D2

ROTATING STARS

D.O. Gough

Institute of Astronomy & Department of Applied Mathematics and Theoretical Physics, University of Cambridge, & National Center for Atmospheric Research, Boulder, Colorado

ABSTRACT

A normal star is a mass of compressible gas of low Prandtl number, held together by its own gravitational force. It generates energy principally in its central regions, either by very slow gravitational contraction or, more commonly, by thermonuclear reactions; this energy is subsequently transported to the surface down a temperature gradient by radiative diffusion or convection.

If the star has no angular momentum the equations governing its motion admit spherically symmetric stationary solutions. But if the star is rotating it is not possible in general for it to exist in a state with no motion other than its rotation; meridional currents are set up which advect angular momentum through the star. The details of this process are not well understood. Stability calculations seem to imply that no stable steady solutions exist.

The distribution of angular momentum in a star depends on a balance between transport by the large scale meridional motions and by the Reynolds stresses of turbulence. Neither theory nor observation is yet sufficiently refined to deduce what that distribution is even in the sun.

INTRODUCTION

The structure and evolution of the majority of stars for most of their life seem to be well represented by theoretical models that ignore the fact that they rotate. This is because the ratio of the centrifugal to gravitational forces, or rotational Froude number, is low, and for many purposes the dynamical effects of rotation produce only a small perturbation from a nonrotating configuration. Nevertheless rotation presents some very interesting fluid dynamical problems which are the subject of this review.

The astrophysical consequences of rotation in stars are not without interest. Rotation affects the spectra of the emitted radiation from which conditions in stellar atmospheres are deduced, and the possibility of rotationally induced mixing, which is discussed below, has important implications concerning the nuclear transmutations that occur in stellar cores. In addition, there are stars which appear to be rotating so rapidly that the dynamical effects of the centrifugal force have a significant influence on their evolution. These matters are not discussed here; instead the reader is referred to the reviews by Strittmatter [1] and Fricke and Kippenham [2], and the IAU colloquium proceedings edited by Slettebak [3].

Stars condense from the interstellar medium and at some point attain a state of hydrostatic equilibrium. They are opaque and hot in the centre, the stored heat maintaining a pressure gradient which balances gravity. Their structure is such that the balance

$$2\,\mathcal{T} + \mathcal{V} = 0 \tag{1.1}$$

is satisfied, where $\mathcal{V}$ is the gravitational potential energy of the star and $\mathcal{T}$ the total kinetic energy, both macroscopic and microscopic [4]. This is the virial theorem of Poincaré and Eddington, and assumes that no body force other than gravity is exerted on the star. Normal stellar material is gaseous and can be assumed to satisfy the usual equations of fluid dynamics. The perfect gas law

$$p = \frac{\mathcal{R}\rho T}{\mu} \tag{1.2}$$

is approximately satisfied, where p, ρ and T are pressure, density and temperature, $\mathcal{R}$ is the gas constant and μ is the mean molecular weight of the gas (mean mass per particle measured in units of the hydrogen atomic mass) which depends upon the chemical composition and the state of ionization. Radiation pressure is usually comparatively small and will be ignored in this discussion.

If all the kinetic energy of the stellar material is in microscopic particle motions (so there is no fluid motion) equation (1.1) may be written

$$3(\gamma - 1)\,\mathcal{U} + \mathcal{V} = 0, \tag{1.3}$$

where $\mathcal{U}$ is the total internal energy of the gas and γ an appropriate mean ratio of principal specific heats. The total energy of the star is

$$\mathcal{E} = \mathcal{U} + \mathcal{V} = -(3\gamma - 4)\,\mathcal{U}. \tag{1.4}$$

It can be shown that for the star to be dynamically stable γ must exceed 4/3. Thus as energy is lost by radiation from the stellar surface, $\mathcal{E}$ decreases and $\mathcal{U}$ increases. The star contracts slowly, liberating gravitational energy which is partly radiated away and partly stored in the compressed heated gas. This process, first discussed by Kelvin and Helmholtz, is controlled by the rate radiation can escape by diffusion from the star. It continues until the centre becomes so hot that nuclear reactions begin to take place. Contraction halts; the star has reached the main sequence, the total energy output, or luminosity, at the surface being supplied entirely by the thermonuclear reactions. Its gross structure now depends primarily on just its mass. On the main sequence stars convert hydrogen, their principal constituent, into helium. This is the longest phase of their evolution, and consequently most of the visible stars in the sky are main sequence stars.

Once hydrogen has been exhausted from the reacting core further gravitational contraction and additional nuclear reactions take place, the details depending on the star's mass and composition, and eventually the star contracts into its final state : a cooling white dwarf supported against gravity by a sea of degenerate electrons, a neutron star of degenerate nuclear matter, or perhaps a black hole.

This review does not attempt to discuss the properties of a star in all stages of its evolution, but concentrates on the main sequence. Magnetic fields, which are undoubtedly important in some stars, are hardly considered; it seems prudent to try to understand first what is probably the simpler dynamics of a nonmagnetic star. Observational techniques are not discussed at all.

If one plots the mean surface rotation rates of main sequence stars as a function of their mass M the most striking feature is an abrupt rise, in the vicinity of 1¼ solar masses, from approximately zero to a value corresponding to a Froude number which is an appreciable fraction of unity [5]. Observations of the distribution of angular velocity across the surface have been made only for the sun, which is a typical slowly rotating star. The angular velocity Ω varies by about 20% from pole to equator, the rotational Froude number $R_{\odot}^3\Omega^2/GM_{\odot}$ being about 2×10^{-5} [6], where $M_{\odot}$ and $R_{\odot}$ are the solar mass and radius and G the gravitational constant. These are the principal main sequence observations a theory of stellar rotation must explain.*

Aside from the astrophysical and fluid dynamical aspects of the solar rotation interest has recently been aroused in connection with gravitation theory. One of the major puzzles of the last century was the discrepancy between the measured precession of the perihelion of the orbit of Mercury and the predictions of Newtonian gravitation theory [7]. One of the proposed explanations was that the sun was oblate, but the observed oblateness [8,9] turned out to be too small to account for the precession [10]. With the advent of General Relativity, however, it was generally agreed that the problem had been resolved, because Einstein's theory predicted the observed result [11]. But more recently Dicke [12] and Roxburgh [13] proposed that the apparent agreement was fortuitous, and that an alternative theory of gravitation [14] based on the scalar-tensor theory of Jordan [15], which predicts a lower precession rate, is correct, the residual precession of Mercury's orbit arising from an assumed oblateness of the sun's figure. Subsequently the shape of the sun's image was remeasured by Dicke and Goldenberg [16, 17]. An oblateness of about the required value was reported, if it is assumed that the oblateness of the image implies a similar oblateness in the shape of the gravitational equipotentials. That this assumption is justified has been argued by Dicke but has not been generally accepted [17,18]. This important issue will be discussed in the concluding section of this review.

---

*Perhaps the most important question of the subject is why stars exist at all, for had they condensed conserving angular momentum to their present size from interstellar material with initial rotation rates typical of the galactic rotation they would not be gravitationally bound. This is a problem in star formation.

In the next section possible steady solutions of the equations governing rotating stars are discussed. Their stability is considered in the subsequent section, and the final section is addressed to how angular momentum is distributed thoughout a star as it evolves. But before that a few remarks about the magnitudes of quantities pertaining to stars are perhaps not out of place.

The sun, which may be considered representative, has a mass $M_\odot = 2 \times 10^{33}$ g, a radius $R = 7 \times 10^{10}$ cm and a luminosity $L_\odot = 4 \times 10^{33}$ erg $\sec^{-1}$. At a distance of about 0.5 $R_\odot$ from the centre its density is about the same as its mean density which is approximately the density of water, and its temperature is about 3×10^6 ^{o}K; the central density and temperature are about 100 g cm^{-3} and 1.5×10^7 ^{o}K. At the visible surface, or photosphere, its density and temperature are about 3×10^{-7} g cm^{-3} and 5800 ^{o}K. The outer 20% by radius is in a state of turbulent convection and the rest is generally believed to be convectively stable. Qualitatively this is a property of all stars at the low mass end of the main sequence (lower main sequence); the higher mass rapidly rotating stars of the upper main sequence have only thin weak convective regions near the surface, and they also have convective central cores.

