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SUMMARY

A curved member under axial load is analyzed using beam column theory to determine nonlinear response and the tangent stiffness associated with small displacements from the nonlinear state. Such a result is suitable for incorporation into a general nonlinear analysis using a corotational coordinate system to describe the rigid body type motion of individual members. Application of the method to buckling problems is also shown. Several examples are given to show the accuracy of the method.

INTRODUCTION

The nonlinear analysis of frame or lattice structures can be accomplished with a variety of approaches with varying degrees of success depending on the problem. In the present paper, attention is given to frame structures that may undergo large displacements but small strain which is often characterized as a very flexible or lightly loaded structure. The approach taken is to use a moving or corotational coordinate system that follows each member so there is no limit on displacement and rotation. Exact solutions of the beam column equations are used for total and incremental forces so that accuracy is achieved with few or no member interior nodes. Such a theory has been developed in reference 1 for straight members. For very flexible structures, member imperfections may be a significant factor in nonlinear response or a member may be designed to have a small curvature. In the present paper, the method of reference 1 is extended to include curved members. The application of the tangent stiffness matrix associated with the incremental forces to buckling problems is also illustrated. Results of nonlinear analysis of a simple arch and imperfect columns are compared with previous work to indicate the accuracy of the approach.

SYMBOLS

A_1,A_2,A_3,A_4	constants of integration in equation (A4)
B	transformation matrix between relative displacements (or associated forces) and incremental displacements (or associated forces) in local coordinates
C	matrix defined by equation (A21)
b_1,b_2,b_3,b_4	bowing functions defined by equations (A9) through (A12)
c_1,c_2	stability functions defined by equations (A6)
c_b	shortening due to bending
D,d	global and local displacement vectors respectively
d_r	local relative displacement vector
EA	axial stiffness of member
EI	bending stiffness of member
e	amplitude of member imperfection
F,f	global and local force vectors respectively
f_r	force vector associated with relative displacements
G	matrix giving contribution of total member loads to incremental local force vector
G_1,G_2	bending extension coupling terms defined by equation (A18)
H	extensional stiffness term defined by equation (A18)
K,k	global and local stiffness matrices respectively

L	member length
L^*	member length after deformation
m_1, m_2	bending moments at ends of member
p	external load on structure in X direction
p_E	euler load of column
p_1, p_2	axial loads at ends of member
$Q = (m_1 + m_2)/L^*$	
q	ratio of axial load to Euler load
q_1, q_2	shear loads at ends of member
R, S	stiffness matrices from reference 5
T	transformation matrix from local coordinates of displaced member to global coordinates
U, u	global and local displacements in X(x) direction
V, v	global and local displacements in Y(y) direction
W	component of T matrix
w	deflection of member
X, Y	global coordinate system
x, y	local coordinate system
β	imperfection parameter defined by equation (A5)
γ	angle between X axis and line passing through member ends
θ	angle between axial and diagonal member
θ_1, θ_2	relative rotations at ends of member
ψ	rotation at a node
ϕ	stability function parameters defined in equation (A6)
Δ	stability function parameters defined in equation (A6)

Subscripts

a	denotes axial member
d	denotes diagonal member
i,j	denote nodes i and j
o	initial value corresponding to zero displacement

Superscripts

T	denotes matrix transpose
'	denotes derivative with respect to q

A δ preceding a variable indicates incremental value

ANALYSIS APPROACH

The equations developed are for plane frames but extension to space frames as was done in reference 2 is straightforward. Consider a member in a plane frame connecting node i to j. After displacement, its position is indicated in figure 1. The analysis problem, illustrated in figure 2, is to determine the forces p_1, m_1, m_2, when the member has undergone the relative deformation of θ_1, θ_2 and u_2. Note that p_2, q_1, q_2 are determined by equilibrium from p_1, m_1, m_2. In addition, the tangent stiffness relating incremental displacements to incremental forces must be determined.

$$\delta f = k \delta d \tag{1}$$

where

$$\delta f^T = [\ \delta p_1,\ \ \delta q_1,\ \ \delta m_1,\ \ \delta p_2,\ \ \delta q_2,\ \ \delta m_2]$$

$$\delta d^T = [\ \delta u_1,\ \ \delta v_1,\ \ \delta \psi_1,\ \ \delta u_2,\ \ \delta v_2,\ \ \delta \psi_2]$$

The matrix k can be assembled into a global stiffness by the transformation

$$K = T^T k T \tag{2}$$

where

$$T = \begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix} \tag{3}$$

$$W = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

where γ is defined by

$$\begin{aligned} \cos\gamma &= (X_j - X_i + U_j - U_i)/L^* \\ \sin\gamma &= (Y_j - Y_i + V_j - V_i)/L^* \end{aligned} \tag{5}$$

with

$$L^* = \sqrt{(X_j - X_i + U_j - U_i)^2 + (Y_j - Y_i + V_j - V_i)^2}$$

In assembling the global stiffness matrix, the forces in the members can be combined with any external load to give the unbalanced force at each node. The negatives of these forces are applied to the structure to achieve a new deflected position and the process is repeated until convergence is achieved. At each stage of the iteration, the relative displacements are determined as shown in figure 1 by

$$\begin{aligned} \theta_1 &= \psi_1 - (\gamma_i - \gamma_0) \\ \theta_2 &= \psi_j - (\gamma_j - \gamma_0) \\ U_2 &= L^* - L \\ U_1 &= 0 \end{aligned}$$

where γ_0 and L ar the value of γ and L* respectively evaluated for $U_i = U_j = V_i = V_j = 0$. The matrix k and expresions for m_1, m_2 and p_1 were derived in reference 1 for straight members. The corresponding development for curved members is given in Appendix A.

