & -0.580837 & 1.64753 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \\ b_{14} \\ b_{14} \\ b_{15} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix}$$

The solution to this equation is

$$B^T = [0 \quad 0 \quad 0 \quad -0.5 \quad -0.588738 \quad -0.357266]$$

where the last m elements is the vector of T_K^{-1} belonging to the controlled

mode.

Selecting the 2×2 matrix A_{B1} for damping, then for $\bar{Q}_{11} = I$, $\bar{Q}_{21} = I$ and $\bar{B}_1^T = [0\ 1]$, the Riccati equation is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -3.11262 \\ 1 & 0 \end{bmatrix} \bar{R}_1 + \bar{R}_1 \begin{bmatrix} 0 & 1 \\ -3.11262 & 0 \end{bmatrix} - \bar{R}_1 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \bar{R}_1 = 0$$

which has a solution

$$\bar{R}_1 = \begin{bmatrix} 5.48385 & 0.10098 \\ 0.10098 & 1.09363 \end{bmatrix}$$

and A_{B1} closed loop is

$$\bar{A}_{B1} = A_{B1} - \bar{B}_1 \bar{Q}_2^{-1} \bar{B}_1^T \bar{R}_1 = \begin{bmatrix} 0 & 1 \\ -3.2136 & -1.09363 \end{bmatrix}$$

To recover the full Riccati matrix, use $R = T^T \bar{R} T$ which gives

$$R = \begin{bmatrix} 2.61365 & 3.07751 & 1.86753 & .0481279 & .0566694 & .0343889 \\ 3.07751 & 3.62369 & 2.19898 & .0566694 & .0667269 & .0404921 \\ 1.86753 & 2.19898 & 1.33441 & .0343889 & .0404921 & .024572 \\ .0481279 & .0566694 & .0343889 & .521233 & .613739 & .372438 \\ .0566694 & .0667269 & .0404921 & .613739 & .722664 & .438536 \\ .0343889 & .0404921 & .024572 & .372438 & .438536 & .266119 \end{bmatrix}$$

The closed loop system matrix is given by $\bar{A} = A - BQ_2^{-1}B^T R$ where the numerical values are

$$\bar{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -9.03486 & 4.95896 & -.0249062 & -.377504 & -.444502 & -.269739 \\ 4.95896 & -11.0483 & 5.97067 & -.444502 & -.52339 & -.317611 \\ -.0249063 & 5.97067 & -13.0178 & -.269739 & -.317611 & -.192737 \end{bmatrix}$$

The eigenvalues of $\overline{A}$ are

$$\lambda_{1,2} = -0.546816 \pm j1.70722$$

$$\lambda_{3,4} = \pm j3.26511$$

$$\lambda_{5,6} = \pm j4.3848$$

which agrees with those of A_{B1} and A_{B2}.

The required feedback control vector u(t) is given by

$$U(t) = -Q_2^{-1}B^T R X(t) = -Q_2^{-1}\overline{B}^T\overline{R}\ T\ X(t) \qquad (3.16)$$

which can now be determined as Q_2, $\overline{B}$ and $\overline{R}$ are known.

4. Optimal Control of the General Canonical State Matrix Form

The procedure described in the proceeding sections considered the undamped state matrix in canonical form. The method can be extended to the general form by utilizing the algebraic theory in Chapters 2 and 5. Consider the damped matrix

$$A = \begin{bmatrix} 0 & I \\ -K & -C \end{bmatrix} \tag{4.1}$$

where $K \epsilon R^{m \times m}$ and $C \epsilon R^{m \times m}$ with no restrictions placed on the forms of these matrices. The optimal control theory can be applied to this form with some modifications in the computational procedure.