Energy is transported through the convectively stable regions of a star principally by radiative diffusion. Except near the surface, the energy flux $\underset{\sim}{F}$ satisfies

$$\underset{\sim}{F} = - K\nabla T , \qquad (1.5)$$

where $K = 4acT^3/3\chi\rho$ is the radiative conductivity; a is the radiation constant, χ the opacity of the gas and c the velocity of light. At a median point in the sun its value is about 10^{15} erg cm^{-1} sec^{-1} $^oK^{-1}$. The kinematic shear viscosity arising from photon momentum transport is $\nu_r = 4aT^4/15\,\chi\rho^2 c \simeq 1$ cm^2 sec^{-1} at a radius r of 0.5 $R_\odot$ and increases with r nearly everywhere. Viscosity due to microscopic particle ion motions ν_i is of order $m_p^{\frac{1}{2}}(kT)^{5/2}/(e^4\rho\ \ln A)$ where $A = (m_p/\rho)(kT/e^2)^3$, m_p and e are the proton mass and charge and k is the Boltzmann's constant; this is about 10 cm^2 sec^{-1} at $r = 0.5$ $R_\odot$ and increases slowly with r.

From these values can be calculated a timescale characteristic of large scale dynamical motions : the time for free fall under gravity from surface to centre. In view of the balance (1.1) this is of the same order as the sound travel time τ_p and is about 1 hr. The thermal diffusion time across a distance $R_\odot$ is equal to the Kelvin-Helmholtz gravitational contraction time $\tau_{KH} \simeq 10^7$ yr and is short compared with the nuclear time $\tau_n \simeq 10^{10}$ yr to convert say 10% of all the hydrogen to helium at the current luminosity. The age of the sun $\tau_\odot$ is 5×10^9 yr.

The Reynolds number associated with the smallest velocity that can be of global significance in this time, $R_\odot/\tau_\odot$, is $R_\odot^2/(\nu_r+\nu_i)\tau_\odot = 3000$. In other words, the characteristic viscous diffusion time is much larger than the age of the sun. For flow velocities comparable with the rotation speed (about 2 km sec^{-1} at the equator) the Reynolds number is of order 10^{16} throughout most of the star. Under most circumstances, therefore, viscous forces can be ignored. Only in the extreme outer regions and in boundary layers might this not be so. The Prandtl number $(\nu_r+\nu_i)\rho C_p/K$, where C_p is the specific heat at constant pressure, is about 10^{-6} at a median point.

STEADY ROTATING CONFIGURATIONS

As with most physical systems it is expedient to seek the steady configurations first. They are comparatively simple to analyse and, if stable, are states towards which one might expect stars to evolve.

In most of the published work steady axisymmetric models are constructed in which the only motion is a rotation about the z axis. The momentum equation then reduces

$$-\nabla p + \rho\nabla\Phi + \rho\Omega^2\underline{\varpi} = 0, \tag{2.1}$$

where $\underline{\varpi}$ is the cylindrical radius vector from the z axis, Ω is the angular velocity and Φ is the gravitational potential which satisfies

$$\nabla^2\Phi = -4\pi G\rho. \tag{2.2}$$

In equation (2.1) viscous forces have been ignored and it has been assumed that no magnetic field is present.

The simplest models to analyse are of white dwarfs. In these stars the electrons are degenerate and a barytropic equation of state is a good approximation. Because the fluid is assumed inviscid the angular velocity distribution is not determined completed by the equations of motion. However, it follows immediately from equation (2.1) that the centrifugal force is derivable from a potential V and that Ω varies only with distance ϖ from the axis of rotation. Accordingly equation (2.1) reduces to

$$\nabla p = \rho\nabla(\Phi + V) \equiv \rho\nabla\Psi, \tag{2.3}$$

from which it follows that p and ρ are constant on surfaces of Ψ, or level surfaces. There remains considerable freedom in the choice of $\Omega(\varpi)$. It can be set by specifying, for example, the angular momentum per unit mass h(m), where m is the mass enclosed in a cylinder ϖ = constant. Then any choice of h(m) leads to a solution. It was pointed out by Ostriker and Mark [19] that $\Omega(\varpi)$ cannot be specified arbitrarily.

An important consequence of rotation concerns the possible masses of white dwarfs. The equation of state of the degenerate white dwarf material is such that in the special case of no rotation there is a maximum mass M_c which can be supported against the self gravity of the star. This was first demonstrated by Chandrasekhar [20] who found

$M_c \approx 1.44\ M_\odot$. The estimate has been revised with the use of a more refined equation of state [e.g. 21], but the principle is unchanged and for a long time it was commonly believed that M_c was an upper limit to the masses of real white dwarfs. However, it was emphasised by Mestel [22], for example, that this limit applies only to nonrotating spherically symmetric configurations, and Ostriker, Bodenheimer and Lynden-Bell [23] suggested that rotating white dwarfs with masses much greater than M_c exist. Because gravitational forces scale with their characteristic length L as L^{-2} and centrifugal forces as L^{-3} for a star whose mass and angular momentum are conserved, a rotating star more massive than M_c would be prevented from collapsing indefinitely by the centrifugal forces. Indeed Ostriker et al. pointed out that were it not for the changes in physics that occur at extremely high densities a strict upper bound to the possible masses of rotating white dwarfs would be removed entirely, whatever the angular momentum.

More interesting from a fluid dynamical point of view are the nondegenerate main sequence stars for which temperature is present in the equation of state. In these stars energy is transported down the temperature gradient from centre to surface. Now configurations in which the only motion is rotation with an associated conservative centrifugal force field do not occur, and much of the wisdom gained from the simple barytropic models is not very useful. In general the energy transport equation does not permit T to be constant on level surfaces. Consequently, in view of equation (1.2), p and ρ cannot both be constant on level surfaces, which implies that the momentum equation (2.3) cannot be accurately satisfied. The unbalanced pressure gradients drive a circulation $\underset{\sim}{u}$ in meridional planes which advects both heat and angular momentum through the star [24-26].

Except in the surface layers of the star the motions induced are so slow that inertial forces are negligible compared with the other terms in the momentum equation so that equation (2.3) is very nearly satisfied. For a given rotation field the circulation velocity can thus be estimated from the energy equation

$$\rho C_p\, \underset{\sim}{u}.\nabla T - \delta\, \underset{\sim}{u}.\nabla p = \rho\varepsilon - \mathrm{div}\, \underset{\sim}{F} \qquad (2.4)$$

simply as that required to transport the requisite amount of heat to maintain p, ρ and T (very nearly) constant on level surfaces. In equation (2.4) $\delta = -\left(\frac{\partial \ln\rho}{\partial \ln T}\right)_p$ and ε is the thermonuclear energy generation rate per unit mass. Thus equation (2.5) may be rewritten approximately :

$$\frac{\rho T C_p}{p}(\nabla - \nabla_{ad})\, \underset{\sim}{u}.\nabla\Psi = \varepsilon - \frac{1}{\rho}\,\mathrm{div}\, \underset{\sim}{F}\,, \qquad (2.5)$$

where $\nabla = \frac{d\ln T}{d\ln p}$ and $\nabla_{ad} = \left(\frac{\partial \ln T}{\partial \ln p}\right)_s$, the derivative being taken at constant specific entropy s, and

$$\mathrm{div}\,\underset{\sim}{F} = -f'(\Psi)|\nabla\Psi|^2 + f(\Psi)\left[\frac{1}{\varpi}\frac{d}{d\varpi}(\varpi^2\Omega^2) - 4\pi G\rho\right], \tag{2.6}$$

where $f(\Psi) = K\frac{dT}{d\Psi}$ and a prime denotes differentiation with respect to the argument. It is immediately clear from (2.5) and (2.6) that $\underset{\sim}{u}$ cannot be zero if Ω is constant, for since all the terms in (2.6) except $\nabla\Psi$ are then constant on level surfaces it would follow that $f'(\Psi) = 0$ and $\varepsilon = 2(\Omega^2/\rho - 2\pi G)f$; in general this cannot be satisfied because ε depends on nuclear physics, and hence on ρ and T but not on Ω, and in particular is essentially zero in all but a central core of the star [27]. There is no general proof that $\underset{\sim}{u} = \underset{\sim}{0}$ when Ω is a function of ϖ alone, though it has been shown to be the case if $\frac{1}{\varpi}\frac{d}{d\varpi}(\varpi^2\Omega^2) - 4\pi G\rho$ vanishes in a region in which $\varepsilon = 0$ [28]. Detailed calculations of the meridional flow were first performed by Sweet [29] who assumed slow uniform rotation.

