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ABSTRACT

To study the nonlinear physics of uniform turbulent shear flow, the unaveraged Navier-Stokes equations are solved numerically. This extends our previous work in which mean gradients were absent. For initial conditions, modified three-dimensional-cosine velocity fluctuations are used. The boundary conditions are modified periodic conditions on a stationary three-dimensional numerical grid. A uniform mean shear is superimposed on the initial and boundary conditions. The three components of the mean-square velocity fluctuations are initially equal for the conditions chosen. As in the case of no shear the initially nonrandom flow develops into an apparently random turbulence at higher Reynolds number. Thus, randomness or turbulence can apparently arise as a consequence of the structure of the Navier-Stokes equations. Except for an initial period of adjustment, all fluctuating components grow with time. The initial equality of the three intensity components is destroyed by the shear, the transverse components becoming smaller than the longitudinal one, in agreement with experiment. Also, the shear creates a small-scale structure in the turbulence. The nonlinear solutions are compared with linearized ones.

INTRODUCTION

This is an extension of our work in which the development of isotropic turbulence was examined numerically.[1] In that work the nonlinear transfer of energy to smaller scales of motion and turbulent dissipation were studied.

Another important process is the production of turbulence by a mean shear. Most turbulent flows, both those occurring in nature and those which are manmade, are in fact shear flows, where the turbulence is produced and maintained by the shear. Because of the added complexity, the nonlinear problem of turbulent shear flow is even more difficult than that of isotropic turbulence. So it is not surprising that little progress has been made in obtaining an analytical solution from first principles. An attempt to obtain a numerical solution would seem to be in order.

Conceptually, the simplest turbulent shear flow (although certainly not the simplest to produce experimentally[2]) is one in which the turbulence is homogeneous and uniformly sheared. At least two significant numerical studies of that type of turbulence have recently been made.[3,4] In both of those studies random initial conditions with a range of eddy sizes were used.

In the spirit of Ref. 1, the present numerical study of uniformly-sheared turbulence starts with simple determinate initial conditions which possess a single length scale. In this way we can study how the turbulence develops from nonturbulent initial conditions. Moreover much higher Reynolds-number flows can be calculated with a given numerical grid when a single length scale is initially present, at least for early and moderate times. Some results using a perturbation series were obtained in Refs. 5 and 6, but the calculations could not be carried very far in time. Moreover the components of the initial intensities were not equal, as they are here.

As will be seen, several interesting results which could not be obtained in the previous work on turbulent shear flow are obtained in the present study. One of the significant findings is that the structure of the turbulence produced in the presence of a strong shear is much finer than that produced in its absence.

THE NONLINEAR PROBLEM

As in Ref. 1, the Navier-Stokes and continuity equations for an incompressible fluid are written in dimensionless form as

$$\frac{\partial \tilde{u}_i}{\partial t} = -\frac{\partial(\tilde{u}_i \tilde{u}_k)}{\partial x_k} - \frac{\partial \tilde{p}}{\partial x_i} + \frac{\partial^2 \tilde{u}_i}{\partial x_k \partial x_k} \tag{1}$$

and

$$\frac{\partial \tilde{u}_k}{\partial x_k} = 0, \tag{2}$$

where

$$\tilde{u}_i = \frac{x_0}{\nu} \tilde{u}_i^{\star}, \quad \tilde{p} = \frac{x_0^2}{\rho \nu^2} \tilde{p}^{\star}, \quad t = \frac{\nu}{x_0^2} t^{\star}, \quad \text{and} \quad x_i = \frac{x_i^{\star}}{x_0}.$$

Note that the stars on dimensional quantities are omitted for corresponding dimensionless quantities. The subscripts can take on the values 1, 2, and 3, and a repeated subscript in a term indicates a summation. The quantity $\tilde{u}_i^{\star}$ is an instantaneous velocity component, $x_i^{\star}$ is a space coordinate, x_0 is a characteristic length, $t^{\star}$ is the time, ρ is the density, ν is the kinematic viscosity, and $p^{\star}$ is the instantaneous pressure. In order to obtain an explicit equation for the pressure, we take the divergence of Eq. (1) and apply the continuity Eq. (2) to get

$$\frac{\partial^2 \tilde{p}}{\partial x_\ell \, \partial x_\ell} = - \frac{\partial^2(\tilde{u}_\ell \tilde{u}_k)}{\partial x_\ell \, \partial x_k}. \tag{3}$$

In the remainder of the analysis it will be found convenient to use the set of Eqs. (1) and (3), rather than Eqs. (1) and (2).

The expression assumed for the initial disturbance is, in dimensionless form

$$\tilde{u}_i = \sum_{n=1}^{3} a_i^n \cos \vec{q}^n \cdot \vec{x} + \delta_{i1} \frac{dU_1}{dx_2} x_2 \tag{4}$$

where

$$a_i^n = \frac{x_0}{\nu} a_i^{*n}, \quad \vec{q}^n = x_0 \vec{q}^{*n},$$

and

$$U_i = \frac{x_0}{\nu} U_i^*.$$

The quantity a_i^{*n} is an initial velocity amplitude or Fourier coefficient of the disturbance, $\vec{q}^{*n}$ is an initial wave number vector, U_i^* is a mean velocity component, and δ_{ij} is the Kronecker delta (equals 1 for $i = j$ and 0 for $i \neq j$). The quantities $\vec{q}^n \cdot \vec{x}$ are dot products $(\vec{q}^n \cdot \vec{x} = q_1^n x_1 + q_2^n x_2 + q_3^n x_3)$. Equation (4) reduces to the initial condition in Ref. 1 for $dU_1/dx_2 = 0$ (no shear). In order to satisfy the continuity condition, Eq. (2),

$$a_i^n q_i^n = 0. \tag{5}$$

For the present work, as in Ref. 1, we set

$$a_i^1 = k(2, 1, 1),\ a_i^2 = k(1, 2, 1),\ a_i^3 = k\ (1, 1, 2),$$
$$q_i^1 = (-1, 1, 1),\ q_i^2 = (1, -1, 1),\ \text{and}\ \ q_i^3 = (1, 1, -1), \qquad (6)$$

where k is a quantity that fixes the Reynolds number. In addition to satisfying continuity, Eqs. (6) give

$$\overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2} = \overline{u_0^2} \qquad (7)$$

at the initial time, where $u_i = \tilde{u}_i - \delta_{i1} U_1$ and the overbars indicate averaged values. Thus, Eqs. (4) and (6) give a particularly simple initial condition, in that we need be concerned with only one component of the mean-square velocity. Note that it is necessary to have at least three terms in the summation in Eq. (4) to satisfy Eq. (7).

In order to carry out the numerical solution of Eqs. (1) and (3) subject to initial conditions (4) and (6), we use a stationary cubical grid with a maximum of 32^3 points and with faces at $x_i = 0$ and 2π. For boundary conditions we assume modified periodicity. That is, we let

$$(\tilde{u}_i)_{x_j=2\pi+b_j} = (\tilde{u}_i)_{x_j=b_j} + \delta_{i1}\, \delta_{j2}\, 2\pi\, dU_1/dx_2 \qquad (8)$$

and

$$(\tilde{p})_{x_j=2\pi+b_j} = (\tilde{p})_{x_j=b_j} \qquad (9)$$

for any b_j. Note that Eq. (8) is not a tensor equation. It is the usual periodicity condition, but with a superimposed mean shear, and is consistent with the initial condition given by Eq. (4). Equation (8) is used to calculate numerical derivatives at the boundaries.

