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## ON THE RADIATION IMPEDANCE OF A RECTANGULAR PISTON

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# ON THE RADIATION IMPEDANCE OF A RECTANGULAR PISTON

by

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Single integral representations for the resistive and reactive components of the radiation impedance appropriate to a rectangular piston are established, thereby enabling a systematic refinement of estimates at both short and long wave lengths. Comparisons with previous analyses are made explicit as well as extensions and corrections thereto.

## §1. Introduction

Acoustical literature contains numerous theoretical analyses pertaining to the characteristics of planar pistons which encompass both time-periodic and transient states of motion. The circular shape has been dealt with in the fullest manner, given the most favorable hypothesis from the standpoint of a boundary value problem, namely that the piston is set in an infinite rigid baffle; and the particularly difficult task here, typical of most radiation or scattering problems, involves the development of precise details at very short wave lengths. Analogous considerations (of a more recent date) bearing on the rectangularly shaped piston differ somewhat in matters of derivation and presentation, without leading to a genuinely efficient or practicable scheme of analytical estimation in the short wave length limit. Thus, if the piston sides are of lengths  $a$ ,  $b$  respectively and its uniform normal motion is

time-periodic at the (angular) frequency  $\omega = kc$ , Chetaev (1951) obtains (by an indirect and complex reduction of multiple integrals) the resistance and reactance formulas (omitting normalization factors)

$$R = 1 - \frac{2}{\pi \eta} \int_0^1 \left[ \text{Ci}(ka\sqrt{\eta^2 + \zeta^2}) + \eta^2 \text{Ci}(kb\sqrt{\frac{1}{\eta^2} + \zeta^2}) \right] (1-\zeta) d\zeta$$

$$+ \frac{1}{\pi \eta} \left[ \text{Ci}(ka) - \frac{\sin ka}{ka} + \frac{\cos ka - 1}{(ka)^2} \right] + \frac{\eta}{\pi} \left[ \text{Ci}(kb) - \frac{\sin kb}{kb} + \frac{\cos kb - 1}{(kb)^2} \right] \quad (1)$$

$$X = \frac{2}{\pi \eta} \int_0^1 \left[ \text{Si}(ka\sqrt{\eta^2 + \zeta^2}) + \eta^2 \text{Si}(kb\sqrt{\frac{1}{\eta^2} + \zeta^2}) \right] (1-\zeta) d\zeta$$

$$- \frac{1}{\pi \eta} \left[ \text{Si}(ka) + \frac{\cos ka - 2}{ka} + \frac{\sin ka}{(ka)^2} \right] - \frac{\eta}{\pi} \left[ \text{Si}(kb) + \frac{\cos kb - 2}{kb} + \frac{\sin kb}{(kb)^2} \right]$$

where

$$\eta = \frac{a}{b} \quad (2)$$

specifies the geometrical aspect ratio and

$$\text{Si } x = \int_0^x \frac{\sin \zeta}{\zeta} d\zeta, \quad \text{Ci } x = - \int_x^\infty \frac{\cos \zeta}{\zeta} d\zeta$$

designates the sine/cosine integral functions. Burnett (1969) recasts the formulas (1) through an integration by parts which eliminates the latter functions, yielding (in a modified notation)

$$R = 1 - \frac{2}{\pi \delta^2} \left[ 1 + \cos \sqrt{n + \frac{1}{n}} \delta + \sqrt{n + \frac{1}{n}} \delta \sin \sqrt{n + \frac{1}{n}} \delta - \cos \sqrt{n} \delta - \cos \frac{\delta}{\sqrt{n}} \right] \\ + \frac{2}{\pi} \sqrt{n} \left[ \int_{\frac{1}{\sqrt{n}}}^{\sqrt{n + \frac{1}{n}}} \sqrt{1 - \frac{1}{n \zeta^2}} \cos \delta \zeta d\zeta + \frac{1}{n} \int_{\frac{1}{\sqrt{n}}}^{\sqrt{n + \frac{1}{n}}} \sqrt{1 - \frac{n}{\zeta^2}} \cos \delta \zeta d\zeta \right]$$