The right and left eigenvectors for the eigenvalue λ_j are given by

$$y_{cj} = \begin{bmatrix} y_j \\ \lambda_j y_j \end{bmatrix} \tag{4.2}$$

and

$$z_{cj} = \begin{bmatrix} (\lambda_j I + C) y_j \\ y_j \end{bmatrix} \tag{4.3}$$

respectively where $A(\lambda_j) y_{cj} = 0$ and $A^T(\lambda_j) z_{cj} = 0$. The eigenprojector P_j for the eigenvector λ_j is defined as

$$P_j = \frac{y_{cj} z_{cj}^T}{z_{cj}^T y_{cj}} = \bar{y}_{cj} \bar{z}_{cj}^T \tag{4.4}$$

where y_{cj}, and z_{cj} denote the unnormalized eigenvectors and the eigenvectors with the bar indicates normalized eigenvectors. The eigenvalues will occur

in complex conjugate pairs thus λ_{2i-1} will be taken as the eigenvalue in the upper half plane with $\mathrm{Im}(\lambda_{2i-1})\geq 0$ and λ_{2i} in the lower half plane. The eigenprojector for a pair of conjugate eigenvalues of the jth mode

$$P_j = P_{2i-1}+P_{2i} = \bar{y}_{c2i-1}\bar{z}^T_{c2i-1}+\bar{y}_{c2i}\bar{z}^T_{c2i} \tag{4.5}$$

or from (4.2) and (4.3)

$$P_j = \begin{bmatrix} y_{2i-1}y^T_{2i-1}(\lambda_{2i-1}I+C^T)+y_{2i}y^T_{2i}(\lambda_{2i}I+C^T) \\ y_{2i-1}y^T_{2i-1}(\lambda^2_{2i-1}I+C^T\lambda_{2i-1})+y_{2i}y^T_{2i}(\lambda^2_{2i}I+C^T\lambda_{2i}) \end{bmatrix}$$

$$\begin{bmatrix} y_{2i-1}\,y^T_{2i-1} + y_{2i}\,y^T_{2i} \\ y_{2i-1}y^T_{2i-1}\lambda_{2i-1}+y_{2i}y^T_{2i}\lambda_{2i} \end{bmatrix} = P_{2i-1}+\bar{P}^*_{2i-1} \tag{4.6}$$

but $\lambda_{2i} = \lambda^*_{2i-1}$ and $y_{2i} = y^*_{2i-1}$. Thus the terms in (4.6) can be combined if desired. The eigenprojector P_j will be a real matrix since $A_j = AP_j$ will be a real matrix.

Suppose now that the first of modes, j = 1,2,...,q, are to be the ones that are to be damped. It then follows that

$$P_D = \sum_{i=1}^{q} P_j \tag{4.7}$$

and if P_{UD} is the undamped projector, it follows that

$$P_{UD} = I - P_D \tag{4.8}$$

The projector P_D can be computed by determining the latent vectors of $A(\lambda) = I\lambda^2+C\lambda+K$ or from the eigenvectors of A.

The matrix T = S+J is needed but this can be computed from P_D. The eigenprojector for the first q modes can also be defined in terms of the

sign matrix S with

$$P_D = \frac{1}{2}(S+I) = \Phi \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \Phi^{-1} \tag{4.9}$$

with

$$S = \Phi \begin{bmatrix} I_q & 0 \\ 0 & -I_{m-q} \end{bmatrix} \Phi^{-1} = \Phi J \Phi^{-1} \tag{4.10}$$

where Φ is the eigenvector matrix of A. To determine S+J, note that

$$S = 2P_D - I \tag{4.11}$$

and thus

$$T = S+J = 2P_D - I + J = 2P_D + 2 \begin{bmatrix} 0 & 0 \\ 0 & -I_{m-q} \end{bmatrix} \tag{4.12}$$

The next step in the algorithm is to find A_{B1} which is the block diagonal matrix that is to be damped by the optimal control strategy. It can be shown that

$$A_{B1} = A_{11} + A_{12}R_{21} \tag{4.13}$$

where A_{11} and A_{12} are partitioned blocks of A with $A_{11} \epsilon R^{q \times q}$ and $A_{12} \epsilon R^{q \times 2m-q}$. These submatrices are determined directly from A. The Riccati function R_{21} must be found but this matrix can be obtained from P_D or T. It can be shown that R_{21} is given by

$$R_{21} = \phi_{21}\phi_{11}^{-1} \tag{4.14}$$

where $\phi_{11} \epsilon C^{q \times q}$ and $\phi_{21} \epsilon C^{q \times 2m-q}$. Partitioning (4.9) and carrying out the algebraic steps gives

sign matrix S with

$$P_D = \frac{1}{2}(S+I) = \Phi \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \Phi^{-1} \tag{4.9}$$

with

$$S = \Phi \begin{bmatrix} I_q & 0 \\ 0 & -I_{m-q} \end{bmatrix} \Phi^{-1} = \Phi J \Phi^{-1} \tag{4.9}$$

where Φ is the eigenvector matrix of A. To determine S+J, note that

$$S = 2P_D - I \tag{4.10}$$

and thus

$$T = S+J = 2P_D - I + J = 2P_D + 2 \begin{bmatrix} 0 & 0 \\ 0 & -I_{m-q} \end{bmatrix} \tag{4.11}$$