The spacial- and time-differencing schemes are essentially those used by Clark et al.[7] and in Ref. 1. That is, for the spacial derivatives in Eqs. (1) and (3) we use centered fourth-order difference expressions.[8] For time-differencing we use a predictor-corrector method with a second-order (leap-frog) predictor and a third-order (Adams-Moulton) corrector.[9] The Poisson equation for the pressure (Eq. (3)) is solved directly (no iteration) by a fast Fourier-transform method. This method preserves continuity quite well ($\nabla\cdot\vec{u} = 0$).

Some of the results will be extrapolated to zero mesh size in an effort to obtain more accuracy. The method of fourth-order extrapolation in Ref. 1, which is consistent with fourth-order differencing, is used here. However to increase the accuracy, more of the grid-point spacings are chosen close to zero: 32^3, 24^3, and 16^3 grid points, rather than the 32^3, 16^3, and 8^3 points in Ref. 1 are used. Equation (10) in Ref. 1 must then be replaced by

$$t_c = 1.6814\, t_1 - 0.7182\, t_2 + 0.0368\, t_3, \tag{10}$$

where t_1, t_2, and t_3 are respectively values of t calculated for 32^3, 24^3, and 16^3 grid points at a fixed $\overline{u_i^2}$, and t_c is corrected value of t at that $\overline{u_i^2}$.

In order to study the processes occurring in sheared turbulence, it is convenient to break the instantaneous velocities and pressures in Eqs. (1) and (3) into mean and fluctuating components; thus, set $\tilde{u}_i = U_i + u_i$ and $\tilde{p} = P + p$. Then, if averages are taken, and the averaged equations are subtracted from the unaveraged ones, we get, for uniformly-sheared homogeneous fluctuations

$$\frac{\partial u_i}{\partial t} = -\delta_{i1}\frac{dU_1}{dx_2}u_2 - \frac{dU_1}{dx_2}x_2\frac{\partial u_i}{\partial x_1} - \frac{\partial}{\partial x_k}(u_i u_k) - \frac{\partial p}{\partial x_i} + \frac{\partial^2 u_i}{\partial x_k \partial x_k} \tag{11}$$

and

$$\frac{\partial^2 p}{\partial x_k \, \partial x_k} = -2 \frac{dU_1}{dx_2} \frac{\partial u_2}{\partial x_1} - \frac{\partial^2 (u_k u_\ell)}{\partial x_k \, \partial x_\ell} \tag{12}$$

For Eqs. (11) and (12) the initial and boundary conditions given by Eqs. (4), (8), and (9) become

$$u_i = \sum_{n=1}^{3} a_i^n \cos \vec{q}^{\,n} \cdot \vec{x}, \tag{13}$$

$$(u_i)_{x_j = 2\pi + b_j} = (u_i)_{x_j = b_j}, \tag{14}$$

and

$$(p)_{x_j = 2\pi + b_j} = (p)_{x_j} \tag{15}$$

for any b_j.

From Eqs. (11) and (12) written at a point P and similar equations written at a point P', we can construct the following two-point correlation equations for uniformly-sheared homogeneous fluctuations[10]:

$$\frac{\partial}{\partial t} \overline{u_i u_j'} = -\delta_{i1} \frac{dU_1}{dx_2} \overline{u_2 u_j'} - \delta_{j1} \frac{dU_1}{dx_2} \overline{u_i u_2'} - \frac{dU_1}{dx_2} r_2 \frac{\partial}{\partial r_1} \overline{u_i u_j'}$$

$$- \frac{\partial}{\partial r_k} (\overline{u_i u_j' u_k'} - \overline{u_i u_k u_j'}) + \frac{\partial}{\partial r_i} \overline{p u_j'} - \frac{\partial}{\partial r_j} \overline{u_i p'} + 2 \frac{\partial^2 \overline{u_i u_j'}}{\partial r_\ell \, \partial r_\ell}, \tag{16}$$

$$\frac{\partial^2 \overline{u_i p'}}{\partial r_k \, \partial r_k} = -2 \frac{dU_1}{dx_2} \frac{\partial \overline{u_i u_2'}}{\partial r_1} - \frac{\partial^2 \overline{u_i u_k' u_\ell'}}{\partial r_k \, \partial r_\ell}, \tag{17}$$

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and

$$\frac{\partial^2 \overline{pu_j'}}{\partial r_k \, \partial r_k} = 2 \frac{dU_1}{dx_2} \frac{\partial \overline{u_2 u_j'}}{\partial r_1} - \frac{\partial^2 \overline{u_\ell u_k u_j'}}{\partial r_\ell \, \partial r_k}. \tag{18}$$

where the unprimed and primed quantities are measured at P and P', respectively, and $r_i = x_i' - x_i$. Equations (16) to (18) show that the two point correlations for uniformly-sheared turbulence are functions only of $\vec{r}$ and not of $\vec{x}$. That is, the turbulence is homogeneous. That is not the case for nonuniformly-sheared turbulence, for instance for U_1 proportional to x_2^2 instead of to x_2.

A single-point correlation equation for $\overline{u_i u_j}$ can be written as

$$\frac{\partial}{\partial t} \overline{u_i u_j} = -\delta_{i1} \frac{dU_1}{dx_2} \overline{u_j u_2} - \delta_{j1} \frac{dU_1}{dx_2} \overline{u_i u_2} - \overline{u_j \frac{\partial p}{\partial x_i}} - \overline{u_i \frac{\partial p}{\partial x_j}} + \overline{u_j \frac{\partial^2 u_i}{\partial x_\ell \, \partial x_\ell}} + \overline{u_i \frac{\partial^2 u_j}{\partial x_\ell \, \partial x_\ell}} \tag{19}$$

Equations (16) to (19) are useful for studying the processes occurring in uniformly-sheared turbulence.

THE LINEARIZED PROBLEM

Equations (11) and (12) are linearized by neglecting the terms $\partial(u_i u_k)/\partial x_k$ and $\partial^2(u_k u_\ell)/\partial x_k \, \partial x_\ell$. The numerical solution, with initial and periodic boundary conditions given by Eqs. (13) to (15), then proceeds as in the nonlinear case.