If a member load is high relative to its buckling load and/or the member has undergone large relative deformations (θ_1, θ_2 large), the approach described above may become inaccurate. One remedy is to put internal nodes in the member. However, this can greatly increase solution times as a result of the increases in matrix size but even more significant is the increase in band width. An alternative approach is to use substructures for individual members, as indicated in figure 3. An arbitrary number of internal nodes is located in the member; three internal nodes are shown in figure 3 for an example. The analysis of the single member is performed by the method described above for the displacement boundary conditions shown in the figure. This will require several iterations to obtain the correct set of internal displacements. If the nodes are numbered starting in the interior so that the ends are numbered last, the tangent stiffness matrix for the converged displacements may be triangulated down to the last two nodes. The remaining matrix is the tangent stiffness matrix for the entire member. The forces at the ends of the member can be determined to complete all the information required for the next iteration in the analysis of the complete structure having nodes only at connections between members.

RESULTS

The present theory has been applied to several problems to assess the accuracy of the analysis. In the following sections, the elastica problem, a column with initial imperfection, the deformation of a shallow arch, and buckling of a lattice truss with member imperfections will be presented.

The Elastica

The elastica problem has a known solution valid for unlimited deflection and so provides a good basis for comparison. Results presented in Table 1 were calculated for the geometry of figure 4 and are compared with the exact solution which is in the form of elliptic integrals from reference 3. One half of a pin ended column was modelled with a symmetry boundary condition in the middle. Results are presented for no internal nodes and for one internal node by use of the substructure analysis. The comparison with the exact result is good to load levels appreciably higher than the buckling load, with the results for one internal node essentially identical to the exact result up to twice the buckling load. Note the large rotations present that correspond to relative rotations well over one radian.

Imperfect Column

The same model as in figure 4 was analyzed with an imperfection equal .01L. The results are shown in figure 5 where the end load is plotted as a function of the lateral deflection. For loads less than the buckling load, the result is in close agreement with the classical amplification factor formula. Above 80% of the buckling load, the two results deviate with the present solution approaching the elastica for higher loadings. The use of a substructure with one internal node shows a slight difference from the result without a substructure at loads above the buckling load indicating that reasonable accuracy has been achieved without an internal node.

Shallow Arch

A clamped shallow arch with a central load has been analyzed and compared with finite element results from reference 4. In figure 6, the load P is plotted as a function of the central deflection U. One node at the point of load application was used. Results without substructuring and with a substructure having one internal node are in close agreement indicating convergence has been achieved. This is confirmed by the close agreement with the finite element results of reference 4. Note that 32 curved elements with three degrees of freedom per node were required for convergence. At the higher loadings where the arch has become inverted, the present results indicate the finite element solution may not be completely converged.

Buckling of a Lattice Truss with Imperfect Diagonal Members

In reference 5 an analysis was made of the buckling of a family of lattice configurations with repetitive geometry. A feature of the method is that the six degrees of freedom of one node are all that is involved in the solution. Figure 7 is an illustration of two of the configurations analyzed. The present theory allows the treatment of configurations with imperfect members. The details of how the analysis of reference 5 is extended to include imperfect members is given in Appendix B.

An example of the application of the theory is given in figure 8. A three element truss, three bays long and simply supported at the ends has been analyzed. Note that a truss with a length that is a multiple of three bays would also buckle at the same load unless the column mode was lower. The buckling load for a perfect truss is presented as a function of length in reference 5. The diagonal members were assumed to have an imperfection as shown on the figure. The buckling load, normalized by the load that would be achieved if the diagonals provided simple support is plotted as a function of the amplitude of the imperfection. The symbols are from a complete analysis of the structure by the computer program of reference 6 which is a general space frame buckling and vibration program based on exact member theory for straight members. The imperfection was modelled by two straight segments with a maximum deviation from a straight line equal to the amplitude of the imperfection. Such a model is somewhat stiffer than the curved shape used in the present analysis which accounts for the slightly higher result. However, the close agreement throughout the large range of imperfection indicates the accuracy of the present approach. For this problem, the imperfection has a very large effect on the buckling load.

CONCLUDING REMARKS

The tangent stiffness matrix for a curved member based on the exact solution of the beam column equation has been developed. The matrix would be suitable for incorporation into a nonlinear analysis that uses a corotational coordinate system. The accuracy of the method as shown by several numerical examples is such that no internal nodes would be required in any member for many applications. In cases where the relative rotation is large or the axial load is approaching or exceeding the individual member buckling load a substructuring method is presented which allows the complete analysis to be accomplished with nodes only at points of load application or at connections to other members. The only limit to such an approach is the excedence of the elastic limit. The tangent stiffness matrix can also be used for calculating buckling loads for certain configurations having imperfect members.

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Applications of the method are shown for several problems to indicate the accuracy of the approach. The problems treated are (1) the elastica, (2) an imperfect column, (3) centrally loaded shallow arch, and (4) buckling of a lattice truss with imperfect members. The benefit of using exact member equations compared to the usual finite element approximation functions was demonstrated clearly with the arch problem. The present method with only one node at the point of load application achieved results comparable to a solution using 32 curved finite elements in conjunction with a corotational coordinate system.

APPENDIX A

NONLINEAR STIFFNESS OF A CURVED MEMBER

The expressions for member stiffness are based on an exact solution of the beam column equation for a member with a sinusoidal deviation from a straight line. The axial force in the member is calculated accounting for shortening due to member curvature. The differential equation for lateral deflection is

$$EIw^{IV} + p_1(w_0'' + w'') = 0 \tag{A1}$$

with boundary conditions

$$\begin{aligned} &w(0) = w(L) = 0 \\ &w'(0) = \theta_1 \qquad w'(L) = \theta_2 \end{aligned} \tag{A2}$$

and

$$w_0 = \sin \pi x/L \tag{A3}$$

The solution for w is

$$w = A_1 + A_2\, x/L + A_3 \cos \phi x/L + A_4 \sin \phi x/L + \frac{qe}{1-q} \sin \pi x/L \tag{A4}$$

where q is the ratio of the applied compressive load to the Euler load

$$q = p_1 L^2/(\pi^2 EI)$$

and

$$\phi = \pi\sqrt{q}$$

The bending moments at each end are found by the usual process to be

$$\begin{aligned} m_1 &= EI[c_1(\theta_1 - \beta) + c_2(\theta_2 + \beta)]/L \\ m_2 &= EI[c_2(\theta_1 - \beta) + c_1(\theta_2 + \beta)]/L \end{aligned} \tag{A5}$$

