D8

DUALITY BETWEEN SURFACE COUPLED INTERFACIAL WAVES FOR ELECTRICALLY CHARGED AND SELF-GRAVITATING DROPS

Markus Zahn
Department of Electrical Engineering
University of Florida
Gainesville, Florida 32611

I. Introduction

Many investigations of electro-fluid mechanical interactions consider only surface electrical forces, such as is the case of an interface supporting a surface charge distribution or of an interface between different dielectrics [1], [2]. For these cases, complications resulting from nonuniform equilibrium electric field intensities and interfacial curvature occur only for non-planar systems. In contrast, recent work has investigated electrohydrodynamic coupling of perfectly insulating fluids supporting volume space charge distributions [3], [4]. Such analysis is appropriate if the fluid is highly insulating and the dynamics of interest occur over time scales which are short compared with those required for initially injected charges to accumulate appreciably at a fluid surface. For this case, the volume space charge imposes a nonuniform electric field distribution even in planar geometry, which must be taken into account when the interfaces move as the interfacial equilibrium electric fields which act on the interfaces also change.

This work examined the propagation and instability characteristics of small signal electro-fluid mechanical space charge and polarization waves for electrohydrodynamic configurations similar to those of the classic fluid problems of the Rayleigh-Taylor instability for superposed charged planar layers, and of interfacial capillary oscillations of charged liquid cylinders (or jets) and charged spherical drops [5]. A systematic approach was developed to handle multi-interfacial systems of incompressible, inviscid, and perfectly insulating fluids in planar, cylindrical, and spherical geometry through the use of a general set of relations for perturbation field and flow variables on the perturbed surfaces of fluid layers having constant properties [3]. Although the methods developed are valid for any geometry, we limit ourselves here to systems initially in spherical equilibrium.

The analysis showed that the electrohydrodynamic coupling for uniformly charged layers could be represented as a purely surface coupled interaction, even though volume Coulomb forces are present. The pertinent electrical equation necessary in this development is Poisson's equation relating the electrical potential ϕ to the charge density q and permittivity ε (assumed constant)

$$\nabla^2\phi = \frac{-q}{\varepsilon} \tag{1}$$

The electrohydrodynamic coupling occurs through the Coulomb force density

$$\overline{f} = -q\nabla\phi \tag{2}$$

The methodical approach developed can be used for any system whose force density is curl free within a homogeneous layer (q = constant) with the resulting coupling occurring through the interfaces as the force density can then be lumped with the hydrodynamic pressure.

This work will also use this duality with the directly analogous equations to (1) and (2) for self-gravitating systems

$$\nabla^2 V = 4\pi G\rho \tag{3}$$

$$\overline{f} = -\rho\nabla V \tag{4}$$

where V is the gravitational potential, ρ is the mass density and G is the universal gravitational constant ($G \approx 6.67 \times 10^{-11}$ nt-m^2/kg^2), to deduce the dynamics and instability characteristics of spherical self-gravitating geometries [6]. It is a simple matter to obtain results for self-gravitating systems for those electrohydrodynamic problems already solved by making the simple substitutions

$$\phi \to V \; ; \; q \to \rho \; ; \; \varepsilon \to -\frac{1}{4\pi G} \tag{5}$$

However, an important distinction between self-gravitating and electrohydrodynamic systems remains. Whereas the charge density can be either positive or negative, the mass density is always positive. Thus there is always a force of attraction between masses, while because of the minus sign difference in Eqs. (1) and (3), like charges repel and opposite charges attract. Because of the absence of "negative mass" there are no gravitational analogs to polarization effects due to differences in permittivity or electrical shielding due to the presence of electrical conductors. The gravitational constant G is independent of material properties. Thus with the differences between electrical and gravitational systems in mind, we can immediately write down the solutions to those gravitational problems analogous to already solved electrohydrodynamic problems using the conversions of Eq. (5).

In this work, we will derive the general electrohydrodynamic "prototype" relations for a spherical shell and then derive and contrast the dispersion characteristics of perfectly conducting and perfectly insulating charged drops. Using duality we will then immediately write down the analogous solutions for self-gravitating drops.

II. Equations of Motion

1. General Development

No matter the geometry, the general equations of motion for an incompressible, inviscid, and perfectly insulating charged fluid with mass density ρ, velocity $\overline{v}$, pressure p, charge density q, permittivity ϵ, and electric field $\overline{E}$ are

Conservation of momentum -

$$\rho \frac{D\overline{v}}{Dt} + \nabla p = q\overline{E} - \frac{1}{2}\overline{E} \cdot \overline{E}\nabla\epsilon \tag{6}$$

Conservation of mass -

$$\frac{D\rho}{Dt} = 0 \; ; \; \nabla \cdot \overline{v} = 0 \tag{7}$$

Maxwell's equations -

$$\nabla \times \overline{E} = 0 \; ; \; \overline{E} = -\nabla\phi \tag{8}$$

Gauss's Law: $$\nabla \cdot (\epsilon\overline{E}) = q \; ; \; \frac{D\epsilon}{Dt} = 0 \tag{9}$$

Conservation of Charge: $$\frac{Dq}{Dt} = 0 \tag{10}$$

In the prototype layer, examples of which are shown in Fig. (1), the liquid is homogeneous such that the equilibrium properties of mass density ρ_Δ, charge density q_Δ, and dielectric constant ϵ_Δ, are constant. Then all quantities are assumed to have small perturbations from the equilibrium. Subject to the constraint of a homogeneous medium the equilibrium variables must obey the time independent form of (6)-(10) and must satisfy the boundary conditions between regions.

