LINEAR FORCE AND MOMENT EQUATIONS FOR AN ANNULAR SMOOTH
SHAFT SEAL PERTURBED BOTH ANGULARLY AND LATERALLY*

J. Fenwick
Rocketdyne Division
Rockwell International
Canoga Park, California 91304

R. DiJulio, M. C. Ek, R. Ehrgott,
H. Green, and S. Shaolian
California State University at Northridge
Northridge, California 91330

SUMMARY

Coefficients are derived for equations expressing the lateral force and pitching moments associated with both planar translation and angular perturbations from a nominally centered rotating shaft with respect to a stationary seal.

The coefficients for the lowest order and first derivative terms emerge as being significant and are of approximately the same order of magnitude as the fundamental coefficients derived by means of Black's equations. Second derivative, shear perturbation, and entrance coefficient variation effects are adjudged to be small.

The outcome of the investigation delineated in this report defines the coefficients of the equations:

$$\tau = A\varepsilon + B\dot{\varepsilon} + C\ddot{\varepsilon} + D\alpha + E\dot{\alpha} + F\ddot{\alpha}$$
$$\bar{F} = a\varepsilon + b\dot{\varepsilon} + c\ddot{\varepsilon} + d\alpha + e\dot{\alpha} + f\ddot{\alpha} \tag{1}$$

The assumptions utilized, mathematical means employed, and conclusions derived thereby will be the objective of this report.

The effects of the additional terms upon a typical rotordynamic system are presented.

INTRODUCTION

The factors leading to increasing pressures, speeds, and temperatures in jet engines, high performance pumps, and turbomachinery for rocket engine applications have been widely noted in the literature, e.g., Rothe (ref. 1).

The increased incidence of stability problems as a function of increased power density, particularly the phenomenon of "subsynchronous whirl," has been noted by many authors.

Among the design features strongly contributing to this phenomenon in a wide variety of high-powered turbomachinery are annular shaft seals. These seals are in general utilized to separate regions of high and low gas or liquid pressure. In addition to their strong effects as pseudo bearings and destabilizing devices, they offer, when

130

properly configured, in many designs the only real possibility for the introduction of damping in rotating systems. Various authors have delineated the extent of both the problems and remedies surrounding these possibilities; for example, Alford (ref. 2), Ek (ref. 3), Childs (ref. 4), and Gunter (ref. 5).

A very significant part of the understanding of the dynamic behavior of annular smooth shaft seals has come from the pioneering work of Henry Black of Heriott Watt University, Edinburgh, United Kingdom, who in a series of papers (refs. 6-12) defined the effect of shaft displacements with both long and short seal assumptions up to and including the second derivative of the displacement. A number of workers have extended his work significantly, these being Jensen (refs. 7 and 8), Hirs (ref. 13), Childs (ref. 4), and Alliare (refs. 5 and 14). The recent trend is to shift to a modification of the bulk flow theory to define more closely a set of equations approximating experimental results [Hirs (ref. 13)].

On many significant recent problems, e.g., the problem on the high-speed rotating machinery for the Space Shuttle Main Engine (SSME) [Ek (ref. 3)], Black's equations were the only analytical representation available for significant computer modeling. Present efforts to extend this work include continuing work by Allaire (ref. 14) at the University of Virginia, Childs (ref. 4) at Texas A&M, and Fleming (ref. 12) of NASA Lewis, as well as experimental work with liquid oxygen and hydrogen at the Rocketdyne Division of Rockwell International.

Figure 1. - Orientation of references axes for seal equations.

This study represents a departure from the general approach being carried on by other investigators in that it explores the effect of pitching moments and angulation in the seal, as shown schematically in Fig. 1.
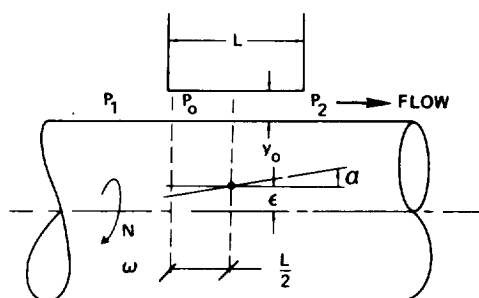
NOMENCLATURE

The nomenclature in general is that employed by Black (ref. 6) and specific nomenclature is introduced as comments in the body of the nomenclature table, in the figures, or presented below. Additional functions are delineated in the text as required.

$\left.\begin{array}{l}\text{A-F}\\\text{a-f}\end{array}\right\}$ = coefficients in force equations

b = gap thickness, $y(x,t)$, inches

D = d/dt, differential operation with regard to time

$\bar{E}$ = shaft elastic modulus

$\bar{F}$ = force on seal, total due to perturbations

I = shaft area moment of inertia, m^4 ($in.^4$)

K = effective bearing spring rate, N/m (lb/in.)

F_n = force on seal, n is defined in the text

H = $PR\pi\sigma/6\lambda(1+\xi+2\sigma)$, general coefficient for moment and force equations $\xi = 0.5$

H_1 = $PR\pi\sigma L/\lambda\gamma$

θ = $\partial F/\partial\varepsilon$, N/m^3 ($lb/in.^3$)

ν = kinematic viscosity, m^2/s ($in.^2/sec$)

131

L = length of seal, m (in.)

ℓ = component spacing on rotor, m (in.)

N = speed of rotation, rpm

P = total pressure difference across seal, Pa (lb/in.^{2})

p = pressure within passage, $p(x,t)$, Pa (lb/in.^{2})

R = shaft radius, m (in.)

R_a = axial Reynolds No. $2Vy_0/\nu$

R_r = rotating Reynolds No. $R\omega y_0/\nu$

T = passage time, L/V, seconds

t = time, variable, seconds

u = axial fluid velocity, $u(x,t)$, m/s (in./sec)

V = mean fluid velocity, m/s (in./sec)

W = weight of rotor, kg (lb)

x = axial distance, independent variable, m (in.)

y,z = radial displacement of rotor mass

α = angular displacement

γ = $1.5 + 2\sigma$, $\xi = 0.5$

δ = weight density, kg/m^3 (lb/m^3)

ε = lateral displacement, $Y(t)$, m (in.)