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where

$$\beta = \frac{e\pi}{L}\,\frac{q}{1-q}$$

and c_1, c_2 are the stability functions defined as

$$c_1 = \Delta\left(\frac{\sin\phi}{\phi} - \cos\phi\right) \qquad c_2 = \Delta\,(1-\cos\phi) \tag{A6}$$

where

$$\Delta = \frac{\pi^2 q}{2(1-\cos\phi) - \frac{\sin\phi}{\phi}}$$

Note that $\cos\phi$ and $\frac{\sin\phi}{\phi}$ are real for ϕ imaginary so the above expressions are valid for negative q. The compressive force p_1 is given by

$$p_1 = -EA\left(\frac{u_2 - u_1}{L} + c_b\right) \tag{A7}$$

where

$$c_b = \frac{1}{2}\int_0^L \left(\frac{dw}{dx}\right)^2 dx$$

$$= b_1(\theta_1+\theta_2)^2 + b_2(\theta_1-\theta_2-2\beta)^2 + b_3(\theta_1-\theta_2)\,\beta_o + b_4\beta_o^2 \tag{A8}$$

and

$$\beta_o = e\pi/L$$

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The b_1 and b_2 are bowing functions derived by Onan in reference 2 who showed the following relations with the stability functions

$$b_1 = \frac{(c_1 + c_2)(c_2 - 2)}{8\pi^2 q} = -\frac{c_1' + c_2'}{4\pi^2} \tag{A9}$$

$$b_2 = \frac{c_2}{8(c_1 + c_2)} = -\frac{c_1' - c_2'}{4\pi^2} \tag{A10}$$

where a prime denotes derivative with respect to q.

The b_3 and b_4 are new bowing functions due to the initial deflection and are given by

$$b_3 = \frac{(c_1 - c_2)}{\pi^2 (1-q)^2} \tag{A11}$$

$$b_4 = -\frac{2b_3 q}{(1-q)} + \frac{1}{4}\left[\frac{1}{(1-q)^2} - 1\right] \tag{A12}$$

Define a loading vector f_r as

$$f_r^T = [p_1, m_1, m_2] \tag{A13}$$

and a displacement vector d_r as

$$d_r^T = [u_2 - u_1, \theta_1, \theta_2] \tag{A14}$$

Incremental values of f_r and d_r may be related by

$$\delta f_r = t\, \delta d_r \tag{A15}$$

The components of the matrix t are obtained as

$$t_{ij} = \frac{\partial f_{ri}}{\partial d_{rj}} + \frac{\partial f_{ri}}{\partial q}\frac{\partial q}{\partial d_{rj}} \tag{A16}$$

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$$t = \begin{bmatrix} \frac{EA}{LH} & \frac{EAG_1}{H} & \frac{EAG_2}{H} \\ & \frac{EI}{L}c_1 + \frac{EALG_1^2}{H} & \frac{EI}{L}c_2 + \frac{EALG_1G_2}{H} \\ \text{Symmetric} & & \frac{EI}{L}c_1 + \frac{EALG_2^2}{H} \end{bmatrix} \tag{A17}$$

where

$$H = 1 + \frac{EAL^2}{\pi^2 EI}[b_1'(\theta_1 + \theta_2)^2 + b_2'(\theta_1 - \theta_2 - 2\beta) + b_3'(\theta_1 - \theta_2)\beta_0 + b_4'\beta_0^2]$$

$$\begin{aligned} G_1 &= -[2b_1(\theta_1 + \theta_2) + 2b_2(\theta_1 - \theta_2 - 2\beta) + b_3\beta_0] \\ &= [c_1'(\theta_1 - \beta) + c_2'(\theta_2 + \beta) - \frac{(c_1 - c_2)\beta_0}{(1 - q)^2}]/\pi^2 \\ G_2 &= -[2b_1(\theta_1 + \theta_2) - 2b_2(\theta_1 - \theta_2 - 2\beta) - b_3\beta_0] \\ &= [c_2'(\theta_1 - \beta) + c_1'(\theta_2 + \beta) + \frac{(c_1 - c_2)\beta_0}{(1 - q)^2}]/\pi^2 \end{aligned} \tag{A18}$$

The parameter H represents a reduced extensional stiffness and G_1, G_2 account for bending-extension coupling. The two definitions of G_1 and G_2 can be shown to be identities by equations (A9) and (A10) and must be true in order for the matrix t to be symmetric. Equations (A9) and (A10) can also be used to calculate all derivatives with respect to q that are required.

At this point the form of the t matrix is identical to that of reference 1. The only difference is that the definition of H and G_i contain extra terms as a result of the member curvature. The transformations necessary to obtain the tangent stiffness matrix are based on the geometry and equilibrium of the member under incremental displacement. The final equations are identical to those in reference 1 but are developed in a slightly different manner. The forces on the member in global coordinates are given by

$$F = T^T B^T f_r \tag{A19}$$

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where

$$B = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1/L^* & 1 & 0 & -1/L^* & 0 \\ 0 & 1/L^* & 0 & 0 & -1/L^* & 1 \end{bmatrix} \quad \text{(A20)}$$

The incremental forces due to a small change in the displacement vector are given by

$$\delta F = T^T B^T \delta f_r + C \delta D \quad \text{(A21)}$$

where

$$C = \left[\frac{\partial(T^T B^T)}{\partial D_1} f_r, \frac{\partial(T^T B^T)}{\partial D_2} f_r, \ldots. \frac{\partial(T^T B^T)}{\partial D_6} f_r\right]$$