and

$$X = \frac{2}{\pi \delta^2} \left[ \sin \sqrt{n + \frac{1}{n}} \delta - \sqrt{n + \frac{1}{n}} \delta \cos \sqrt{n + \frac{1}{n}} \delta + \delta \left( \sqrt{n} + \frac{1}{\sqrt{n}} \right) - \sin \sqrt{n} \delta - \sin \frac{\delta}{\sqrt{n}} \right] \\ - \frac{2}{\pi} \sqrt{n} \left[ \int_{\frac{1}{\sqrt{n}}}^{\sqrt{n + \frac{1}{n}}} \sqrt{1 - \frac{1}{n \zeta^2}} \sin \delta \zeta d\zeta + \frac{1}{n} \int_{\frac{1}{\sqrt{n}}}^{\sqrt{n + \frac{1}{n}}} \sqrt{1 - \frac{n}{\zeta^2}} \sin \delta \zeta d\zeta \right] \quad (3)$$

where

$$\delta = k \sqrt{ab}. \quad (4)$$

Burnett writes: "Despite the deceptively simple appearance of the integrals in (3), this author was unable to further reduce them in terms of known tabulated functions. They can be expressed in terms of confluent hypergeometric functions but the various forms investigated do not correspond to any known functions. Since Chetaev enjoyed the assistance of two very able mathematicians, Tikhanov and Samarskii, yet left his solution in the form of (1), it might be construed that there is no easy reduction of these integrals." In keeping with such observations he proceeds to a numerical evaluation of the representations (3) on the basis of an accurate quadrature technique (further explained in Burnett and Soroka, 1972) and to develop an analytical technique, suitable for large  $\delta$ , which "becomes prohibitively tedious" in its execution; no specific

estimates of the quantities  $R$  and  $X$  applicable when  $\delta \gg 1$  are furnished, however. Stepanishen (1977) deduces the radiation impedance by complex Fourier transformation of the (previously known) impulse response function of the piston (which is itself a Fourier transform of the impedance); and notes that the ensuing representation can be brought into full accord with the one due to Burnett. He relates the impedance to particular (parameter-dependent) values of a complex integral

$$C(x, y) = \int_0^y \frac{e^{i\pi \coth \xi}}{\coth^2 \xi} d\xi \quad (5)$$

and connects the asymptotic behavior of  $C$ , as  $x \rightarrow \infty$ , with those of  $R$  and  $X$  for  $ka \gg 1$ . The given estimate

$$C(x, y) \sim (\pi/2x)^{1/2} e^{i(x + \pi/4)} + O(x^{-3/2}), \quad x \gg 1 \quad (5a)$$

(attributed to an elementary application of stationary phase techniques) should involve a (lower order) term, namely  $O(x^{-1})$ , and this necessitates a change in the stated versions of  $R$  and  $X$  at high frequencies (which is made explicit in §3). Stepanishen also obtains a general series expansion for  $C(x, y)$  in terms of Bessel functions and notes that the corresponding representations for  $R$ ,  $X$  which follow are conveniently approximated at low frequencies. A like degree of reduction which leads to integral expressions for the mutual radiation impedance of square and rectangular pistons in a rigid infinite baffle has been achieved by Arase (1964), although the nature of theoretical estimates at short wave lengths is not considered.



It is the purpose of this paper to show that the radiation impedance of a rectangular piston admits a representation with single integrals which are related to cylinder functions and are appropriately constituted for purposes of systematic analytical estimation at both high and low frequencies. The straightforward manner of arriving at this representation is detailed in §2 and estimates obtained therefrom are recorded in §3; a separate account is intended for the analogous handling of impedance functions pertaining to rectangular panels or beams, whose prescribed normal velocities have a non-uniform modal form.

# §1. Formulation and Transformation of the Impedance Function

The half-space radiation impedance,  $Z$ , of a piston with a coplanar rigid baffle, as inferred from the integral of the time-average product of the fluctuating pressure and (uniform) normal velocity over its finite surface area  $S$ , takes the form

$$\frac{Z}{\rho c} = -\frac{ik}{2\pi} \int_S \frac{\exp(ik\sqrt{(x-x')^2 + (y-y')^2})}{\sqrt{(x-x')^2 + (y-y')^2}} dx dy dx' dy' \quad (6)$$

where  $\rho, c$  designate the equilibrium density and sound speed, respectively, of the adjacent medium and a complex time factor  $e^{-ikct}$  applies. If  $S$  has a rectangular shape and