The next step in the algorithm is to find A_{B1} which is the block diagonal matrix that is to be damped by the optimal control strategy. It can be shown that

$$A_{B1} = A_{11} + A_{12}R_{12} \tag{4.12}$$

where A_{11} and A_{12} are partitioned blocks of A with $A_{11} \epsilon R^{q \times q}$ and $A_{12} \epsilon R^{q \times 2m-q}$. These submatrices are determined directly from A. The Riccati function R_{21} must be found but

$$P_D = \begin{bmatrix} \phi_{11}(\phi_{11}-\phi_{12}\phi_{22}^{-1}\phi_{21})^{-1} & -\phi_{11}(\phi_{11}-\phi_{12}\phi_{22}^{-1}\phi_{21})^{-1}\phi_{12}\phi_{22}^{-1} \\ \phi_{21}(\phi_{11}-\phi_{12}\phi_{22}^{-1}\phi_{21})^{-1} & \phi_{21}(\phi_{21}-\phi_{22}\phi_{12}^{-1}\phi_{11})^{-1} \end{bmatrix} \tag{4.15}$$

thus*

$$R_{21} = P_{D21}\, P_{D11}^{-11} \tag{4.16}$$

and

$$R_{12} = -P_{D11}^{-1}\, P_{D12} \cdot \tag{4.17}$$

The latter matrix, R_{12}, is not needed unless T^{-1} is sought in which case T^{-1} can be written down directly with

$$T^{-1} = \frac{1}{2}\begin{bmatrix} I & -R_{12} \\ R_{21} & -I \end{bmatrix} \tag{4.18}$$

The computational procedure for the damped system matrix is as follow:

a) Select the modes to be damped.

b) Compute the right eigenvectors for the selected modes.

c) Using the latent vectors obtained from the eigenvectors, compute the eigenprojector for the selected modes.

d) Compute T and R_{21}.

e) Form the block matrix A_{B1}.

f) Using the Laub algorithm [32] compute the algebraic Riccati solution for the control law of A_{B1}.

g) Compute $u(t) = -Q_2^{-1}B^T RX(t)$ and implement the control.

---

* It is assumed that P_{D11} is not singular.

5. Conclusion

The development of a procedure for optimal control of selected modes of a second-order lambda matrix has been given in this section. Although the development has been abbreviated and did not cover the most general model there appears to be no significant problems for the general case. It has been shown that the control vector for a rather small second-order lambda matrix can be determined so that damping is added to the lowest modes. The other modes remain undamped.

The mode decoupling procedure used with the optimal control strategy avoids the problem of mode "spill-over". The algorithm as presented does not give the design the option of placing system eigenvalues at desired location although this can be achieved with some modifications to the theory provided that the designer give up some restrictions on the final form of the closed loop system matrix.

The calculations carried out in this chapter resulted from several specific computer programs which could be combined for a general package. Since software development is a time-consuming task, no effort was made the develop such a package at this time.

Appendix A

The eigenprojectors $P_{i,m_i-k-1} \in C^{mn \times mn}$, as defined in (3.2), can be obtained from

$$P_{i,m_i-q_i-k} = \lim_{\lambda \to \lambda_i} W_R \{ \frac{1}{k!} \frac{d^{(k)}}{d\lambda^{(k)}} [(\lambda-\lambda_i)^{m_i-q_i+1} (\lambda I-J)^{-1}] \} W_L^T \qquad (A.1)$$

$$k = 0,1,\ldots,m_i-q_i$$

where W_R and W_L are the right and left eigenvector matrices. The inverse of $(\lambda I-J)$ is given by