Because the fluid layer is homogeneous the charge density, mass density, and permittivity remain constant in spite of the fluid motion. This is to be expected, because any transport of material into a given region leads to a transport of material which has the same properties as that previously occupying the given region. This statement only applies to those portions of the fluid not swept out by interfacial motions. If a point of interest is adjacent to an interface, an excursion of the interface could result in an abrupt change of properties. However, if surface deflections are considered at a given instant, all properties everywhere between interfaces are uniform.

Denoting perturbation variables with primes, we take the divergence of the linearized form of (6), to yield the set of perturbation equations

$$\nabla^2\phi' = 0$$
$$\nabla^2\pi' = 0 \; ; \; \pi' = p' + q_\Delta\phi' \tag{11}$$

Thus regardless of the geometry, the problem reduces to solutions of

Laplace's equation, both for the perturbation potential and for the modified pressure π'.

2. Generalized Relations For a Spherical Prototype Layer

We consider the lower prototype layer shown in Fig. (1) with the understanding that the picture is a cross section of a spherical shell. In equilibrium the uniformly charged fluid extends over the range $\beta \leq r \leq \alpha$. We denote all variables at the inner interface with a superscript β and all variables at the outer interface with a superscript α. The radial perturbation displacement's are ξ^β and ξ^α. When these displacements are zero the equilibrium distributions are

$$p_o + q_\Delta \Phi_0 = \text{constant}$$

$$\Phi_0 = \frac{-q_\Delta r^2}{6\epsilon_\Delta} + \frac{\text{constant}_1}{r} + \text{constant}_2 \tag{12}$$

$$\bar{E} = \left[\frac{q_\Delta r}{3\epsilon_\Delta} + \frac{\text{constant}_1}{r^2}\right]\bar{i}_r$$

All perturbation variables are assumed of the form

$$\pi' = \text{Re}\hat{\pi}(r)P_n^m(\cos\psi)\exp j[\omega t - m\theta] \; ; \; m > 0 \; , \; n > 0 \; , \; m \leq n \tag{13}$$

where $P_n^m(\cos\psi)$ are Legendre functions depending on the azimuthal angle ψ.

Substituting into Eq. (11) yields

$$\hat{\pi}(r) = A_1 r^n + A_2 r^{-(n+1)} \tag{14}$$

where A_1 and A_2 are constants to be determined from the boundary conditions

$$\hat{v}_r(\beta) = j\omega\hat{\xi}^\beta$$
$$\hat{v}_r(\alpha) = j\omega\hat{\xi}^\alpha \tag{15}$$

From Eq. (6) we know, that the perturbation velocity is relating to $\hat{\pi}(r)$ as

$$\hat{v}_r(r) = -\frac{1}{j\omega\rho_\Delta}\frac{d\hat{\pi}}{dr} \tag{16}$$

so that the interfacial displacements are related to the parameters A_1 and A_2 as

$$\begin{bmatrix} \hat{\xi}^{\alpha} \\ \hat{\xi}^{\beta} \end{bmatrix} = \frac{1}{\rho_{\Delta}\omega^2} \begin{bmatrix} n\alpha^{n-1} & -(n+1)\alpha^{-(n+2)} \\ n\beta^{n-1} & -(n+1)\beta^{-(n+2)} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tag{17}$$

For the purposes of our analysis, the inverse relations are needed. Once A_1 and A_2 are determined in terms of $\hat{\xi}^{\alpha}$ and $\hat{\xi}^{\beta}$, substitution into Eq. (14) yields the relationship between the interfacial modified pressures and the interfacial displacements as

$$\begin{bmatrix} \hat{\pi}(\alpha) \\ \hat{\pi}(\beta) \end{bmatrix} = \begin{bmatrix} F(\alpha,\beta) & G(\alpha,\beta) \\ G(\beta,\alpha) & F(\beta,\alpha) \end{bmatrix} \begin{bmatrix} \hat{\xi}^{\alpha} \\ \hat{\xi}^{\beta} \end{bmatrix} \tag{18}$$

where

$$F(x,y) = \frac{\left(\frac{x}{y}\right)^n \frac{x^2}{n} + \left(\frac{y}{x}\right)^n \frac{xy}{n+1}}{\left[\left(\frac{x}{y}\right)^n x - \left(\frac{y}{x}\right)^n y\right]} (\omega^2\rho_{\Delta})$$

$$G(x,y) = \frac{-(2n+1)y^2}{n(n+1)\left[\left(\frac{x}{y}\right)^n x - \left(\frac{y}{x}\right)^n y\right]} (\omega^2\rho_{\Delta}) \tag{19}$$

