(NASA-TM-84941) TYPE 2 RADIO BURSTS, INTERPLANETARY SHOCKS AND ENERGETIC PARTICLE EVENTS (NASA) 30 p HC A03/MF A01 CSCL 03B

N83-17435

Unclas
G3/90 08216

NASA

Technical Memorandum 84941

Type II Radio Bursts, Interplanetary Shocks and Energetic Particle Events

H. V. Cane and R. G. Stone

NOVEMBER 1982

National Aeronautics and
Space Administration

Goddard Space Flight Center
Greenbelt, Maryland 20771

ABSTRACT

Using the ISEE-3 radio astronomy experiment data we have identified 37 interplanetary (IP) type II bursts in the period September 1978 to December 1981. We list these events and the associated phenomena. The events are preceded by intense, soft X ray events with long decay times (LDEs) and type II and/or type IV bursts at meter wavelengths. The meter wavelength type II bursts are usually intense and exhibit herringbone structure. The extension of the herringbone structure into the kilometer wavelength range results in the occurrence of a shock accelerated (SA) event. The SA event is an important diagnostic for the presence of a strong shock and particle acceleration. The majority of the interplanetary type II bursts are associated with energetic particle events. Our results support other studies which indicate that energetic solar particles detected at 1 A.U. are generated by shock acceleration.From a preliminary analysis of the available data there appears to be a high correlation with white light coronal transients. The transients are fast i.e. velocities greater than 500 km/sec.

TYPE II RADIO BURSTS, INTERPLANETARY SHOCKS AND ENERGETIC PARTICLE EVENTS

H. V. CANE* AND R. G. STONE

Laboratory for Extraterrestrial Physics

NASA Goddard Space Flight Center, Greenbelt, Maryland 20771

*Research Associate - University of Maryland

ABSTRACT

Using the ISEE-3 radio astronomy experiment data we have identified 37 interplanetary (IP) type II bursts in the period September 1978 to December 1981. We list these events and the associated phenomena. The events are preceded by intense, soft X ray events with long decay times (LDEs) and type II and/or type IV bursts at meter wavelengths. The meter wavelength type II bursts are usually intense and exhibit herringbone structure. The extension of the herringbone structure into the kilometer wavelength range results in the occurrence of a shock accelerated (SA) event. The SA event is an important diagnostic for the presence of a strong shock and particle acceleration. The majority of the interplanetary type II bursts are associated with energetic particle events. Our results support other studies which indicate that energetic solar particles detected at 1 A.U. are generated by shock acceleration.From a preliminary analysis of the available data there appears to be a high correlation with white light coronal transients. The transients are fast i.e. velocities greater than 500 km/sec.

I. INTRODUCTION

The type II burst results from plasma emission generated by a shock as it propagates out through the solar corona. The frequency drift rate of the burst is related to the shock's velocity. Ground-based observations of type II bursts pertain to coronal heights less than about 5 solar radii. A shock typically takes 2 days to reach the earth at 215 solar radii where it causes a sudden commencement geomagnetic storm . In situ observations of shocks provide information about shock properties including the velocity, but such observations have been made primarily by earth orbiting satellites. Thus shock properties in the region between 5 and 215 solar radii are not well studied. Satellite experiments operating at low radio frequencies can access this region by remotely observing interplanetary type II bursts. The value of such observations are limited in part by our current understanding of the radio emission process.

Two events were reported from observations from IMP-6 during the previous period of solar maximum (Malitson et al., 1973,1976). Boischot et al. (1980) reported the detection of a number of events from the Voyager spacecraft and there have been accounts of detections from the Prognoz-8 satellite (Pinter et al., 1982). However the ISEE-3 radio astronomy experiment (Knoll et al., 1978), which operates over a frequency range of 2 MHz to 30 kHz, has obtained the most complete and detailed set of observations of interplanetary (IP) type II bursts thus far available. The experiment is more sensitive than previous experiments and because of its orbit is much less troubled by terrestrial kilometric radiation. In addition the experiment can observe the sun continuously. An initial paper reporting the detection of 12 events was prepared in 1980 and published in 1982 (Cane et al. 1982).

Cane et al. (1981) reported on a new class of kilometer wavelength bursts, the shock accelerated (SA) events, also related to energetic shocks. The SA event is the low frequency continuation of the herringbone structure associated with meter wavelength type II bursts. At kilometer wavelengths these events precede the IP type II burst. The SA event allows the determination of the start of the type II event low in the corona. The sequence of events is illustrated in figure 1 (from Cane et al., 1981). There is no frequency coverage between about 20 and 2 MHz. The time difference between the end of the meter wavelength type II burst and the observation of a type II burst at 2 MHz is of the order of 1/2 hour. However the rapid drift of the SA event means that the time difference between the 2 MHz SA event and the start of the meter wavelength type II burst is of the order of a minute. Thus associations between meter wavelength and kilometer wavelength type II bursts can be made unambiguously.

In this paper we present information on the IP type II events and associated phenomena. Since the writing of the previous paper our understanding of the data has greatly improved and the sample of events has increased three-fold. It is therefore timely to provide an up-date and more comprehensive description.

II. DATA ANALYSIS

The ISEE-3 radio astronomy data shows numerous slow drift features in the dynamic spectra (plots of intensity as a function of frequency and time). The

ORIGINAL PAGE IS
OF POOR QUALITY

majority of these are short-lived with very slow drift rates and we have no explaination for their origin. A smaller number last for many hours but because they commence in the middle of our frequency range we have no way of determining a likely starting time at the sun. Our list of type II bursts has been restricted to those events which drift through the data at a rate consistent with known shock velocities and which are preceeded by an SA event. Thus we can identify the start of the event at the sun. Whereas other events may be related to solar shocks we include only those which are clearly the kilometer wavelength counterpart of the meter wavelength phenomenom.

In table 1 we list the IP type II events. As in our previous catalog there are two categories. Category 1 events have been unambiguously identified with a sudden commencement. For the most part the type II emission is discernable over the frequency range at which the events are best observed namely 500-80 kHz and therefore the events are observed for many hours. The reason for these upper and lower frequency bounds will be discussed later. The events marked with an asterisk are not as well observed because of other activity occurring at the same time, which limits the detectibility of the burst.

Category 2 events are those bursts which are not followed by a sudden commencement or only last for a few hours. Some of the events not followed by a sudden commencement are associated with flares far from central meridian and the shock probably was not extensive enough in heliographic longitude to intercept the earth.

There exist a number of candidate IP type II events which are not listed.

One event on Dec 5, 1981 was excluded because there was no ground-based data to corroborate the presence of a strong shock . However a particle event was detected as was a sudden commencement and it is likely that the event originated behind the west limb. Another event on May 10, 1981 was excluded because the start of the event at the sun could not be determined very accurately. It is probable that this event was associated with an east limb coronal transient observed by the P78-1 coronograph. For a number of events there was corroborative ground-based data but the low frequency data was of poor quality or the event was observed only over a very small frequency range.