$$-\frac{a}{2} < x, x' < \frac{a}{2}, \quad -\frac{b}{2} < y, y' < \frac{b}{2}$$

the quadruple integral (6) can be reduced, via the transformations

$$\xi = x - x', \quad \eta = x + x', \quad \zeta = y - y', \quad \mu = y + y'$$

to the double integral

$$\frac{Z}{g_{cab}} = -\frac{2ik}{ab} \int_0^a d\xi \int_0^b d\zeta (a-\xi)(b-\zeta) \frac{e^{ik\sqrt{\xi^2+\zeta^2}}}{\sqrt{\xi^2+\zeta^2}} \quad (7)$$

that is usefully resolved into a pair spanning the triangular subdomains of the rectangle  $0 < \xi < a$ ,  $0 < \zeta < b$  whose specifications in polar coordinates are

$$0 < \rho = \sqrt{\xi^2 + \zeta^2} < a \sec \vartheta, \quad 0 < \vartheta = \tan^{-1} \frac{\zeta}{\xi} < \tan^{-1} \frac{b}{a}$$

and

$$0 < \rho < b \csc \vartheta, \quad \tan^{-1} \frac{b}{a} < \vartheta < \frac{\pi}{2};$$

after affecting the elementary  $\rho$ -integration this yields

$$\begin{aligned} \frac{Z}{g_{cab}} = 1 - \frac{2i}{\pi ab k^2} \left\{ (a+b)k - i + \int_0^{\tan^{-1} \frac{b}{a}} (ka \sin \vartheta - kb \cos \vartheta + 2i \sin \vartheta \cos \vartheta) e^{ika \sec \vartheta} d\vartheta \right. \\ \left. + \int_{\tan^{-1} \frac{b}{a}}^{\pi/2} (-ka \sin \vartheta + kb \cos \vartheta + 2i \sin \vartheta \cos \vartheta) e^{ikb \csc \vartheta} d\vartheta \right\} \end{aligned} \quad (8)$$

with only single integrals remaining.

In the limit  $b \rightarrow \infty$  the above representation implies that

$$\frac{\hat{Z}}{\rho c a} = 1 - \frac{2i}{\pi k a} + \frac{2i}{\pi k a} \int_0^{\pi/2} \cos \vartheta e^{i k a \sec \vartheta} d\vartheta \quad \left( \hat{Z} = \frac{Z}{b} \right)$$

and

$$\begin{aligned} \frac{R}{\rho c a} &= \operatorname{Re} \left\{ \frac{\hat{Z}}{\rho c a} \right\} = 1 - \frac{2}{\pi k a} \int_0^{\pi/2} \cos \vartheta \sin(k a \sec \vartheta) d\vartheta \\ &= 1 - \frac{2}{\pi k a} \int_1^{\infty} \frac{\sin k a \zeta}{\zeta^2 \sqrt{\zeta^2 - 1}} d\zeta \end{aligned} \quad (9)$$

on invoking the change of variable  $\zeta = \sec \theta$ . It is evident that the latter result describes the radiation resistance for an infinite strip piston of width  $a$  given by the analogue of (6), namely

$$\frac{R}{\rho c a} = \frac{k}{2a} \operatorname{Re} \int_{-a/2}^{a/2} H_0^{(1)}(k|x-\pi'|) d\pi d\pi'$$

in terms of a Hankel function or, alternatively, by

$$\frac{R}{\rho c a} = \frac{k}{\pi a} \operatorname{Re} \int_{-\infty}^{\infty} \frac{1 - \cos \xi a}{\xi^2} \frac{d\xi}{\sqrt{k^2 - \xi^2}} \quad (10)$$

when use is made of the representation

$$H_0^{(1)}(k|x-\pi'|) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-\pi')}}{\sqrt{k^2 - \xi^2}} d\xi$$

with a contour that passes above/below the branch points at  $\xi = \pm k$ , respectively, so that

$$\arg \sqrt{k^2 - \xi^2} = \begin{cases} 0, & |\xi| < k \\ \pi/2, & |\xi| > k \end{cases}$$