λ = friction coefficient, $\Delta p = \dfrac{\lambda L}{r_n}\dfrac{V^2}{2g}$,

$r_n = \dfrac{\text{area}}{\text{wetted perimeter}} , \dfrac{b}{2}$

$= 0.079\, R_a^{-1/4}\left[1 + \left(\dfrac{7}{8}\dfrac{R_r}{R_a}\right)^2\right]^{3/8}$

loss factor per Yamada (ref. 15)

μjk = functions of σ, coefficients for the perturbation terms

μ = absolute viscosity, Pa-s (lbf-sec/in.2)

ξ = entry loss coefficient = 0.5 nominally

ρ = mass density, $\dfrac{\delta}{g}$, kg-s2/m4 (lb-sec2/in.4)

σ = $\lambda L/y_0$

τ = pitching moment taken about $x = \dfrac{L}{2}$, N-m (lb-in.)

τ_0 = pitching moment at $x = 0$, N-m (lb-in.)

ϕ = $\partial F/\partial \alpha$, N/rad (lb/rad)

χ = $\partial \tau/\partial \varepsilon$, N (in.-lb/in.)

ψ = $\partial \tau/\partial \alpha$, N-m (in.-lb/rad)

ω = circular frequency, rad/sec

$\bar{\omega}$ = design speed, rad/sec

ASSUMPTIONS

In the following equations, assumptions are made as follows:

1. The system (shaft within seal) is assumed to be centered.

2. Perturbations about the nominal centered position are small.

3. Whenever two second order terms are multiplied, they are assumed to be higher order, i.e., the product approaches zero.

4. Short seal theory is assumed, i.e., perturbations have negligible effect in the tangential direction.

5. The period of time required for a particle of fluid to travel from the inlet to the discharge is small compared to the frequency of the system.

6. There is no change of properties in the working fluid as it passes through the process.

7. The entrance coefficient into the seal is assumed to be 0.5.

132

8. The effect of fluid shear variation on the shaft moment is neglected in the fundamental equations.

9. Total linearity is assumed, i.e., any of the effects can be added individually with no interdependence between individual effects.

10. Tangential fluid velocity equal to one-half the shaft tangential velocity is developed immediately at the seal entrance.

11. Perturbations are planar; however, Yamada's (ref. 15) work in defining λ for rotating shaft effects applies.

APPROACH

The process of deriving the fundamental equations departs from that used by Black (ref. 6) in three important respects: (1) clearance is expressed as a function of both x and α, (2) the equations for continuity and velocity are completely rewritten as a consequence of (1), and (3) the order of integration is reversed. In addition, the first derivations lead to a result in which all perturbations, moments, and forces are taken at x = 0. This is done because of great simplification of the boundary conditions. These are then translated to perturbations about x = L/2, and in the case of the moments, τ_o is translated to $\tau|_{L/2}$. Explanations of each step and necessary symbol derivations occur as needed in the text.

DERIVATION OF FUNDAMENTAL EQUATIONS

We first consider two plates of length L and unit width, nominally y in. apart, with the top plate fixed, the bottom plate perturbed: perturbations are positive upward and counterclockwise, as are forces and moments. Flow is positive to the right (Fig. 2). No fluid moves tangentially or normal to the surfaces of the plates.
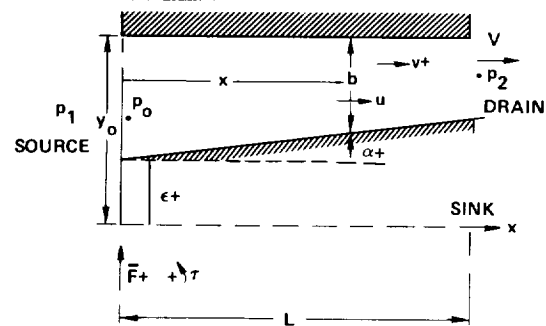
Figure 2. - Derivation of fundamental equations.

The steady-state pressure distribution through the flow process is reflected by the following relationships:

$$P = P_1 - P_2 = \rho(1.5 + 2\sigma)\frac{V^2}{2} \tag{2}$$

$$P_1 - P_o = \frac{1.5}{2}\rho V^2 \tag{3}$$

The fundamental equation for head loss through the flow process is

$$\frac{\partial h}{\partial x} = -\frac{\lambda}{b}\frac{u^2}{g} \tag{4}$$

From the Navier-Stokes equations for unsteady flow,

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + \frac{\lambda u^2}{b} \tag{5}$$

133

If a control volume of length dx, and bounded by the two plates, is now considered, the plates can be thought of as perturbed streamlines. The general continuity equation for an unsteady stream tube of fixed length ds is

$$\frac{\partial(\rho\vec{A})}{\partial t} + \frac{\partial}{\partial s}(\rho\vec{u}\cdot\vec{A}) = 0 \tag{6}$$

Since ρ = constant, $\vec{A} = b$, $\vec{u}\ \vec{A} = ub$, $s = x$,

$$\frac{\partial b}{\partial t} + \frac{\partial(ub)}{\partial x} = 0 \tag{7}$$

or

$$b\frac{\partial u}{\partial x} + u\frac{\partial b}{\partial x} + \frac{\partial b}{\partial t} = 0 \tag{8}$$

Now, approximately

or

$$\left.\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\alpha}{y_o}u + \frac{\dot{\varepsilon}}{y_o} + \frac{x}{y_o}\dot{\alpha} \\[2mm]
\frac{\partial u}{\partial x} - \frac{\alpha}{y_o}u &= \frac{\dot{\varepsilon} + x\dot{\alpha}}{y_o}
\end{aligned}\right\} \tag{9}$$

and $U(t) = V + v(t)$,

$$u = \frac{1}{y_o}\left(\dot{\varepsilon}x + \frac{x^2}{2}\dot{\alpha}\right) + V + v + \frac{V\alpha x}{y_o} \tag{10}$$

Higher order terms (HOT) are dropped. Now, integrate and set the following boundary conditions:

$$\text{at} \quad x = 0,\ p = p_o = p_1 - \frac{1.5}{2}\rho(V^2 + 2V\sigma)$$

and

$$\text{at} \quad x = L,\ p = p_2$$

$$-\frac{1}{\rho}p = \left(\dot{v} + \frac{2V\lambda}{y_o}v\right)x + \frac{x^2}{2y_o}\ddot{\varepsilon} + \left(\frac{Vx}{y_o} + \frac{\lambda V x^2}{y_o^2}\right)\dot{\varepsilon}$$

$$+ \frac{\lambda V^2 x}{y_o^2}\varepsilon + \frac{x^3}{6y_o}\dddot{\alpha} + \left(\frac{Vx^2}{y_o} + \frac{\lambda V x^3}{3y_o^2}\right)\dot{\alpha} \tag{11}$$

$$+ \left(\frac{V^2 x}{y_o} + \frac{3}{2}\frac{\lambda V^2 x^2}{y_o^2}\right)\alpha + \frac{\lambda V^2 x}{y_o} + c(t)$$