The table lists a number of phenomena examined in conjunction with the low frequency data. In general we have used data published in Solar Geophysical Data (SGD) and apart from Culgoora dynamic spectra, have not made use of original data sets. In the main our study has been restricted to phenomena which occur high in the corona. We have not used radio data outside the meter wavelength band. Decameter wavelength information was not used because observatory coverage in this region of the spectrum is at best limited and often rendered unuseable because of interference.

The time given in the second column is the start of the meter wavelength type II burst or, if no type II was reported, the start of the SA event. In the latter case the time is enclosed in brackets.

(i) Hα observations

Most events have been associated with an Hα flare. The flare was determined using the start of the meter wavelength type II burst which occurs within a few minutes of the maximum in Hα (Roberts, 1959). All the events with good identification were bright flares of importance 1 or greater.

The flare location is also shown in table 1. The question mark denotes an assumed behind-the-limb flare. The longitudinal distribution of flares associated with IP type II bursts is shown in table 2 and figure 2.

(ii) Soft X-rays

The 1-8 A soft X-ray class has been estimated from the daily graphs presented in SGD. The majority of the X-ray events are intense and have decays longer than 4 hours i.e they are long duration events (LDEs). The 'Y' in the column after the X ray class indicates an LDE event was observed. Thirty-five of the 37 events were seen in soft X-rays. Of the two remaining events, one occurred during an X-ray data gap and the other has been attributed to an event behind the west limb. Twenty-four of the associated X-ray events have long decays and a further 9 have decays between about 2 and 4 hours. Two events had decays less than 2 hours. These are associated with slow, category 2 IP type II bursts.

For the year 1981 we have catalogued all soft X-ray events whose 1-8 A class was greater than M4 and with a duration longer than 4 hours. Of 18 events 10 were associated with IP type II bursts. Six further events were associated with SA events only i.e. not followed by a type II burst. The remaining 2 events occurred during ISEE-3 data gaps. It would appear that intense LDE X-ray events correlate well with strong shocks, many of which produce IP type II bursts.

(iii) Meter wavelength radio emission

Eighteen of the 37 events are preceded by a meter wavelength type II/IV burst pair, 10 events by a type IV burst and 9 by a type II burst. Reports of continuum are included under the classification of type IV. The annotations 'W' and 'P' mean 'weak' and 'possible' respectively. Most of the meter wavelength type II bursts are classified as intense. The dynamic spectra for events also observed by the Culgoora observatory exhibit complex behaviour with herringbone structure. We believe that all IP type II events are associated with meter wavelength activity. As discussed in the introduction to this section, the only candidate events not associated with meter wavelength activity are probably behind-the-limb events.

The reported occurrences of type III bursts associated with the meter wavelength type II and/or type IV events have been listed. The intensity class is given and the time interval between the start of the meter wavelength type II burst, or the SA event, and the reported start of the type III activity. The annotation 'D' implies that the type III activity occurred during the meter type II or type IV burst. Nine of the IP type II events are not preceded within 25 mins by, or associated with, any type III activity. The statistics do not include single bursts (i.e. type IIIb) or ongoing storm activity. For an additional 4 events the type III activity commenced after the start of the type II burst and may be herringbone structure. Only one event is preceded by intense type IIIG/V activity. This event occurred on July 23 1980, during Culgoora observing time and an examination of the data reveals the possible presence of two type II bursts. The first event occurred shortly after the type III/V burst. The second event commenced a few minutes later and is the event which continued to kilometer wavelengths.

(iv) Coronal transients

As shown in table 2 there were less IP type II bursts in 1980 than in 1979 or 1981. None of the 37 events have been associated with a coronal transient observed by the SMM coronagraph. Conversely the P-78 coronagraph, which began operating in late March 1979, has observed many transients of which a number have been associated with IP type II events. We have indicated whether a transient was seen or not with 'Y'. The 'g' indicates a data gap. These gaps will be filled in as additional data is made available. From the comparison to date all but one IP type II event has a fast (velocity greater than 500 km/sec) transient associated with it. For one event, marked with a question mark, it is unclear whether a transient did occur. A study to determine the correlation between fast transients and IP type II bursts is underway.

(v) Energetic particles

In table 1 we have included the magnitude of the associated particle events. The data is the count rates from the >18 Mev/n detector onboard ISEE-3 (T.T. von Rosenvinge, private communication). Intensity classes 1,2,3 correspond to count rates greater than 1, 10, 100 counts/sec respectively. Thirty-two of the 37 events were associated with energetic particle events even though many of the flare sites were not well connected. For most western events the particle onset time is within an hour or two of the start of the solar activity. For some eastern events the delay is as long as 10 hours.

Three events are considerably delayed (of the order of 24 hours) but have been associated with the solar event because of the absence of other candidate flares. These 'delayed events' are marked with a question mark.

In figure 3 we show the kilometer wavelength activity associated with particle events during a period of 4 months. The figure suggests that larger particle events can be associated with IP type II bursts whereas smaller events can usually be associated with an SA event not followed by an IP type II burst. The association of a particle event with an SA event allows unambiguous identification of the associated flare, because of the positional information obtained with the low frequency radio experiment.

(vi) Sudden commencements and shocks

Sudden commencements were associated with all category 1 events by definition and 11 of the category 2 events. Shocks were detected at ISEE-3 approximately 30 minutes before the SC. The radio astronomy experiment also detects the shocks in situ by the increase in the low frequency (LF) continuum (Hoang et. al, 1980) corresponding primarily to the increase in the ambient density at the spacecraft.

(vii) Transit velocities

The transit velocities of the shocks have been deduced from the time interval between the start of the event (in column 2) and the arrival of the shock at 1 AU as determined by the sudden commencement.

For the 22 category 1 events a mean transit time of approximately 2 days and a mean transit velocity of 840 km/sec is obtained. With the inclusion of the 11 category 2 events followed by sudden commencements we find a mean transit velocity of 800 km/sec. The velocity distribution is shown in figure 4. No event had a transit velocity greater than 1100 km/sec (corresponding to a transit time less than 1.5 days).

In figure 5 we show the distribution of transit velocities as a function of flare longitude. The dashed line shows the mean transit velocity for 6 ranges of longitude. The distribution in longitude is reasonably uniform. The in velocity of shocks from the limbs relative to those from near central meridian is at most 20%, suggesting that to a first approximation most shocks expand isotropically. Individual events may expand anisotropically such as the event of September 14, 1979. This event had a transit velocity of 440 km/sec. It was the last of 4 large flares from the same active region and thus the shock was propagating into a very disturbed corona. The intensity of the event at kilometer wavelengths was comparable to that from events with transit velocities near 900 km/sec. (We show in another paper that the intensity of an IP type II burst is a function of shock velocity). This suggests that radial above the flare site (E90) a transit velocity of the order of 900 km/sec would have been determined.