Note that the analysis implicitly assumes the interfacial displacements to be small as Eqs. (15) and (16) were evaluated at the equilibrium positions (α,β) rather than at the interfaces themselves $(\alpha + \xi^{\alpha}, \beta + \xi^{\beta})$. Fortunately, because the velocity itself is a perturbation, the difference between evaluating it at the interface or at the equilibrium position is second order in the perturbation amplitudes. This illustrates the general approach used in linearized surface deformation problems. The boundary condition at the moving interface is replaced by one at the equilibrium position of the boundary, thus greatly simplifying the analysis.

The analysis is still not complete for as the interfaces deform, in addition to perturbing all variables, the equilibrium quantities acting on the interfaces also change. Thus to compute the total first order change in all variables evaluated at the interface, linear changes of equilibrium quantities must be included. For example, the total linear changes in the pressures are:

$$\hat{p}^{\alpha} = \hat{p}(\alpha) + \left.\frac{dp_0}{dr}\right|_{(r=\alpha)} \hat{\xi}^{\alpha}$$

$$\hat{p}^{\beta} = \hat{p}(\beta) + \left.\frac{dp_0}{dr}\right|_{(r=\beta)} \hat{\xi}_{\beta} \tag{20}$$

Similarly for the potentials

$$\hat{\phi}^{\alpha} = \hat{\phi}(\alpha) + \left.\frac{d\phi_0}{dr}\right|_{(r=\alpha)} \hat{\xi}^{\alpha} = \hat{\phi}(\alpha) - E^{\alpha}\hat{\xi}^{\alpha}$$

$$\hat{\phi}^{\beta} = \hat{\phi}(\beta) + \left.\frac{d\hat{\phi}_0}{dr}\right|_{(r=\beta)} \hat{\xi}^{\beta} = \hat{\phi}(\beta) - E^{\beta}\hat{\xi}^{\beta} \tag{21}$$

Using these definitions we have

$$\hat{p}^{\alpha} \equiv \hat{\pi}(\alpha) - q_{\Delta}\hat{\phi}(\alpha) + E^{\alpha}q_{\Delta}\hat{\xi}^{\alpha}$$

$$\hat{p}^{\beta} \equiv \hat{\pi}(\beta) - q_{\Delta}\hat{\phi}(\beta) + E^{\beta}q_{\Delta}\hat{\xi}^{\beta} \tag{22}$$

$$\varepsilon_{\Delta}\hat{e}_r^{\alpha} = \varepsilon_{\Delta}\,\hat{e}_r(\alpha) + \varepsilon_{\Delta}\left.\frac{dE_0}{dr}\right|_{r=\alpha}^{\hat{\xi}^{\alpha}}$$

$$\varepsilon_{\Delta}\hat{e}_r^{\beta} = \varepsilon_{\Delta}\,\hat{e}_r(\beta) + \varepsilon_{\Delta}\left.\frac{dE_0}{dr}\right|_{r=\beta}^{\hat{\xi}^{\beta}} \tag{23}$$

Then using Eqs. (20) - (22) we obtain the generalized mechanical relations evaluated at the interfaces

$$\begin{bmatrix}\hat{p}^{\alpha}\\ \hat{p}^{\beta}\end{bmatrix} = \begin{bmatrix}F(\alpha,\beta) & G(\alpha,\beta)\\ G(\beta,\alpha) & F(\beta,\alpha)\end{bmatrix}\begin{bmatrix}\hat{\xi}^{\alpha}\\ \hat{\xi}^{\beta}\end{bmatrix} - q_{\Delta}\begin{bmatrix}\hat{\phi}^{\alpha}\\ \hat{\phi}^{\beta}\end{bmatrix} \tag{24}$$

where F and G are given in Eq. (19)

Similar operations are performed in the solutions of Laplace's equation for the perturbation potential $\hat{\phi}(r)$ to yield the electrical relations at the interfaces

$$\begin{bmatrix}\varepsilon_{\Delta}\hat{e}_r^{\alpha}\\ \varepsilon_{\Delta}\hat{e}_r^{\beta}\end{bmatrix} = \varepsilon_{\Delta}\begin{bmatrix}B(\alpha,\beta) & C(\alpha,\beta)\\ C(\beta,\alpha) & B(\beta,\alpha)\end{bmatrix}\begin{bmatrix}\hat{\phi}^{\alpha} + E^{\alpha}\hat{\xi}^{\alpha}\\ \hat{\phi}^{\beta} + E^{\beta}\hat{\xi}^{\beta}\end{bmatrix} + \varepsilon_{\Delta}\begin{bmatrix}\left.\frac{dE_0}{dr}\right|_{r=\alpha}^{\hat{\xi}^{\alpha}}\\ \left.\frac{dE_0}{dr}\right|_{r=\beta}^{\hat{\xi}^{\beta}}\end{bmatrix} \tag{25}$$

