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THE FISSION THEORY OF BINARY STARS

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Abstract

About half the stars in the sky are binary stars. The fission theory, proposed by Poincàré in 1885, tries to explain the occurrence of binary stars by a natural process of evolution of a single star; in virtue of radiating energy, a rotating, axisymmetric liquid mass becomes highly flattened, becomes unstable, and continues its evolution along a series of progressively more elongated ellipsoids. When these become unstable, further evolution is supposed to take place along a series of "pear-shaped" figures, having a constriction in the middle. When this constriction has become deep enough, the figure consists effectively of a pair of detached masses orbiting one another.

Part of this picture lay on firm mathematical grounds; the rest depended on the outcome of a series of mathematical problems. By the 1920's, these problems had been worked out, their solutions were adverse to Poincàré's picture, and the fission theory became dormant. A recent reformulation of the theory, to bring it more nearly in line with newer astrophysical information, promises a different outcome to the analogous series of mathematical problems, and has awakened the fission theory.

INTRODUCTION

About half the stars in the sky are not individual stars at all, but binary stars, i.e., pairs of stars in orbit about a common center of mass. This figure (and the true fraction may be substantially greater than one-half; cf. [1]) is so large that it is not possible to regard the binary star as a freak occurrence without meaning for an understanding of the broad outlines of cosmogony. An explanation must be sought in the general framework of stellar origins and evolution.

Several suggestions as to the origin of binary and multiple systems have been made. Some of these, particularly those requiring two or more nearby points at which the interstellar medium begins the process of star formation, present mathematical and formulational difficulties so great that it has not as yet proved possible to analyze them even approximately. They must be regarded as speculative at present. Two others have proved more or less tractable mathematically: these are the capture theory and the fission theory. It is widely agreed (1) that the capture theory is not capable of producing binary systems in anything like the large numbers observed.

The subject of this article is the fission theory. Briefly expressed, this theory provides a mechanism whereby a single, rotating star evolves into a pair of stars orbiting one another. The remaining sections are devoted to describing the mechanism in detail, explaining some of the criticisms that have been made against the fission theory during its long history, and how these criticisms are affected by recent theoretical developments. The last section is an assessment of the present status of the fission theory.

ELLIPSOIDAL FLUID MASSES

The model-context in which the fission theory is discussed is that of the Maclaurin spheroids and the Jacobi ellipsoids. We briefly describe these and certain generalizations of them here, referring to (2) for a fuller description and derivations.

An ellipsoidal fluid mass of uniform density ρ, rotating with a uniform angular velocity Ω, having semiaxes $a_1 \geq a_2 \geq a_3$, and subjected only to the force of its own gravitation, is a figure of relative equilibrum if

$$a_2 = a_1 \quad \text{and} \quad \Omega^2 = 2(A_1 - \frac{a_3^2}{a_1^2}A_3) \tag{1}$$

or if

$$a_1^2a_2^2A_{12} = a_3^2A_3 \quad \text{and} \quad \Omega^2 = 2B_{12}, \tag{2}$$

where

$$A_i = \pi G\rho a_1a_2a_3 \int_0^\infty \frac{du}{(a_i^2+u)\{(a_1^2+u)(a_2^2+u)(a_3^2+u)\}^{1/2}} \qquad (i=1,2,3,)$$

and

$$A_{12} = \frac{A_1 - A_2}{a_2^2-a_1^2}, \quad B_{12} = A_2 - a_1^2A_{12}.$$

Here G is the universal constant of gravitation.

Equation (1) represents a sequence of spheroids, the Maclaurin spheroids. There is one such equilibrum figure for each value of a_3/a_1 between zero and one (or for each value of the eccentricity e of meridian sections in the same interval). Equation (2) represents a sequence of ellipsoids with unequal axes, the so-called Jacobi ellipsoids. It can be shown that for each value of a_2/a_1 between 0 and 1, there is a unique value of a_3/a_1 satisfying the first of equations (2). This gives the curve in the $(a_2/a_1)(a_3/a_1)$-plane marked $\Lambda = 0$ in Figure 1.

At the point along the Jacobi sequence where $a_2/a_1 = 1$, $a_3/a_1 = 0.5827$ (or $e = 0.8127$) and the two series have a member in common. This is the "point of bifurcation" to which reference will be made later.