The equivalence of (9) and (10) is affirmed on noting the relation

$$\begin{aligned} \operatorname{Re} \int_{-\infty}^{\infty} \frac{1 - e^{i\xi a}}{\xi^2} \frac{d\xi}{\sqrt{k^2 - \xi^2}} &= \operatorname{Re} \left\{ \frac{2}{i} \int_k^{\infty} \frac{1 - e^{i\xi a}}{\xi^2} \frac{d\xi}{\sqrt{\xi^2 - k^2}} + \frac{\pi a}{k} \right\} \\ &= \frac{\pi a}{k} - \frac{2}{k^2} \int_1^{\infty} \frac{\sin ka\zeta}{\zeta^2 \sqrt{\zeta^2 - 1}} d\zeta \end{aligned}$$

which expresses the consequence of displacing the original integral into the upper half of the  $\xi$ -plane and allowing for the pole singularity at  $\xi = 0$ .

When  $a, b \neq \infty$ , the respective changes of variable  $\zeta = \sec \theta$  and  $\zeta = \csc \theta$  convert the integrals in (8) to the forms

$$\begin{aligned} I_1 &= \int_0^{\tan^{-1} \frac{b}{a}} (ka \sin \theta - kb \cos \theta + 2i \sin \theta \cos \theta) e^{ika \sec \theta} d\theta \\ &= \int_1^{\sqrt{1 + (\frac{b}{a})^2}} \left( \frac{ka}{\zeta^2} - \frac{kb}{\zeta^2 \sqrt{\zeta^2 - 1}} + \frac{2i}{\zeta^3} \right) e^{ika\zeta} d\zeta \\ &= i \left( e^{ika} - \frac{1}{1 + (\frac{b}{a})^2} e^{ik\sqrt{a^2 + b^2}} \right) - kb \int_1^{\sqrt{1 + (\frac{b}{a})^2}} \frac{e^{ika\zeta}}{\zeta^2 \sqrt{\zeta^2 - 1}} d\zeta \end{aligned} \quad (11)$$

and

$$\begin{aligned}
 I_2 &= \int_{\tan^{-1} \frac{b}{a}}^{\pi/2} (-ka \sin \theta + kb \cos \theta + 2i \sin \theta \cos \theta) e^{ikb \csc \theta} d\theta \\
 &= i \left( e^{ikb} - \frac{1}{1+(\frac{a}{b})^2} e^{ik\sqrt{a^2+b^2}} \right) - ka \int_1^{\sqrt{1+(\frac{a}{b})^2}} \frac{e^{ikb\zeta}}{\zeta^2 \sqrt{\zeta^2-1}} d\zeta.
 \end{aligned} \tag{12}$$

Hence

$$\begin{aligned}
 \frac{Z}{\rho cab} &= 1 - \frac{2i}{\pi ab k^2} \left\{ (a+b)k - i + I_1 + I_2 \right\} \\
 &= 1 - \frac{2i}{\pi ab k} (a+b) - \frac{2}{\pi ab k^2} + \frac{2}{\pi ab k^2} (e^{ika} + e^{ikb} - e^{ik\sqrt{a^2+b^2}}) \\
 &\quad + \frac{2i}{\pi ka} \int_1^{\sqrt{1+(\frac{b}{a})^2}} \frac{e^{ika\zeta}}{\zeta^2 \sqrt{\zeta^2-1}} d\zeta + \frac{2i}{\pi kb} \int_1^{\sqrt{1+(\frac{a}{b})^2}} \frac{e^{ikb\zeta}}{\zeta^2 \sqrt{\zeta^2-1}} d\zeta
 \end{aligned} \tag{13}$$

and thus, utilizing the relation

$$\frac{d\zeta}{\zeta^2 \sqrt{\zeta^2-1}} = d \sqrt{1 - \frac{1}{\zeta^2}}, \quad \zeta > 1$$

$$\begin{aligned}
 \frac{Z}{\rho cab} &= 1 - \frac{2i}{\pi ab k} (a+b) - \frac{2}{\pi ab k^2} + \frac{2}{\pi ab k^2} (e^{ika} + e^{ikb} - e^{ik\sqrt{a^2+b^2}}) \\
 &\quad + \frac{2i}{\pi kab} \sqrt{a^2+b^2} e^{ik\sqrt{a^2+b^2}} \\
 &\quad + \frac{2}{\pi} \int_1^{\sqrt{1+(\frac{b}{a})^2}} \sqrt{1 - \frac{1}{\zeta^2}} e^{ika\zeta} d\zeta + \frac{2}{\pi} \int_1^{\sqrt{1+(\frac{a}{b})^2}} \sqrt{1 - \frac{1}{\zeta^2}} e^{ikb\zeta} d\zeta
 \end{aligned} \tag{14}$$