where

$$B(x,y) = \frac{\left[n\left(\frac{x}{y}\right)^{n} + (n+1)\left(\frac{y}{x}\right)^{n+1}\right]}{\left[\left(\frac{y}{x}\right)^{n} y - \left(\frac{x}{y}\right)^{n} x\right]}$$

$$C(x,y) = -\frac{(2n+1)\left(\frac{y}{x}\right)}{\left[\left(\frac{y}{x}\right)^{n} y - \left(\frac{x}{y}\right)^{n} x\right]} \tag{26}$$

The general relations of (24) and (25) greatly simplify in various limits. As the inner radius goes to zero ($\beta \to 0$), so that the shell becomes a drop, the terminal relations reduce to

$$\hat{p}^{\alpha} = \frac{\omega^2 \rho_{\Delta\alpha}}{n}\hat{\xi}^{\alpha} - q_{\Delta}\hat{\phi}^{\alpha}$$

$$\lim_{\beta \to 0} \quad \varepsilon_{\Delta}\hat{e}_r^{\alpha} = -\frac{\varepsilon_{\Delta} n}{\alpha}\left(\hat{\phi}^{\alpha} + E^{\alpha}\hat{\xi}^{\alpha}\right) + \varepsilon_{\Delta}\left.\frac{dE_0}{dr}\right|_{r=\alpha}\hat{\xi}^{\alpha} \tag{27}$$

As the outer radius becomes very large, ($\alpha \to \infty$), the terminal relations become

$$\hat{p}^{\beta} = \frac{-\omega^2 \rho_{\Delta\beta}}{(n+1)}\hat{\xi}^{\beta} - q_{\Delta}\hat{\phi}^{\beta}$$

$$\lim_{\alpha \to \infty} \quad \varepsilon_{\Delta}\hat{e}_r^{\beta} = \frac{\varepsilon_{\Delta}(n+1)}{\beta}\left(\hat{\phi}^{\beta} + E^{\beta}\hat{\xi}^{\beta}\right) + \varepsilon_{\Delta}\left.\frac{dE_0}{dr}\right|_{r=\beta}\hat{\xi}^{\beta} \tag{28}$$

Equations (24) and (25) are useful because they relate the interfacial variables of pressure, displacement, electrical potential and electrical displacement, which appear in interfacial boundary conditions. The boundary conditions for all cases include mass and electric potential continuity, the second condition being equivalent to the continuity of the tangential component of electric field, as well as an interfacial force balance. For perfectly conducting interfaces, the perturbation potential must be zero but the surface charge imposes a surface force density. Because there are no electric fields inside the perfectly conducting fluids, there are no polarization effects. In contrast, perfectly insulating fluids with volume charge have nonzero interfacial potential and although free charge can make no contribution to an interfacial surface force density, there is a surface polarization force if the permittivity of the fluid and its surroundings differ. For non-perfect conductors (including perfect insulators) no surface charge can be allowed on

any interface for an inviscid analysis to be meaningful, as electrical shear stresses would accompany such a surface charge density, and our model would then necessarily have to include viscosity (or some other mechanism) to balance this shear stress [7], [8]. This requires the normal component of electric displacement to be continuous across an interface for non-perfect conductors. An inviscid formulation is allowed only if the interface has no surface charge or if the interface is perfectly conducting, as then the electric field terminates perpendicular to the interface, resulting in no electrical shear force. However, the same systematic techniques have been applied to a viscous prototype layer in planar geometry so that these limitations may be removed [9]. In general, considering viscous fluids greatly increases the mathematical complexity of the analysis but yet has no effect on the conditions for instability [10].

III. Stability and Dynamics of Charged Spherical Drops

The modes of oscillation and instability of charged, initially spherical drops are obtained for small perturbations by using the generalized relations of (24) and (25). Rayleigh's limit determines the maximum amount of charge distributed around the surface of an isolated perfectly conducting drop for it to be stable [11]. The isolated uniformly charged, perfectly insulating drop has historical significance through the liquid drop model for fission of the nucleus proposed by Bohr and Wheeler [12]. In this section we wish to generalize both these classic problems by immersing the charged spherical drops within a uniformly charged region. Figure 2 describes the geometry with the understanding that the picture is the cross section of a spherical system.

In spherical geometry, all perturbations due to small signal interfacial motions are assumed of the form

$$\xi = \mathrm{Re}\ \hat{\xi}\ \exp[j(\omega t - m\theta)]P_n^m(\cos\psi)\ ;\ m > 0\ ,\ n > 0\ ,\ m \leq n \tag{29}$$

For spherical drops we only need to know these relations in the two limits where the inner radius tends to zero, $\beta \to 0$, and when the outer radius gets very large, $\alpha \to \infty$, which are described by Eqs. (27) and (28).