These figures of relative equilibruim were shown by Riemann (3) to be special solutions of a much more general system of equations. Riemann considered the problem of finding the most general motions of a self-gravitating fluid of uniform density compatible with the assumption that the free surface remains an ellipsoid. This leads to motions of uniform vorticity relative to the rotating reference frame in which the ellipsoidal surface is at rest. Moreover, there is no requirement that the figures be in a steady-state: a system of ordinary differential equations determining the semiaxes and the parameters of the motion is obtained. These equations can take the form

$$
\begin{aligned}
\ddot{a}_1 &= -2a_1A_1 + \frac{2p_c}{\rho a_1} + \frac{2K_1{}^2}{(a_1-a_2)^3} + \frac{2K_2{}^2}{(a_1+a_2)^3},\\
\ddot{a}_2 &= -2a_2A_2 + \frac{2p_c}{\rho a_2} + \frac{2K_1{}^2}{(a_2-a_1)^3} + \frac{2K_2{}^2}{(a_1+a_2)^3},\\
\ddot{a}_3 &= 2a_3A_3 + \frac{2p_c}{\rho a_3},
\end{aligned}
\qquad (3)
$$

with K_1 and K_2 constants depending on initial conditions. To these equations must be adjoined the equation of mass conservation $a_1a_2a_3\rho$ = constant, and a constitutive equation relating the pressure at center p_c to other variables; if the fluid is incompressible, this relation is ρ=constant and can be used to eliminate the central pressure from equations (3). We observe, however, that in Riemann's more general formulation, the fluid need not be incompressible, i.e., ρ may be a function of time.

The system (5) allows equilibrium solutions, obtained by setting the time-derivative terms equal to zero. When mapped out in Figure 1, this family of equilibrium figures is found to occupy the horn-shaped region bounded above and below by the curves marked $K_2 = 0$ and $K_1 = 0$, respectively, and on the right by the segment of the Maclaurin series between $e = 0$ and $e = 0.9529$.

RESUMÉ OF THE FISSION THEORY

Although the earliest ideas of the fission hypothesis are attributable to Lord Kelvin (4) the theory only emerged in a complete form with the appearance of a remarkable memoir by Poincaré in 1885 (5). One of the outcomes of Poincaré's work was a description of how a single self-gravitating mass (a star or planet) might become a double system (a binary star or a planet-sattelite system). The mechanism, explained in the model-context of the Maclaurin spheroids and Jacobi ellipsoids, operates as follows.

Imagine a Maclaurin spheroid with an eccentricity e less than 0.8127. Suppose it contracts in virtue of radiating energy away, but so slowly as not to disburb the relative equilibrium. As it contracts, it becomes more flattened in virtue of angular momentum conservation. Ultimately it reaches the point of bifurcation where $e = 0.8127$. Beyond this point, the Maclaurin spheroid is known to be "secularly unstable," i.e., unstable if viscosity is present. Supposing the latter so, further evolution cannot proceed along the Maclaurin sequence, but must proceed along the Jacobi sequence which is known to be secularly stable. Next a further point of bifurcation, where a new, so-called pear-shaped series, branches off the Jacobi series, is attained. Assuming the Jacobi series secularly unstable past the new point of bifurcation and the pear-shaped series secularly stable, further evolution must proceed along the latter series.

In Poincaré"s original memoir, a sketch of the pear-shaped figure was given showing it to have the shape suggested by its name. The two ends of the figure are rather thicker than the central portion, which appears constricted. Poincaré suggested that this constriction narrows as evolution along the pear-shaped sequence continues. When it has narrowed to the extent that the figure consists essentially of a pair of detached masses connected by a narrow neck, the system is a binary system.

This picture that Poincaré painted rested on a basis of solid mathematical reasoning up to and including the computations implying the existence of the pear-shaped sequence that branches off the Jacobi sequence. Two important elements in his description, however, were conjectures as to the outcomes of certain complicated problems left open to subsequent research. One of these conjectures was that the pear-shaped series is secularly stable. This conjecture was taken up by Darwin (6), Liapounov (7), and Jeans (8). Although Darwin initially concluded stability, the others instability, Jeans further detected a minor error in Darwin's computations which, when corrected, also led to the conclusion of instability.

The other principal conjecture was that the instability along the Jacobi sequence was a secular and not a dynamical instability, implying that the ensuing motions take place on the viscous timescale T_v rather than on the much shorter dynamical timescale. This requires further explanation.