It is verifiable at once that the representations for  $\text{Re}(Z/\rho_{cab})$  and  $-\text{Im}(Z/\rho_{cab})$  inferred from (14) correspond exactly with the Burnett forms (1) of the (unnormalized) piston resistance and reactance. Furthermore, the prototype integral in (13),

$$\int_1^{\sqrt{1+\delta^2}} \frac{e^{i\pi\zeta}}{\zeta^2\sqrt{\zeta^2-1}} d\zeta = \int_0^{y=\cosh^{-1}\sqrt{1+\delta^2}} \frac{e^{i\pi\cosh\xi}}{\cosh^2\xi} d\xi$$

$$= C(\pi, y),$$

duplicates the one which Stepanishen defines and employs in an equivalent impedance representation. A straightforward derivation of the hitherto available formulas which constitute the basis for theoretical and numerical predictions relating to the rectangular piston impedance is thus completed; and a different manner of handling same, with patent advantages at short wave lengths, now merits description.

### §3. Conversion and Estimation of the Impedance Function

Let the integral

$$I(\pi, y) = C(\pi, \cosh^{-1} y) = \int_1^y \frac{e^{i\pi\zeta}}{\zeta^2\sqrt{\zeta^2-1}} d\zeta, \quad y > 1 \quad (15)$$

which enters into (13) be rewritten in the composite fashion

$$I(\pi, y) = \int_1^\infty \frac{e^{i\pi\zeta}}{\zeta^2\sqrt{\zeta^2-1}} d\zeta - \int_y^\infty \frac{e^{i\pi\zeta}}{\zeta^2\sqrt{\zeta^2-1}} d\zeta \quad (16)$$

and consider the one-variable function that figures,

$$F(x) = \int_1^{\infty} \frac{e^{ix\zeta}}{\zeta^2 \sqrt{\zeta^2 - 1}} d\zeta, \quad x > 0 \quad (17)$$

with the initial value

$$F(0) = \int_0^{\infty} \operatorname{sech}^2 \vartheta d\vartheta = 1. \quad (18)$$

Sequential differentiations with respect to  $x$  yield the relations

$$\frac{dF}{dx} = i \int_1^{\infty} \frac{e^{ix\zeta}}{\zeta \sqrt{\zeta^2 - 1}} d\zeta \quad (19)$$

and

$$\frac{d^2 F}{dx^2} = - \int_1^{\infty} \frac{e^{ix\zeta}}{\sqrt{\zeta^2 - 1}} d\zeta = -\frac{\pi i}{2} H_0^{(1)}(x) \quad (20)$$

where  $H_0^{(1)}(x)$  is the Hankel function; thus, on integrating (20) and taking note of the condition  $dF/dx \rightarrow 0$ ,  $x \rightarrow \infty$ , it follows that

$$\frac{dF}{dx} = \frac{\pi i}{2} \left[ 1 - \int_0^x H_0^{(1)}(\zeta) d\zeta \right].$$

Another integration supplies the original function, viz.

$$\begin{aligned}
 F(x) &= F(0) + \frac{\pi i x}{2} - \frac{\pi i}{2} \int_0^x d\sigma \int_0^\sigma H_0^{(1)}(\zeta) d\zeta \\
 &= 1 + \frac{\pi i x}{2} - \frac{\pi i}{2} \left\{ x \int_0^x H_0^{(1)}(\zeta) d\zeta - \int_0^x \zeta H_0^{(1)}(\zeta) d\zeta \right\} \\
 &= 1 + \frac{\pi i x}{2} - \frac{\pi i}{2} x \int_0^x H_0^{(1)}(\zeta) d\zeta + \frac{\pi i}{2} \left\{ x H_1^{(1)}(x) - \lim_{\epsilon \rightarrow 0} \epsilon H_1^{(1)}(\epsilon) \right\} \\
 &= \frac{\pi i}{2} x \left\{ \int_x^\infty H_0^{(1)}(\zeta) d\zeta + H_1^{(1)}(x) \right\} \tag{21a} \\
 &= \frac{\pi i}{2} x \left\{ \int_x^\infty \frac{1}{\zeta} d(\zeta H_1^{(1)}(\zeta)) + H_1^{(1)}(x) \right\} \\
 &= \frac{\pi i}{2} x \int_x^\infty \frac{H_1^{(1)}(\zeta)}{\zeta} d\zeta, \tag{21b}
 \end{aligned}$$