1. Perfectly Conducting Drop With Surface Charge Within a Uniformly Charged Region

We first consider a configuration similar to that in Fig. (2) with a perfectly conducting spherical drop of radius R, mass density ρ_1 and surface tension γ with uniformly distributed surface charge density σ_f immersed within a perfectly insulating charged fluid of infinite extent with charge density q_2 and mass density ρ_2. The equilibrium electric field distribution is

$$E_r = \frac{\sigma_f R^2}{\varepsilon_2 r^2} + \frac{q_2(r - R^3/r^2)}{3\varepsilon_2} \qquad r > R \tag{30}$$

so that

$$E_1 = 0\ ,\ E_2 = \frac{\sigma_f}{\varepsilon_2}\ ,\ \left.\frac{dE_2}{dr}\right|_{r=R} = \frac{-2\sigma_f}{\varepsilon_2 R} + \frac{q_2}{\varepsilon_2} \tag{31}$$

Because the drop is perfectly conducting there are no electric fields inside ($E_1 = 0$). The perturbation interfacial boundary conditions are

$$\begin{aligned} &\hat{\xi}_1 = \hat{\xi}_2 \equiv \hat{\xi}\ ,\ \hat{\phi}_1 = \hat{\phi}_2 \equiv 0 \\ &\hat{p}_1 - \hat{p}_2 + \varepsilon_2 E_2 \hat{e}_{r2} - \left(\frac{\gamma}{R^2}\right)(n-1)(n+2)\hat{\xi} = 0 \end{aligned} \tag{32}$$

From Eqs. (27) and (28) we obtain

$$\hat{p}_1 = \rho_1 \omega^2 R\hat{\xi}/n$$

$$\hat{p}_2 = -\rho_2 \omega^2 R\hat{\xi}/(n+1) \tag{33}$$

$$\varepsilon_2 \hat{e}_{r2} = \varepsilon_2 E_2 (n+1)\hat{\xi}/R + \varepsilon_2 \left.\frac{dE_2}{dr}\right|_{r=R} \hat{\xi}$$

Substitution of Eq. (33) into Eq. (32) yields the dispersion equation as

$$\left(\frac{\omega^2 R^3}{\gamma}\right)\left(\frac{\rho_1}{n} + \frac{\rho_2}{(n+1)}\right) = (n-1)(n+2) - \left(\frac{\sigma_f^2 R}{\varepsilon_2 \gamma}\right)\left(n - 1 + \frac{q_2 R}{\sigma_f}\right) \tag{34}$$

We see that the self-field term proportional to σ_f^2 due to the interaction of the surface charge with its own field is always destabilizing, while the imposed field term proportional to $q_2\sigma_f$ due to the interaction of the space charge q_2 with the field due to σ_f can be stabilizing if q_2 and σ_f are of opposite sign. If $\sigma_f = 0$, there is no electromechanical coupling.

We examine (34) in Rayleigh's limit where there is no space charge in the outer region ($q_2 = 0$), so that Eq. (34) can be written as

$$\left(\frac{\omega^2 R^3}{\gamma}\right)\left(\frac{\rho_1}{n} + \frac{\rho_2}{(n+1)}\right) = (n-1)\left(n + 2 - \frac{\sigma_f^2 R}{\varepsilon_2 \gamma}\right) \tag{35}$$

We see that the $n = 0$ mode is not allowed if $\rho_1 \neq 0$ due to the incompressibility of the drop, and that the $n = 1$ mode is neutrally stable. As σ_f is increased, the first mode to become unstable is $n = 2$ with critical total charge $Q_T (Q_T = 4\pi R^2 \sigma_f)$

$$Q_T = 8\pi R(\varepsilon_2 \gamma R)^{1/2} \tag{36}$$

However if the drop has negligible density so that $\rho_1 = 0$, which is the case of a bubble within an infinite fluid with surface charge on the bubble fluid interface then the $n = o$ solution is allowed. In the absence of electrical forces, surface tension makes this case unstable by acting to collapse the bubble. However, with surface charge present, Coulombic repulsion could stabilize the system if

$$\sigma_f^2 R/\varepsilon_2 \gamma > 2 \tag{37}$$

For finite q_2, the $n = 1$ mode will be stable only if $q_2\sigma_f < 0$.

2. Perfectly Insulating Drop with Volume Charge Within a Uniformly Charged Region

We now consider a perfectly insulating spherical drop supporting a constant volume charge distribution within a uniformly charged region, as in Fig. (2). We include the possibility of having a point charge of value Q at r = o, and give the drop a dielectric constant different from the surrounding region to include polarization forces.