We can obtain an analytical solution for unbounded linearized fluctuations by using unbounded three-dimensional Fourier transforms as in Ref. 10. The solution does not satisfy constant periodic boundary conditions. Instead of working with the averaged equations used in Ref. 10, it is instructive to work with the unaveraged ones, and use the initial condition given by Eq. (13), instead of those in Ref. 10. In this case the Fourier transforms must be generalized functions (a series of delta functions), but the method of solution is the same as that in Ref. 10. Since u_2 is the velocity component which is critical in maintaining the fluctuations against the dissipation,[11,12] we will, for brevity, calculate only that component. Equation (11) for u_2 and Eq. (12), when linearized, are independent of u_1 and u_3. The solution obtained by using the initial condition (13) is

$$u_2 = \sum_{n=1}^{3} U_2^n \cos\left(\vec{q}^{\,n} \cdot \vec{x} - a q_1^n t x_2\right) \tag{20}$$

$$p = \sum_{n=1}^{3} p^n \sin\left(\vec{q}^{\,n} \cdot \vec{x} - a q_1^n t x_2\right) \tag{21}$$

where

$$U_2^n = \frac{a_2^n q^{n^2}}{q^{n^2} - 2aq_1^n q_2^n t + a^2 q_1^{n^2} t^2} \exp\left[-t\left(q^{n^2} - a q_1^n q_2^n t + \frac{1}{3} a^2 q_1^{n^2} t^2\right)\right], \tag{22}$$

$$p^n = \frac{-2a a_2^n q_1^n q^{n^2}}{\left(q^{n^2} - 2aq_1^n q_2^n t + a^2 q_1^{n^2} t^2\right)^2} \exp\left[-t\left(q^{n^2} - a q_1^n q_2^n t + \frac{1}{3} a^2 q_1^{n^2} t^2\right)\right], \tag{23}$$

$a = dU_1/dx_2$, $q^{n^2} = q_1^{n^2} + q_2^{n^2} + q_3^{n^2}$, and the a_i^n and q_i^n are as given in the initial conditions (Eqs. (13) and (6)). Mean values are obtained by integrating over all space. For instance,

$$\overline{pu_2} = \sum_{n=1}^{3} \frac{1}{2} p^n U_2^n. \tag{24}$$

It is clear from the form of Eqs. (20) and (21) that the solution does not satisfy constant periodic boundary conditions. By omitting the term $-(dU_1/dx_2)x_2\partial u_i/\partial x_1$ as well as the nonlinear terms in Eqs. (11) and (12), we can, however, obtain a simple analytical solution which satisfies those conditions:

$$u_2 = \sum_{n=1}^{3} a_2^n \exp\left[t\left(2a\,\frac{q_1^n q_2^n}{q^{n^2}} - q^{n^2}\right)\right] \cos \vec{q}^{\,n}\cdot\vec{x}, \tag{25}$$

$$p = -\sum_{n=1}^{3} \frac{2aq_1^n a_2^n}{q^{n^2}} \exp\left[t\left(2a\,\frac{q_1^n q_2^n}{q^{n^2}} - q^{n^2}\right)\right] \sin \vec{q}^{\,n}\cdot\vec{x}. \tag{26}$$

This solution is useful for checking the numerical calculations and for studying the effect of the term $(dU_1/dx_2)x_2\partial u_2/\partial x_1$ on the fluctuations.

For discussing the linearized case for constant periodic boundary conditions, it is convenient to convert Eqs. (11) and (12) to a spectral form by taking their three-dimensional Fourier transforms. This gives, for u_2, on neglecting nonlinear terms,

$$\frac{\partial\varphi_2^n}{\partial t} = aq_1^n \sum_{\kappa_2'} \frac{1}{\kappa_2'}\,\varphi_2^n(\kappa_1,\kappa_2 - \kappa_2',\kappa_3) - \left(q_1^{n^2} + \kappa_2^2 + q_3^{n^2}\right)\varphi_2^n + \frac{2aq_1^n\kappa_2\varphi_2^n}{q_1^{n^2} + \kappa_2^2 + q_3^{n^2}}, \tag{27}$$

where

$$\varphi_2^n(\vec{\kappa}) = \frac{1}{8\pi^3}\int_{-\pi}^{\pi} dx_2 \iint_{-\infty}^{\infty} u_2^n(\vec{x})e^{-i\vec{\kappa}\cdot\vec{x}}\,dx_1\,dx_3 \tag{28}$$

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or

$$u^n(\vec{x}) = \sum_{k_2=-\infty}^{\infty} \iint_{-\infty}^{\infty} \varphi_2^n(\vec{\kappa}) e^{i\vec{\kappa}\cdot\vec{x}} \, d\kappa_1 \, d\kappa_3, \tag{28a}$$

$$u_2 = \sum_{n=-3}^{3} u_2^n, \quad \varphi_2 = \sum_{n=-3}^{3} \varphi_2^n, \tag{29}$$

$\vec{\kappa}$ is the wavenumber vector, and φ_2 is the Fourier transform of u_2. Note that a finite transform is used in the x_2-direction in order to satisfy periodic boundary conditions at $x_2 = -\pi, \pi$. The initial condition on φ_2^n is given by

$$\left(\varphi_2^n\right)_0 = \frac{1}{2} a_2^n \, \delta_{\kappa_2 q_2^n} \, \delta\left(\kappa_1 - q_1^n\right) \delta\left(\kappa_3 - q_3^n\right) \tag{30}$$

where $\delta(x)$ is the Dirac delta function $\left(\int_{-\infty}^{\infty} \delta\left(\kappa_1 - q_1^n\right) f(\kappa_1) d\kappa_1 = f\left(q_1^n\right)\right)$ and δ_{qr} is again the Kronecker delta $\left(\delta_{\kappa_2 q_2^n} f(\kappa_2) = f\left(q_2^n\right)\right)$.

RESULTS AND DISCUSSION

As mentioned earlier, the u_2-component of the velocity fluctuation (in the direction of the mean velocity gradient) is crucial in maintaining the turbulence against the dissipation.[11,12] Therefore when, for brevity, only one component of the velocity fluctuation is discussed, that component is chosen as u_2. More will be said about the maintenance of the turbulence later.

Figure 1 shows the evolution of $u_2/\overline{u_0^2}^{1/2}$ at a fixed point in space for a high Reynolds number, as calculated from the full nonlinear equations. In spite of the nonrandom initial conditions (Eqs. (13) and (6)), the velocity fluctuations have the appearance of those for a random turbulence. The dashed curves for $u_2/\overline{u_0^2}^{1/2}$ are for initial conditions perturbed approximately 0.1 percent. The perturbed curves at first follow the unperturbed ones but eventually depart sharply. Although the appearance of the curves in Figs. 1(a) and in 1(b) differs considerably, the perturbed curves in the two figures take about the same length of time to break away from the unperturbed ones. A very small perturbation of initial conditions causes a large change in the values of u_2 except at small times. On the other hand the root-mean-square values of the velocities change smoothly with time and are unaffected by the perturbation of the inital conditions. These features are characteristic of turbulence. (Although the root-mean-square curve in Fig. 1(a) appears hori-zontal, it eventually goes smoothly to zero when extended.)

A striking feature of the curves for $u_2/\overline{u_0^2}^{1/2}$ in Fig. 1 is the small scale structure exhibited for sheared turbulence (Fig. 1(b)) when compared with the structure for no shear (Fig. 1(a)). This shear-related small-scale structure is produced by the term $-(dU_1/dx_2)x_2\,\partial u_i/\partial x_1$ in Eq. (11), from which is obtained the term $-(dU_1/dx_2)r_2\partial\overline{u_i u_j'}/\partial r_1$ in the correlation Eq. (16). If we take the Fourier transform of that term we get the mean-gradient transfer term in the spectral equation corresponding to Eq. (16).[10] Its effect in transferring energy to small-scale components is similar to that of the nonlinear transfer term (the Fourier transform of the triple-correlation term in Eq. (16)). The production of the small-scale structure by the shear might be thought of as due to a stretching of the random vortex lines in the turbulence by the mean gradient.