The instability along the Maclaurin sequence that sets in at $e = 0.8127$ (at the point of bifurcation where the Jacobi sequence branches off) is a secular instability only; i.e., if viscosity is absent, the Maclaurin sequence is stable down to $e = 0.9529$ (where the curve $K_1 = 0$ intersects the Maclaurin line $a_2/a_1 = 1$; cf. Fig. 1). If viscosity is present, the Maclaurin spheroids are unstable for $e > 0.8127$, and the e-folding time is the viscous diffusion time T_v. Poincaré conjectured that the same situation prevailed along the Jacobi sequence. The reason, or rather, the hope, behind this conjecture was the conviction that, if the instability were dynamical, rapid motions, on the dynamical timescale would ensue, and it would not be possible to infer the subsequent behavior on the basis of equations of equilibrium: the full, dynamical equations would then have to be used. But this conjecture that the instability is secular and not dynamic is also wrong, as Cartan showed in 1924 (9).

CRITICISMS OF THE THEORY

The adverse outcomes to the two problems left open by Poincaré appeared to destroy the theoretical foundations of the fission theory, because the fluid mass no longer has any stable state toward which it can evolve, and its behavior must indeed be dynamical. While the result of this dynamical behavior may yet be a binary system ([10], [11]), the arguments used to infer this are of a highly speculative character. Moreover, there are further criticisms of the theory. It may be useful to list the principal criticisms:

1. The Jacobi sequence is dynamically unstable at the point where the pear-shaped sequence bifurcates.

2. The pear-shaped sequence is unstable.

3. The theory refers to incompressible masses, whereas stars are gaseous.

4. The theory relies on the presence of viscosity, implicitly assuming that the viscous timescale T_v is short compared to the contraction timescale T_c, whereas the opposite is true (12).

This list is by no means complete, but is perhaps sufficient to make one wonder why there remains any interest in the fission theory. One reason, no doubt, is the vague feeling that the adverse conclusions are in some measure due to the unrealistic character of the model, and that the qualitative picture may yet be right. Recent developments support this feeling.

RECENT DEVELOPMENTS

Many of the criticisms of the fission theory can be answered. In this section we concentrate on the four criticisms listed in the preceding section (for a fuller discussion, see ref. [13]).

The first criticism in that list is a criticism only because of the conviction that the occurrence of a dynamical instability requires solving the full, dynamical equations to follow up its consequences. Now, this need not be the case where two timescales are involved (13). Recent work on similar, but mathematically simpler, problems of this kind shows that motion may always take place on the slow timescale, except for a very short time interval during which the evolutionary path shifts from one stable branch of equilibrium solutions to another (14), (15).

Turning to the second criticism in that list, the instability of the pear-shaped sequence, we observe that the question of stability or instability is very sensitive to the change in energy on going from the ellipsoidal to the pear-shaped figure. In a star, an important contribution to the total energy is made by the internal energy. This contribution is suppressed by the assumption (criticism 3) of incompressibility. Hence criticisms 2. and 3. may be closely related, and reformulation of the problem that answers criticism 3. may well yield a conclusion of stability rather than instability for the pear-shaped sequence.

A form of the theory free of the fourth criticism has recently been given (13), (16). It also answers criticism 3. to the extent of allowing for internal energy, as well as gravitational and kinetic energy. It is formulated in the context of the Riemann ellipsoids with ρ a function of time. Instead of evolving along the Maclaurin-Jacobi sequence as in the classical theory, the fluid mass evolves along the Maclaurin series to the point marked O in Figure 1, and thereafter along the series marked $K_1 = 0$. Instability (analogous to that of Jacobi sequence) sets in at the point marked L_1. Hence the evolution is qualitatively similar to that of the classical theory, at least to the point where the ellipsoidal sequence encounters a point of bifurcation. The dashed line in Figure 1 represents a sample trajectory that starts out almost, but not quite, axisymmetric.

PRESENT STATUS

Insofar as the four criticisms explicitly dealt with are concerned, it would appear that none of them need apply in the reformulated version of the theory referred to in the preceding section, although the question of the stability of the analogue of the pear-shaped sequence has yet to be settled.

Other criticisms can be made, and it may not be possible to answer them all to the critic's satisfaction, so the question whether the fission theory is or is not a viable explanation for the occurrence of binary stars may never have a universally accepted answer. We can, however, say the following: whereas it appeared some years ago that the fission theory may have been incompatible with the laws of dynamics, this no longer appears to be the case.