Therefore

$$c(t) = -\frac{p_1}{p_o} + \frac{1.5}{2}\rho(v^2 + 2Vv)$$

134

But

$$P_1 - P_2 = P = \rho(1.5 + 2\sigma)\frac{V^2}{2}$$

$$0 = \left[\dot{v} + \frac{V}{L}(1.5 + 2\sigma)v\right] + \frac{\sigma}{2\lambda}\left[\ddot{\varepsilon} + \frac{V}{L}2(1 + \sigma)\dot{\varepsilon} + \frac{V^2}{L^2}2\sigma\varepsilon\right]$$

$$+ \frac{L\sigma}{6\lambda}\left[\ddot{\alpha} + \frac{V}{L}2(3 + \sigma)\dot{\alpha} + \frac{V^2}{L^2}3(2 + 3\sigma)\alpha\right] \tag{12}$$

If now, $p = p_{steady} + \Delta p(t)$, then Eq. (11) becomes

$$-\frac{\Delta p}{\rho} = \left(\dot{v} + 2\frac{V\lambda}{y_o}v\right)x + \frac{x^2}{2y_o}\ddot{\varepsilon} + \left(\frac{Vx}{y_o} + \frac{\lambda Vx^2}{y_o}\right)\dot{\varepsilon}$$

$$+ \frac{\lambda V^2 x}{y_o^2}\varepsilon + \frac{x^3}{6y_o}\ddot{\alpha} + \left(\frac{Vx^2}{y_o} + \frac{\lambda Vx^3}{3y_o}\right)\dot{\alpha} \tag{13}$$

$$+ \left(\frac{V^2 x}{y_o} + \frac{3}{2}\frac{\lambda V^2 x^2}{y_o^2}\right)\alpha + 1.5\ Vv$$

DERIVATION AND SUMMARY OF FORCE AND PITCHING MOMENT EQUATIONS

The symbolism and form of the equations for a shaft is:

$$\left.\begin{array}{l}\tau = A\varepsilon + B\dot{\varepsilon} + C\ddot{\varepsilon} + D\alpha + E\dot{\alpha} + F\ddot{\alpha} \\ \bar{F} = a\varepsilon + b\dot{\varepsilon} + c\ddot{\varepsilon} + d\alpha + e\dot{\alpha} + f\ddot{\alpha}\end{array}\right\} \tag{14}$$

The fundamental assumption of linear independence and superposition is made, i.e.;

$$\left.\begin{array}{l}F_1 = a\varepsilon + b\dot{\varepsilon} + c\ddot{\varepsilon} \\[4pt] F_2 = d\alpha + e\dot{\alpha} + f\ddot{\alpha} \\[4pt] \tau_1 = A\varepsilon + B\dot{\varepsilon} + C\ddot{\varepsilon} \\[4pt] \tau_2 = D\alpha + E\dot{\alpha} + F\ddot{\alpha} \\[4pt] \bar{F} = F_1 + F_2\ ,\ \tau = \tau_1 + \tau_2\end{array}\right\} \tag{15}$$

In the above, all displacements, moments, and forces are taken about $x = L/2$. However, the first solutions will be taken about $x = 0$, and the results transferred to

$x = L/2$. The subscript convention at $x = 0$ will be

$$F_o = F_{10} + F_{20}\ ,\ \tau_o = \tau_{10} + \tau_{20} \tag{16}$$

135

For the cases about x = 0, certain simplifying parameters will be defined: the resulting form of the equations will be:

$$F_{10} = a_o \epsilon + b_o \dot{\epsilon} + c_o \ddot{\epsilon} = H \ (\mu_{10}\epsilon + \mu_{11} \ T\dot{\epsilon} + \mu_{12} \ T^2\ddot{\epsilon}) \tag{17}$$

$$F_{20} = d_o \alpha + e_o \dot{\alpha} + f_o \ddot{\alpha} = H \ (\mu_{20}\alpha + \mu_{21} \ T\dot{\alpha} + \mu_{22} \ T^2\ddot{\alpha}) \tag{18}$$

$$\tau_{10} = A_o \epsilon + B_o \dot{\epsilon} + C_o \ddot{\epsilon} = H_1 \ (\mu_{30}\epsilon + \mu_{31} \ T\dot{\epsilon} + \mu_{32} \ T^2\ddot{\epsilon}) \tag{19}$$

$$\tau_{20} = D\alpha + E\dot{\alpha} + F\ddot{\alpha} = H_1 \ (\mu_{40}\alpha + \mu_{41} \ T\dot{\alpha} + \mu_{42} \ T^2\ddot{\alpha}) \tag{20}$$

where $T = L/V$, and H, H_1, and $\mu_{j,k}$ will be defined as part of the derivation.

FORCE AND PITCHING MOMENT EQUATIONS FOR A SHAFT

The definition of pressure perturbations seen in Eq. (13) is for two plates of unit width. Utilizing these, the following force and pitching equations can be written by integrating around a perturbed shaft as in Fig. 1. For example,

$$\frac{F_o}{R\pi\rho} = \int_0^L \Delta p \ dx = - \left\{ \left(\dot{v} + \frac{2V\lambda v}{y_o} \right) \frac{L^2}{2} + \frac{L^3}{6y_o} \ddot{\epsilon} + \left(\frac{VL^2}{2y_o} + \frac{\lambda VL^3}{3y_o} \right) \dot{\epsilon} \right.$$
$$+ \frac{\lambda v^2 L^2}{2y_o^2} \epsilon + \frac{L^4 \ddot{\alpha}}{24 y_o} + \left(\frac{VL^3}{3y_o} + \frac{\lambda VL^4}{12 y_o} \right) \dot{\alpha}$$
$$\left. + \left(\frac{v^2 L^2}{2y_o} + \frac{\lambda v^2 L^3}{2y_o^2} \right) \alpha + 1.5 \ VvL \right\} \tag{21}$$

$$\frac{F_o}{R\pi\rho} = - \left\{ \frac{VL}{2} \left[\frac{L}{V} \dot{v} + (2\sigma + 3)v \right] + \frac{v^2 L}{6 y_o} \left[\frac{L^2}{V^2} \ddot{\epsilon} + \frac{L}{V} (3 + 2\sigma)\dot{\epsilon} + 3\sigma\epsilon \right] \right.$$
$$\left. + \frac{v^2 L^2}{24 y_o} \left[\frac{L^2}{V^2} \ddot{\alpha} + \frac{L}{V} (8 + 2\sigma)\dot{\alpha} + 12(1 + \sigma)\alpha \right] \right\} \tag{22}$$