The magnitude of this meridional Eddington-Sweet circulation velocity u can be estimated from equations (2.5) and (2.6) to be [30]

$$u \sim \lambda \frac{L}{Mg}\left(\frac{1}{\nabla - \nabla_{ad}}\right)\frac{\bar{\rho}}{\rho}, \tag{2.7}$$

where M, L and $\bar{\rho}$ are the mass, luminosity and mean density of the star, g is gravitational acceleration and $\lambda = R\Omega^2/g$ is the Froude number. This estimate has been confirmed by detailed computations for certain specified nonuniform rotation laws with λ small [31] by expanding in powers of λ about the nonrotating configuration; the case of uniform rotation is singular but appears to yield similar velocities [29, 32].

So far it has been implicitly assumed that the stellar fluid is chemically homogeneous. Nuclear transmutations in the cores of main sequence stars generate gravitationally stable gradients in the mean molecular weight μ. According to Mestel [33] these prevent the Eddington-Sweet circulation from penetrating the core by deflecting them in a thin viscous boundary layer.

It has been of some concern that the Eddington-Sweet circulation velocity, in the present approximation, diverges at the surface of the star where the density is almost zero. This arises because the heat capacity of the gas per unit volume is too low to transport at low velocities the flux of heat required to satisfy equations (2.5) and (2.6). Smith [34] pointed out that excessive demands were being made by the diffusion approximation (1.5) near the stellar surface, where it isn't valid, and showed that the situation was alleviated somewhat by using a more careful treatment of radiative transfer for the outer layers [see also ref. 35]. The velocities were still so great near the surface that the assumptions of the procedure, such as the neglect of the inertia terms, are hardly plausible, and Smith concluded that in any case the flow would be liable to shear instabilities.

Smith's analysis, in common with earlier work, assumed an inexorable uniform rotation, a weak poloidal magnetic field having been invoked to preserve it. In the absence of such a field, advection of angular momentum by meridional circulation is inevitable, and a state of nonuniform rotation would rapidly be set up in the surface layers. Osaki [36] argued, without proof, that in that event readjustment of the angular velocity distribution would be such as to reduce the meridional circulation velocity in the stellar atmosphere to zero.

Another place where the circulation is unable to advect heat efficiently is at the edge of a convection zone where $\nabla = \nabla_{ad}$, this time because the slowly rising or falling fluid, which maintains pressure balance with its surroundings, cannot modify the temperature difference between it and its environment. Once again equation (2.5) demands an infinite velocity unless div $\underset{\sim}{F} = \rho\varepsilon$ when $\nabla = \nabla_{ad}$, which is not generally the case for an arbitrarily specified distribution of angular momentum. Osaki [37] pointed out once again that this implies that a steady solution does not exist for the assumed rotation law, and that appropriate readjustments must take place.

From (2.7) can be estimated the timescale for stellar material to traverse a distance comparable with the radius R of the star :

$$\tau_{ES} = \frac{R}{u} \simeq \lambda^{-1}\tau_{KH} \simeq 2 \times 10^{-3}\left(\frac{M}{M_\odot}\right)\left(\frac{R}{R_\odot}\right)^{-1} \lambda^{-1}\tau_n . \tag{2.8}$$

For the sun $\tau_{ES} \approx 100\ \tau_n$, if it assumed to rotate with about its surface angular velocity throughout, which suggests that little advection of angular momentum by Eddington-Sweet currents is taking place. This conclusion must be treated with some caution, however, because the estimate (2.8) is very approximate and there is evidence [38] that it overestimates the circulation time somewhat. For the rapidly rotating upper main sequence stars $\tau_{ES} \approx 10^{-2}\ \tau_n$ and considerable readjustment of the rotation field must have occurred.

The advection of angular momentum by meridional currents has not been discussed in great detail, though Sakurai [38] has considered the early stages of evolution of a simplified solar model from a state of rigid rotation. A steady state with circulation must have $\varpi^2\Omega$ constant on streamlines, which implies that Ω is generally greater near the rotation axis. However, Mestel [30] has suggested that a star that conserves its angular momentum will redistribute it in such a way as to choke off completely the meridional flow. Accordingly one might seek (nonconservative) rotation laws which drive no Eddington-Sweet circulation. Schwarzschild [39], Roxburgh [40,41] and Clement [42] have attempted this, but their models are not entirely consistent. However even ~~if consistent models are obtained,~~ either with or without circulation, it is not at all obvious that a star could actually evolve to such a state. Furthermore, as is discussed in the next section, it seems likely that all steady solutions are unstable.

This discussion has concentrated on the rotation of regions that are stable to convection. In convection zones too meridional currents are likely to arise except for very special distributions of angular momentum [43], but here the dynamics is complicated by the influence of the rotation on the convection. It is common practice to assume the Reynolds stresses act to force rigid rotation [39-42], though other assumptions have been made when discussing the convective envelope of the sun [see final section].

There has been no detailed study of nonaxisymmetric configurations of the type discussed by Lebovitz [44] using an equation of state that is realistic for stars.

STABILITY

The discussion will be limited to instabilities that arise directly from the rotation of the star. Dr Lebovitz has already explained what is known about fission [44] and I shall dwell on it no further. Of the modes of instability that remain the most widely studied have been the axisymmetric ones on a length scale small compared with the characteristic scale of variation of the basic 'equilibrium' state, and which derive from the Rayleigh instability.

Rayleigh [45] argued, by analogy with the instability of a density stratified fluid under gravity and by energy considerations, that a uniform incompressible inviscid fluid, steadily rotating between two coaxial cylinders, is stable to axisymmetric perturbations if

$$\frac{d}{d\varpi}(\varpi^2\Omega)^2 > 0 \tag{3.1}$$

everywhere in the fluid, and is unstable if (3.1) is violated anywhere. For an inhomogeneous incompressible fluid the criterion is

$$\frac{d}{d\varpi}(\rho\varpi^4\Omega^2) > 0. \tag{3.2}$$

It is possible to perform a proper perturbation analysis to establish that criterion (3.2) is both necessary and sufficient for stability to axisymmetric perturbations [46]. Generalization to the more complicated flows encountered in stellar models is difficult, however, and one must resort to other arguments. A powerful method which is relatively simply to handle is to use the principle of virtual work : an infinitesimal virtual adiabatic displacement ξ of the fluid is imagined to occur and the energy difference δE between the final and initial states is computed. If $\delta E > 0$ for all possible $\underset{\sim}{\xi}$

there is no energy available in the fluid to drive any disturbance, and the basic state is stable. On the other hand, if displacements ξ exists for which $\delta E < 0$ instability cannot be deduced (unless $\delta E < 0$ ~~for all ξ) unless it can be demonstrated that~~ δE is negative for a realizable disturbance which satisfies the equations of motion. However, even though instability cannot be strictly proved, in most of the special cases that have been investigated instability has been found. The method has been used by Høiland [47] who used d'Alembert's principle to estimate the frequencies of linearized axisymmetric adiabatic perturbations of a purely rotating flow. The criterion for the stability of such a flow is that the quadratic form in ξ:

$$I \equiv \left[\frac{1}{\varpi^3}\frac{\partial C^2}{\partial \varpi} + g_\varpi \frac{\partial \ln q}{\partial \varpi}\right]\xi^2 + \left[\frac{1}{\varpi^3}\frac{\partial C^2}{\partial z} + g_\varpi \frac{\partial \ln q}{\partial z} + g_z \frac{\partial \ln q}{\partial \varpi}\right]\xi + g_z \frac{\partial \ln q}{\partial z} \quad (3.3)$$

is positive definite everywhere. Here g_ϖ and g_z are the ϖ and z components of the apparent gravitational acceleration g' (which includes the centrifugal acceleration) C is the circulation $\varpi^2\Omega$ and q is the potential density [i.e. the density the fluid would have if brought to a standard pressure; thus the difference dq in potential density between two neighbouring points is $d\rho - \left(\frac{\partial \rho}{\partial p}\right)_s dp = \frac{\rho}{C_p}\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_p ds$ if the fluid is chemically homogeneous]. The form $I(\xi)$ is positive definite if the conditions

$$\Lambda_1 \equiv \frac{1}{\varpi^3}\frac{\partial C^2}{\partial \varpi} + g_\varpi \frac{\partial \ln q}{\partial \varpi} > 0 \quad (3.4)$$

and

$$\Lambda_2 \equiv (\nabla q \wedge \nabla C^2) \cdot (g' \wedge \varpi) > 0 \quad (3.5)$$

are simultaneously satisfied. In deriving the second condition the curl of the momentum equation governing the steady state in the form

$$g' \wedge \nabla \ln q = \frac{1}{\varpi^3}\frac{\partial C^2}{\partial z} e_\phi \quad (3.6)$$

was used, where e_ϕ is a unit toroidal vector. These conditions were obtained earlier from an approximate analysis by Solberg [48] who considered the motion of a displaced parcel of fluid in pressure equilibrium with its surroundings.

The analysis of Høiland provides sufficient but not necessary conditions for stability. A more recent derivation, which amounts almost to the same thing, is presented by Fricke and Smith [49] using a variational principle derived by Lynden-Bell and Ostriker [50]. Using the variational principle is potentially more powerful, for if it can be shown that the eigenfunctions of the linear stability problem form a complete set it is a simple matter to show without solving the equations that conditions (3.4) and (3.5) are also necessary.

In certain special cases conditions (3.4) and (3.5) reduce to previously known criteria. When $\Omega = 0$, $\Lambda_2 = 0$ and equation (3.6) implies that ∇q and $\underset{\sim}{g}$ are parallel. Condition (3.4) then reduces to the well known requirement that for stability the potential density must decrease upwards; for a compressible atmosphere this is the condition for convective stability [51, 52]. For rotation independent of z in the absence of gravity, Λ_2 is again zero; if the flow is incompressible $q = \rho$ and (3.4) is just the condition (3.2).

In the more general baroclinic case Λ_2 is not zero. As pointed out by Høiland [47], since ∇q and $\underset{\sim}{g}'$ are not parallel there is a region of directions for a displacement $\underset{\sim}{\xi}$ within which $(\underset{\sim}{\xi}.\underset{\sim}{g}')(\underset{\sim}{\xi}.\nabla q) < 0$, so that such a displacement would liberate energy via the buoyancy. Similarly there is a region of directions for $\underset{\sim}{\xi}$ satisfying $(\underset{\sim}{\xi}.\underset{\sim}{\varpi})(\underset{\sim}{\xi}.\nabla C^2) < 0$ within which rotational energy could be liberated. The inequalities (3.4) and (3.5) are simply the conditions that these two regions do not intersect, and that differential rotation and the combined effect of gravity and the centrifugal force on the density stratification always act in opposition. They are satisfied if the circulation increases away from the rotation axis in surfaces of constant q [53], or equivalently if q increases in the direction of apparent gravity in surfaces of constant C.

In the discussion above perturbations $\delta\Phi$ in the gravitational potential have been ignored. This is a good approximation for disturbances of short wavelength, but is not necessarily so otherwise. Taking $\delta\Phi$ into consideration does not alter the criterion for convective stability in the absence of rotation because here the neutral modes of the marginal state generate no density perturbations and so $\delta\Phi = 0$ [54,55]. But in a rotating configuration one would not expect that to be the case if q is not constant in the equilibrium configuration. Fricke and Smith [49] have shown that the perturbation $\delta\Phi$ is never stabilizing, so criteria (3.4) and (3.5) are not even sufficient for stability to large scale axisymmetric disturbances, and Fricke [56] has constructed a cylindrically symmetric example in which perturbations in gravitational potential are destabilizing.

Very different stability criteria are obtained when the perturbations are no longer constrained to be adiabatic. Now thermal diffusion can weaken the buoyancy forces responsible for stabilizing adiabatic disturbances, and provided there is some driving force, sufficiently slow motion cannot be prevented. This was pointed out by Yih [57] who demonstrated that in the absence of gravity the steady flow of a low Prandtl number fluid between two rotating coaxial cylinders can be unstable if the Rayleigh criterion (3.1) is not satisfied, even though (3.2) is. He showed also that under some circumstances viscosity can destabilize an otherwise stable flow if $\frac{d\rho}{d\varpi} < 0$.

A consistent analysis for baroclinic flows has not been published, though one can guess at the stability criteria. If (3.4) and (3.5) are not simultaneously satisfied there are directions in which both buoyancy and rotation tend to drive a displaced fluid element. Thermal diffusion acts to reduce buoyancy, but does not change its sign, so instability is likely to occur. Again, perturbations in the gravitational potential are being ignored. If (3.4) and (3.5) are satisfied, then the criterion for stability can be estimated by setting the buoyancy terms in (3.3) to zero [cf 53]: $I(\xi)$ is then positive definite if and only if

$$\frac{\partial C^2}{\partial \varpi} \geqslant 0 \quad \text{and} \quad \frac{\partial C^2}{\partial z} = 0. \tag{3.7}$$

These conditions were first obtained by Goldreich and Schubert [58] and Fricke [56], using a plane wave approximation to the eigenfunctions of the linear stability analysis. It was suggested that the conditions were necessary and sufficient for stability. In addition to Rayleigh's criterion (3.1) it is also required that Ω be independent of z if the star is to be stable. This condition is automatically satisfied if $\nabla q = 0$ in the equilibrium configuration, but not otherwise : ∇q was set to zero in (3.3) because it was argued that thermal diffusion destroys buoyancy in a sufficiently slowly moving flow, even though it is not zero in the equilibrium state.

The growth of the instabilities discussed by Yih, Goldreich and Schubert, and Fricke is controlled by thermal diffusion, though the energy source is the rotation. The dynamics is essentially the same as that of the Eddington-Sweet circulation. Nevertheless the growth time is much shorter than the Eddington-Sweet circulation time because these modes can occur with very short wavelength; the reduction of the growth rates by buoyancy cannot be eliminated entirely, however, because at extremely short wavelengths viscous forces come into play. It appears, therefore, that criteria (3.7) are likely to apply even to steady configurations with meridional circulation.

The instability of Goldreich and Schubert and Fricke which seems to occur when $\partial\Omega/\partial z \neq 0$ has a profound implication concerning rotating stars: since it appears that there are no steady solutions to the equations governing the structure of a nondegenerate star for which Ω is independent of z, any steady state that might exist must be unstable. Although no proof of the instability has yet been found, it seems likely that the result is correct. The arguments supporting it predict also the instability demonstrated by Yih, which is hardly different, and can also be used to predict similar diffusively controlled instabilities such as salt fingers [59], whose existence has been well established [60].