Although we first discussed a mean-gradient transfer term in 1961,[10] the present results give the first graphic demonstration of the effectiveness of that term in producing a small-scale structure in turbulence. Since that is a linear effect, we can study it either by the full nonlinear solutions already considered in Fig. 1 (which contain linear as well as nonlinear effects), or by linearized solutions. Velocity fluctuations obtained by the latter are plotted in Fig. 2 (Eq. (20)). The presence of small structure in the curves for $dU_1/dx_2 = 4434$, and its absence in those for $dU_1/dx_2 = 0$ are apparent. The curve for no shear decays monotonically to zero when extended. This is in contrast to the nonlinear case in Fig. 1(a) for no shear, where at least larger fluctuations are present. The linearized curves for $dU_1/dx_2 = 4434$ in Fig. 2 follow closely the nonlinear ones in Fig. 1(b) for small times. Likewise the linearized curves in Fig. 2 for periodic boundary conditions follow closely those for unbounded conditions for small times. For larger times the fluctuations for unbounded conditions continue to decay, whereas those for constant periodic boundary conditions grow. The development of structure in the curves for unbounded conditions is produced by the term $aq_i^n tx_2$ in the argument of the cosine in Eq. (20) ($a = dU_1/dx_2$). This term arises from the term $-ax_2\ \partial u_2/\partial x_1$ in Eq. (11), as is evident from its absence in Eq. (25), where the term $-ax_2\ \partial u_2/\partial x_1$ has been neglected. For constant periodic boundary conditions, small-scale structure in the fluctuations or the transfer of energy between wave numbers is produced by the term containing the summation over κ_2' in Eq. (27). That term is the Fourier transform of $-ax_2\ \partial u_2/\partial x_1$ in Eq. (11). From its form we see that it can produce a complicated inter-wavenumber interaction. The quantity φ_2^n at each κ_2 interacts with φ_2^n at every other allowable κ_2. Evidently a difference between the solutions for unbounded conditions and those for constant periodic conditions is that only fluctuations at integral values of κ_2 are possible when periodic conditions are imposed, whereas for unbounded conditions, fluctuations are possible at all values of κ_2.

Although the linear term $-ax_2\ \partial u_2/\partial x_1$ is effective in producing oscillations, even in the absence of nonlinear effects (Fig. 2), the curves lack the random appearance of those in Fig. 1(b). Evidently the only way we can have a linear turbulent solution is to put the turbulence in the initial conditions, as in Ref. 10. Both $-ax_2\ \partial u_2/\partial x_1$ and the nonlinear terms in Eq. (11) are necessary to produce the small-scale turbulence in Fig. 1(b) from nonrandom initial conditions. The former acts like a chopper which chops the flow into small-scale components. While the latter also do that, their most visible effect here is to produce randomization. As in Ref. 1, the randomization might occur as a result of the presence of strange attractors in the flow, by proliferation of eddies or harmonic components (with the loss of identity of the individual eddies), or by both (see Ref. 1 for a discussion of these possibilities).

According to Eq. (20), the manufacture of small-scale fluctuations takes place only in the x_2- direction. Figure 3 shows how this has taken place at a moderate time for a linearized case. A similar plot for the nonlinear case is shown in Fig. 4. The randomizing effect of the nonlinear terms is evident.

Figure 5 shows $\overline{u_2^2}^{1/2}/u_0$ for the nonlinear case, plotted against x_1, rather than against x_2 as in Fig. 4. The curves show some development of small-scale structure in the x_1- direction due to the interaction of the directional components in the nonlinear case. For the linearized flows development of structure occurs only in the x_2- direction.

Cross-correlation coefficients $\overline{u_i u_j}/\overline{u_i^2}^{1/2}\ \overline{u_j^2}^{1/2}$ $(i \neq j)$ are plotted against dimensionless time for the nonlinear case in Fig. 6. Although $\overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2}$ at $t = 0$, the initial cross correlations are not zero but are all positive and equal. However because of the apparent randomization of the flow $\overline{u_2 u_3}$ and $\overline{u_1 u_3}$ approach zero as time increases. On the other hand the values of the turbulent shear stress $\overline{u_1 u_2}$ change from positive to negative and remain negative because of the dynamics of the imposed mean shear. The presence of the mean velocity gradient dU_1/dx_2 causes u_1 to be likely negative when u_2 is positive, so that $\overline{u_1 u_2}$, the correlation between the two, is negative. The waviness in the curves in Fig. 6, as well as that in some of the curves in later figures (e.g., Fig. 9), is probably caused by non-random structure in the flow, possibly that produced by the linear term $-(dU_1/dx_2)x_2\, \partial u_i/\partial x_1$ in Eq. (11) (Fig. 2).

The evolution of the mean-square components of the velocity fluctuations is plotted in Fig. 7. After an initial adjustment period all of the components increase with time, in agreement with experiment[13] and the numerical results in Ref. 3. The numerical results in Ref. 4, on the other hand, show $\overline{u_2^2}$ and $\overline{u_3^2}$ decreasing at all times. Those results are apparently related to the use of periodic boundary conditions on a sheared numerical grid rather than on a stationary one. Reference 3 also used a sheared grid but applied a correction (the grid was remeshed), so that the results are similar to the present ones in Fig. 7 for a stationary grid. Our $\overline{u_1^2}$ - component is the largest of the three, $\overline{u_2^2}$ is smallest, and $\overline{u_3^2}$ lies slightly above $\overline{u_2^2}$, in agreement with experiment[13] and previous numerical results.

The effect of discretization error on the numerical results for $\overline{u_2^2}$ is shown in Fig. 8. Curves are plotted for 16^3, 24^3, and 32^3 grid points, together with a fourth-order extrapolation to zero grid-point spacing (an infinite number of grid points). The differences between the results for 32^3 points and the fourth-order extrapolation are small, but increase somewhat at large times.

Figure 9 shows the evolution of velocity pressure-gradient correlations. (Parts of some of the curves are omitted to avoid confusion.) The velocity pressure-gradient terms in the one-point correlation equation (Eq. (19)), together with the production terms, are responsible for maintaining the turbulence against the dissipation (given by the last two terms in Eq. (19)). There are no production terms in the equations for $\partial\overline{u_2^2}/\partial t$ and $\partial\overline{u_3^2}/\partial t$ $\left(\delta_{i1}\ \overline{u_j u_2}\ \partial U_1/\partial x_2 \text{ and } \delta_{j1}\ \overline{u_i u_2}\ \partial U_1/\partial x_2 \text{ are zero}\right)$. Thus $\overline{u_2^2}$ and $\overline{u_3^2}$ generally receive energy only from the $\overline{u_1^2}$ - component, whose equation has a nonzero production term. Equation (19) shows that in order to do that, $\overline{u_j\ \partial p/\partial x_i}$ and $\overline{u_i\ \partial p/\partial x_j}$ must be negative for $i = j = 2, 3$ and positive for $i = j = 1$. Figure 9 shows that is actually the case for constant periodic boundary conditions except for an initial adjustment period, so that the turbulence is maintained (Fig. 7). The maintenance of the $\overline{u_2^2}$ - or u_2-component is particularly critical because if u_2 goes to zero, the Reynolds shear stress $\overline{u_1 u_2}$ in the production term of the $\overline{u_1^2}$ - equation (see Eq. 19)) will go to zero and there will be nothing to keep the turbulence going. All the components will then eventually decay. That is what happens in the linearized analysis for unbounded turbulence in Fig. 9 and Ref. 10.