Further research along the lines of working out, and testing the stability of, the analogue of the pear-shaped sequence should do much to clarify the situation.

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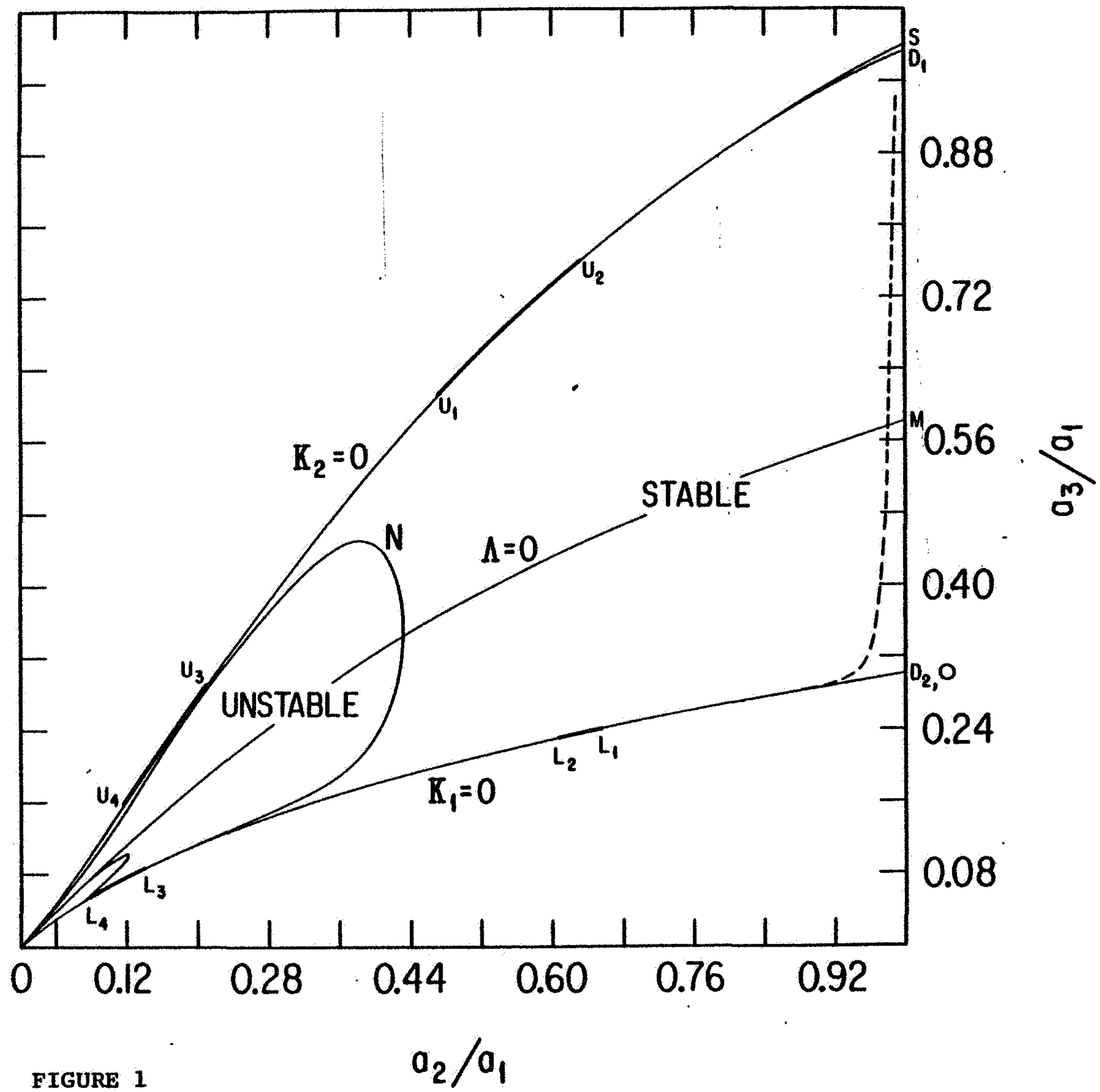

S
D1
0.88
0.72
0.56
0.40
0.24
0.08
M
D2, O
U2
U1
K2 = 0
N
Λ = 0
STABLE
U3
UNSTABLE
U4
L2
L1
K1 = 0
L3
L4
0
0.12
0.28
0.44
0.60
0.76
0.92
a2/a1
a3/a1

FIGURE 1