III. DISCUSSION

We have examined the correlations between IP type II bursts and other solar phenomena. IP type II bursts and SA events correlate well with energetic particle events and this is consistent with theoretical (Ramaty et al., 1980) and observational evidence (Gloeckler et al., 1976) which suggest that flare produced shocks are responsible for the production of solar cosmic rays.

The association of LDE's with white-light transients has been shown by Sheeley et al. (1975) and Kahler (1977). The association between coronal mass ejection events and proton events was shown by Kahler et al. (1978). The association between particle events and LDE X-ray events is discussed by Nonnast et al. (1982). Since the current models suggest that energetic particles are shock accelerated all the above phenomena i.e. transients, proton events and LDE X-ray events, should be associated with strong shocks. Strong shocks are confirmed by our associations of these phenomena with IP type II bursts.

The typical starting frequency of the fundamental of meter wavelength type II bursts is 70-100 MHz (Kundu, 1965). An initial investigation of Culgoora dynamic spectra shows that the starting frequency of the meter wavelength burst associated with many of the IP type II events and the events producing an SA event alone, is probably well below 70 MHz. This result can be deduced from the observation of a number of SA events for which there was no associated meter wavelength type II burst. The observation of an SA event at 2 MHz implies the presence of a shock at coronal heights below the 2 MHz plasma level. The type II burst from which the SA event originates must occur above 2 MHz. See Figure 1 for clarification. If no event is detected at meter wavelengths the type II burst must occur in the frequency range between 20 and 2 MHz. This means that for those SA events not associated with a meter wavelength type II burst, which includes about 30% of the events followed by an IP type II burst, the type II commenced below 20 MHz. This result suggests that the shocks which survive to the IP medium are formed high in the corona where presumably the absence of closed field lines facilitates their escape. The alternative possibility that the shocks are formed at lower heights but are not producing detectable radio emission seems less likely.

The SA events and IP type II bursts not associated with a meter wavelength type II burst are associated with a type IV burst. We suggest that the observation of a type IV burst is a good indicator of the presence of a shock and that if no meter wavelength type II burst is detected that a type II burst may have commenced below the lowest frequency available to ground based observers.

The standard sequence of events in a large flare is often illustrated as consisting of two distinct stages (Wild, Smerd and Weiss, 1963). A group of intense type III bursts is shown to occur within a few minutes of the start of the flare. These are followed about five minutes later by a type II and a type IV burst. As can be seen from table 1, the sequence of meter wavelength activity associated with the IP type II events is extremely varied. Although the type II/IV burst pair occurs about 50% of the time there are also events with type II and no type IV and vice versa. In addition the type III bursts can commnce before or during the type II burst and are not always a separate entity. More importantly, for 9 events type III activity is completely absent. This complete absence of type III bursts was also noted by Svestka and Fritzova-Svestkova (1974) who studied the meter wavelength activity associated with large proton events. The idea of a standard sequence of events is misleading.

We interpreted the transit velocity distribution as a function of heliographic longitude as indicating that, on the average, interplanetary shocks propagate isotropically. This agrees with the results of Chao and

Lepping (1974) who found that "the average shock surface in the ecliptic plane near the earth's orbit lies on a circle centred at the sun with a radius of 1 AU". The fact that many of the shocks are detected at the earth and yet originate in regions far from central meridian, indicates the huge angular extent covered by such shocks. Essentially unambiguous associations can be made between shocks detected at or near earth and the responsible flare region because of the presence of the IP type bursts and the SA events.

Figure 2 suggests that there might be an E-W asymmetry in the location of flares associated with IP type II bursts. There were 15 events from sites east of E30 as against 9 events west of W30. We point out that because of the dependence of shock structure on the Archimedian spiral of the interplanetary magnetic field, we might expect any asymmetry to be in the eastern direction i.e. to favour eastern flares. The western portion of shocks are expected to have more highly compressed magnetic field than the eastern portion and to be well defined quasi-perpendicular shocks. This geometry has been invoked to explain the east-west asymmetry in the magnitude of Forbush decreases (Barnden, 1973). However the sample of available events is too small to establish the statistical significance of this result as yet.

IV CONCLUSION

We have identified 37 IP type II bursts and listed the associated phenomena. The following results were obtained;

1. IP type II bursts are associated with meter wavelength type II and/or type IV bursts, intense LDE X-ray and energetic particle events and probably with coronal transients.

2. A number of events have no associated type III activity at meter wavelengths.

3. The starting frequencies of the associated meter wavelength type II bursts may be lower than average.

The unambiguous identification of shocks detected at 1 AU with a source location on the sun provides the following results;

4. The mean transit velocity of the more energetic solar shocks is 800 km/sec corresponding to a transit time of about 2 days.

5. To a first approximation the shocks propagate isotropically.

We thank T. T. von Rosenvinge for providing ISEE-3 particle data, R. Howard for making available P78-1 coronagraph data and R. T. Stewart for sending Culgoora dynamic spectra. We are grateful to E. W. Cliver for a critical review of this paper.

ORIGINAL PAGE IS
OF POOR QUALITY

REFERENCES

Barnden, L. R. Proc. 13th Int. Cosmic Ray Conf., 1973, 2, 1277.

Boischot, A., Riddle, A. C., Pearce, J. B., and Warwick, J. W. 1980, Solar Phys., 65, 397.

Cane, H. V., Stone, R. G., Fainberg, J., Stewart, R. T., Steinberg, J. L. and Hoang, S. 1981, Geophys. Res. Lett., 8, 1285.

Cane, H. V., Stone, R. G., Fainberg, J., Steinberg, J. L. and Hoang, S. 1982, Solar Phys., 78, 187.

Chao, J. K. and Lepping, R. P. 1974, J. Geophys. Res., 79, 1799.

Gloeckler, G., Sciambi, R. K., Fan, C. Y., and Hovestadt, D. 1976, Ap. J. (Letters), 209, L93.

Hoang, S., Steinberg, J. L., Epstein, G., Tilloles, P., Fainberg, J. and Stone, R. G. 1980, J. Geophys. Res., 85, 3419.

Kahler, S. 1977, Astrophys. J., 214, 891.

Kahler, S., Hildner, E. and van Hollebeke, M. A. I. 1978, Solar Phys., 57, 429.

Knoll, R., Epstein, F., Huntzinger, G., Steinberg, J. L., Fainberg, J., Grena, F., Mosier, S. R. and Stone, R. G. 1979, IEEE Trans. Geosc. Electronics, GE-16, 199.

Kundu, M. R. 1965, Solar Radio Astronomy, Interscience, New York.

Malitson, H. H., Fainberg, J. and Stone, R. G. 1973, Astrophys. Letters, 14, 111.

Malitson, H. H., Fainberg, J. and Stone, R. G. 1976, Space Sci. Rev., 19, 511.

Nonnast, J. H., Armstrong, T. P. and Kohl, J. W. 1982, J. Geophys. Res., 87, 4327.