A comparison between the nonlinear results for $\overline{u_2^2}$ and various linearized solutions is given in Fig. 10. The same initial conditions are used for all the cases (Eqs. (4) or (13) and (6)). For all of the results, except those for the unbounded linearized case (obtained by using unbounded Fourier transforms (Eq. (20)), the crucial $\overline{u_2^2}$ component eventually increases, so that the turbulence or fluctuations are maintained. In the unbounded linearized case $\overline{u_2^2}$ decreases at all times. That was expected, since the $\overline{u_2^2}$ - results for that case in Refs. 10 and 11 (for different initial conditions) decreased at all times. Somewhat unexpected are the linearized results for constant periodic boundary conditions, which show that the fluctuations are maintained for those cases. Whereas Fig 9. shows that in the unbounded case the velocity pressure-gradient correlations remove energy from the $\overline{u_2^2}$ - component and cause the fluctuations to decay as in Ref. 10, the imposition of constant periodic boundary conditions changes the sign of those correlations and brings energy into $\overline{u_2^2}$, so that the fluctuations are maintained. Equation (25), which satisfies periodic boundary conditions, shows that, at least when the term $-(dU_1/dx_2)x_2\ \partial u_i/\partial x_1$ in Eq. (11) is neglected, $\overline{u_2^2}$ increases at large times if $2aq_1^n q_2^n > q^{n^4}$ for at least one n.

Comparison of the linearized case for periodic boundary conditions in Fig. 10 with the corresponding nonlinear case shows that the nonlinear terms have a stabilizing influence. That is, the values of $\overline{u_2^2}$ increase more slowly for the nonlinear case. Moreover comparison of the curve for the linearized case with periodic boundary conditions and with the term $-(dU_1/dx_2)x_2\ \partial u_2/\partial x_1$ in Eq. (11) missing (Eq. (25)) with the corresponding curve for that term included shows that the presence of that term also has a stabilizing influence. Since neglect of that term is equivalent to neglecting[10] the mean-gradient-transfer term in the spectral equation for $\overline{u_2^2}$, we can consider the latter term as stabilizing. Thus both the nonlinear transfer term associated with triple correlations and the linear mean-gradient transfer term in the spectral equation for $\overline{u_2^2}$ are stabilizing. The reason is that both terms transfer energy to small eddies where it is dissipated more easily.

It is of interest that the one-point correlation equation for $\partial\overline{u_i u_j}/\partial t$ (Eq. (19)) contains neither a term associated with velocity-gradient transfer nor with nonlinear transfer. That is, both of those processes give zero direct contribution to the rate of change of $\overline{u_i u_j}$; they only change the distribution of energy among the various spectral components or eddy sizes. This spectral transfer, of course, still affect the way in which $\overline{u_i u_j}$ evolves (see Fig. 10). Even though Eq. (19) contains no transfer terms, the transfer of energy among the various spectral components of the velocity alters the terms that do appear in Eq. (19), so that $\partial\overline{u_i u_j}/\partial t$ is affected indirectly. That is not a small effect.

The modified linear solution given by Eqs. (25) and (26) (dash-dot-dot curve in Fig. 10) is the simplest solution in which the fluctuations can be maintained against the dissipation. In obtaining it the only mean-gradient term retained in the equations for u_2 (Eqs. (11) and (12), $i = 2$) is $-2(dU_1/dx_2)\partial u_2/\partial x_1$, a source term in the Poisson equation for the pressure. If that term is also neglected, u_2 decays and, as discussed earlier, all of the components of the fluctuations decay. Moreover, as shown in Fig. 10 and already discussed, the term $-(dU_1/dx_2)x_2\ \partial u_i/\partial x_1$ in Eq. (11) is stabilizing, so it is of no help in maintaining the fluctuations. Thus, at least in the linearized case, the presence of the source term $-2(dU_1/dx_2)\partial u_2/\partial x_1$ in the Poisson equation for the pressure is necessary for maintaining the fluctuations. That term should play a similar important role in the maintenance of nonlinear turbulence, although in that case it is hard to separate the linear effects from the nonlinear ones. In particular, the role of the nonlinear source term in the Poission equation for the pressure remains unclear, although it may have an effect similar to that of the linear source term.

Figures 11 and 12 show the approach to isotropy of nonlinear uniformly sheared turbulence when the shear is suddenly removed. Although the shear produces considerable anisotropy, the components $\overline{u_i^2}$ of the mean-square fluctuation approach equality upon removal of the shear and remain equal. The velocity pressure-gradient correlations in Eq. (19) are thus successful in transferring energy among the various directional-components in such a way that equality of the $\overline{u_i^2}$ is produced. We note that $\overline{u_2^2}$ continues to increase for a short time after the shear is removed, probably because it receives energy from both $\overline{u_1^2}$ and $\overline{u_3^2}$.

In addition to equality of the $\overline{u_i^2}$, zero cross-correlations $\overline{u_i u_j}$ are required for isotropy. Figure 12 shows that $\overline{u_1 u_2}$, which is nonzero when the turbulence is sheared, approaches zero when the shear is removed, and along with the other cross-correlations, remains close to zero. The destruction of $\overline{u_1 u_2}$, apparently by nonlinear randomization effects, occurs over a finite time period rather than instantaneously on removal of the shear.

Another expected effect of removal of the mean shear is that the small scale structure produced by the term $-(dU_1/dx_2)x_2\ \partial u_i/\partial x_1$ in Eq. (11) should die out. According to Fig. 13, that occurs almost immediately when dU_1/dx_2 goes to zero, evidently because of the large fluctuating shear stresses between the small-scale eddies.

CONCLUDING REMARKS

According to our results for both sheared and unsheared flow at higher Reynolds number, the nonlinear structure of the Navier-Stokes equations is such that an apparently random turbulence can develop from nonrandom initial conditions. The presence of a mean gradient, in addition to producing a non-zero turbulent shear stress, produces small-scale fluctuations in the flow.

These can be attributed to a mean-gradient transfer term in the spectral equation for the velocity fluctuations,[10] or to an equivalent term in Eq. (16) or (11). However the small-scale fluctuations produced by that term alone (linear solution) are essentially nonrandom. Evidently the only way we can have a linear turbulent solution is to put the turbulence in the initial conditions, as in Ref. 10. In order to produce the observed small-scale turbulence from nonrandom initial conditions, the presence of both the linear mean-gradient transfer term and the nonlinear terms in the equations is necessary. The former term, or its equivalent in Eq. (16) or (11), acts like a chopper which chops the flow into small-scale components, and the latter terms, while they also produce small-scale components, act most visibly here as randomizers.