where the versions (21a) and (21b) are particularly apt. Thus, the former leads, in conjunction with the relations

$$\int_0^\infty H_0^{(1)}(\zeta) d\zeta = 1$$

and

$$\frac{2}{\pi} \int_0^x H_0^{(1)}(\zeta) d\zeta = x \left\{ H_0^{(1)}(x) \left( \frac{2}{\pi} - S_1(x) \right) + S_0(x) H_1^{(1)}(x) \right\}$$



to the generally valid result

$$F(x) = \frac{\pi i}{2} x \left\{ 1 + H_1^{(1)}(x) - x H_0^{(1)}(x) + \frac{\pi x}{2} (H_0^{(1)}(x) S_1(x) - H_1^{(1)}(x) S_0(x)) \right\}, \quad x > 0 \quad (22)$$

involving Hankel,  $H_0^{(1)}(x)$ , and Struve,  $S_{0,1}(x)$ , functions of the zero<sup>th</sup> and first orders.

For large values of  $x$  it is expedient to utilize (21b) and write

$$\begin{aligned} F(x) &= \frac{\pi i}{2} x \int_x^\infty \frac{d(-H_0^{(1)}(\zeta))}{\zeta} \\ &= \frac{\pi i}{2} H_0^{(1)}(x) - \frac{\pi i}{2} x \int_x^\infty \frac{H_0^{(1)}(\zeta)}{\zeta^2} d\zeta \\ &= \frac{\pi i}{2} H_0^{(1)}(x) - \frac{\pi i}{2} x \int_x^\infty \frac{d(\zeta H_1^{(1)}(\zeta))}{\zeta^3} \\ &= \frac{\pi i}{2} H_0^{(1)}(x) + \frac{\pi i}{2x} H_1^{(1)}(x) - \frac{3\pi i}{2} x \int_x^\infty \frac{H_1^{(1)}(\zeta)}{\zeta^3} d\zeta \\ &= \frac{\pi i}{2} H_0^{(1)}(x) + \frac{\pi i}{2x} H_1^{(1)}(x) - \frac{3\pi i}{2x^2} H_0^{(1)}(x) \\ &\quad + \frac{9\pi i}{2} x \int_x^\infty \frac{H_0^{(1)}(\zeta)}{\zeta^4} d\zeta, \end{aligned} \quad (23)$$

with the evident capability of securing additional terms that contain higher reciprocal powers of  $x$ .

Repeated integration by parts yields, as regards the two-variable function of (16),

$$\begin{aligned}
 \int_y^\infty \frac{e^{ix\zeta}}{\zeta^2 \sqrt{\zeta^2 - 1}} d\zeta &= \frac{1}{ix} \int_y^\infty \frac{d(e^{ix\zeta})}{\zeta^2 \sqrt{\zeta^2 - 1}} \\
 &= \frac{ie^{ixy}}{xy^2 \sqrt{y^2 - 1}} + \frac{i}{x} \int_y^\infty e^{ix\zeta} \frac{d}{d\zeta} \left( \frac{1}{\zeta^2 \sqrt{\zeta^2 - 1}} \right) d\zeta \\
 &= \frac{ie^{ixy}}{xy^2 \sqrt{y^2 - 1}} - \frac{e^{ixy}}{x^2} \frac{d}{d\zeta} \left( \frac{1}{\zeta^2 \sqrt{\zeta^2 - 1}} \right)_{\zeta=y} - \frac{1}{x^2} \int_y^\infty e^{ix\zeta} \frac{d^2}{d\zeta^2} \left( \frac{1}{\zeta^2 \sqrt{\zeta^2 - 1}} \right) d\zeta
 \end{aligned} \tag{24}$$

and the number of explicit terms which involve (higher and higher) reciprocal powers of  $x$  ( $y > 1$ ) can be readily found.