and similarly,

$$\frac{\tau_o}{R\pi\rho} = - \left\{ \frac{VL^2}{3} \left(\frac{L}{V} \dot{v} + 2\sigma v + (1.5)^2 v \right) + \frac{Lv^2\sigma}{8\lambda} \left[\frac{L^2}{V^2} \ddot{\epsilon} + \frac{8L}{V} \left(\frac{1}{3} + \frac{\sigma}{4} \right) \dot{\epsilon} + \frac{8\sigma}{3} \epsilon \right] \right.$$
$$\left. + \frac{L^2 v^2 \sigma}{30\lambda} \left[\frac{L^2}{V^2} \ddot{\alpha} + \frac{30L}{V} \left(\frac{1}{4} + \frac{\sigma}{15} \right) \dot{\alpha} + 30 \left(\frac{1}{3} + \frac{3\sigma}{8} \right) \alpha \right] \right\} \tag{23}$$

After eliminating v, $\dot{v}$ between these equations (23-24) and (12-13), the forces and torques can be defined. This is done piece-wise, then finally superposed.

136

FORCE DUE TO LATERAL DISPLACEMENTS

This will be expressed in the form

$$F_{10} = a_o \epsilon + b_o \dot{\epsilon} + c_o \ddot{\epsilon} = H (\mu_{10}\epsilon + \mu_{11} T\dot{\epsilon} + \mu_{12} T^2\ddot{\epsilon}) \tag{24}$$

From Eq. (12), the equation expressing $v, \dot{v}$ as a function of ϵ, $\dot{\epsilon}$, $\ddot{\epsilon}$, is, with $(a, \dot{\alpha}, \ddot{\alpha} = 0)$

$$0 = \left[\dot{v} + \frac{V}{L} (1.5 + 2\sigma)v \right] + \frac{\sigma}{2\lambda} \left[\ddot{\epsilon} + \frac{V}{L} 2(1 + \sigma)\dot{\epsilon} + \frac{V^2}{L^2} 2\sigma\epsilon \right] \tag{25}$$

This can be expressed as

$$0 = [TD + \gamma]v + \frac{V}{L} \frac{\sigma}{2\lambda} [T^2D^2 + 2(1 + \sigma)TD + 2\sigma]\epsilon \tag{26}$$

Now, from Eq. (19), with $\alpha, \dot{\alpha}, \ddot{\alpha} = 0$

$$\frac{F_{10}}{R\pi\rho} = - \left\{ \frac{VL}{2} [TD + (2\sigma + 3)]v + \frac{V^2L}{6y_o} [T^2D + (3 + 2\sigma)TD + 3]\epsilon \right\} \tag{27}$$

Eliminating v, $\dot{v}$ between these two equations (26-27) with $TD \ll \gamma$, eliminating (TD) of third and higher power,

$$F_{10} = H \{\mu_{10}\epsilon + \mu_{11} T\dot{\epsilon} + \mu_{22} T^2\ddot{\epsilon}\} \tag{28}$$

where

$$H = \frac{\sigma\pi RP}{6\lambda\gamma} , \quad \gamma = (1.5 + 2\sigma)$$

$$\mu_{10} = \frac{9\sigma}{\gamma}$$

$$\mu_{11} = \frac{(9 + 12\sigma + 4\sigma^2) \gamma - 9\sigma}{\gamma^2}$$

$$\mu_{12} = \frac{(8\sigma^3 + 18\sigma^2 + 18\sigma)}{\gamma^3}$$

This agrees with Black (ref. 6) except that in μ_{12}, Black has 19σ; 18σ is correct.

FORCE DUE TO AN ANGULAR PERTURBATION

This will be expressed in the form

$$F_{20} = H (\mu_{20}\alpha + \mu_{21} T\dot{\alpha} + \mu_{22} T^2\ddot{\alpha}) \tag{29}$$

Proceeding as before,

$$H = \frac{PR\pi\sigma}{6\lambda\gamma} \quad , \quad \gamma = 1.5 + 2\sigma$$

$$\mu_{20} = \frac{3L(2\sigma^2 + 6\sigma + 3)}{\gamma}$$

$$\mu_{21} = \frac{L}{\gamma^2} [\gamma(2\sigma^2 + 11.5\sigma + 12) - (6\sigma^2 + 18\sigma + 9)]$$

$$\mu_{22} = \frac{L}{\gamma^3} [\gamma^2(4.25 + 2\sigma) - \gamma(2\sigma^2 + 11.5\sigma + 12) + (6\sigma^2 + 18\sigma + 9)]$$

PITCHING MOMENT DUE TO A TRANSLATION

The pitching moment due to a translation will be expressed in the form

$$\tau_{10} = A_o \varepsilon + B_o \dot{\varepsilon} + C_o \ddot{\varepsilon} = H_1 (\mu_{30}\varepsilon + \mu_{31} T\dot{\varepsilon} + \mu_{32} T^2\ddot{\varepsilon}) \tag{30}$$

The equation expressing the boundary condition is (26)

$$0 = [TD + \gamma]v + \frac{V}{L}\frac{\sigma}{2\lambda} [T^2D^2 + 2(1 + \sigma)TD + 2\sigma]\varepsilon \tag{31}$$

From Eq. (21), $\alpha, \dot{\alpha}, \ddot{\alpha} = 0$,

$$\frac{\tau_{10}}{R\pi\rho} = - \left\{ \frac{VL^2}{3} (TD + 2\sigma + 1.5^2)v \right.$$

$$\left. + \frac{L}{8} v^2 \frac{\sigma}{\lambda} \left[T^2D^2 + 8\left(\frac{1}{3} + \frac{\sigma}{4}\right) TD + \frac{8\sigma}{3} \right] \varepsilon \right\} \tag{32}$$

Eliminating $v, \dot{v}$, as before,

$$\tau_{10} = H_1 (\mu_{30}\dot{\varepsilon} + \mu_{31} T\dot{\varepsilon} + \mu_{32} T^2\ddot{\varepsilon}) \tag{33}$$

where

$$H_1 = \frac{R\pi\sigma LP}{\lambda\gamma} \quad , \quad \gamma = 1.5 + 2\sigma$$

$$\mu_{30} = \frac{\sigma}{2\gamma}$$

$$\mu_{31} = \frac{1}{\gamma} \left[\frac{\sigma^2}{3} + \frac{3\sigma}{4} + \frac{1}{2} \right] - \frac{\sigma}{2\gamma^2}$$

$$\mu_{32} = \frac{1}{\gamma} \left(\frac{\sigma}{3} + \frac{3}{8} \right) - \frac{1}{\gamma^2} \left(\frac{\sigma^2}{3} + \frac{3\sigma}{4} + \frac{1}{2} \right) + \frac{\sigma}{2\gamma^3}$$

138

The above pitching moment is about x = 0.