In all of the cases calculated with constant periodic boundary conditions, including both linear and nonlinear flows, the velocity pressure-gradient correlations are successful in distributing the energy among the directional components (Fig. 9), so that the turbulence or the fluctuations are maintained. This is in spite of the presence of a production term in the equation for only one of the components. Both the linear mean-gradient transfer term and the nonlinear terms mentioned in the last paragraph have a stabilizing effect. That is, they cause the fluctuations to increase at a slower rate. It is shown that, at least for the linearized solution with constant periodic boundary conditions, a mean-gradient source term in the Posisson equation for the pressure is necessary for maintaining the fluctuations against the dissipation. That term should play a similar important role in the maintenance of nonlinear turbulence, although in that case it is hard to separate the linear effects from the nonlinear ones. In particular, the role of the nonlinear source term in the Poisson equation for the pressure remains unclear, although it may have an effect similar to that of the linear source term. For the linearized unbounded solution (obtained by using unbounded Fourier transforms) the fluctuations decay, as expected from the results of Ref. 10.

When the mean-velocity gradient is suddenly removed, the turbulent shear stress goes to zero in a finite time period, and the velocity pressure-gradient correlations cause the turbulence to attain the isotropic state. The intensities of the directional components become and remain equal. In addition, the small-scale structure produced by the mean-gradient transfer term quickly vanishes.

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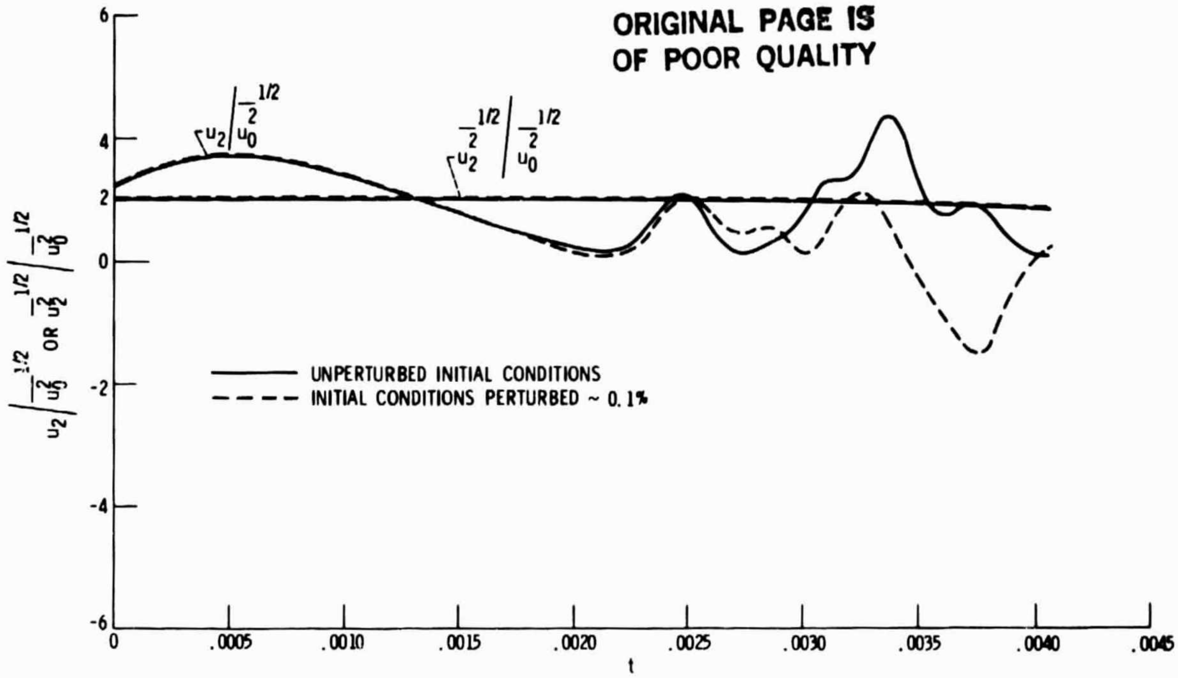
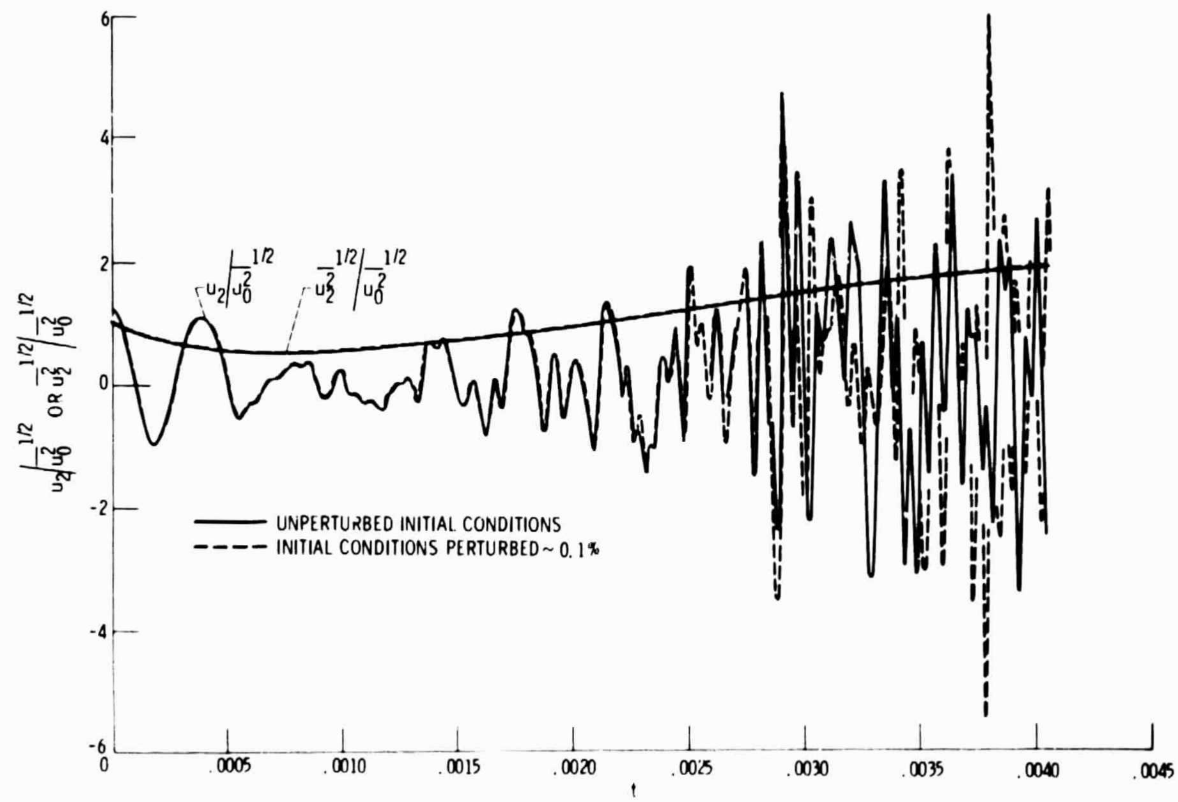


(a) $dU_1/dx_2 = 0$

Figure 1. - Effect of uniform shear on calculated evolution of nonlinear turbulent velocity fluctuations (normalized by initial condition) for a high Reynolds number $\left(\overline{u_0^2}^{1/2} x_0/\nu = 1108\right)$. Root-mean-square fluctuations are spacially averaged. Δx_i $\pi/16$. $x_1 = x_2 = 9\pi/8$, $x_3 = 3\pi/8$ for unaveraged fluctuations.