Accordingly, the impedance integral (15) is realized in terms of a single variable component, with alternative and explicit cylinder function forms, and a two-variable 'incomplete' cylinder function component; the former, which takes into account the full singularity of the original integral at its lower limit, can be estimated on the basis of (extensive) cylinder function asymptotics while a numerical evaluation of the latter is furthered by writing

$$\begin{aligned}
 \int_y^\infty \frac{e^{ix\zeta}}{\zeta^2 \sqrt{\zeta^2 - 1}} d\zeta &= \int_y^\infty \frac{e^{ix\zeta}}{\zeta^2} \left( \frac{1}{\sqrt{\zeta^2 - 1}} - \frac{1}{\zeta} + \frac{1}{2\zeta^3} - \frac{3}{8\zeta^5} + \dots \right) d\zeta \\
 &\quad + \int_y^\infty e^{ix\zeta} \left( \frac{1}{\zeta^3} - \frac{1}{2\zeta^5} + \frac{3}{8\zeta^5} - \dots \right) d\zeta
 \end{aligned}$$

and expressing  $\int_y^\infty \zeta^{-n} e^{ix\zeta} d\zeta$  in terms of tabulated sine/cosine integral functions.

When the simplest asymptotic developments of the Hankel function  $H_0^{(1)}(x)$ ,  $x \gg 1$ , are employed in (23) and the latter combined with (16) and (24) the result is

$$I(x, y) \simeq i \sqrt{\frac{\pi}{2x}} e^{i(x - \frac{\pi}{4})} \left(1 - \frac{9i}{8x}\right) - \frac{i e^{i\pi y}}{y^2 \sqrt{y^2 - 1}} \left(\frac{1}{x} - i \frac{3y^2 - 2}{y(y^2 - 1)} \frac{1}{x^2}\right) + O\left(\frac{1}{x^{5/2}}\right), \quad x \gg 1, \quad y > 1 \quad (25)$$

only the leading term,  $O(x^{-1/2})$ , appears explicitly in the estimate (5a) proposed by Stepanishen. Applying (25) to deduce a short wave length characterization of the impedance from (13) it follows that

$$\frac{Z}{\rho_{cab}} \sim 1 - \frac{2i}{\pi ab k} (a+b) - \sqrt{\frac{2}{\pi}} \frac{e^{i(ka - \frac{\pi}{4})}}{(ka)^{3/2}} \left(1 - \frac{9i}{8ka}\right) - \sqrt{\frac{2}{\pi}} \frac{e^{i(kb - \frac{\pi}{4})}}{(kb)^{3/2}} \left(1 - \frac{9i}{8kb}\right) - \frac{2}{\pi ab k^2} (e^{ika} + e^{ikb}) - \frac{2i}{\pi} e^{ik\sqrt{a^2+b^2}} \frac{(a^2+b^2)^{3/2}}{(kab)^3} + O(k^{-7/2}) \quad (26)$$

whence

$$ka \gg 1$$

$$\frac{R}{\rho_{cab}} \sim 1 - \sqrt{\frac{2}{\pi}} \left\{ \frac{\cos(ka - \pi/4)}{(ka)^{3/2}} + \frac{\cos(kb - \pi/4)}{(kb)^{3/2}} \right\} - \frac{2}{\pi ab k^2} (1 - \cos ka - \cos kb) - \frac{9}{8} \sqrt{\frac{2}{\pi}} \left\{ \frac{\sin(ka - \pi/4)}{(ka)^{5/2}} + \frac{\sin(kb - \pi/4)}{(kb)^{5/2}} \right\} + \frac{2}{\pi} \frac{(a^2+b^2)^{3/2}}{(kab)^3} \sin k\sqrt{a^2+b^2} + O(k^{-7/2}) \quad (27)$$

and

$$\begin{aligned} \frac{X}{\rho c a b} \sim & \frac{2}{\pi a b k} (a+b) + \sqrt{\frac{2}{\pi}} \left\{ \frac{\sin(ka - \pi/4)}{(ka)^{3/2}} + \frac{\sin(kb - \pi/4)}{(kb)^{3/2}} \right\} - \frac{2}{\pi a b k^2} (\sin kb + \sin ka) \\ & - \frac{9}{8} \sqrt{\frac{2}{\pi}} \left\{ \frac{\cos(ka - \pi/4)}{(ka)^{5/2}} + \frac{\cos(kb - \pi/4)}{(kb)^{5/2}} \right\} + \frac{2}{\pi} \frac{(a^2 + b^2)^{3/2}}{(kab)^3} \cos k\sqrt{a^2 + b^2} \\ & + O(k^{-7/2}) \end{aligned} \quad (28)$$

in the same limit.