Using equation (A15), equation (A21) can be written as

$$\delta F = T^T B^T t \ \delta d_r + C \ \delta D \quad \text{(A22)}$$

The incremental relative displacements are related to the incremental local displacements by

$$\delta d_r = B \delta d \quad \text{(A23)}$$

which can be written in terms of global displacements as

$$\delta d_r = B T \delta D \quad \text{(A24)}$$

Performing the differentiation indicated in equation (A21) and using equation A(24), equation (A22) can be written as

$$\delta F = T^T (B^T t B + G) T \ \delta D \quad \text{(A25)}$$

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where

$$G = \frac{1}{L^*}\begin{bmatrix} 0 & Q & 0 & 0 & -Q & 0 \\ & -P & 0 & -Q & P & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & Q & 0 \\ & \text{symmetric} & & & -P & 0 \\ & & & & & 0 \end{bmatrix} \qquad \text{(A26)}$$

and

$$Q = (m_1 + m_2)/L^* \qquad \text{(A27)}$$

Thus the global tangent stiffness can be obtained from the conventional coordinate transformation oerating on a matrix k defined as

$$k = B^T t B + G \qquad \text{(A28)}$$

The matrix k is symmetric with the independent elements given as follows

$$k_{11} = k_{14} = k_{44} = \frac{EA}{LH}$$

$$k_{12} = k_{15} = k_{45} = -k_{24} = \frac{EA}{L^*H}(G_1 + G_2) + \frac{m_1 + m_2}{L^{*2}}$$

$$k_{13} = -k_{34} = \frac{EAG_1}{H}$$

$$k_{16} = -k_{46} = \frac{EAG_2}{H}$$

$$k_{22} = k_{55} = -k_{25} = \frac{1}{L^{*2}}[2(c_1+c_2)\frac{EI}{L} + \frac{EAL}{H}(G_1+G_2)^2] - \frac{P}{L^*}$$

$$k_{23} = k_{35} = \frac{1}{L^*}[(c_1+c_2)\frac{EI}{L} + \frac{EAL}{H}G_1(G_1+G_2)] \qquad \text{(A29)}$$

$$k_{26} = -k_{56} = \frac{1}{L^*}[(c_1+c_2)\frac{EI}{L} + \frac{EAL}{H}G_2(G_1+G_2)]$$

$$k_{33} = c_1\frac{EI}{L} + \frac{EAL}{H}G_1^2$$

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$$k_{36} = c_2 \frac{EI}{L} + \frac{EAL}{H} G_1 G_2$$

$$k_{66} = c_1 \frac{EI}{L} + \frac{EAL}{H} G_2$$

The sequence of solution steps when θ_1, θ_2 and $u_2 - u_1$ are known is to solve equations (A7) and (A8) for p_1 which can be done iteratively. Then m_1 and m_2 can be determined for equation (A5) and the stiffness matrix k from equation (A24). If the interior deflection of the member is desired, it can now be determined rather easily from equation (A4) using the following expressions for the constants of integration.

$$\begin{aligned} A_1 &= -A_3 = m_1/p_1 \\ A_2 &= (m_1 + m_2)/p_1 \\ A_4 &= -A_2 + L(\theta_1 - \beta) \end{aligned} \qquad \text{(A28)}$$

APPENDIX B

BUCKLING OF LATTICE STRUCTURES WITH IMPERFECT MEMBERS

For certain structural configurations having the proper symmetry, the initial nonlinear state may be rather simple consisting of shortening of individual members due to mutual restraint of adjoining members no end rotation occurs. Examples of such structures are shown in figure 7 which was taken from reference 5 where the buckling of these configurations was studied. The cylindrical configurations when compressed axially, shorten without bending even if individual members are not straight provided all similar members have the same imperfection. Of interest may be the load level at which such a structure bifurcates from this initial state. The tangent stiffness matrix for the imperfect member with $\theta_1 = \theta_2 = 0$ could be used in a conventional buckling analysis to determine this bifurcation point. Considerable simplification for this case is possible which will be illustrated by giving the changes to the analysis of reference 5 which are required to treat imperfect members.

The analysis of reference 5 applies to a repetitive structure so that any imperfection must be identical for each group of members. For this analysis one imperfection, a displacement normal to a cylinder containing the nodes, is assumed to exist in the diagonal members and another in the ring members. Only the upper half of the 12 x 12 three dimensional stiffness matrix is required in the analysis so the stiffness terms associated with deflections are taken in the plane of the imperfection from the present planar analysis. In reference 5 the upper half of the complete member stiffness matrix is partitioned into two 6 x 6 matrices R and S. The following are the changes required from the usual representation in order to account for the imperfection

$$R_{11} = -S_{11} = EA/LH \tag{B1}$$

$$R_{55} = c_1 EI/L + EALG_1^2/H \tag{B2}$$

$$S_{55} = c_2 EI/L - EALG_1^2/H \tag{B3}$$

$$R_{15} = -S_{15} = -S_{51} = \frac{EAG_1}{LH} \tag{B4}$$

If there were no imperfection $H=1$ and $G_1=0$ and equations (B1) to (B4) are identical to the standard expression for straight members used in reference 5. The global stiffness matrix K which yields the buckling eigenvalue given in reference 5 in terms of the elements of R and S is still applicable for the modified terms given in equations (B1) to (B3). However, in reference 5, R_{15}, S_{15}, S_{51} which are defined by equation (B4), were all zero so that additions to the global stiffness matrix K must be made as follows

$$\Delta K_{14} = [4S_{15}s_\theta c_\theta s_\varepsilon s_\eta]_d$$

$$\Delta K_{15} = [4(R_{15} + S_{15}c_\varepsilon c_\eta)c_\theta^2 c_\alpha]_d$$