Pinter, S., Grigorieva, V. P., Kecskemety, K. and Kudela, K. 1982, Hungarian Academy of Sciences, KFK1-182-20.

Ramaty, R., et al., 1980, in Solar Flares, ed. P. A. Sturrock (Boulder: Colorado Associated University Press), Chap. 4, 117.

Roberts, J. A. 1959, Aust. J. Phys., 12, 327.

Sheeley, N. R., Jr., Bohlin, J. D., Brueckner, G. E., Purcell, J. D., Scherrer, V. E., Tousey, R., Smith, J. B., Jr., Speich, D. M., Tandberg-Hanssen, E., Wilson, R. M., DeLoach, A. C., Hoover, R. B., and McGuire, J. P. 1975, Solar Phys., 45, 377.

Svestka, Z. and Fritzova-Svestkova, L. 1974, Solar Phys., 36, 417.

Wild, J. P., Smerd, S. F., and Weiss, A. A., 1963, Ann. Rev. Astron. Astrophys., 1, 291.

TABLE 1

INTERPLANETARY TYPE II BURSTS

A. CATEGORY I EVENTS

DATE			(i) START	FLARE POSITION	(ii) X-RAY		(iii) METER II	IV	III	DT	(iv) TR	(v) P	(vi) SC	(vii) TR.VEL.
1978	SEPT 23		0958	N35 W50	X1	Y	2	2	2	+3D	N	3	25/0718	920
	NOV 10		0110	N17 E01	M2	Y	3	2	-		N	2	12/0100	870
	DEC 11		(1943)	S16 W50	X1	Y	-	3	3	-11D	N	1	14/0127	770
1979	APR 03	*	(0110)	S25 W14	M4	Y	-	2	V	D	Y	2	05/0155	850
	APR 23	*	0156	W100?	-	-	P	P	1	+4D	Y	-	24/2357	900
	JULY 04	*	1920	N11 E36	M2	Y	1	3	-		N	1?	06/1930	870
	AUG 18		1412	E90?	X6	Y	3	3	-		N	3	20/0625	1030
	AUG 26		(1800)	N05 W11	X2	Y	-	2	3	-85D	Y	2	29/0459	700
	SEPT 14		0700	E90?	X2	Y	3	-	2	-37	N	3	18/0546	440
1980	JULY 17	*	(0548)	S11 E06	M5	?	-	2	3	D	N	3	18/1925	1100
	JULY 23		0100	S19 E20	M9	?	2	2	3	-4	N	1	25/1111	710
	OCT 15		0523	N19 E54	M2	?	3	V	V	-12D	N	3	18/0114	610
	NOV 22		0554	S17 E40	M2	Y	P	-	2	-3	N	-	24/2257	640
1981	APR 01	*	0138	S43 W53	X2	Y	2	3	3	-5	Y	2	03/0347	836
	APR 24		1355	N14 W50	X5	Y	3	3	3	-1D	N	3	26/0812	990
	MAY 08		2233	N09 E43	M7	Y	1	2	-		Y	2	10/2208	870
	MAY 13	*	0357	N12 E58	X1	Y	-	1	1	D	Y	1	14/1856	1060
	MAY 16		0825	N10 E16	X1	Y	2	3	3	-14	N	3	17/2302	1080
	AUG 07	*	(1904)	S10 E34	M4	Y	-	3	3	-3	N	2?	10/0434	730
	OCT 07		2259	S13 E90	X4	Y	3	1	1	-3	N	3	10/1434	650
	OCT 12		0627	S17 E30	X3	Y	3	3	-		N	3	13/2241	1030
	DEC 27		0251	S13 E16	C7	Y	-	2	1	-5	N	2	29/0455	830

ORIGINAL PAGE IS OF POOR QUALITY

TABLE 1 (cont.)

B. CATEGORY 2 EVENTS

DATE			(i) START	FLARE POSITION	(ii) X-RAY	(iii) METER II	IV	III	DT	(iv) TR	(v) P	(vi) SC	(vii) TR.VEL.
1978	OCT	01	0718	S13 E57	M9 Y	3	3	3	-2D	ε	1	04/0047	630
1979	FEB	16	0150	N16 E59	ε ε	3	1	2	+5D	ε	1	18/0304	840
	MAR	01	1018	S23 E58	X1 ?	2	-	2	-2D	ε	2	04/0445	620
	MAR	09	1027	N17 E80	X1 ?	3	-	3	-5D	ε	1	-	
	MAR	11	1055	S24 W76	M1 N	3	-	3	-1S	ε	-	15/0556	460
	AUG	20	0915	N05 E77	X2 Y	3	1	2	+1D	ε	1	-	
1980	APR	04	1503	N27 W35	M5 Y	3	3	-		Y	2	06/1059	950
	NOV	11	1729	S11 W69	M5 N	3	2	3	-2	?	1	14/1153	630
	NOV	23	1048	N13 W23	M2 Y	3	2	3	-3	Y	2	26/0422	710
	NOV	28	(0950)	S13 E62	M2 Y	-	2	2	D	ε	1?	-	
1981	APR	26	(1130)	N12 W74	X1 Y	-	3	1	-4	ε	1	-	
	MAY	14	0844	N20 E32	M2 ?	3	-	2	-5	ε	-	16/0532	940
	NOV	09	1242	? ?	M4 ?	2	-	-		ε	ε	11/1238	870
	NOV	22	0729	N12 W21	M1 ?	1	1	-		ε	1	25/0229	620
	DEC	09	(1900)	N12 W16	M4 ?	-	2	-		ε	2	12/0144	760

ORIGINAL PAGE IS
OF POOR QUALITY

TABLE 2

FLARE DISTRIBUTION

A. CATEGORY 1

Longitude range	1978	1979	1980	1981	Total
E90 - 61	-	2	-	1	3
E60 - 31	-	1	2	2	5
E30 - 01	1	-	2	4	7
W0 - 30	-	2	-	-	2
W31 - 60	2	-	-	2	4
W61 - 90	-	1	-	-	1
Total	3	6	4	9	22

B. CATEGORY 2

Longitude range	1978	1979	1980	1981	Total
E90 - 61	-	2	-	-	2
E60 - 31	1	2	1	1	5
E30 - 01	-	-	-	-	-
W00 - 30	-	-	1	2	3
W31 - 60	-	-	1	-	1
W61 - 90	-	1	1	1	3
Unknown				1	
Total	1	5	4	5	15

Figure Captions

Figure 1. A schematic representation of the relationship between meter wavelength type II activity with herringbone structure and the activity observed at kilometer wavelengths. Only the long wavelength elements of the herringbone structure are shown (from Cane et al., 1981).