The discussion so far has been concerned solely with axisymmetric instabilities. Less progress has been made with nonaxisymmetric instabilities because they are more difficult to analyse. Rayleigh [61] showed that steady shear flow of a homogeneous incompressible inviscid fluid in which the motion is either rectilinear or pure rotation is stable to two dimensional infinitesimal perturbations in the plane of the shear unless the vorticity somewhere has a turning point. Fjørtoft [62] showed that this turning point must be a maximum in the magnitude of the vorticity. General necessary and sufficient conditions have not been obtained [63] and it appears that each flow must be analysed separately. However, the results suggest that a concentration of vorticity that isn't bounded too closely by rigid walls is unstable.

Vertical density stratification can stabilize a horizontal flow with velocity $u(\zeta)$ where ζ is a vertical coordinate, provided there is no thermal diffusion and the locally defined Richardson number

$$Ri = - g'\frac{dq}{d\zeta} / \left(\frac{du}{d\zeta}\right)^2 \tag{3.8}$$

is everywhere greater than $\frac{1}{4}$ [64,65]. Thermal diffusion reduces the influence of a gravitationally stable density distribution, just as in Yih's problem, and instabilities can develop on a diffusion timescale [66]. Guesses at how these results generalize to more complicated flows must be made with caution because the dynamics of shear instability is not thoroughly understood. One is warned, for example, by Huppert's [67] result that a gravitationally stable density gradient can destabilize an otherwise stable shear.

It is argued by Zahn [68] that shear instabilities provide an important angular momentum mixing mechanism in rotating stars. Zahn pointed out that all flows at very high Reynolds number appear to be unstable, and shear in isentropic surfaces in particular will not be stabilized by a gravitationally stable density gradient. The important distinction between flows characterized by a high but finite Reynolds number and inviscid flows is that the former can generate vorticity with viscous stresses, though in stars this must occur in thin boundary layers if a resulting instability is to be generated in a time short enough to be interesting. In terrestrial conditions it is usually against a rigid wall that vorticity is so created. This cannot be the case in stars, but internal boundary layers such as those discussed by Mestel [33] in the vicinity of a near discontinuity in chemical composition can arise, and coupling with other motions such as convection may be important. If shear instabilities do commonly occur, then, as Zahn stresses, angular momentum transport resulting from them will generally dominate any transport by the diffusivily controlled instabilities discussed above.

EVOLUTION OF THE ROTATION OF STARS

Since it appears that no stable steady state of a nondegenerate star exists, stars must evolve to states of time dependent motion so long as there is nuclear energy enough to keep them on the main sequence. The state to which a star evolves finally, however, after all of its available nuclear fuel has been exhausted must be a state of minimum energy, which is rigid body rotation. This assumes, of course, that an equilibrium state is available to it and that it has not collapsed into a black hole.

The angular momentum distribution in a main sequence star is therefore determined by the competing transports due to large scale meridional circulation and the smaller scale probably turbulent motions arising from the rotational instabilities discussed in the previous section. In addition, angular momentum is transported by convection and by large scale oscillations, if they are present.

To what distribution of angular momentum does a star evolve ? The only star for which we have direct evidence is the sun. Observations of the surface show that the equator rotates faster than the poles by about 20%, the precise amount varying somewhat with time, and that the rotation of features directly related to variations in the magnetic field is greater than the results from Doppler measurements. Magnetic field variations are produced by fluid motions beneath the photosphere, where the kinetic energy density exceeds the magnetic energy density, so the latter observation suggests that the angular velocity of the sun at least near the surface increases with depth (though Yoshimura [69] has argued against this interpretation). This is not inconsistent with Dicke's [12] conclusion that his measured oblateness of the radiant intensity of the solar disk results from an oblateness in the gravitational potential caused by rapid rotation of the interior. The situation has been reviewed recently by Gilman [70].

Little is known about angular momentum transport by motions resulting from the instabilities to which a star is subject. It is usually assumed that turbulence in a convection zone, for example, acts on large scale motion in the manner of molecular viscosity, forcing the zone towards a state of rigid rotation. Durney [71,72], Gilman [73] and Gierasch [74], for example, have studied models of the solar convection zone in which meridional circulation is acted upon by such a turbulent viscosity with a view to explaining the observed differential rotation. The circulation is driven by the horizontal pressure gradients resulting from rotational distortion, just as in radiative zones. In Durney's models an assumed preferred rotational inhibition of the convective heat flux in polar regions contributed further to these gradients [cf 75]. The models can produce equatorial acceleration at the surface, though the dependence of angular velocity on depth is not in all cases that suggested by the observations, and in all cases but Durney's latest model the variations in surface energy flux from pole to equator are too great to be consistent with observation. The models are not consistently coupled to the radiative interior and, as Gierasch has emphasized, this may have led to serious errors in the results.

Another possible source of error is the assumption that the Reynolds stresses act like a scalar viscosity. This could be a reasonable approximation only if the turbulence were isotropic, but since buoyancy and rotation both impose preferred directions on the forces driving the flow, anisotropy in the Reynolds stresses is inevitable. Wasiutyński [76] and Biermann [77] pointed out that anisotropy arising from buoyancy would drive nonuniform rotation in stars, and Kippenhahn [78], Cocke [79] and Köhler [80] subsequently employed simple anisotropic stress tensors to compute differentially rotating solar envelopes. Models with equators rotating faster than the poles can be constructed, with an appropriate choice of turbulent stresses, but in the latest ones [80] the angular velocity decreases with depth. The anisotropy induced by the rotation itself has been discussed by Gough and Lynden-Bell [81] who argued that there is a weak tendency for turbulence to force motion on a larger scale towards a state of no vorticity. Experiments were performed to support their arguments, but later Strittmatter, Illingworth and Freeman [82] showed that they had been misinterpreted; the vorticity expulsion hypothesis remains unconfirmed. If it is true, however, and if all other transport mechanisms are ignored, the Reynolds stresses would act to distribute the angular velocity in the solar convection zone such that it decreases with latitude and increases with depth. The ideas that led to vorticity expulsion have not been developed enough for Reynolds stresses to be calculated.

Busse [83,84] has studied the dynamics of the convection itself by expanding the solutions of the equations of motion about the marginally stable state. The conditions are highly idealized and so the results are not directly applicable to solar convection, though it is hoped that some aspects of the solar dynamics are revealed. In the models, the motion develops an anisotropy which leads to increasing surface angular velocity from pole to equator.

A completely consistent picture of the rotation of the solar convection zone is not yet at hand, but considerable progress has clearly been made in the last decade.

In computations of the structure of rotating stars, convective cores are invariably assumed to be constrained by large isotropic turbulent stresses. Tayler [43] has discussed the effect of the anisotropy that must be present and concluded that for practical purposes it is probably not necessary to take into account the anisotropy in energy flow due to convection, but that one must, of course, worry about the distribution of angular momentum.

Angular momentum transport resulting from the development of rotationally driven instabilities has been discussed less. Kippenhahn [85] pointed out that the transport of angular momentum by the diffusive modes discovered by Yih, Goldreich, Schubert and Fricke would lead to the formation of a shear which would become unstable to nonaxisymmetric instabilities. He argued that this would limit the development of the diffusive modes, and by a simple-minded argument estimated the time scale for gross readjustment of a star's angular momentum to be at least the thermal diffusion (Kelvin-Helmholtz) time τ_{KH}. James and Kahn [86,87] looked at the development of the instability more carefully and concluded that the shear would be concentrated in narrow corridors. Thus there would be separate regions of diffusively controlled motion and dynamically unstable shear. This is not dissimilar to the separate layers of fingers and convection in thermohaline convection [60,88,89]. James and Kahn estimated that the time for global readjustment of angular momentum is as great as the Eddington-Sweet circulation time τ_{ES}. It appears that the timescale obtained was as long as τ_{ES} because the possibility that motion in the diffusive region be on the small viscously controlled scale realized by salt fingers was overlooked. If fingering does occur, angular momentum should be transported globally in a time shorter than τ_{ES} and possibly τ_{KH}. However such argument by analogy may be too naive. The vectorial nature of angular momentum and the complicated geometry may act such that this comparatively simple layered structure cannot persist, and the growth time of the large scale diffusive instabilities, namely τ_{ES}, may then be the shortest time in which global angular momentum redistribution can occur.