TORQUE DUE TO AN ANGULAR PERTURBATION

After proceeding as before, Eq. (20) is expressed in the form

$$\tau_{20} = H_1 \left(\mu_{40}\alpha + \mu_{41} T\dot{\alpha} + \mu_{42} T^2\ddot{\alpha} \right) \tag{34}$$

where

$$H = \frac{PR\pi\sigma L}{\gamma\lambda}$$

$$\mu_{40} = \frac{L^2}{45\gamma} [22.5\sigma^2 + 50.67\sigma + 22.5]$$

$$\mu_{41} = \frac{L^2}{45\gamma^2} [\gamma(8\sigma^2 + 39.75\sigma + 33.75) - (22.5\sigma^2 + 50.67\sigma + 22.5)]$$

$$\mu_{42} = \frac{L^2}{45\gamma^3} [8\sigma + 14.25)\gamma^2 - \gamma(8\sigma^2 + 39.75\sigma + 33.75)$$

$$+ (22.5\sigma^2 + 50.67\sigma + 22.5)]$$

TRANSLATION OF EQUATIONS TO x = L/2

The translation of the solutions for perturbations about x = L/2 can now be written. These involve the α terms only. An x rotation at α = L/2, can be considered as the sum of an α perturbation at x = 0 plus a displacement of $-\alpha \frac{L}{2}$, so that at L/2 there is a pure α perturbation.

$$F_2 = F_{20} - F_{10}\left(-\alpha \frac{L}{2}\right) = d_o\alpha + e_o\dot{\alpha} + f_o\ddot{\alpha} - a_o \frac{L}{2}\alpha - b_o \frac{L}{2}\dot{\alpha} - c_o \frac{L}{2}\ddot{\alpha}$$

$$= \left(d_o - a_o \frac{L}{2}\right)\alpha + \left(e_o - b_o \frac{L}{2}\right)\dot{\alpha} = \left(f_o - c_o \frac{L}{2}\right)\ddot{\alpha} \tag{35}$$

and

$$\tau_{20} = \left(D_o - A_o \frac{L}{2}\right)\alpha + \left(E_o - B_o \frac{L}{2}\right)\dot{\alpha} + \left(F_o - C_o \frac{L}{2}\right)\ddot{\alpha} \tag{36}$$

These are both for α rotated at x = 1/2, but the moment is still at x = 0. The next step is to translate the resultant force and moment to x = L/2. The fundamental relationships are $(F = F_o)$,

$$\tau = \tau\big|_{L/2} = \tau_o - F \frac{L}{2} = \tau_{10} + \tau_{20} - (F_1 + F_2) \frac{L}{2} \tag{37}$$

To summarize, these become:

Forces:

$$F_1 = a_o\epsilon + b_o\dot{\epsilon} + c_o\ddot{\epsilon} \quad , \quad \text{since } a_o = a, \; b_o = b, \; c_o = c$$

$$F_2 = \left(d_o - a_o \frac{L}{2}\right)\alpha + \left(e_o - b_o \frac{L}{2}\right)\dot{\alpha} + \left(f_o - c_o \frac{L}{2}\right)\ddot{\alpha} \tag{38}$$

139

Moments

$$\tau_1 = \tau_{10} - F_1 \frac{L}{2} = \left(A_o - a_o \frac{L}{2}\right)\varepsilon + \left(B_o - b_o \frac{L}{2}\right)\dot{\varepsilon} + \left(C_o - c_o \frac{L}{2}\right)\ddot{\varepsilon}$$

$$\tau_2 = \tau_{20} - F_2 \frac{L}{2} = \left(D_o - A_o \frac{L}{2} - c_o \frac{L}{2} + a_o \frac{L^2}{4}\right)\alpha$$

$$+ \left(F_o - B_o \frac{L}{2} - e_o \frac{L}{2} + b_o \frac{L^2}{4}\right)\dot{\alpha}$$

$$+ \left(F_o - C_o \frac{L}{2} - f_o \frac{L}{2} + c_o \frac{L^2}{4}\right)\ddot{\alpha} \tag{39}$$

where

$$a_o = H \mu_{10} \,, \quad b_o = H \mu_{11} T \,, \quad c_o = H \mu_{12} T^2$$

$$d_o = H \mu_{20} \,, \quad e_o = H \mu_{21} T \,, \quad f_o = H \mu_{22} T^2$$

and

$$A_o = H_1 \mu_{30} \,, \quad B_o = H_1 \mu_{31} T \,, \quad C_o = H_1 \mu_{32} T^2$$

$$D_o = H_1 \mu_{40} \,, \quad E_o = H_1 \mu_{41} T \,, \quad F_o = H_1 \mu_{42} T^2$$

The symbols are previously defined. And finally,

$$\tau = A\varepsilon + B\dot{\varepsilon} + C\ddot{\varepsilon} + D\alpha + E\dot{\alpha} + F\ddot{\alpha} \left.\begin{array}{c} \\ \\ \end{array}\right\}$$

$$F = a\varepsilon + b\dot{\varepsilon} + c\ddot{\varepsilon} + d\alpha + e\dot{\alpha} + f\ddot{\alpha} \tag{40}$$

where

$$A = A_o - a_o \frac{L}{2} \,, \quad B = B_o - b_o \frac{L}{2} \,, \quad C = C_o - c_o \frac{L}{2}$$

$$D = D_o - A_o \frac{L}{2} - d_o \frac{L}{2} + a_o \frac{L^2}{4}$$

$$E = E_o - B_o \frac{L}{2} - e_o \frac{L}{2} + b_o \frac{L^2}{4}$$

$$F = F_o - C_o \frac{L}{2} - f_o \frac{L}{2} + c_o \frac{L^2}{4}$$

and where

$$a = a_o \,, \quad b = b_o \,, \quad c = c_o$$

$$d = d_o - a_o \frac{L}{2} \,, \quad e = e_o - b_o \frac{L}{2} \,, \quad f = f_o - c_o \frac{L}{2}$$

EVALUATION OF SEAL COEFFICIENTS

The geometry and parameter values shown in Table I were used to evaluate a sample rotor. For comparative purposes, the properties of two fluids, water and steam under supercritical conditions were used.