(b) $dU_1/dx_2 = 4434$.

Figure 1. Concluded.

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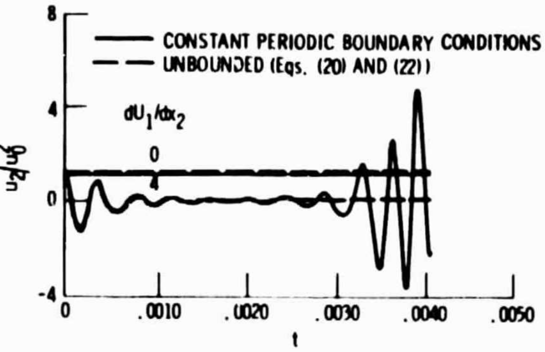

Figure 2. - Calculated evolution of linearized velocity fluctions (normalized by initial condition). $dU_1/dx_2 = 4434$, $\overline{u_0^2}^{1/2} x_0/\nu = 1108$, $x_1 = x_2 = 9\pi/8$, $x_3 = 3\pi/8$, $\Delta x_1 = \pi/16$.

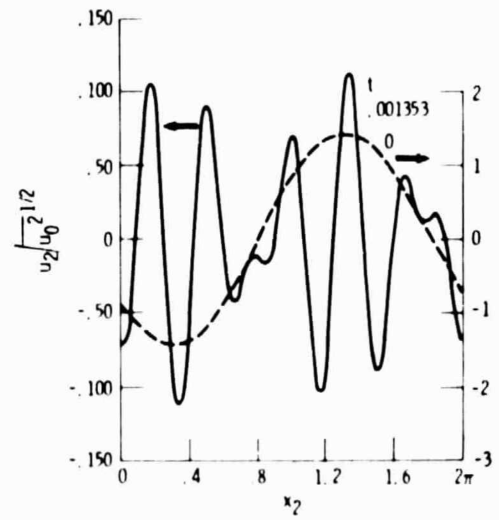

Figure 3. - Linearized solution for $u_2/\overline{u_0^2}$ vs x_2 for unbounded fluctuations (Eq. (20)). $x_1 = 9\pi/8$, $x_3 = 3\pi/8$, $dU_1/dx_2 = 4434$, $\overline{u_0^2}^{1/2} x_0/\nu = 1108$.

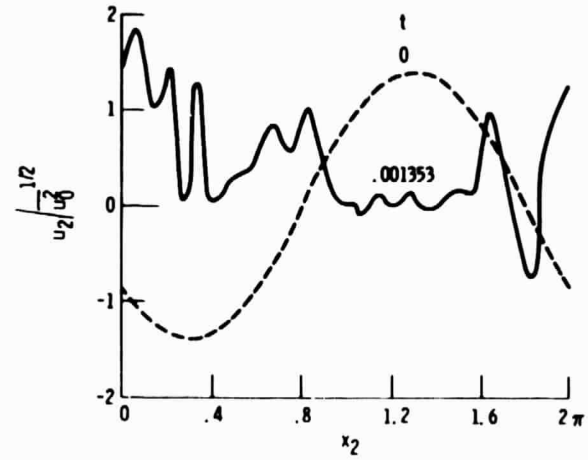

Figure 4. - Nonlinear solution for $u_2/\overline{u_0^2}^{1/2}$ vs x_2. $x_1 = 9\pi/8$, $x_3 = 3\pi/8$, $dU_1/dx_2 = 4434$. $\overline{u_0^2}^{1/2} x_0/\nu = 1108$. $\Delta x_i = \pi/16$.

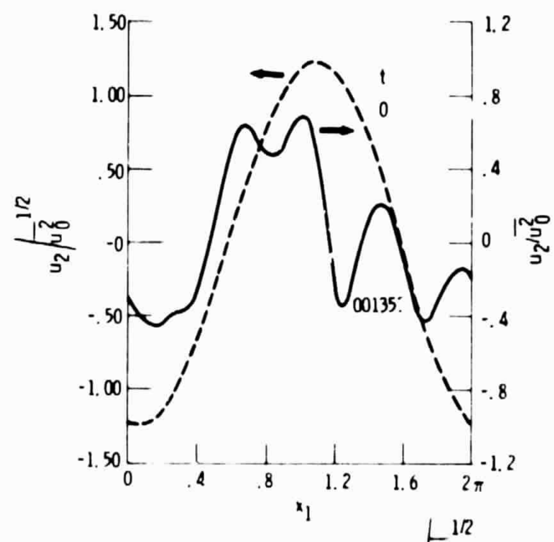

Figure 5. - Nonlinear solution for $u_2/\overline{u_0^2}^{1/2}$ vs x_1. $x_2 = 9\pi/8$, $x_3 = 3\pi/8$, $dU_1/dx_2 = 4434$. $\overline{u_0^2}^{1/2} x_0/\nu = 1106$. $\Delta x_i = \pi/16$.

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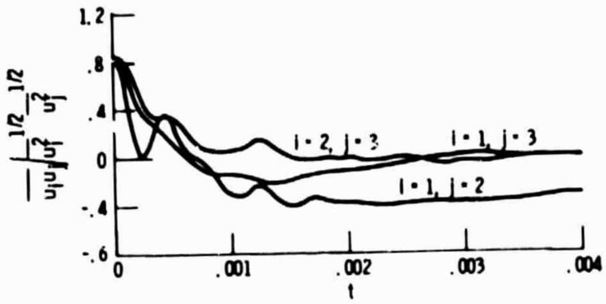

Figure 6. - Calculated cross-correlation coefficients plotted against dimensionless time. $dU_1/dx_2 = 4434$, $\overline{u_0^{*2}}^{1/2} x_0/\nu = 1108$, $\Delta x_i = \pi/16$.

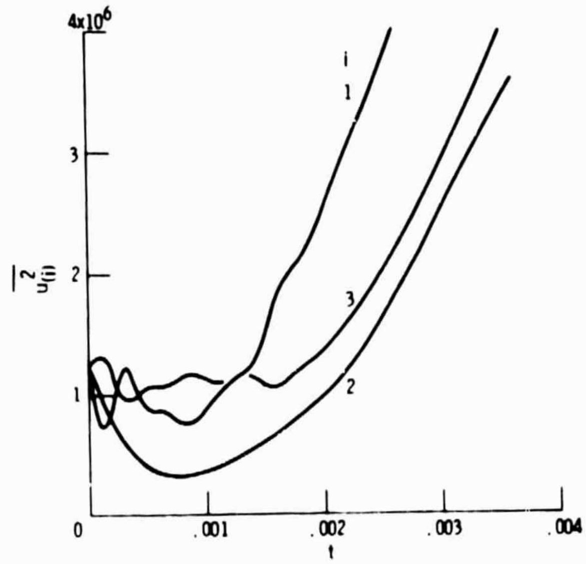

Figure 7. - Calculated evolution of mean-square velocity components. $dU_1/dx_2 = 4434$, $\overline{u_0^{*2}}^{1/2} x_0/\nu = 1108$, $\Delta x_i = \pi/16$.