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$$\Delta K_{16} = [-4S_{15}c_\theta^2 s_\alpha s_\varepsilon s_\eta]_d$$

$$\Delta K_{24} = [(-4(R_1 + S_{15}c_\varepsilon c_\eta)s_\theta^2 c_\alpha]_d + [-2(R_{15} + S_{15}c_\tau)c_\alpha]_r$$

$$\Delta K_{25} = [-4S_{15}c_\theta s_\theta c_\alpha^2 s_\varepsilon s_\eta]_d$$

$$\Delta K_{26} = [-4S_{15}c_\theta s_\theta c_\alpha s_\varepsilon s_\eta]_d$$

$$\Delta K_{34} = [4S_{15}s_\theta^2 s_\alpha c_\varepsilon s_\eta]_d$$

$$\Delta K_{35} = [-4S_{15}c_\theta s_\theta c_\alpha s_\alpha s_\varepsilon c_\eta]_d$$

$$\Delta K_{36} = [4S_{15}c_\theta s_\theta s_\alpha^2 s_\varepsilon s_\eta]_d$$

where ΔK_{ij} is the addition that must be made to K_{ij} with the notation the same as reference 5.

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Table 1. The Elastica Problem of Figure 4.

P/P_E	X/L[1]	X/L[2]	X/L[3]	y/L[1]	y/L[2]	y/L[3]	ψ[1]	ψ[2]	ψ[3]
							degrees		
1.015	.970	.971	.970	.220	.212	.218	20	19.37	19.86
1.063	.881	.888	.881	.422	.410	.422	40	38.90	39.94
1.152	.741	.757	.741	.593	.576	.593	60	58.35	59.98
1.213	.560	.589	.558	.719	.699	.719	80	77.71	79.99
1.518	.349	.398	.348	.792	.770	.790	100	96.97	99.98
1.884	.123	.198	.125	.803	.788	.800	120	116.12	119.95
2.541	-.107	-.001	-.102	.750	.753	.745	140	135.14	139.86
4.029	-.340	-.208	-.324	.625	.668	.617	160	154.05	159.69
9.116	-.577	-.459	-.529	.421	.527	.426	176	169.47	175.63

1 Reference 3

2 No interior node

3 1 interior node

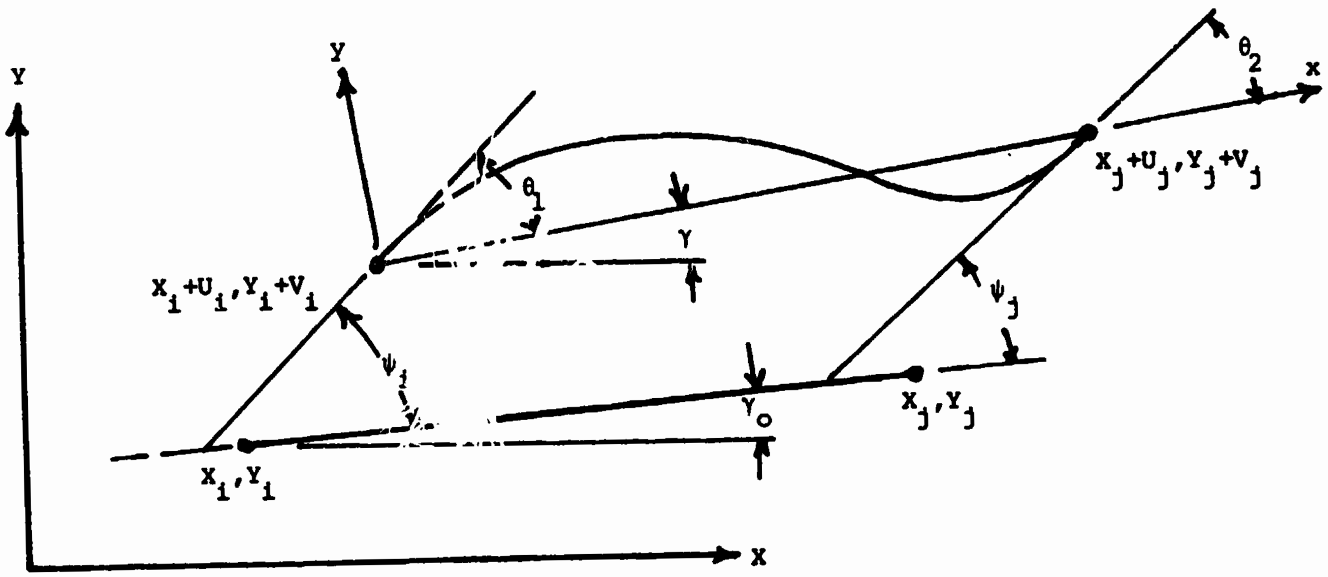

Figure 1. - Typical member before and after displacement.

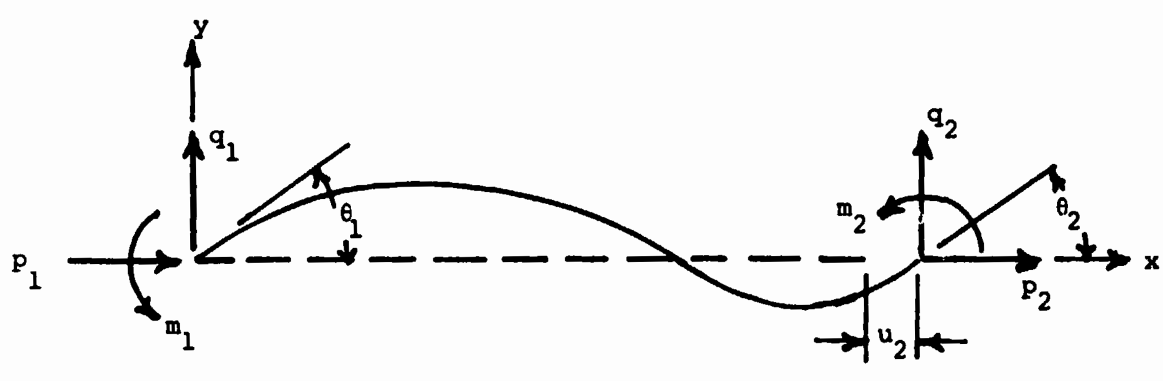

Figure 2. - Forces on member and relative displacements.