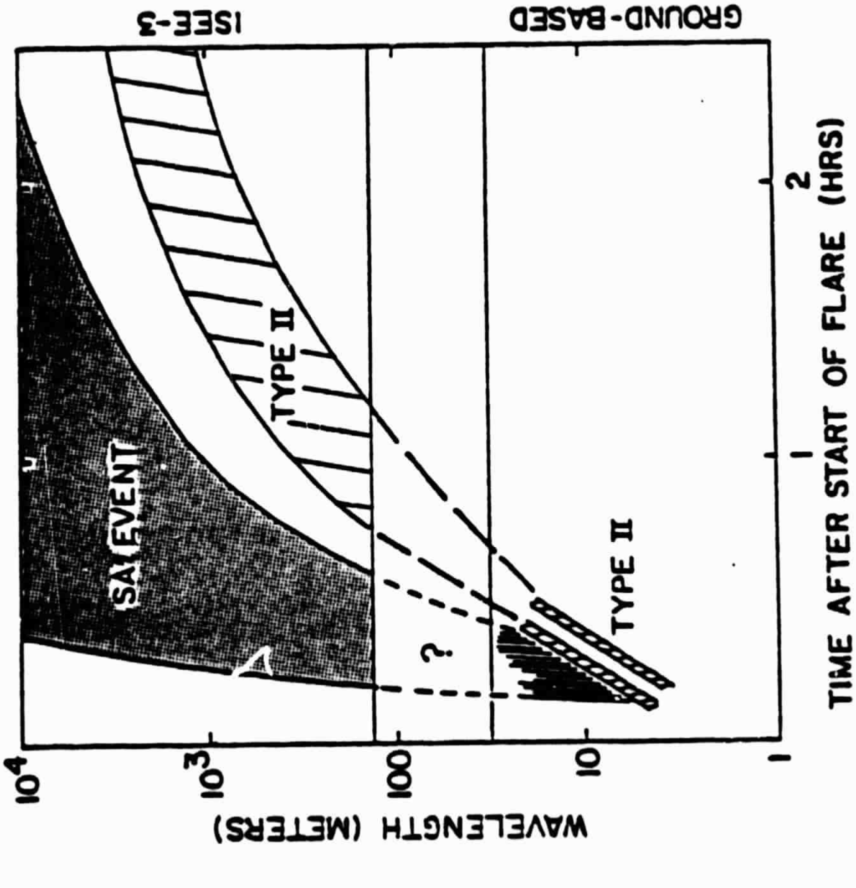
Figure 2. Histograms of the distribution of flare longitudes of the flares associated with the IP type II bursts.

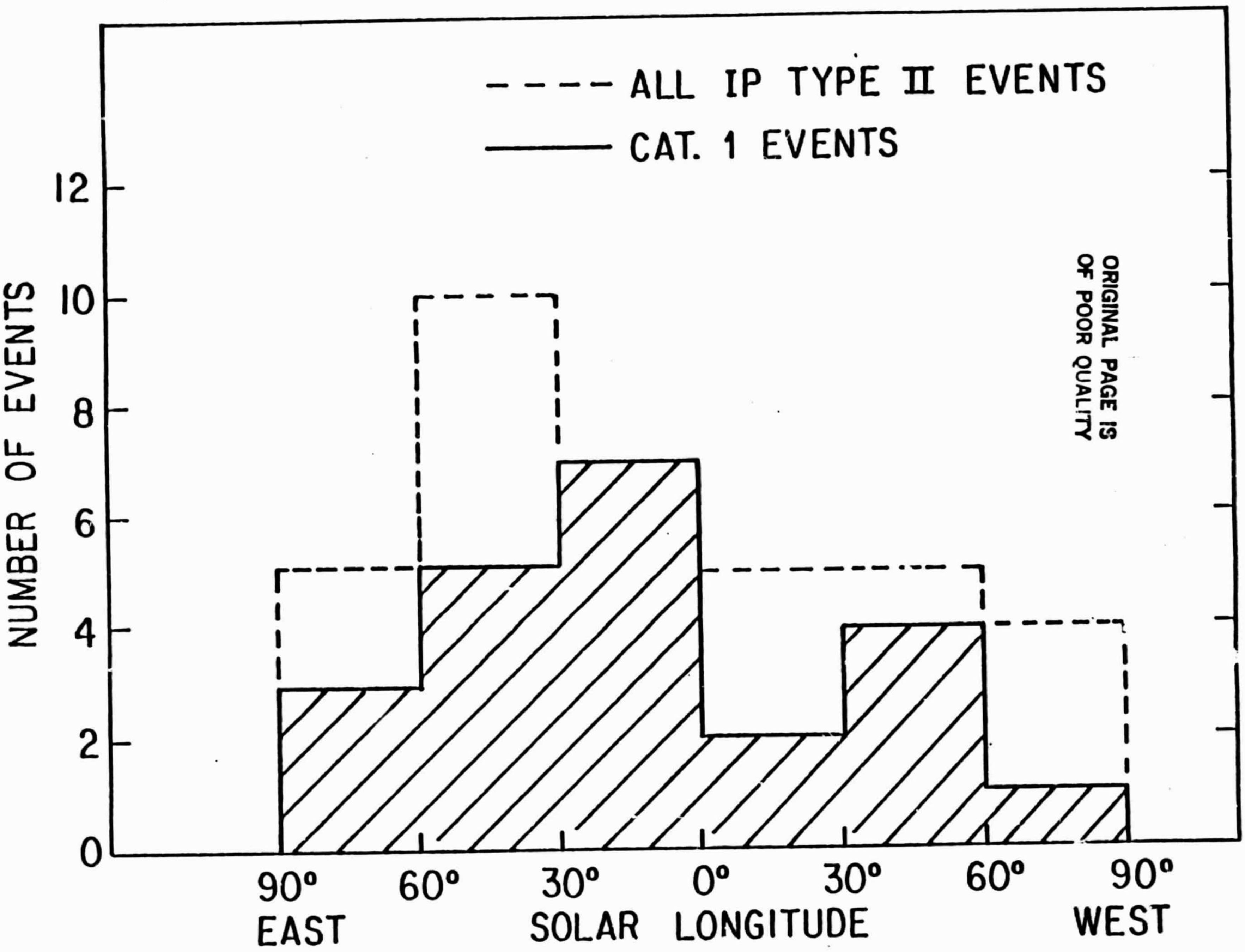
Figure 3. Count rate of the >18 Mev/nucleon detector on ISEE-3 (courtesy of T. T. von Rosenvinge). The occurrences of IP type II bursts and SA events are indicated.

Figure 4. Histogram of the distribution of transit velocities of the shocks associated with the IP type II events.