These expressions for the radiation resistance and reactance of a rectangular piston incorporate two orders of magnitude, namely  $k^{-5/2}$  and  $k^{-3}$ , beyond those previously given and correct, in the terms of order  $k^{-2}$ , the analogous approximations of Stepanishen; thus, his contributions

$$- \frac{2}{\pi a b k^2} \cos k\sqrt{a^2 + b^2} \quad \text{and} \quad \frac{2}{\pi a b k^2} \sin k\sqrt{a^2 + b^2}$$

in  $R$  and  $X$ , respectively, are eliminated by taking into account an order  $1/x$  term missing from (5a) and present in (24) [cf. Appendix].

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### Appendix

In a paper published subsequently to that concerning the rectangular piston, Stepanishen (1978) cites the theorem: Let  $\varphi(\zeta)$  be  $N$  times continuously differentiable in  $\alpha < \zeta < \beta$ . Let  $0 < \lambda < 1$ ,  $0 < \mu < 1$ ; then, as  $x \rightarrow +\infty$ ,

$$\begin{aligned} & \int_{\alpha}^{\beta} e^{ix\zeta} (\zeta-\alpha)^{\lambda-1} (\beta-\zeta)^{\mu-1} \varphi(\zeta) d\zeta \\ &= \sum_{n=0}^{N-1} \frac{\Gamma(n+\lambda)}{n! x^{n+\lambda}} \exp\left\{i\left(\alpha x + \frac{1}{2}\lambda\pi + \frac{1}{2}n\pi\right)\right\} \frac{d^n}{d\zeta^n} [(\beta-\zeta)^{\mu-1} \varphi(\zeta)]_{\zeta=\alpha} \\ &+ \sum_{n=0}^{N-1} \frac{\Gamma(n+\mu)}{n! x^{n+\mu}} \exp\left\{i\left(\beta x - \frac{1}{2}\mu\pi + \frac{1}{2}n\pi\right)\right\} \frac{d^n}{d\zeta^n} [(\zeta-\alpha)^{\lambda-1} \varphi(\zeta)]_{\zeta=\beta} + O\left(\frac{1}{x^N}\right). \end{aligned}$$

As regards the integral  $I(x,y) = \int_1^y \frac{e^{ix\zeta}}{\zeta^2 \sqrt{\zeta^2-1}} d\zeta$ , in particular, where the identifications

$$\alpha = 1, \beta = y, \lambda = \frac{1}{2}, \mu = 1 \quad \text{and} \quad \varphi(\zeta) = \frac{1}{\zeta^2 \sqrt{\zeta+1}}$$

apply, it is found that

$$\begin{aligned} \int_1^y \frac{e^{ix\zeta}}{\zeta^2 \sqrt{\zeta^2-1}} d\zeta &= \sum_{n=0}^{N-1} \frac{\Gamma(n+\frac{1}{2})}{n! \chi^{n+1/2}} \exp \left\{ i \left( \chi + \frac{\pi}{4} + \frac{n\pi}{2} \right) \right\} \frac{d^n}{d\zeta^n} \left( \frac{1}{\zeta^2 \sqrt{\zeta+1}} \right)_{\zeta=1} \\ &+ \sum_{n=0}^{N-1} \frac{\Gamma(n+1)}{n! \chi^{n+1}} \exp \left\{ i \left( \chi y - \frac{\pi}{2} + \frac{n\pi}{2} \right) \right\} \frac{d^n}{d\zeta^n} \left( \frac{1}{\zeta^2 \sqrt{\zeta^2-1}} \right)_{\zeta=y} \\ &+ O\left(\frac{1}{\chi^N}\right) \end{aligned}$$

and the expected agreement with (24) obtains.