Throughout this discussion I have had in mind that the angular momentum of a star is conserved. This is not always so. For example, Kraft [5] has shown that the rotation rates of solar-type main sequence stars decrease with age. This appears to be a result of angular momentum loss at the surface associated with mass loss in stellar winds [90].

Stars like the sun have surrounding their photospheres high temperature coronae which are thought to be maintained by the dissipation of energy from mechanical waves which originated in their convection zones below. The coronal pressure is too great merely to hold up the outer atmosphere hydrostatically [91], and gas streams outwards exerting a torque on the subphotospheric regions of the star via a magnetic field. This process has been studied by Ferraro and Bhatia [92], Modisette [93], Weber and Davis [94] and Mestel [95,96], and for the sun leads to significant deceleration in a time comparable with its age. The mass lost is a negligible fraction of the stellar mass.

The depth of the convection zones of main sequence stellar models decreases abruptly with mass at about $M = 1\frac{1}{4}\ M_{\odot}$; and it has been suggested [e.g. 90] that only deep convection zones can support strong coronae and stellar winds. Consequently it is plausible that only stars at the low mass end of the main sequence have been significantly decelerated on the main sequence, which would explain at least in part the strange dependence of angular velocity on mass mentioned in the introduction. However Kraft's [5] observations of solar-type stars in the Pleiades group, which have arrived on the main sequence only recently, also show an abrupt change in surface rotation rate. It is possible that this is a result of very different angular momentum distributions inside upper and lower main sequence stars. Alternatively it reflects different total angular momentum, which must either have been present at the time of formation of the stars or is a product of Kelvin-Helmholtz contraction to the main sequence. During part of the gravitational collapse phase all stars have extensive convective envelopes, and many are probably convective throughout, so conditions seem ripe for something like the stellar wind deceleration mechanism to operate very efficiently. There are no reliable estimates of the angular momentum that should be lost by this mechanism, but since the contraction time decreases with stellar mass it is plausible that the low mass stars have been slowed down the most. It is perhaps worth mentioning at this point that vorticity expulsion in a fully convective star implies angular momentum expulsion [81], and estimates of the expulsion rate during gravitational contraction based on analogy with magnetic field expulsion [97] leads to a sharp break in the main sequence rotation rate at just about the place where it is observed.

The continual extraction of angular momentum from the sun adds yet further complexity to the problem of determining its present state of rotation. The magnetic field in the solar atmosphere is too weak to be dynamically important much below the photosphere, and so can transmit a torque only to the outer regions of the sun. Dicke [12] claimed that therefore only the outer convection zone has been decelerated, because viscous diffusion would take about 10^{13} yr to affect the core, which is much longer than the age of the sun (about 5×10^{9} yr). The argument is invalid, for aside from the existence of the angular momentum transport processes more efficient than viscosity, discussed above, the changing centrifugal force field near the surface induces pressure inbalance which drives currents like the Eddington-Sweet circulation, causing the entire sun to respond faster than it would by viscous diffusion alone. This process is called spin-down, and has been reviewed recently by Benton and Clark [98].* It occurs in homogeneous incompressible fluids in rigid containers, angular momentum being exchanged between fluid and container in thin viscous boundary layers which were first discussed by Ekman [99]; Einstein [100], Prandt [101] and Bondi and Lyttleton [102] studied the problem further. The sun is not in a rigid container, so the details must be somewhat different, but estimates of the time to spin down the solar core based on this analysis and postulating a turbulent Ekman layer at the base of the convection zone [103], or on an analysis treating the convection zone as a rigid porous medium [104], suggest values considerably less than the solar age. However, aside from the details of the coupling of the convection zone to the radiative interior, these estimates must be treated cautiously because they were obtained without taking the stratification of the sun into account.

For a while there was some dispute over the effect of gravitationally stable thermal stratification [98] but now it appears, as one would naively expect, that thermal diffusion can in general remove any buoyancy tending to prevent the spin-down currents. The timescale of the resulting flow depends on the properties of the fluid, and linearized theory suggests that it is certainly bounded above by τ_{ES} [98,105,106]. Analogous results hold when stratification is by chemical composition, but in stars the spin-down timescales are then much greater because particle diffusion acts more slowly than thermal diffusion. Experiments are in general agreement with the theoretical findings [107]. Compressibility appears to make no qualitative difference to the conclusions [108].

The stability and spin-down arguments all lead one to the conclusion that global angular momentum readjustments occur in stars in a time which is at most the Eddington-Sweet time, but is probably much less. In any case, if the sun arrived on the main sequence rotating ten or twenty times faster than its surface is now, as Dicke [12,18] believes, angular momentum will already have been redistributed at least throughout its chemically homogeneous envelope. Approximate calculations by Sakurai [38] considering the evolution of the large scale circulation in an initially uniformly rotating solar model are consistent with this conclusion. If, on the other hand, the entire sun has always rotated on the main sequence at a rate comparable with its present surface value the issue is less clear. It can be argued that gravitationally stable gradients in chemical composition generated in the core by nuclear reactions might have prevented the core from spinning down. This is possible, though it has been suggested that the core is periodically mixed [109], which would render this conclusion less likely.

From a fluid dynamical point of view it is therefore quite uncertain what the angular momentum history of the sun has been, though in balance the evidence favours that a global readjustment has taken place on the main sequence. This does not necessarily imply that the solar core is not in a state of comparatively rapid rotation, however, for advection of angular momentum by meridional currents generally leads

*The problem usually studied is one where the external torque increases the rotation rate. This is called spin-up.

to higher angular velocity near the rotation axis [cf 38]. This is opposed by turbulent mixing which may tend to make the angular velocity more uniform.

Observational evidence that may have some bearing on the degree of mixing that has taken place concerns the abundances of lithium and beryllium. Beryllium is destroyed by nuclear reactions under conditions prevailing within the inner 50% by radius of the sun. The agreement of the solar surface abundance of beryllium with the relative abundance in the chondritic meteorites [e.g. 110] might lead one to believe that significant mixing of the sun's surface layers with the interior has not occurred. On the other hand lithium, which is destroyed inside a radius of about 0.6 $R_\odot$, is almost absent in the surface. By the same argument it appears that mixing down to this level has occurred, and since it is below the bottom of the convection zone (which is about 0.8 $R_\odot$) rotationally induced mixing of some kind has presumably taken place. There is no clear fluid dynamical reason why partial mixing to this degree should occur. Dicke [111] has concluded that this apparent lack of complete mixing implies that the solar core is not coupled to the surface, and is therefore still rotating at its initial rapid rate. Contrarily, Goldreich and Schubert [58] argued that incomplete mixing is indicative of slow rotation throughout the entire main sequence history of the sun, and that the core is therefore unlikely to be rotating rapidly. The problem is unresolved, and is complicated by the possibility that element separation within the fluid obscures the implications surface abundance measurements have on the degree of mixing that has occurred [112].

Finally, let us return to the solar oblateness measurements. Dicke and Goldenberg [16,17] have measured the oblateness of only the radiative intensity distribution in the solar image, and interpreted this as a direct consequence of a similar oblateness in the gravitational equipotentials. This interpretation has been questioned [113-116] and it was suggested in particular that effects of radiative transfer and intensity inhomogeneities in the diffuse solar atmosphere may have been responsible for the apparent oblateness. Recently Hill and Stebbins [117] have measured the shape of the sun, paying careful attention to surface nonuniformities [118]. They found time varying brightness fluctuations sufficient to account for Dicke and Goldenberg's measurement without recourse to assuming any deviation from the spherical shape, and during a period when the brightness was nearly uniform they inferred an oblateness no greater than what one would expect if the sun were rotating at approximately the surface angular velocity throughout. No doubt the controversy is not yet over, but at present it cannot be said that the sun provides convincing evidence against the theory of general relativity.

