

Thermal Background Noise Limitations

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Modern detection systems are increasingly limited in sensitivity by the background thermal photons which enter the receiving system. This paper derives expressions for the fluctuations of detected thermal radiation. Incoherent and heterodyne detection processes are considered. The paper is intended to be tutorial in style. Many good references to the subject of photon detection statistics are given.

I. Introduction

An important consideration in evaluating and designing sensitive receiving systems for astronomy, deep space communications, and other related activities (such as the search for extraterrestrial intelligence), in which the goal is to detect the presence of a weak signal in the presence of noise, is the evaluation of the noise. In general, noise is due to the combined presence of both the signal and various sources of unwanted noise. The unwanted noise may arise in the detector itself, from the immediate surroundings of the detector (i.e., emission from the telescope), or from the (background) emission which enters the telescope and is intercepted by the detector. Background emission may originate from the atmosphere, from nearby objects, or from the cosmic background itself.

When the detection device is a square-law detector, the noise manifests itself as fluctuations in the output voltage (or current) from the detector $V(t)$, where

$$V(t) = rP(t) \quad (1)$$

In this expression $P(t)$ is the instantaneous power incident on the detector, and r is a constant of proportionality.

A measure of the fluctuations in $V(t)$ about its mean is the variance defined by

$$\begin{aligned} \text{var} [V(t)] &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} (V(t) - \bar{V})^2 dt \\ &= r^2 \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} (P(t) - \bar{P})^2 dt \quad (2) \end{aligned}$$

This parameter is frequently encountered in estimating the signal-to-noise ratio of a system. In general, a calculation of the variance requires a knowledge of both the signal and noise, as well as the frequency response of the system.

Frequently, there is a desire to know the variance when $P(t)$ is due to a finite band of thermal radiation at temperature T .

In particular, this result is useful when the dominating source of noise is bandlimited thermal radiation whose power greatly exceeds the signal power. This situation is commonly referred to as the background limited noise condition. Numerous approximations of the background limited noise appear in the literature, especially in the thermal limit, where $h\nu/kT \ll 1$ and in the quantum limit where $h\nu/kT \gg 1$. However, the more general expressions which describe thermal radiation over the entire range of $h\nu/kT$ are not as readily available. In this note, I derive some useful expressions related to the detection of thermal radiation and discuss some of the results. The purpose of the paper was primarily to acquaint the author with the trends of background limited noise. It is printed here in the anticipation that others may find the discussion useful. Many important points related to the detection process and signal/noise ratios are not covered in this note. In particular, I omitted all discussions of the type of detector involved in the detector process. The reader is referred to the Bibliography for additional discussions.

II. Derivations

A. Photons as Independent Particles

We assume initially that the radiation field incident on the detector is a stream of statistically independent photons. For this situation, each photon produces a power

$$p(t) = h\nu \delta(t - t_0) \quad (3)$$

and an energy given by

$$E = \int_{-\infty}^{+\infty} p(t) dt = h\nu \quad (4)$$

Equation (3) is not strictly correct since photons are bosons and consequently do not occupy available energy states independently. In particular, the statistics of arrival times of photons at a detector indicate that photons are "bunched". Nevertheless, we will continue with this assumption and apply a correction for the bunching later on. The justification for this approach is threefold: (1) it is relatively straightforward; (2) it introduces the terms in the results in a clear manner; and (3) the analysis leads to the quantum or particle-like noise term.

Since the total power received by the detector is a linear superposition of the sum of all the photons reaching the detector during the interval $(0, \theta)$, we can write for the total power received by the detector when K photons arrive

$$P_K = \sum_{k=1}^K h\nu \delta(t - t_k) \quad (5)$$

where t_k is the arrival time of the k photon. These arrival times are random and unknown.