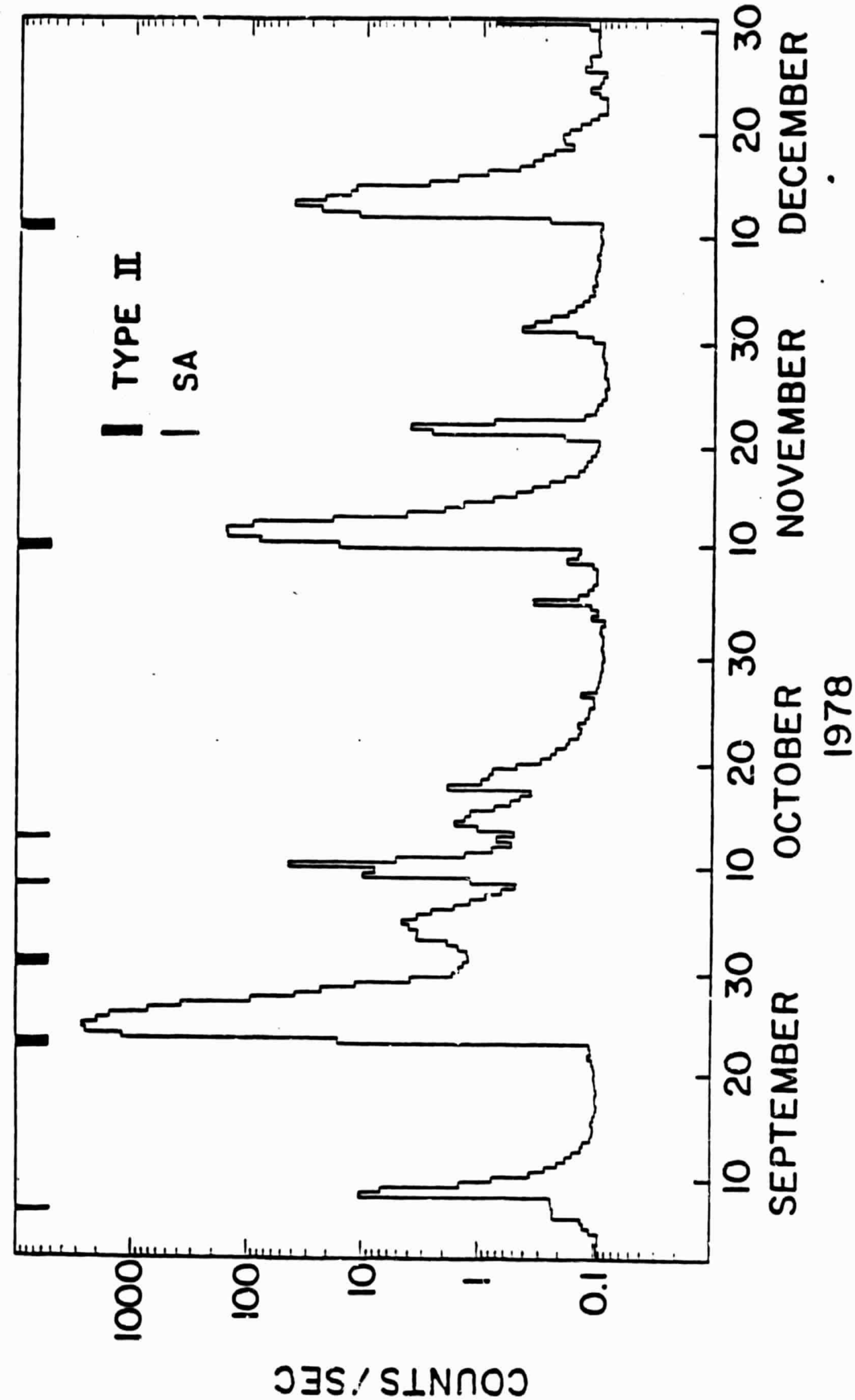
Figure 5. Shock transit velocities shown as a function of the longitude of the associated flare.