TABLE I. - GEOMETRY AND PARAMETERS

SEAL	
R — RADIUS	7.62 cm (3 INCHES)
L — LENGTH	2.54 cm (1 INCH)
Y_0 — CLEARANCE	0.254 mm (0.01 INCH)
N — SHAFT SPEED	30,000 RPM
p — PRESSURE DROP	344.6 N/cm^2 (500 PSI)
μ — VISCOSITY (STEAM)	2.067 X 10^{-9} N·S/cm^2 (3 X 10^{-9} LB·SEC/IN.^{2})
δ — DENSITY (STEAM)	13.84 Kg/m^3 (0.0005 LB/IN.^{3})
μ — VISCOSITY (WATER)	1.013 X 10^{-5} N·S/cm^2 (1.47 X 10^{-5} LB·SEC/IN.^{2})
δ — DENSITY (WATER)	996.5 Kg/m^3 (0.036 LB/IN.^{3})
ROTOR	
Ē — ELASTIC MODULUS	2.067 X 10^7 N/cm^2 (3 X 10^7 LB/IN.^{2})
I — AREA MOMENT OF INERTIA	24.97 cm^4 (0.6 IN.^{4})
K — BEARING SPRING RATE	1.75 X 10^6 N/cm (10^6 LB/IN.)
ℓ — COMPONENT SPACING	15.24 cm (6 IN.)
W — ROTOR WEIGHT	36.28 Kg (80 LB)
G — ℓ^3K/EI	

As indicated in ref. 6, if all dynamic terms are retained in the coefficient formulation, the results are in the form of a frequency-dependent cubic divided by a first-order term. A Taylor's expansion was used for the denominator to obtain a simple polynomial form for the coefficients. Questions remain as to how many terms should be retained and what range of frequencies are allowed before the approximation degrades.

To answer the question of higher order terms, the equations for the seals were established in the cubic form and evaluated for the conditions of Table I. The resulting transfer functions are shown in Table II. From these examples, it is obvious that the only affects that may legitimately be considered are those of stiffness and damping. Only those two affects were considered in the evaluation that follows.

Seal coefficients were evaluated over a wide range of speed by assuming that pressure drop across the seal varied as the square of speed. The density for water was assumed to be constant but the density of steam was assumed to be proportional to pressure drop. The spring rates were found to vary with speed squared for both water and steam. The damping coefficients, however, varied with speed for the case of water and speed squared for steam. The restoring force and torque as a function of shaft translation and angulation are therefore described by

$$\begin{Bmatrix} F(\omega) \\ \tau(\omega) \end{Bmatrix} = \begin{bmatrix} \theta & \phi \\ \chi & \psi \end{bmatrix} \begin{Bmatrix} \varepsilon \\ \alpha \end{Bmatrix}$$

141

TABLE II. - SUMMARY OF SEAL COEFFICIENTS AT 30,000 rpm

WATER

θ (N/cm) = 7.84 X 10^4 $\left(1 + \frac{S}{548.3}\right)\left(1 + \frac{S}{7347}\right)\left(1 + \frac{S}{12100}\right)/\left(1 + \frac{S}{7930}\right)$ ≈ 7.84 X 10^4 + 150.2S

ϕ (N) = 6.806 X 10^5 $\left(1 + \frac{S}{3053}\right)\left(1 + 1.948\left(\frac{S}{10440}\right) + \left(\frac{S}{10440}\right)^2\right)/\left(1 + \frac{S}{7930}\right)$ ≈ 6.806 X 10^5 + 264.1S

χ (N) = -3.323 X 10^4 $\left(1 + \frac{S}{7949}\right)\left(1 + \frac{S}{1057}\right)/\left(1 + \frac{S}{7930}\right)$ ≈ -3.323 X 10^4 -31.43S

ψ (N-cm) = -1.2308 X 10^5 $\left(1 - \frac{S}{4482}\right)\left(1 + \frac{S}{8318}\right)\left(1 + \frac{S}{14611}\right)/\left(1 + \frac{S}{7930}\right)$ ≈ -1.2308 X 10^5 + 19.76S

STEAM

θ (N/cm) = 9.327 X 10^4 $\left(1 + \frac{S}{5335}\right)\left(1 + \frac{S}{48219}\right)\left(1 + \frac{S}{85632}\right)/\left(1 + \frac{S}{42063}\right)$ ≈ 9.327 X 10^4 + 18.29S

ϕ (N) = 1.0163 X 10^6 $\left(1 + \frac{S}{26295}\right)\left(1 + 1.872\left(\frac{S}{84780}\right) + \left(\frac{S}{84780}\right)^2\right)/\left(1 + \frac{S}{42063}\right)$ ≈1.0163 X 10^6 + 25.71S

χ (N) = -3.949 X 10^4 $\left(1 + \frac{S}{5880}\right)\left(1 + \frac{S}{45365}\right)/\left(1 + \frac{S}{42063}\right)$ ≈ -3.949 X 10^4 - 8.524S

ψ N-cm) = 3.536 X 10^5 $\left(1 + \frac{S}{41200}\right)\left(1 - \frac{S}{80663}\right)\left(1 + \frac{S}{233705}\right)/\left(1 + \frac{S}{42063}\right)$ ≈ -3.536 X 10^5 + 2.688S

where S = jω

Where operating point variations are described by

$$\begin{bmatrix} \theta & \phi \\ \chi & \psi \end{bmatrix} = \left(\frac{\omega}{\underline{\underline{\omega}}}\right)^2 \begin{bmatrix} a & d \\ A & D \end{bmatrix} + \left(\frac{\omega}{\underline{\underline{\omega}}}\right)^2 \begin{bmatrix} b\frac{d}{dt} & e\frac{d}{dt} \\ B\frac{d}{dt} & E\frac{d}{dt} \end{bmatrix}_{steam} + \left(\frac{\omega}{\underline{\underline{\omega}}}\right) \begin{bmatrix} b\frac{d}{dt} & e\frac{d}{dt} \\ B\frac{d}{dt} & E\frac{d}{dt} \end{bmatrix}_{water}$$

ROTOR MODEL

A simple rotor model was necessary that would exhibit both translation and angulation at the seals. A flexible massless shaft was chosen with a mass load at the center. Seals were placed at each end of the shaft and an ideal bearing was located halfway between each seal and the central mass. Properties for the rotor model are contained in Table I.