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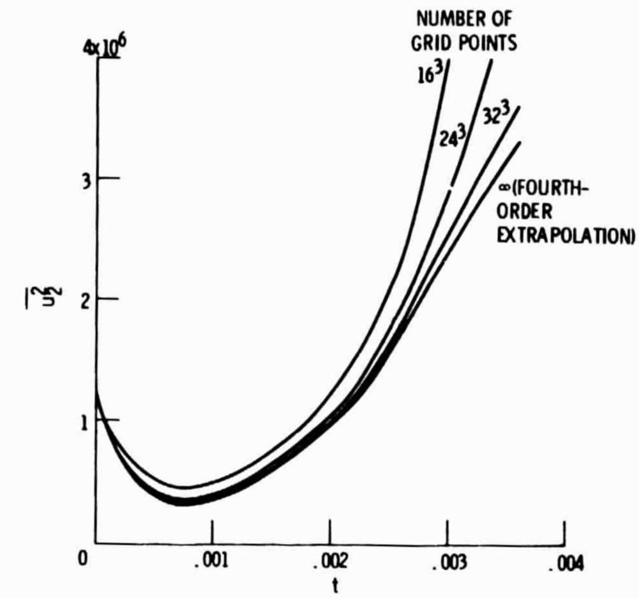

Figure 8. - Effect of numerical grid-point spacing on $\overline{u_2^2}$. $dU_1/dx_2 = 4434$. $\overline{u_0^{\circ 2}}^{1/2} x_0/\nu = 1108$.

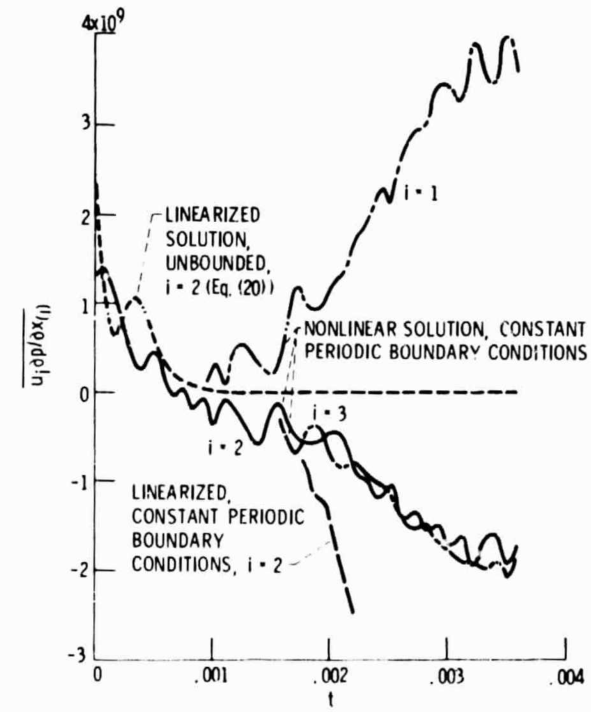

Figure 9. - Calculated evolution of velocity pressure-gradient correlations. $dU_1 dx_2 = 4434$. $\overline{u_0^{\circ 2}}^{1/2} x_0/\nu = 1108$. $\Delta x_i = \pi/16$.

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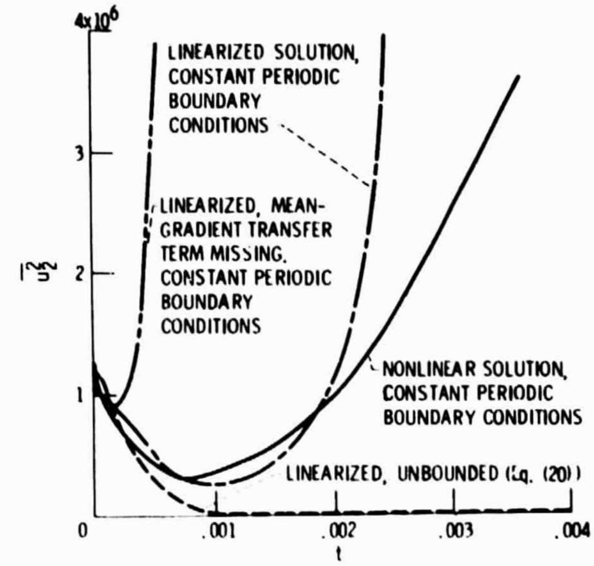

Figure 10. - Evolution of $\overline{u_2^2}$ for various linear and nonlinear solutions. $dU_1/dx_2 = 4434$, $\overline{u_0^{*2}}^{1/2} x_0/\nu = 1108$.

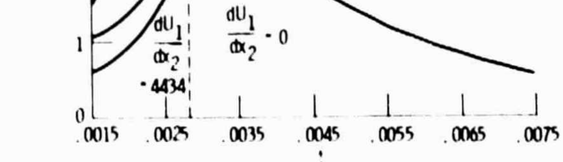

Figure 11. - Calculated approach to isotropy of uniformly-sheared turbulence upon sudden removal of the shear. $\overline{u_0^{*2}}^{1/2} x_0/\nu = 1108$.

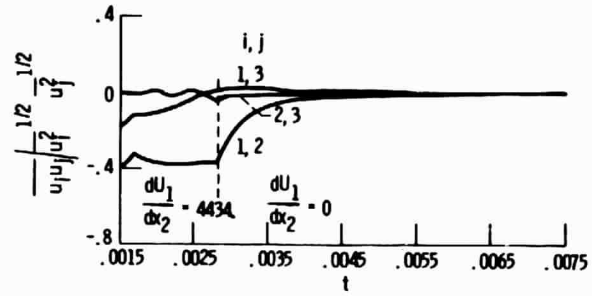

Figure 12. - Calculated evolution of cross-correlation coefficients upon sudden removal of uniform shear. $\overline{u_0^{*2}}^{1/2} x_0/\nu = 1108.$

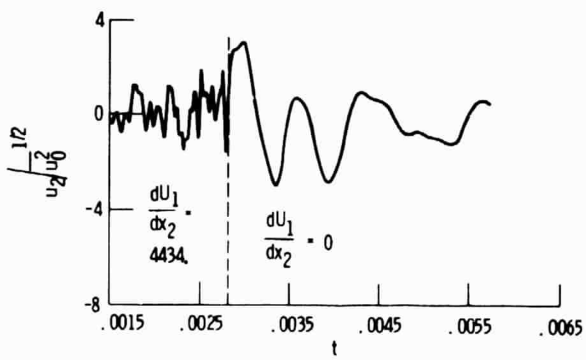

Figure 13. - Effect of removal of uniform shear on structure of turbulence. $\overline{u_0^{*2}}^{1/2} x_0/\nu = 1108.$