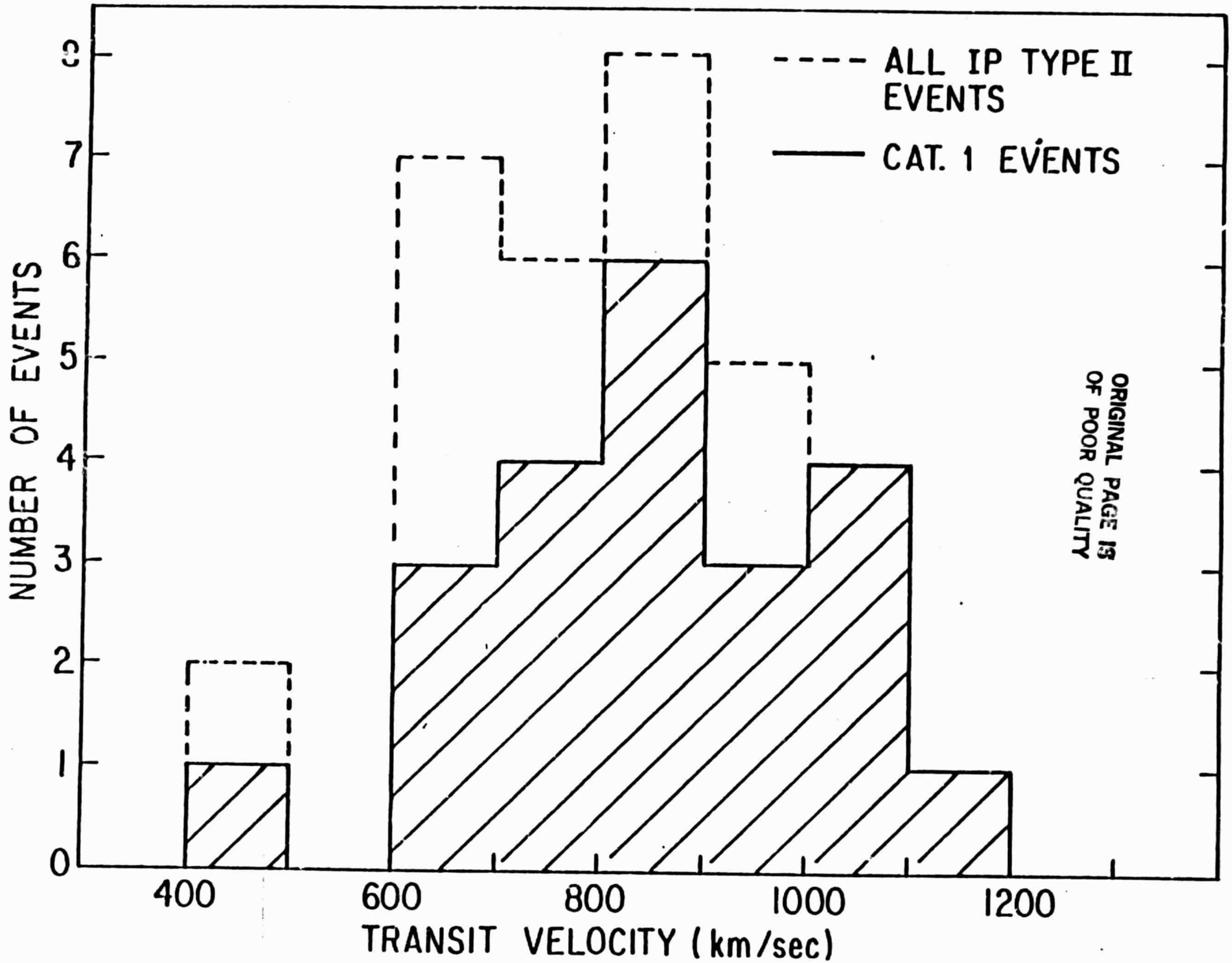
ORIGINAL PAGE IS
OF POOR QUALITY



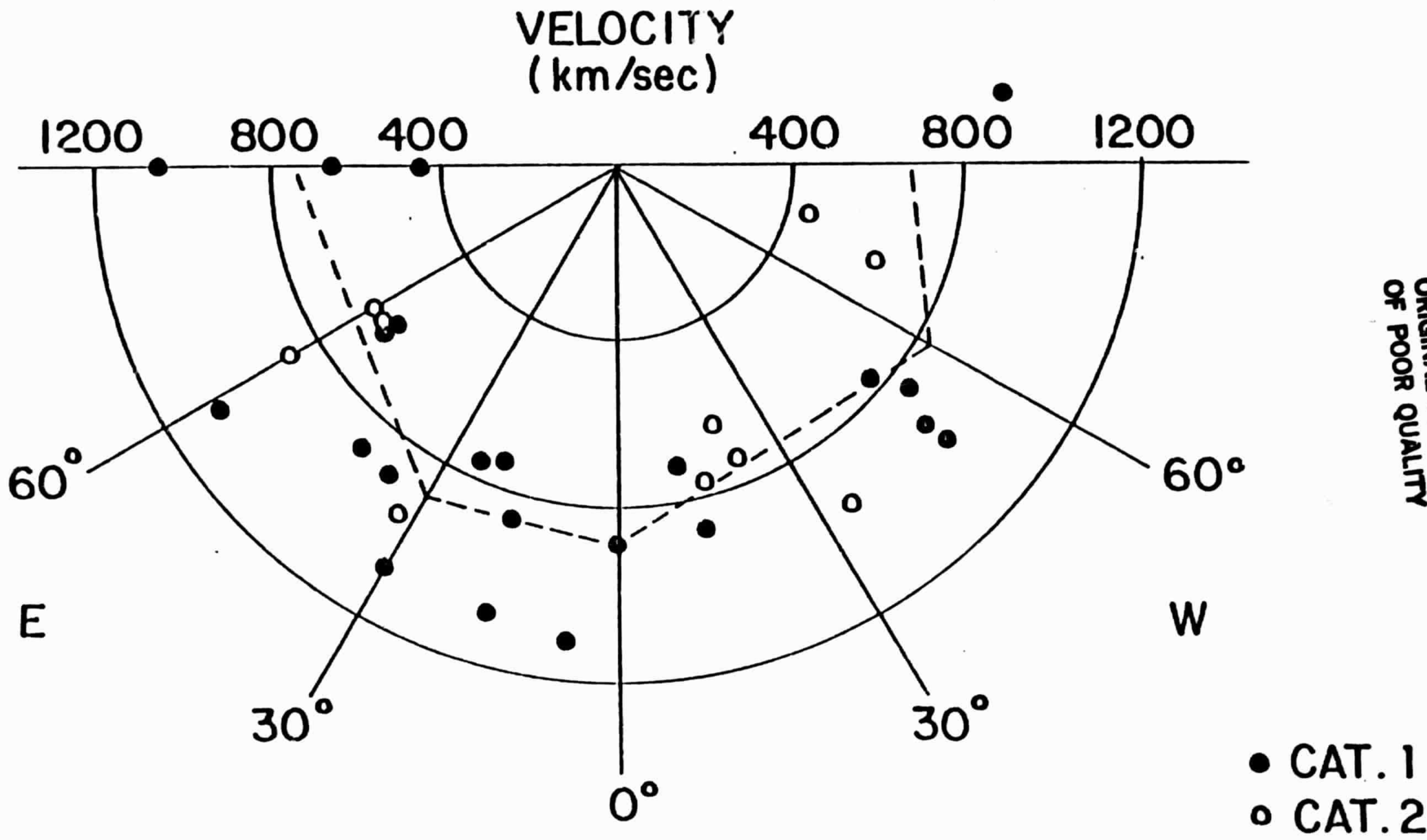


ALL IP TYPE II EVENTS
CAT. 1 EVENTS
NUMBER OF EVENTS
0
2
4
6
8
10
12
90°
EAST
60°
30°
0°
30°
60°
90°
WEST
SOLAR LONGITUDE
ORIGINAL PAGE IS
OF POOR QUALITY


COUNTS/SEC
1000
100
10
1
0.1
TYPE II
SA
10
20
30
SEPTEMBER
OCTOBER
NOVEMBER
DECEMBER
1978



NUMBER OF EVENTS
0
1
2
3
4
5
6
7
8
400
600
800
1000
1200
TRANSIT VELOCITY (km/sec)
ALL IP TYPE II EVENTS
CAT. 1 EVENTS


VELOCITY
(km/sec)
1200
800
400
400
800
1200
60°
60°
E
W
30°
30°
0°
● CAT. 1
○ CAT. 2

NASA TECHNICAL MEMORANDUM

NASA TM-76994

LONG PERIODIC TERMS IN THE SOLAR SYSTEM

P. Bretagnon

(NASA-TM-76994) LONG PERICDIC TEEMS IN THE SOLAR SYSTEM (National Aeronautics and Space Administration) 26 p HC A03/MF A01 CSCL 03B

N83-17434

Unclas

G3/90 08205

Translation of "Termes à longues périodes dans le système solaire", Astronomy and Astrophysics, Vol. 30, No. 1, Jan. 1974, pp. 141-154.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546 JUNE 1982

ORIGINAL PAGE IS OF POOR QUALITY

STANDARD TITLE PAGE

1. Report No.: NASA TM -76994
2. Government Accession No.:
3. Recipient's Catalog No.:
4. Title and Subtitle: LONG PERIODIC TERMS IN THE SOLAR SYSTEM
5. Report Date: June 1982
6. Performing Organization Code:
7. Author(s): P. Bretagnon, Bureau des Longitudes Observatoire, Paris, France
8. Performing Organization Report No.:
10. Work Unit No.:
9. Performing Organization Name and Address: Leo Kanner Associates, Redwood City, California 94063
11. Contract or Grant No.: NASW-3541
13. Type of Report and Period Covered: Translation
12. Sponsoring Agency Name and Address: National Aeronautics and Space Administration, Washington, D.C. 20546
14. Sponsoring Agency Code:

15. Supplementary Notes

Translation of "Termes à longues périodes dans le système solaire", Astronomy and Astrophysics, Vol. 30, No. 1, Jan. 1974, pp. 141-154

(A74-21142)

16. Abstract

The long period variations of the first eight planets in the solar system are studied. First, the Lagrangian solution is calculated and then the long period terms with fourth order eccentricities and inclinations are introduced into the perturbation function. A second approximation was made taking into account the short period terms' contribution, namely the perturbations of first order with respect to the masses. Special attention was paid to the determination of the integration constants.