The equilibrium electric field distribution is

$$E_r = \begin{cases} \dfrac{Q}{4\pi\varepsilon_1 r^2} + \dfrac{q_1 r}{3\varepsilon_1} & r < R \\ \dfrac{Q}{4\pi\varepsilon_2 r^2} + \dfrac{(q_1 - q_2)R^3}{3\varepsilon_2 r^2} + \dfrac{q_2 r}{3\varepsilon_2} & r > R \end{cases} \tag{38}$$

which yields for the interfacial fields and field gradients

$$E_1 = \frac{Q}{4\pi\varepsilon_1 R^2} + \frac{q_1 R}{3\varepsilon_1} \; ; \; \left.\frac{dE_1}{dr}\right|_{r=R} = \frac{-Q}{2\pi\varepsilon_1 R^3} + \frac{q_1}{3\varepsilon_1}$$

$$E_2 = \frac{Q}{4\pi\varepsilon_2 R^2} + \frac{q_1 R}{3\varepsilon_2} \; ; \; \left.\frac{dE_2}{dr}\right|_{r=R} = \frac{-Q}{2\pi\varepsilon_2 R^3} + \frac{2q_1}{3\varepsilon_2} + \frac{q_2}{\varepsilon_2} \tag{39}$$

The interfacial boundary conditions result in the relations

$$\hat{\xi}_1 = \hat{\xi}_2 \equiv \hat{\xi} \; , \; \hat{\phi}_1 = \hat{\phi}_2 \equiv \hat{\phi}$$

$$\varepsilon_1 \hat{e}_{r1} = \varepsilon_2 \hat{e}_{r2} \tag{40}$$

$$\hat{p}_1 - \hat{p}_2 + \varepsilon_2 E_2 \hat{e}_{r2} - \varepsilon_1 E_1 \hat{e}_{r1} - \left(\frac{\gamma}{R^2}\right)(n-1)(n+2)\hat{\xi} = 0$$

From the general relations of Eqs. (27) and (28) we obtain

$$\hat{p}_1 = \rho_1 \omega^2 R \hat{\xi}/n - q_1 \hat{\phi}$$

$$\hat{p}_2 = -\rho_2 \omega^2 R \hat{\xi}/(n+1) - q_2 \hat{\phi} \tag{41}$$

$$\varepsilon_1 \hat{e}_{r1} = -(\varepsilon_1 n/R)(\hat{\phi} + E_1\hat{\xi}) + \varepsilon_1 \left.\frac{dE_1}{dr}\right|_{r=R} \hat{\xi}$$

$$\varepsilon_2 \hat{e}_{r2} = \frac{\varepsilon_2(n+1)}{R}(\hat{\phi} + E_2\hat{\xi}) + \varepsilon_2 \left.\frac{dE_2}{dr}\right|_{r=R} \hat{\xi} \qquad (42)$$

Substitution of Eqs. (41) and (42) into Eq. (40) yields the formidable dispersion relation

$$\left(\frac{\omega^2 R^3}{\gamma}\right)\left(\frac{\rho_1}{n} + \frac{\rho_2}{(n+1)}\right) = (n-1)(n+2)$$

$$+ \frac{(q_1 - q_2)R^3}{\gamma[\varepsilon_2 + (\varepsilon_1 + \varepsilon_2)n]}\,[q_1 - q_2 - (2n+1)\varepsilon_2 E_2/R] \qquad (43)$$

$$+ \frac{(\varepsilon_2 - \varepsilon_1)R^2 E_2}{\gamma[\varepsilon_2 + (\varepsilon_1 + \varepsilon_2)n]}\left[\frac{q_1\varepsilon_2(n+1)}{\varepsilon_1} + q_2 n - \frac{E_1}{R}\left[2[\varepsilon_2 + (\varepsilon_1 + \varepsilon_2)n] + n(n+1)(\varepsilon_2 - \varepsilon_1)\right]\right]$$

In Eq. (43), the last term on the right represents the polarization surface force due to the difference in permittivities and the fields from the space and point charges. The other electric term represents Coulomb forces acting even in the absence of a polarization. For simplicity, we examine Eq. (43) in various limits.

BOHR LIMIT ($\varepsilon_1 = \varepsilon_2 \equiv \varepsilon$, $q_2 = 0$, $Q = 0$).

In the liquid drop model of fission of the nucleus, Bohr and Wheeler considered an isolated uniformly charged spherical drop under surface tension [12]. Sufficient charging of the drop results in instability, which in the context of the nucleus amounts to nuclear fission. Under these conditions Eq. (43) can be reduced to

$$\left(\frac{\omega^2 R^3}{\gamma}\right)\left(\frac{\rho_1}{n} + \frac{\rho_2}{(n+1)}\right) = \frac{(n-1)}{(2n+1)}\left[(2n+1)(n+2) - \frac{2q_1^2 R^3}{3\varepsilon\gamma}\right] \qquad (44)$$

The $n = o$ mode is not allowed for finite ρ_1 due to the drop incompressibility. The $n = 1$ mode is neutrally stable, while the first mode to become unstable as q_1 is increased occurs for $n = 2$ with total charge Q_T($Q_T = 4\pi R^3 q_1/3$)

$$Q_T = 8\pi R(5\varepsilon\gamma R/6)^{1/2} \qquad (45)$$

We see from Eq. (45) compared to Eq. (36) that a perfectly conducting drop could support slightly more total charge before becoming unstable than the

perfectly insulating drop.