ACKNOWLEDGEMENTS

I am grateful to Dr R.C. Smith and Dr J.-P. Zahn for useful discussions. Much of this review was prepared during a visit to the National Center for Atmospheric Research, which is sponsored by the National Science Foundation; I am grateful to Dr P.A. Gilman for the hospitality of the Advanced Study Program.

REFERENCES

1. Strittmatter, P.A. 1969 Stellar Rotation Ann. Rev. Astr. Ap, 7, 665

2. Fricke, K.J. and R. Kippenhahn 1972 Evolution of rotating stars Ann. Rev. Astr. Ap, 10, 45

3. Slettebak, A. (ed) 1970 Stellar Rotation (Proc. IAU colloquium, Columbus, Ohio; Sep. 1969) (Reidel, Dordrecht)

4. Chandrasekhar, S. 1939 An introduction to the study of stellar structure (Univ. Chicago Press)

5. Kraft, R.P. 1967 Studies in stellar rotation. V. The dependence of rotation on age among solar-type stars Ap J., 150, 551

6. Howard, R. and Harvey, J. 1970 Spectroscopic determinations of solar rotation Solar Phys., 12, 23

7. Leverrier, U.J. 1859 Ann. Obs. Paris, 5, 104

8. Airy, G.B. 1862 On the circularity of the sun's disk Mon. Not. Roy. astr. Soc., 22, 80

9. Auwers, A. 1891 Astr. Nachr., 128, 360

10. Newcomb, S. 1895 Fundamental constants of Astronomy (U.S Gov. Printing Office, Washington)

11. Einstein, A. 1915 Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie Sitzungsber. Preuss. Akad. Wiss., 831

12. Dicke, R.H. 1964 The sun's rotation and relativity Nature, 202, 432

13. Roxburgh, I.W. 1964 Solar rotation and the perihelion advance of the planets Icarus, 3, 92

14. Brans, C. and R.H. Dicke 1961 Mach's principle and a relativistic theory of gravitation Phys. Rev., 124, 925

15. Jordan, P. 1948 Fünfdimensionale Kosmologie Astr. Nachr., 276, 193

16. Dicke, R.H. and H.M. Goldenberg 1967 Solar oblateness and General Relativity Phys. Rev. Lett., 18, 313

17. Dicke, R.H. and H.M. Goldenberg 1974 The oblateness of the sun Ap J. Suppl., 27, 131

18. Dicke, R.H. 1970 Internal rotation of the sun Ann. Rev. Astr. Ap, 8, 297

19. Ostriker, J.P. and J.W.-K. Mark 1968 Rapidly rotating stars. I. The self-consistent-field method Ap J., 151, 1075

20. Chandrasekhar, S. 1931 The maximum mass of ideal white dwarfs Ap J., 74, 81

21. Salpeter, E.E. and T. Hamada 1961 Models for zero-temperature stars Ap J., 134, 683

22. Mestel, L. 1965 The theory of white dwarfs Stars and Stellar Systems (ed. L.H. Aller and D.B. McLaughlin, Chicago Univ. Press) 8, 297

23. Ostriker, J.P., P. Bodenheimer and D. Lynden-Bell 1966 Equilibrium models of differentially rotating zero-temperature stars Phys. Rev. Lett., 17, 816

24. Eddington, A.S. 1925 Circulating currents in rotating stars Observatory, 48, 73

25. Vogt, H. 1925 Zum Strahlungsgleichgewicht der Sterne Astr. Nachr., 223, 229

26. Eddington, A.S. 1929 Internal circulation in stars Mon. Not. Roy. astr. Soc., 90, 54

27. von Zeipel, H. 1924 Festschrift für H. von Seelinger, p 144

28. Roxburgh, I.W. 1966 On stellar rotation III. (Appendix) Mon. Not. Roy. astr. Soc., 132, 201

29. Sweet, P.A. 1950 The importance of rotation in stellar evolution Mon. Not. Roy. astr. Soc., 110, 548

30. Mestel, L. 1965 Meridian circulation in stars Stars and Stellar Systems (ed. L.H. Aller and D.B. McLaughlin, Chicago Univ. Press) 8, 465

31. Baker, N. and R. Kippenhahn 1959 Untersuchungen über rotierende Sterne Zs f. Ap, 48, 140

32. Smith, R.C. 1966 A note on meridian circulation in rotating stars Zs f. Ap, 63, 166

33. Mestel, L. 1953 Rotation and stellar evolution Mon. Not. Roy. astr. Soc., 113, 716

34. Smith, R.C. 1970 Meridional circulation in rotating stellar atmospheres Mon. Not. Roy. astr. Soc., 148, 275

35. Brand, D. and R.C. Smith 1971 Meridional circulation in rotating stellar atmospheres - II Mon. Not. Roy. astr. Soc., 154, 293

36. Osaki, Y. 1966 On the atmosphere of a rotating star Publ. Astr. Soc. Japan, 18, 7

37. Osaki, Y. 1972 Meridional circulation in a rotating star Publ. Astr. Soc. Japan, 24, 509

38. Sakurai, T. 1972 The evolution of the solar inner rotation by the Eddington-Sweet type circulation under the influence of the solar wind torque Publ. Astr. Soc. Japan, 24, 153

39. Schwarzschild, M. 1947 On stellar rotation II Ap J., 106, 427

40. Roxburgh, I.W. 1964 On stellar rotation I. The rotation of upper main-sequence stars Mon. Not. Roy. astr. Soc., 128, 157

41. Roxburgh, I.W. 1964 On stellar rotation II. The rotation of lower main sequence stars Mon. Not. Roy. astr. Soc., 128, 237

42. Clement, M.J. 1969 Differential rotation in stars on the upper main sequence Ap J., 156, 1051

43. Tayler, R.J. 1973 Convection in rotating stars Mon. Not. Roy. astr. Soc., 165, 39

44. Lebovitz, N.R. 1974 The fission theory of binary stars Ap J., 190, 121 (see also the article in this volume)

45. Lord Rayleigh 1916 On the dynamics of revolving fluids Proc. Roy. Soc., A93, 148 (also Scientific Papers, 6, 447, CUP Cambridge 1920)

46. Synge, J.L. 1933 The stability of heterogeneous liquids Trans. Roy. Soc. Canada, Third Ser., 27 III, 1

47. Høiland, E. 1941 On the stability of the circular vortex Avhandl. utgitt Norske Vidensk.-Akad. Oslo I, Mat.-naturv. Kl. No 11

48. Solberg, H. 1936 Le mouvement d'inertie de l'atmosphère stable et son rôle dans la théorie des cyclones Procès-verbaux de l'Assn de météor., Edinbourg, Sept. 1936

49. Fricke, K.J. and R.C. Smith 1971 On global dynamical stability of rotating stars Astr. Ap, 15, 329

50. Lynden-Bell, D. and J.P. Ostriker 1967 On the stability of differentially rotating bodies Mon. Not. Roy. astr. Soc., 136, 293

51. Reye, T. 1872 Die Wirbelstürme, Tornados und Wettersäulen (Gegenius, Halle)

52. Schwarzschild, K. 1906 Ueber das Gleichgewicht der Sonnenatmosphäre Gottinger Nachr., Math-phys Kl., 41

53. Zahn, J.-P. 1974 Rotational instabilities and stellar evolution Hightlights in Astronomy, 3

54. Lebovitz, N.R. 1965 On Schwarzschild's criterion for the stability of gaseous masses Ap J., 142, 229

55. Lebovitz, N.R. 1965 On the onset of convective instability Ap.J., 142, 1257

56. Fricke, K.J. 1968 Instabilität stationärer Rotation in Sternen Zs f. Ap, 68, 317

57. Yih, C.-S. 1961 Dual role of viscosity in the instability of revolving fluids of variable density Phys. Fluids, 4, 806