The restoring force due to the shaft, bearings, and seals due to the lateral deflection (Y or Z) at the mass is given by

$$\bar{F}/Y = \bar{N}/\bar{D} \tag{41}$$

where

$$\bar{N} = -2K\left[1 + \left(1 + \frac{4}{3}G\right)\frac{\theta}{K} + \frac{3}{2}\frac{G}{\ell K}(\phi + \chi) + \frac{2G}{\ell^2 K}\psi + \frac{5}{12}\left(\frac{G}{\ell K}\right)^2 (\theta\psi - \phi\chi)\right]$$

142

$$\bar{D} = \left(1 + \frac{G}{3}\right) + \left(\frac{8}{3} + \frac{7}{36} G\right) \frac{G\theta}{K} + \left(2 + \frac{1}{4} G\right) \frac{G}{\ell K} (\phi + \chi)$$

$$+ \left(2 + \frac{5}{12} G\right) \frac{G\psi}{\ell^2 K} + \left(\frac{2}{3} + \frac{G}{18}\right)\left(\frac{G}{\ell K}\right)^2 (\theta\psi - \phi\chi)$$

Inserting values for the structural properties and for the frequency dependent journal coefficients, we obtain a dynamic force-deflection relation at the mass.

$$\frac{\bar{F}}{Y} = - \left[\frac{a_1 + b_1 \frac{d}{dt} + c_1 \frac{d^2}{dt^2}}{d_1 + e_1 \frac{d}{dt} + f_1 \frac{d^2}{dt^2}} \right] \qquad (42)$$

Since the journal equations were developed for a condition where the flow is ostensibly axial, we assume that they are fixed to Couette coordinates. These equations must be transformed to rotor coordinates to determine the whirl orbit. Now assume, steady circular motion of the shaft in the y-z plane. Transforming the force-deflection equations to coordinates rotating with the shaft, we obtain

$$\begin{bmatrix} d_1 - \frac{\omega^2}{4} f_1 & + \frac{\omega}{2} e_1 \\ - \frac{\omega}{2} e_1 & d_1 - \frac{\omega^2}{4} f_1 \end{bmatrix} \begin{Bmatrix} F_y \\ F_z \end{Bmatrix} = \begin{bmatrix} -\left(a_1 - \frac{\omega^2}{4} c_1\right) & - \frac{\omega}{2} b_1 \\ + \frac{\omega}{2} b_1 & -\left(a_1 - \frac{\omega^2}{4} c_1\right) \end{bmatrix} \begin{Bmatrix} Y \\ Z \end{Bmatrix} \qquad (43)$$

The equations for the rotor with the central mass unbalanced a distance Δ from the axis of rotation in the y direction are

$$- \omega^2 \frac{W}{g} y = F_y + \Delta \frac{W}{g} \omega^2$$

$$- \omega^2 \frac{W}{g} z = F_z \qquad (44)$$

Solutions for shaft motion due to the unbalanced mass at a particular speed (ω) are obtained by first scaling the seal coefficients that were obtained at design speed ($\bar{\omega}$). Then solutions of equations (41-44) produce the normalized orbit radius, or amplification ratio

$$\frac{\sqrt{Y^2 + Z^2}}{\Delta}$$

EVALUATION OF ROTOR RESPONSE

Normalized rotor deflection vs shaft speed is shown in Fig. 3 and 4 for the steam and water journals, respectively. Rotor response considering only the trans-lation coefficient (α) is shown for reference. The rotor response with the complete
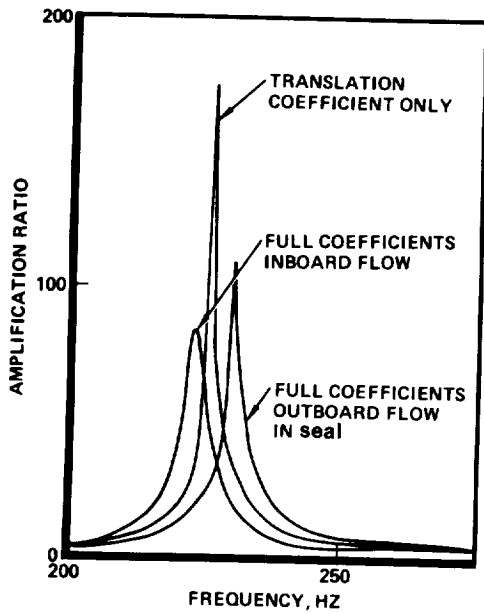
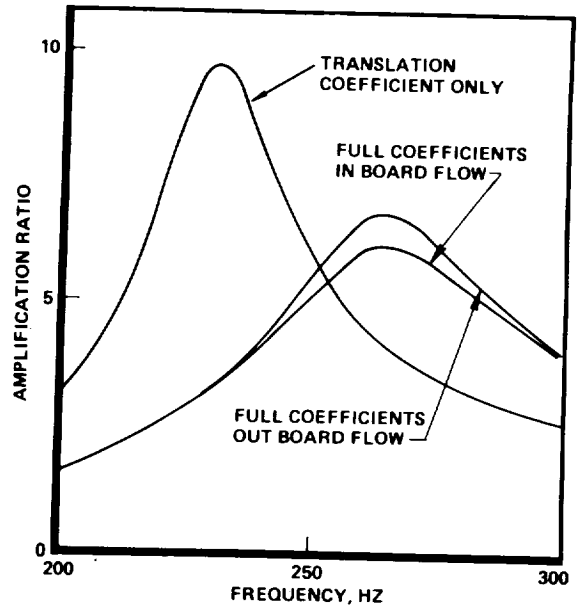Figure 3. - Rotor response with steam as fluid.

Figure 4. - Rotor response with water as fluid.

set of forces are shown for cases where the flow is outboard through the seals and where the flow is reversed. Reversed flow (inboard flow direction) was simulated by changing the sign of the off diagonal coefficients (ϕ and ψ).

To evaluate sensitivity of the various coefficients, cases with the off-diagonal terms increased by 10 percent were run as well as a case where the diagonal coefficient (ψ) was increased by 10 percent. A summary of these cases in terms of critical speed and effective damping (ξ/ξ_{cr}) are contained in Table III.