NO POLARIZATION EFFECTS ($\varepsilon_1 = \varepsilon_2 \equiv \varepsilon$)

There are no polarization forces if the dielectric constants of the two regions are the same. Generalizing the Bohr limit by allowing finite q_2 and Q simplifies Eq. (43) to

$$\omega^2\left(\frac{\rho_1}{n} + \frac{\rho_2}{(n+1)}\right) = \left(\frac{\gamma}{R^3}\right)(n-1)(n+2) + \frac{(q_1 - q_2)^2}{\varepsilon(2n+1)} - \frac{(q_1 - q_2)}{\varepsilon}\left(\frac{q_1}{3} + \frac{Q}{4\pi R^3}\right) \tag{46}$$

The second term on the right of Eq. (46) is a self-field term and is always stabilizing while the last term is an imposed field term and can be either stabilizing or destabilizing depending on the relative charge polarities. A sufficient condition for stability is for the last term in (46) to be negative.

NEUTRAL DROP ($q_1 4\pi R^3/3 + Q = 0$)

If the total charge within the spherical drop is zero, the equilibrium interfacial electric field is also zero. Under this condition even if the permittivities of the two regions are different, there would be no polarization effects as the interfacial electric field is zero. However, due to the electric field gradient, there is still an electromechanical coupling so that Eq. (46) is still appropriate with the last term being zero. This system is stabilized by the space charge. In fact, for the case of a bubble where $\rho_1 = 0$ the destabilizing nature of surface tension for the n = o mode can be opposed so that the system becomes stable if the difference of charge densities are of sufficient magnitude such that

$$(q_1 - q_2)^2 \geq 2\varepsilon\gamma/R^3 \tag{47}$$

UNIFORM SPACE CHARGE DENSITY THROUGHOUT ($q_1 = q_2$)

The interplay of the polarizability and the space charge is demonstrated by considering a case in which the respective space charges are equal ($q_1 = q_2$). Then the second term on the right of Eq. (43) drops out. If the relative permittivities were also equal, there would be no electromechanical effect.

IV. Gravitational Stability of Spherical Drops

For the space charge analysis, the mass density only contributed through inertial terms. For the self-gravitating counterpart, the mass density will also contribute via the force law of Eq. (4). We now consider the gravitational analog to Fig. 2 where a fluid sphere with mass density ρ_1 is immersed within an infinite medium of mass density ρ_2. A point mass M is at the origin r = 0. The equilibrium gravitational field in the negative radial direction at the interface is

$$g = \frac{MG}{R^2} + \frac{4\pi G\rho_1 R}{3} \tag{48}$$

We can use the results obtained for a perfectly insulating drop with volume charge within a uniformly charged region by using the dual relations of Eq. (5), substituted into Eq. (43) with $\varepsilon_1 = \varepsilon_2$ to obtain the self-gravitating dispersion relation

$$\omega^2\left(\frac{\rho_1}{n} + \frac{\rho_2}{(n+1)}\right) = \left(\frac{\gamma}{R^3}\right)(n-1)(n+2) - \frac{4\pi G(\rho_1 - \rho_2)^2}{(2n+1)} + 4\pi G(\rho_1 - \rho_2)\left(\frac{\rho_1}{3} + \frac{M}{4\pi R^3}\right) \tag{49}$$

KELVIN'S MODES

Kelvin first considered this problem in the limit, M = 0, $\rho_2 = 0$, and $\gamma = 0$ [13]. Then Eq. (49) describes the oscillation frequencies of an inviscid liquid globe under its own gravitational field.

$$\omega^2 = \frac{8\pi G\rho_1 n(n-1)}{3(2n+1)} \tag{50}$$

HOLLOW DROP

In the other extreme, a spherical void (M = 0, $\rho_1 = 0$) within a large medium is always unstable for n = 0 and n = 1 as the right hand side of Eq. (49) is always negative and thus unstable. Unlike the space charge dual, self-gravitating forces can never stabilize a spherical bubble.

V. Concluding Remarks

Because the perturbation electrical and gravitational force was curl free within a homogeneous layer, the broad class of interactions involving liquids with discrete stratifications in mass density, charge density, and permittivity are representable as surface coupled interactions. The transfer relations derived are useful since they relate interfacial variables that appear in the boundary conditions. The techniques developed here may also be used for other perturbation volume forces which are curl free within a homogeneous layer, or for surface forces which only act at an interface. Any system which is described by Poisson's equation and a Coulomb force law can be immediately solved using similar dual relations as in Eq. (5).