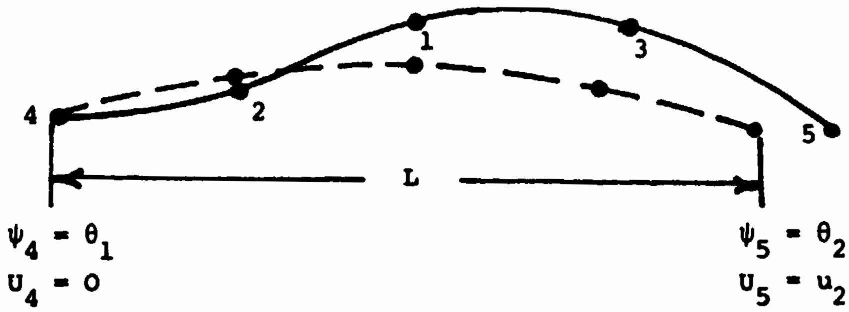

Figure 3. - Member substructure analysis for three internal nodes.

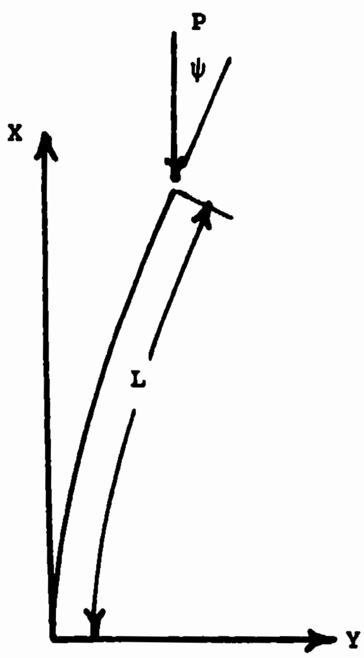

Figure 4. - The elastica problem.

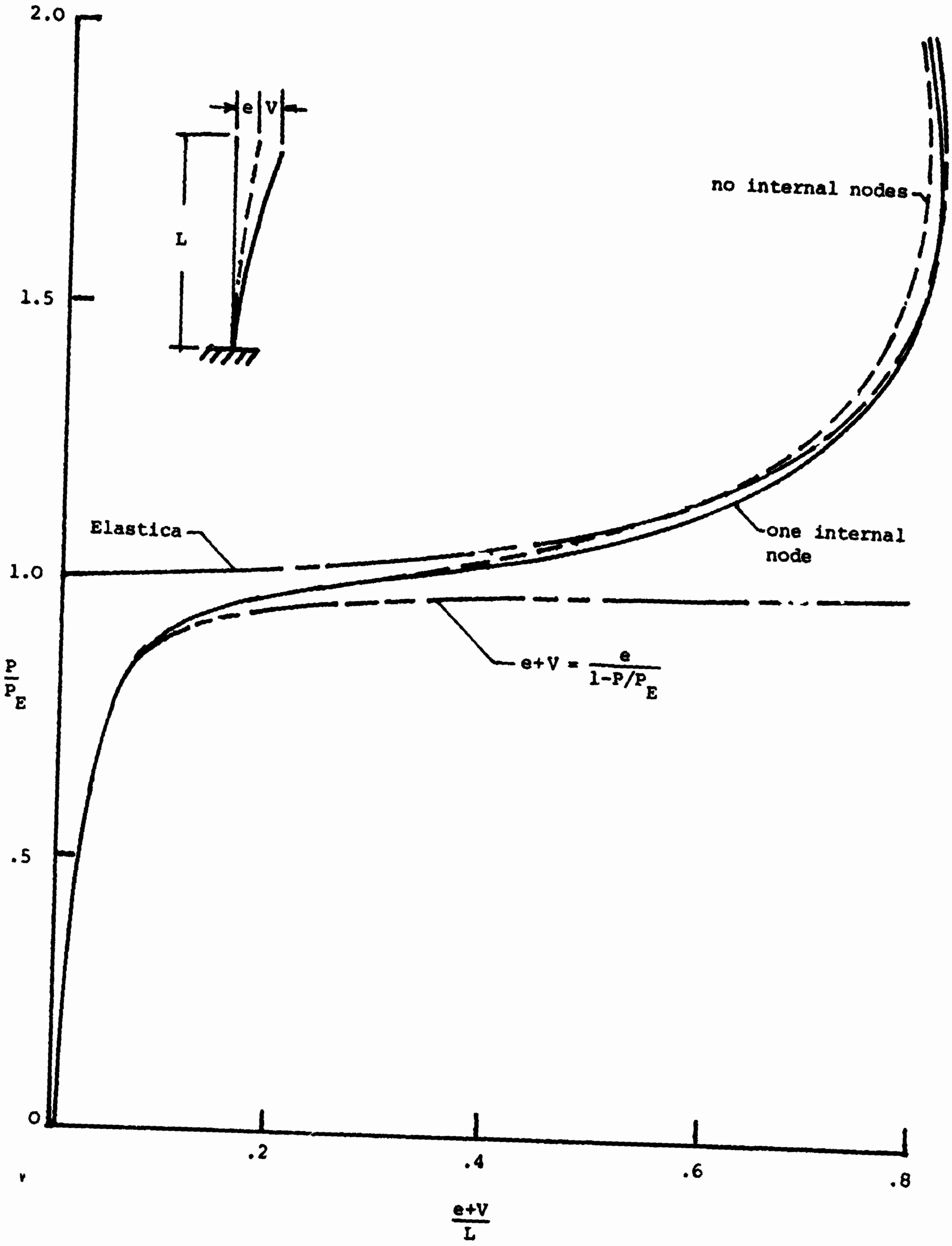

Figure 5. - Deflection of a column with initial imperfection.