TABLE III. - SUMMARY OF CRITICAL SPEED CALCULATIONS

SEAL STIFFNESS MATRIX	STEAM		WATER	
	SPEED (RPM)	DAMPING	SPEED (RPM)	DAMPING
$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	13266	0.0%	13266	0.0%
$\begin{bmatrix} \theta & 0 \\ 0 & 0 \end{bmatrix}$	13585	0.28%	13780	5.12%
$\begin{bmatrix} \theta & \phi \\ \chi & \psi \end{bmatrix}$	13807	0.46%	15540	7.58%
$\begin{bmatrix} \theta & -\phi \\ -\chi & \psi \end{bmatrix}$	13323	0.60%	15510	8.47%
$\begin{bmatrix} \theta & 0 \\ 0 & \psi \end{bmatrix}$	13584	-1.2%	14880	−11.2%
$\begin{bmatrix} \theta & 1.1\phi \\ 1.1\chi & \psi \end{bmatrix}$	13830	0.77%	15900	6.36%
$\begin{bmatrix} \theta & \phi \\ \chi & 1.1\psi \end{bmatrix}$	13806	0.32%	15410	7.87%

DISCUSSION OF THE RESULTS

When angulation affects are included, as well as the translation, the critical speed is shifted slightly and the whirl radius is significantly reduced.

Off diagonal seal coefficients are very significant; without them, the effect of angulation is destablilzing. This indicates that the centers of pressure for rotation and for translation do not coincide. Reversing the journal flow (to the inboard direction) actually increased effective damping for the cases investigated.

The second order "mass" affects of the journals were negligible and for whirl the mass affects may be deleted.

For the two fluids that were considered in the journal, spring rates were very similar but damping coefficients were an order of magnitude higher with water. This is evidenced by an order of magnitude increase in damping at critical speed for that journal. For both fluids the spring rate varies with seal pressure drop and therefore speed squared. The damping coefficient varies with speed for an incompressible fluid and the square of speed for a compressible fluid.

CONCLUDING REMARKS

From a study of the results, the following conclusions have been reached:

1. Journal forces associated with the second time derivative of shaft motion should be ignored. The coefficients are very small and are significant only in a frequency range where the approximations required to obtain them are questionable.

2. Forces associated with the first-time derivative are very important.

3. Pitching moments and angulation effects are as significant as translation.

4. The net effect of moment and angulation is a slight shift in critical speed and a significant decrease in peak amplitude.

5. While the partial of moment with respect to angulation appears to be destabilizing, off diagonal coefficients result in net stabilization. This result is due to a lack of coincidence between the centers of pressure for translation and rotation.

6. Seals, where the flow is outboard, showed lower damping than where the seal flow was inboard. Higher critical speeds, where a different mode shape is involved, may not exhibit the same trend.

7. Damping forces at speeds below the design value are reduced more for a compressible fluid than for an incompressible one.

8. Moments due to variation in fluid shear along the shaft surface are very small.

9. Coefficient accuracy is not known in the absence of experimental data. Specifically this approach assumes full turbulent flow with a rotation at $\omega/2$ throughout the seal. Black (ref. 11) and Hirs (ref. 13) comment on this. For short seals, a significant error may occur because of the slow development of tangential velocity.

REFERENCES

1. Rothe, K.: "Turbopump Configuration Selection for the Space Shuttle Main Engine," ASME Paper No. 74 FE-23, Joint Fluid Engineering Conference, Montreal, Canada, May 1974.

2. Alford, J. S.: "Protecting Turbomachinery From Self-Excited Rotor Whirl," _Journal of Engineering for Power_, Trans. ASME, Series A, October 1965, pp. 333-344.

3. Ek, M. C.: "Solving Subsynchronous Whirl in the High-Pressure Hydrogen Turbomachinery of the SSME," _Journal of Spacecraft and Rockets_, Vol. 17, No. 3, May-June 1980, p. 208.

4. Childs, D. W.: "The Space Shuttle Main Engine High-Pressure Fuel Turbopump Rotordynamic Instability Problem," Gas Turbine Conference, Philadelphia, Pa, 27-31 March 1977, ASME Paper No. 77-GT-49.

5. Allaire, P. E., Lee, C. C., and Gunter, D. J.: "Dynamics of Short Eccentric Plain Seals With High Axial Reynolds Number," _Journal of Spacecraft and Rockets_, Vol. 15, No. 6, 1978, pp. 341-347.

6. Black, H. F.: "Effects of Hydraulic Forces in Annular Pressure Seals on the Vibrations of Centrifugal Pump Rotors," _Journal of Mechanical Engineering Science_, Vol. 11, No. 2, 1969, pp. 206-213.

7. Black, H. F. and Jenssen, D. N.: "Effects of High Pressure Ring Seals on Pump Rotor Vibrations," Fluids Engineering Division, ASME, 71-WA/FE-38, 1971.

8. Black, H. F. and Jenssen, D. N.: "Dynamic Hybrid Bearing Characteristics of Annular Controlled Leakage Seals," Institution of Mechanical Engineers, Vol. 184, Pt. 3N, 1970, pp. 92-100.

9. Black, H. F. and Cochrane, E. A.: "Leakage and Hybrid Bearing Properties of Serrated Seals in Centrifugal Pumps," Paper G5, 6th International Conference on Fluid Sealing, Munich, 1973, pp. 61-70.

10. Black, H. F.: Lateral Stability and Vibrations of High-Speed Centrifugal Pump Rotors," Dynamics of Rotors, International Union of Theoretical and Applied Mechanics Symposium, Lyngby/Denmark, 12-16 August 1974.

11. Black, H. F., Allaire, P. E., and Barrett, L. E.: "Inlet Flow Swirl in Short Turbulent Annular Seal Dynamics," 9th International Conference on Fluid Sealing, BHRA Fluid Engineering, Leeuwenhorst, Netherlands, April 1981.

12. Fleming, D.: "Effective Spring Rates of Tapered Seals," NASA Lewis Turbomachinery Work Shop, March 1981.

13. Hirs, G. G.: "A Bulk Flow Theory for Turbulence in Lubricant Films," _Journal of Lubrication Technology_, Trans. ASME, Series F, Vol. 95, No. 2, April 1973, pp. 137-146.

14. Allaire, P. E., Gunter, E. J., Lee, C. C., and Barrett, L. E.: "Load Capacity and Hybrid Coefficients for Turbulent Interstage Seals," Report No. UVA/528; 49/ME76/103, September 1976.

15. Yamada, Y., "Resistance of Flow Through an Annulus With an Inner Cylinder Rotating," Bulletin JSME, Vol. 5, No. 18, 1962, pp. 302-310.