58. Goldreich, P. and G. Schubert 1967 Differential rotation in stars Ap J., 150, 571

59. Stern, M.E. 1960 The "salt-fountain" and thermohaline convection Tellus, 12, 172

60. Turner, J.S. 1973 Buoyancy effects in fluids (CUP, Cambridge)

61. Lord Rayleigh 1880 On the stability or instability of certain fluid motions Proc. London Math. Soc., 11, 57 (also Scientific Papers, 1, 474, CUP Cambridge 1899)

62. Fjørtoft, R. 1950 Application of integral theorems in deriving criteria of stability for laminar flows and for the baroclinic circular vortex Geofys. Publ., 27, No 6

63. Drazin, P.G. and L.N. Howard 1966 Hydrodynamic stability of parallel flow of inviscid fluid Adv. Appl. Mech., 9, 1

64. Miles, J.W. 1961 On the stability of heterogeneous shear flows J. Fluid Mech., 10, 496

65. Howard, L.N. 1961 Note on a paper by John W. Miles J. Fluid Mech., 10, 509

66. Jones, C.A. 1975 The effect of diffusion on stratified shear flow (to be published)

67. Huppert, H.E. 1973 On Howard's technique for perturbing neutral solutions of the Taylor-Goldstein equation J. Fluid Mech., 57, 361

68. Zahn, J.-P. 1975 Differential rotation and turbulence in stars Problems in stellar hydrodynamics (19th Liège Intern. Ap Colloq., July 1974)

69. Yoshimura, H. 1971 Complexes of activity of the solar cycle and very large scale convection Solar Phys., 18, 417

70. Gilman, P.A. 1974 Solar rotation Ann. Rev. Astr. Ap, 12

71. Durney, B.R. 1971 Differential rotation, meridional velocities, and pole-equator difference in temperature of a rotating convective spherical shell Ap J., 163, 353

72. Durney, B.R. 1974 On the sun's differential rotation: it's maintenance by large-scale meridional motions in the convection zone Ap J., 190, 211

73. Gilman, P.A. 1972 Nonlinear Boussinesq convective model for large scale solar circulations Solar Physics, 27, 3

74. Gierasch, P.J. 1974 The differential rotation of the solar surface Ap J., 190, 199

75. Weiss, N.O. 1965 Convection and differential rotation of the sun Observatory, 85, 37

76. Wasiutyński, J. 1946 Studies in hydrodynamics and structure of stars and planets Ap Norv., 4

77. Biermann, L. 1951 Bemerkungen über das Rotationsgesetz in irdischen und stellaren Instabilitätszonen Zs f. Ap, 28, 304

78. Kippenhahn, R. 1963 Differential rotation in stars with convective envelopes Ap J., 137, 664

79. Cocke, W.J. 1967 On the solar differential rotation and meridional currents Ap J., 150, 1041

80. Kohler, H. 1970 Differential rotation caused by anisotropic turbulent viscosity Solar Physics, 13, 3

81. Gough, D.O. and D. Lynden-Bell 1968 Vorticity expulsion by turbulence : astrophysical implications of an Alka-Seltzer experiment J. Fluid Mech., 32, 437

82. Strittmatter, P.A., G. Illingworth and K.C. Freeman 1970 A note on the vorticity expulsion hypothesis J. Fluid Mech., 43, 539

83. Busse, F.H. 1970 Differential rotation in stellar convection zones Ap J., 159, 629

84. Busse, F.H. 1973 Differential rotation in stellar convection zones II Astr. Ap, 28, 27

85. Kippenhahn, R. 1969 Secularly unstable differential rotation in stars Astr. Ap, 2, 309

86. ~~James, R.A. and F.D. Kahn 1970~~ Non-linear limiting of the Goldreich-Schubert instability Astr. Ap, 5, 232

87. James, R.A. and F.D. Kahn 1971 Angular momentum transport in stars, following the Goldreich-Schubert instability Astr. Ap, 12, 332

88. Stern, M.E. and J.S. Turner 1969 Salt fingers and convecting layers Deep-Sea Res., 16, 497

89. Shirtcliffe, T.G.L. and J.S. Turner 1970 Observations of the cell structure of salt fingers J. Fluid Mech., 41, 707

90. Schatzman, E. 1962 A theory of the role of magnetic activity during star formation Ann. d'Ap, 25, 18

91. Parker, E.N. 1963 Interplanetary dynamical processes (Interscience, New York)

92. Ferraro, V.C.A. and V.B. Bhatia 1967 Corotation and solar wind in the solar corona and interplanetary medium Ap J., 147, 220

93. Modisette, J.L. 1967 Solar wind induced torque on the sun J. Geophys. Res., 72, 1521

94. Weber, E.J. and L. Davis Jr 1967 The angular momentum of the solar wind Ap J., 148, 217

95. Mestel, L. 1968 Magnetic braking by a stellar wind -I Mon. Not. Roy. astr. Soc., 138, 359

96. Mestel, L. 1968 Magnetic braking by a stellar wind - II Mon. Not. Roy. astr. Soc., 140, 177

97. Weiss, N.O. 1966 The expulsion of magnetic flux by eddies Proc. Roy. Soc., A293, 310

98. Benton, E.R. and A. Clark Jr 1974 Spin-up Ann. Rev. Fluid Mech., 6, 257

99. Ekman, V.W. 1905 On the influence of the earth's rotation on ocean-currents Ark. Math. Astr. Fys., 2, No 11

100. Einstein, A. 1916 Elementare Theorie der Wasserwellen und des Fluges Naturwiss., 4, 509

101. Prandtl, L. 1931 Abriß der Stromungslehre (Friedr. Vieweg & Sohn, Braunschweig) p103

102. Bondi, H. and R.A. Lyttleton 1948 On the dynamical theory of the rotation of the earth Proc. Camb. Phil Soc., 44, 345

103. Howard, L.N, D.W. Moore and E.A. Spiegel 1967 Solar spin-down problem Nature, 214, 1297

104. Bretherton, F.P. and E.A. Spiegel 1968 The effect of the convection zone on solar spin-down Ap J., 153, L77

105. Sakurai, T., A. Clark Jr and P.A. Clark 1971 Spin-down of a Boussinesq fluid of small Prandtl number in a circular cylinder J. Fluid Mech., 49, 753

106. Clark, A., Jr 1973 The linear spin-up of a strongly stratified fluid of small Prandtl number J. Fluid Mech., 60, 561

107. Buzyna, G. and Veronis, G. 1971 Spin-up of a stratified fluid : theory and experiment J. Fluid Mech., 50, 579

108. Clark, A., Jr, P.A. Clark, J.H. Thomas and N.-H. Lee 1971 Spin-up of a strongly stratified fluid in a sphere J. Fluid Mech., 45, 131

109. Dilke, F.W.W. and D.O. Gough 1972 The solar spoon Nature, 240, 262

110. Wallerstein, G. and P.S. Conti 1969 Lithium and beryllium in stars Ann. Rev. Astr. Ap, 7, 99

111. Dicke, R.H. 1972 Rotation and lithium abundance in solar-type stars Ap J., 171, 331

112. Schatzman, E. and S. Vauclair 1974 Radiation pressure and the lithium-beryllium problem Comments Ap Sp. Sci., 6, 159

113. Ingersoll, A.P. and E.A. Spiegel 1971 Temperature variation and the solar oblateness Ap J., 163, 375

114. Chapman, G.A. and A.P. Ingersoll 1972 Photospheric faculae and the solar oblateness Ap J., 175, 819

115. Dicke, R.H. 1972 Faculae and the solar oblateness Ap J., 175, 831

116. Dicke, R.H. 1974 Faculae and the solar oblateness II Ap.J., 190, 187

117. Hill, H.A. 1974 The intrinsic visual oblateness of the sun (submitted to Ap J.)

118. Hill, H.A., P.D. Clayton, D.L. Patz, A.W. Healy, R.T. Stebbins, J.R. Oleson and C.A. Zanoni 1974 Solar oblateness, excess equatorial brightness, and relativity (submitted to Phys. Rev. Lett.)