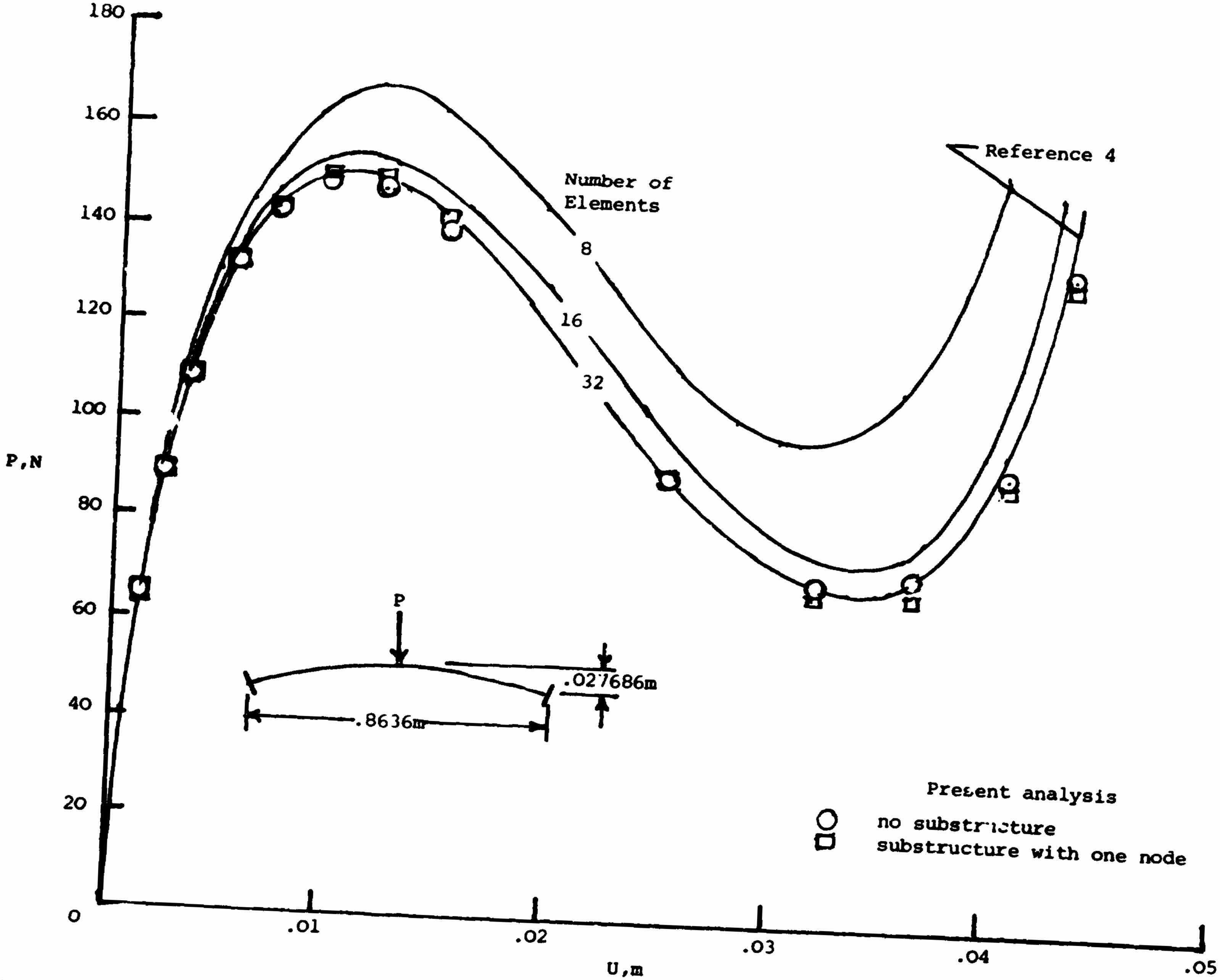

Figure 6. - Deflection of a clamped shallow arch under central loading.