Series such as that given by Eq. (5) can be evaluated using Campbell's theorem, which is discussed in detail by Rice (Ref. 1). Campbell's theorem states that (1) the average value of $P(t)$ (averaged over both time and over all values of k) is given by

$$\overline{P(t)} = R \int_{-\infty}^{\infty} p(t) dt \quad (6)$$

and (2) the variance is given by

$$\text{var } [P(t)] = \Delta P^2 = R \int_{-\infty}^{+\infty} p^2(t) dt \quad (7)$$

where R is the average number of photon arrivals per second. The principal quantity of interest in this note is the variance, for this quantity is related to the precision of physical measurements.

To evaluate the variance, we need to evaluate both R and the integral

$$\int_{-\infty}^{+\infty} p^2(t) dt$$

R can be calculated directly from Eq. (6), using Eq. (4) and Planck's radiation law to calculate the average power. Planck's law gives for the radiance of a blackbody radiator at temperature T and frequency ν

$$B = \frac{2 h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} \quad (8)$$

where

B = radiance, $\text{W m}^{-2} \text{Hz}^{-1} \text{rad}^{-2}$

h = Planck's constant ($= 6.63 \times 10^{-34}$ joule sec)

ν = frequency, Hz

c = velocity of light, ($= 3 \times 10^8$ m/sec $^{-1}$)

k = Boltzmann's constant, ($= 1.38 \times 10^{-23}$ joule K^{-1})

T = temperature, K

Thus a blackbody source of emission at temperature T produces a power given by

$$P = A\Omega B d\nu \quad (9)$$

where Ω is the solid angle subtended by the source, A is the effective collecting area of the detector (telescope), and $d\nu$ is the bandwidth of the received radiation. Thus we have for R

$$R = \frac{A\Omega B}{h\nu} = 2A\Omega \left(\frac{\nu}{c}\right)^2 \frac{d\nu}{e^{h\nu/kT} - 1} \quad (10)$$

To evaluate the integral in Eq. (7), we use Parseval's theorem to rewrite it as

$$\Delta P^2 = 2R \int_0^\infty |S(f)|^2 df \quad (11)$$

where $S(f)$ is the Fourier transform of $p(t)$. Hence we have

$$|S(f)| = \int_{-\infty}^{+\infty} p(t) e^{-2\pi ift} dt = h\nu \quad (12)$$

The variance or mean square value of the power fluctuation is derived by combining Eqs. (11) and (12) to yield

$$\Delta P^2 = 2R(h\nu)^2 \int df \quad (13)$$

If there is a postdetection filter in the system which passes a range of frequencies Δf , then the power content of the fluctuating power is given by

$$\Delta P^2 = 2R(h\nu)^2 \Delta f \quad (14)$$

Substituting for R in Eq. (14), we obtain

$$\Delta P^2 = 4A\Omega \left(\frac{\nu}{c}\right)^2 \frac{1}{e^{h\nu/kT} - 1} (h\nu)^2 d\nu \Delta f \quad (15)$$

It is of interest to note that the familiar shot noise equation can be derived from Eq. (14) by substituting $E = h\nu$ for the photon energy and $P = Rh\nu$ for the average power delivered to the detector. This result is a consequence of the fact that shot noise is related to the statistics of independent particles (our initial assumption), which obey Poisson statistics. Thus we have

$$\Delta P = (2EP\Delta f)^{1/2} \quad (16)$$

B. Photons as Bosons

We now consider the problem of the nonindependence of the photons. In a Bose-Einstein system composed of weakly interacting photons at thermal equilibrium, the average number of photons in a given energy state (the occupancy) is given by (Ref. 2)

$$\bar{n} = \frac{1}{e^{h\nu/kT} - 1} \quad (17)$$

and the mean square fluctuations are given by

$$(n - \bar{n})^2 = \bar{n}(\bar{n} + 1) \quad (18)$$

An analysis which takes into account the Bose-Einstein statistics leads to a variance given by (Refs. 3, 4)

$$\Delta P^2 = 4\Omega A \left(\frac{\nu}{c}\right)^2 (h\nu)^2 n(n+1) d\nu \Delta f \quad (19)$$