$$(\lambda I-J)^{-1} = \begin{bmatrix} (\lambda I-J_1)^{-1} & & & \\ & (\lambda I-J_2)^{-1} & & \\ & & \ddots & \\ & & & (\lambda I-J_r)^{-1} \end{bmatrix} \qquad (A.2)$$

where $J_i \in C^{m_i \times m_i}$ is the Jordan block associated with the eigenvalue λ_i of multiplicity m_i. For a defective matrix, the Jordan block J_i can have the following structure

$$J_i = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & \ddots & & \\ & & \ddots & 1 & \\ & & & \lambda_i & \\ & & & & \lambda_i \ 1 \\ & & & & \ddots \ \ddots \\ & & & & \ddots \ 1 \\ & & & & \lambda_i \end{bmatrix} \qquad (A.3)$$

(q_i-1 rows; q_i-1 columns; m_i-q_i+1 rows; m_i-q_i+1 columns)

where q_i is the number of primary eigenvectors and m_i-q_i the number of generalized eigenvectors associated with λ_i. It follows that $(\lambda I-J_i)^{-1}$ is given by

$$(\lambda I-J_i)^{-1} = \begin{bmatrix} (\lambda-\lambda_i)^{-1} & & & & & & \\ & (\lambda-\lambda_i)^{-1} & & & & & \\ & & \ddots & & & & \\ & & & (\lambda-\lambda_i)^{-1} & & & \\ & & & & (\lambda-\lambda_i)^{-1} & (\lambda-\lambda_i)^{-2} & \cdots & (\lambda-\lambda_i)^{q_i-m_i-1} \\ & & & & & (\lambda-\lambda_i)^{-1} & & \\ & & & & & & \ddots & \\ & & & & & & & (\lambda-\lambda_i)^{-1} \end{bmatrix} \tag{A.4}$$

The bracketed term of (A.1) for $(\lambda I-J_i)$ gives

$$(\lambda-\lambda_i)^{m_i-q_i+1}(\lambda I-J_i)^{-1} = \begin{bmatrix} (\lambda-\lambda_i)^{m_i-q_i} & & & & \\ & \ddots & & & \\ & & (\lambda-\lambda_i)^{m_i-q_i} & & \\ & & & (\lambda-\lambda_i)^{m_i-q_i} & 1 \\ & & & & \ddots \\ & & & & & (\lambda-\lambda_i)^{m_i-q_i} \end{bmatrix} \tag{A.5}$$

It is not difficult to evaluate (A.1) for $k = 0,1,\ldots,m_i-q_i$. The eigenprojector P_{i,m_i-q_i} is then given by

$$P_{i,m_i-q_i} = W_R \left[\begin{array}{c|ccc} 0 & & & \\ \hline 0 & \cdots & 0 & 1 \\ 0 & & 0 & \\ \vdots & \ddots & & \\ 0 & & & \\ \hline & & & 0 \end{array}\right] W_L^T = W_R \text{ sh } F_{i,m_i-q_i} W_L^T \tag{A.6}$$

$$P_{i,0} = W_R \begin{bmatrix} 0 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 0 \end{bmatrix} W_L^T = W_R \, sh \, F_{i,0} \, W_L^T \qquad (A.7)$$

The shifting matrices are important in computing eigenprojectors of a matrix provided that the eigenvalues and eigenvectors are known along with the multiplicity of each eigenvalue and the degeneracy of each Jordan Block.

APPENDIX B

The Kronecker product is a useful mathematical tool for systems theory and can be defined as follows. Let $A_{p\times q}$ and $B_{r\times t}$ then the Kronecker product $A \otimes B$ is

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1q}B \\ a_{21}B & & & \vdots \\ \vdots & & & \vdots \\ a_{p1}B & \cdots & \cdots & a_{pq}B \end{bmatrix} . \qquad (B.1)$$

An excellent treatment of Kronecker algebra has been given by Brewer [14]. If the matrices $N_{n\times n}$, $M_{m\times m}$, $C_{q\times s}$ and $D_{t\times \ell}$ are considered the following properties are of interest in this paper:

$$\text{i)} \quad (N \otimes M)^{-1} = N^{-1} \otimes M^{-1}$$

$$\text{ii)} \quad (A \otimes B)(C \otimes D) = (AC) \otimes (BD) . \qquad (B.2)$$

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