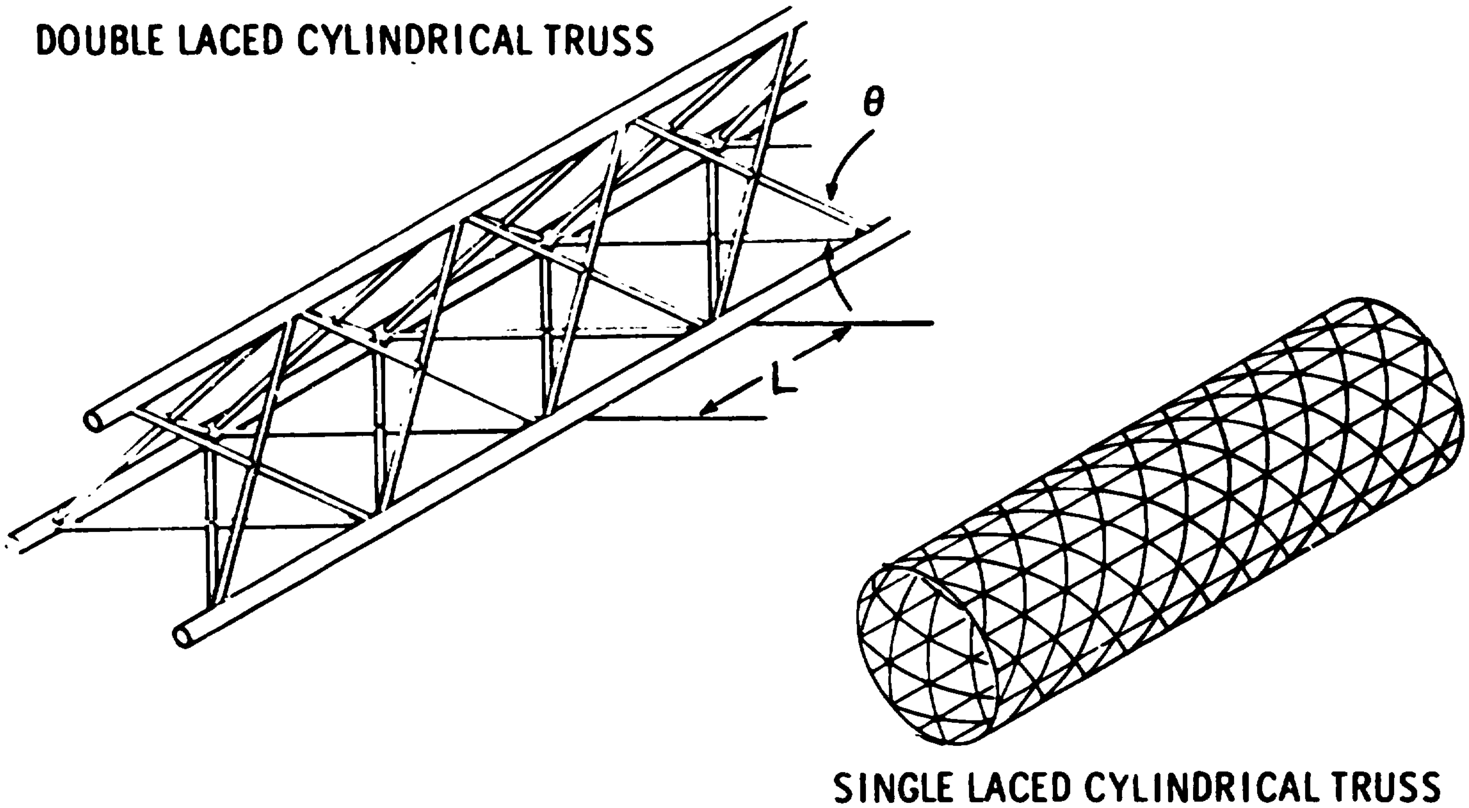

Figure 7. - Repetitive lattice structures.

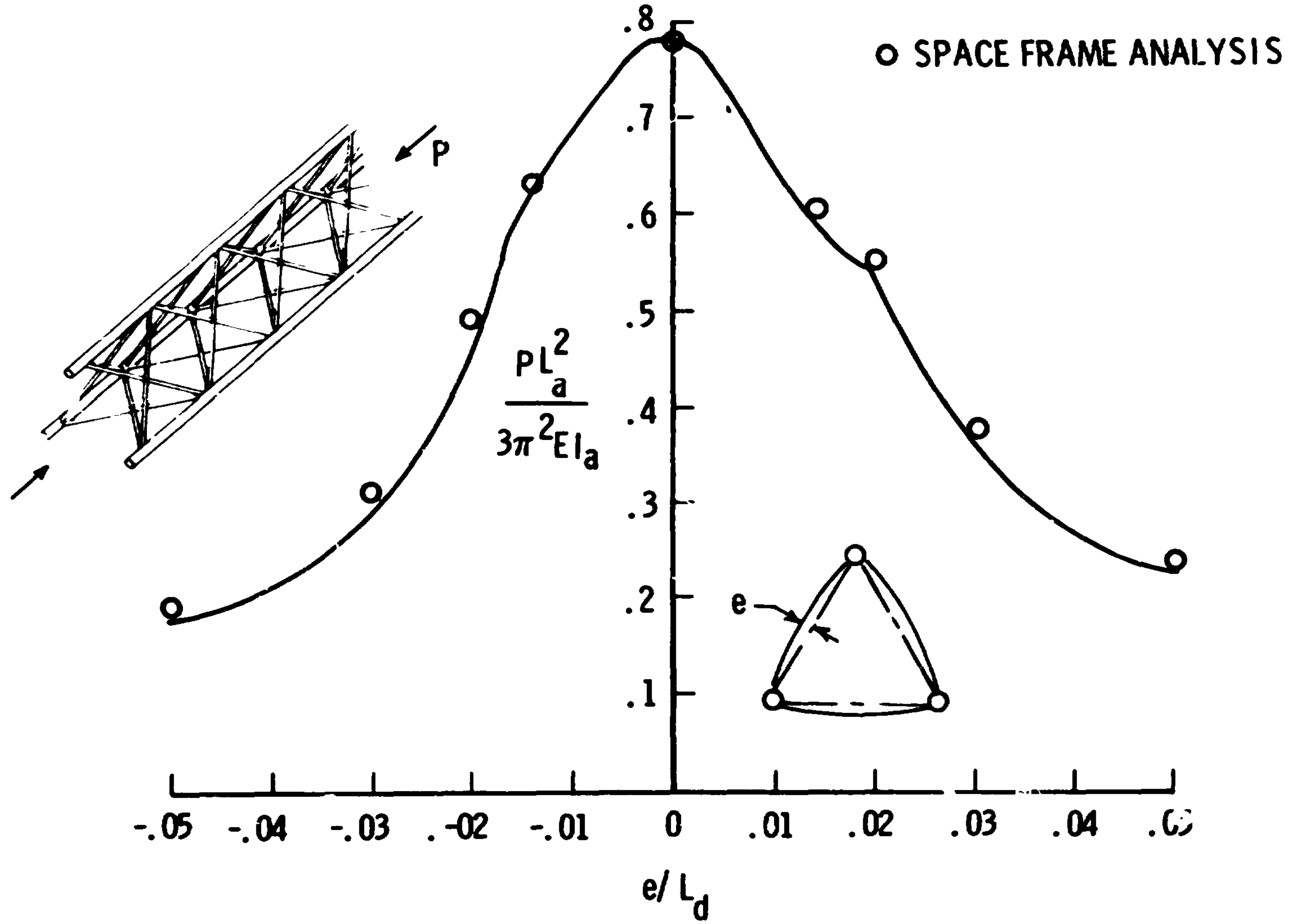

Figure 8. - Buckling of three element double laced cylindrical truss. $\frac{EA_d}{EA_a} = .05$, $\frac{L^2 EA_a}{EI_a} = 1600$, $\theta = 45°$