Comparing this expression with Eq. (15) reveals that the fluctuations are higher by $(1+n)$. The independent photon calculations are corrected for Bose-Einstein statistics by multiplying by the factor $(n+1)$. The term $(n+1)$ is known as the Bose factor. This factor is a monotonically decreasing function of $h\nu/kT$, approaching unity for large values of $h\nu/kT$. Photons behave like individual particles in this spectral region. In the Rayleigh-Jeans limit ($kT \gg h\nu$), the fluctuations are larger than those of independent particles. This increased noise power is due to the wavelike nature of photons. It results from each mode in the field beating with itself to produce a mean square power fluctuation. In the Rayleigh-Jeans limit, the variance becomes

$$\Delta P^2 = 4\Omega A \left(\frac{\nu}{c}\right)^2 (kT)^2 d\nu \Delta f \quad (20)$$

We obtain the classical low-frequency approximation

$$\Delta P = P_0 \sqrt{\frac{2}{d\nu\tau}} \quad (21)$$

by taking

$$\Omega A \left(\frac{\nu}{c}\right)^2 = 1$$

and by defining $P = kT d\nu$ and $\Delta f = 1/2t$.

C. Noise Equivalent Power

A standard measure of sensitivity, used in the optical and infrared spectral regions, is the noise equivalent power (NEP),

defined as the incident signal power required to produce a detector signal equal to the rms noise power in a 1-Hz post-detection bandwidth. We have from Eq. (19)

$$NEP = \left[4\Omega A \left(\frac{\nu}{c}\right)^2 (h\nu)^2 n(n+1) d\nu \right]^{1/2} \quad (22)$$

The units of NEP are watts/ $\sqrt{\text{Hz}}$, although occasionally authors incorrectly drop the $\sqrt{\text{Hz}}$ term. The reference bandwidth $d\nu$, field of view Ω , and detector area A should be specified with each NEP.

The NEP reduces to the following expression in the thermal limit

$$NEP = 2kT \sqrt{\Omega A \left(\frac{\nu}{c}\right)^2 d\nu} \quad (23)$$

D. Heterodyne Detection

We consider here a square law mixer as defined by Eq. (1). For a heterodyne receiver, $P(t)$ is given by (e.g., Blaney, Ref. 5)

$$P(t) = [(2P_L)^{1/2} \cos \omega_L t + (2P_B)^{1/2} \cos(\omega_B t - \phi)]^2 \quad (24)$$

where P_L is the local oscillator power (single polarization), P_B is the signal power, and ϕ is the phase difference between the local oscillator and the signal. As in our previous discussion, we assume that P_B arises from the radiation from a thermal source inside the antenna pattern. Hence we are interested in the noise fluctuations due to coherently detected thermal radiation.

The resultant form of $P(t)$ on dropping high-frequency terms is

$$P(t) \propto P_L + P_B + 2\sqrt{P_L P_B} \cos[(\omega_L - \omega_B) - \phi] \quad (25)$$

The first two terms create a dc current in the mixer while the last term produces the IF current. The variance of $P(t)$ includes the sum of the variances of the local oscillator power, the background power, and the cross product of the local oscillator voltage and the background voltage. The resultant noise power has the form

$$\Delta P^2 \propto h\nu P_L d\nu + h\nu P_L A \Omega \left(\frac{\nu}{c}\right)^2 \bar{n} d\nu + (NEP)^2 \Delta f \quad (26)$$

Two important points to note about this equation are the following. The variance due to the local oscillator power is due

only to the random arrival times of the photons. Poisson statistics apply since the phase of the local oscillator power is fixed. The second point is that the local oscillator power is a multiplicative factor in the first two terms but not the third. Since the "signal" in a heterodyne system is multiplied by the local oscillator power, the local oscillator power may be increased to the point where the background power term is negligible. For this condition we have the following

$$\Delta P^2 \propto P_L h\nu d\nu + P_L h\nu A \Omega \left(\frac{\nu}{c}\right)^2 \bar{n} d\nu \quad (27)$$

III. Discussion

The principal results derived above for the rms noise power which results from the detection of blackbody radiation are as follows:

Incoherent detection (rms power)

$$\Delta P = 2h\nu \sqrt{\bar{n}(\bar{n}+1)} \sqrt{\Omega A \left(\frac{\nu}{c}\right)^2 d\nu \Delta f} \quad (28)$$

Heterodyne detection (spectral power)

$$\Delta P \propto h\nu \left[1 + A \Omega \left(\frac{\nu}{c}\right)^2 \bar{n} \right] d\nu \quad (29)$$

These equations are frequently interpreted in terms of either the NEP or in terms of an equivalent temperature. The equivalent temperature representation of the incoherent detection process follows from taking the input and output bandwidths to be equal and the beam filling factor to be unity. It follows that Eqs. (28) and (29) can then be written as

$$\Delta P = 2h\nu \sqrt{\bar{n}(\bar{n}+1)} d\nu \quad (30)$$

$$\Delta P \propto h\nu (\bar{n}+1) d\nu \quad (31)$$

Noting that these equations are now in the form of a spectral density times a bandwidth, the equivalent temperature is defined as follows:

$$T_E = \frac{2h\nu}{k} \sqrt{\bar{n}(\bar{n}+1)} \quad (32)$$

$$T_E = \frac{h\nu}{k} (\bar{n}+1) \quad (33)$$

Figures 1 and 2 show equivalent temperatures T_E for incoherent and heterodyne detectors. We note from these figures

that in the thermal limit the equivalent temperature equals the source temperature for both the incoherent and linear amplifier cases. However, in the quantum limit, the linear amplifier noise increases like $h\nu$, whereas the incoherent noise power decreases. Thus, incoherent noise power is much less than the

noise power following a heterodyne detector if the frequency is sufficiently high and the background emission low. In the radio spectral region, the 2.7 K cosmic background radiation provides a natural noise source and there is no disadvantage of using a heterodyne detector (or a linear amplifier).

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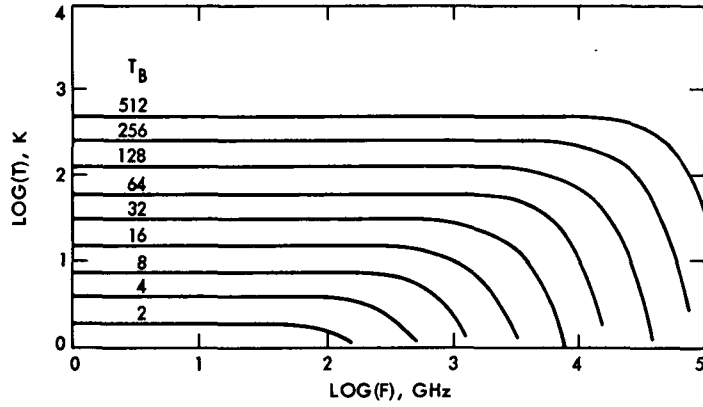


Fig. 1. Equivalent temperatures of incoherent detected thermal noise for a number of different background temperatures for 2 to 512 K

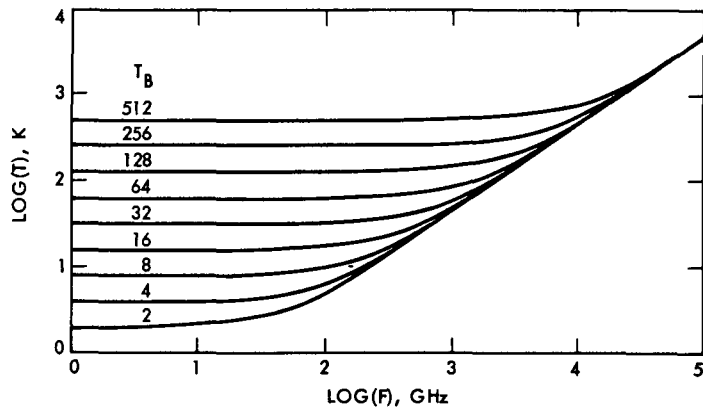


Fig. 2. Equivalent temperatures of heterodyne detected thermal noise for a number of different background temperatures from 2 to 512 K