Since the fluids are modeled as inviscid, force equilibrium at the interface required only a normal stress balance. No electrical shear stresses are allowed because of the absence of viscous shear forces to oppose the electrical forces. A more general analysis which would allow electrical shear forces must include viscosity. Then the mechanical terminal relations must include relations between the shear and normal forces and the shear and normal interfacial displacements. Force equilibrium at the interface requires both a normal and shear stress balance. This results in the mechanical transfer relations becoming 4 x 4 matrices, rather than the simpler 2 x 2 matrices considered here, greatly increasing the algebraic complexity.

ACKNOWLEDGMENT

This work was supported by the National Science Foundation under Grant No. GK-37594.

REFERENCES

1. J. R. Melcher, "Field Coupled Surface Waves," M.I.T. Press, Cambridge, Mass. (1963).
2. E. B. Devitt and J. R. Melcher, "Surface Electrohydrodynamics with High Frequency Fields," Phys. Fluids 8, No. 6 (June 1965).
3. M. Zahn and J. R. Melcher, "Space Charge Dynamics of Liquids," Phys. Fluids 15, No. 7 (July 1972).
4. M. Zahn, "Dynamics of Stratified Liquids in the Presence of Space Charge," Phys. Fluids 15, No. 8 (August 1972).
5. M. Zahn, "Space Charge Coupled Interfacial Waves," Phys. Fluids 17, No. 2 (February 1974).
6. M. Zahn, "Surface Coupled Interfacial Waves for Self-Gravitating Systems," Phys. Fluids, (to be published).
7. J. R. Melcher and W. J. Schwarz, "Interfacial Relaxation Overstability In a Tangential Electric Field," Phys. Fluids 11, No. 12 (December 1968).
8. J. R. Melcher and C. V. Smith, Jr., "Electrohydrodynamic Charge Relaxation and Interfacial Perpendicular-Field Instability," Phys. Fluids 12, No. 4 (April 1969).
9. M. Zahn and R. J. Sojka, "Space Charge Dynamics of Viscous Liquids," Phys. Fluids, (to be published).
10. R. J. Turnbull and J. R. Melcher, "Electrohydrodynamic Rayleigh-Taylor Bulk Instability," Phys. Fluids 12, No. 6 (June 1969).
11. Lord Rayleigh, "On the Equilibrium of Liquid Conducting Masses Charged with Electricity," Phil. Mag. 14, p. 184 (1882).
12. N. Bohr and J. A. Wheeler, "The Mechanism of Nuclear Fission," Phys. Rev. 56, p. 426 (Sept. 1939).
13. S. Chandrasekhar, "Hydrodynamic and Hydromagnetic Stability," Clarendon Press, Oxford (1961).

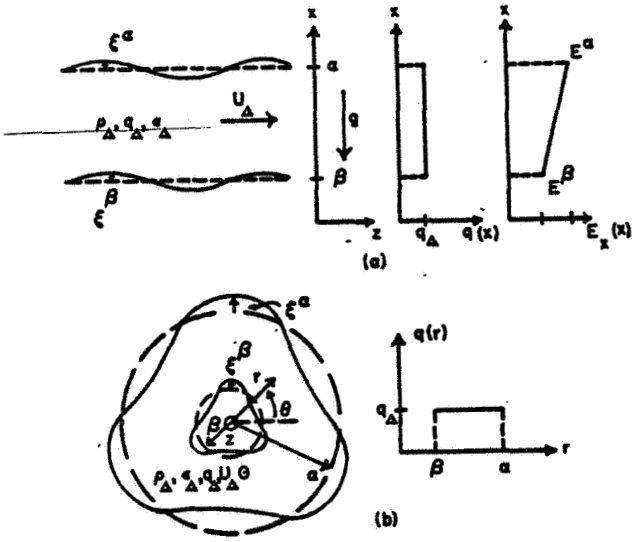

Fig. 1 Prototype layers of incompressible, inviscid, perfectly insulating fluids, supporting uniformly distributed charge densities, (a) planar geometry, (b) cylindrical or spherical geometry.

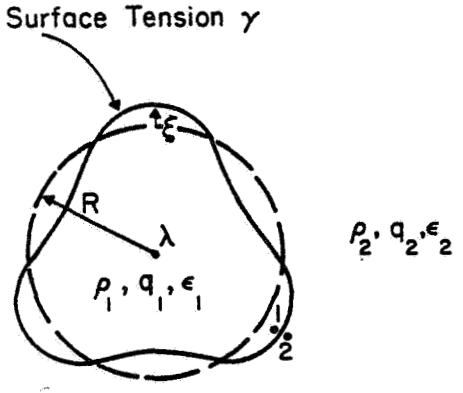

Fig. 2. Cross section of an initially spherical drop of radius R with charge density q_1, mass density ρ_1 and permittivity ϵ_1 placed within an infinite medium with charge density q_2, mass density ρ_2 and permittivity ϵ_2. A point charge ($Q = \lambda$) is at r = 0.