The relative importance of the different contributions is shown. It is useless for example to introduce the long period terms of fifth order if no account has been taken of the short period terms. Meanwhile, the terms that have been neglected would not introduce large changes in the integration constants. Even so, the calculation should be repeated with higher order short period terms and fifth order long periods.

17. Key Words (Selected by Author(s)):

18. Distribution Statement: Unlimited-Unclassified

19. Security Classif. (of this report): Unclassified
20. Security Classif. (of this page): Unclassified
21. No. of Pages: 24
22. Price:

LONG PERIODIC TERMS IN THE SOLAR SYSTEM

P. Bretagnon
Bureau des Longitudes, Observatoire
17, rue E. Deutsch de la Meurthe
F-75014 Paris, France

Received June 27, 1973 /141*

SUMMARY [English language summary from the original text]

We have studied the long period variations of the eight planets of the solar system (Pluto is excluded). We first calculated the Lagrange solution. We then introduced the long period terms of fourth order in excentricities and inclinations in the disturbing function. In a second approximation we took into account the contribution of the short period terms which provide the perturbations of the first order with respect to the masses. We have paid special attention to the problem of the determination of the integration constants.

We began with the expansion of the disturbing function R [formula (1)]. We used the variables $h = e \sin \varpi$, $K = \cos \varpi$, $p = \sin \frac{i}{2} \cos \Omega$, $q = \sin \frac{i}{2} \cos \Omega$ and obtained expression (3) for the disturbing function and the equations of Lagrange (4).

In the Lagrange method, one retains only the second order terms of the quantities h, k, p, q of the so called long period part of the disturbing function. The resolution of the system of differential equations thus obtained gives the solution of Lagrange (5). The corresponding integration constants are given in Tables 2, 3, 4 and 5.

We later introduced the long period terms of the disturbing function, of fourth order in the quantities h, k, p, q. These terms give rise to third order terms in the Eq. (6) for the variables h_u, for example. We then substitute rumerically the Lagrange solution

*Numbers in the margin indicate pagination in the foreign text.

in these third order terms and hence obtain the form (8) of the equation for dh_u/dt.

In a second approximation, we also introduced the short period terms of the disturbing function. The masses are substituted numerically and the terms thus found are indentical in form to those arising from long period terms of fourth order of the disturbing function and are directly added to the Eq. (8).

To solve the systems of Eq. (8) and (9), we used the Krylov-Bogolioubov method, which consists in seeking a solution of the form (11) with a modification of the frequencies given by (12). Through (12) and derivation of (11) we obtain (13). In addition, the substitution of (11) into (8) and (9) gives (14), so that we get the two expressions (13) and (14) for dh_u/dt and dk_u/dt; their third order parts are given in (15) by identification. It is then possible to determine the quantities $M_{u,\psi,\theta}$, $N_{u,\psi,\theta}$, B_j and C_j introduced in (11) and (12).

The solutions are given by (16) and (17) and in Tables 8 to 13.

The comparison between Tables 3 and 8 shows that the integration constants have been greatly modified, particularly for the planets Mercury, Venus, Earth and Mars. This is due to the importance of third order terms for these planets. Table 9 gives the modifications B_i and C_i of the frequencies as well as the new values of these frequencies: $\tilde{g}_i = g_i + B_i$; $\tilde{s}_i = s_i + C_i$. Tables 10 and 11 show the amplitude of the Lagrange solution calculated with the new constants; Tables 12 and 13 show the amplitudes $M_{u,\psi,\theta}$ and $N_{u,\psi,\theta}$ of the arguments of higher order.

This work displays the relative importance of the different contributions: it is, for example, useless to introduce the long period terms of fifth order if one has not taken into account the short period terms. We have included the major contributions; the neglected terms would not introduce large modifications of the

constants of integration. However, the calculation should be repeated including long period terms of fifth order and short period terms of higher order.

Key words: planetary theory, secular perturbations

There have been several studies of long period terms in the solar system· Stockwell, Harzer (1895), Hill (1897), and more recently Brouwer and van Woerkom (1950) and Anolik et al. (1969).

Brouwer and van Woerkom calculated the Lagrange solution for the eight planets and in particular investigated the Jupiter-Saturn case. This was a continuation of the work of Hill, who had determined a mean perturbation function on the basis of Le Verrier's findings. /142
Brouwer and van Woerkom used this perturbation function, which had been extended to sixth order excentricities and inclinations, for Jupiter-Saturn. It is difficult, however, to determine the accuracy of their result because Hill empirically established some of the coefficients.

Anolik et al. dealt with the eight planet case by introducing all the perturbation function's long period terms up to fourth order excentricities and inclinations.

Our goal was to evaluate the significance of the various long period terms according to their origin. We too dealt with only the eight planet problem. Pluto was neglected for several reasons. First of all, the generally accepted mass of Pluto, which previously had been 1/360,000 the solar mass, is now 1/1,8000,000 with a large uncertainty:

$$\frac{m_{\odot}}{m_p} = 1800000 \pm 600000.$$

Moreover, Pluto's radius vector can be less than Neptune's, with the result that expansions in α, the ratio of semimajor axes, of the perturbation function, are no longer convergent. Finally, the introduction of Pluto's influence causes the appearance of very large resonances between Neptune and Pluto whose physical character is unclear.

Lastly, we calculated the eight planet Lagrangian solution and then introduced the perturbation function's fourth order terms as well as the contribution of the short period terms of first order with respect to the masses. In addition, we particulary concentrated on the problem of determining the integration constants because of the significance of the terms modifying the Lagrangian solution.

The expansions of perturbation function R that we used are those constructed by Chapront at the Bureau des Longitudes. They take the fomr of analytical expansions in e and sin i/2, where e represents the excentricity and i the inclination of the orbital plane relative to the plane of origin.

$$R = \sum_{r,s,t,u,j} Q(\alpha) e_I^r e_E^s \times \left(\sin\frac{i_I}{2}\right)^t \times \left(\sin\frac{i_E}{2}\right)^u \cos\Phi_j \qquad (1)$$

with

$$\Phi_j = j_1\lambda_I + j_2\lambda_E + j_3\varpi_I + j_4\varpi_E + j_5\Omega_I + j_6\Omega_E .$$

λ being the planet's longitude, ϖ the argument of the perihelion, and Ω the argument of the node. The subscript I refers to the inside planet and E to the outside one. The perturbation function's long period portion is that part for which λ_I and λ_E are absent, i.e. in which $j_1 = j_2 = 0$. The summation with respect to the small quantities e_I, e_E, $\sin i_{I/2}$, $\sin i_{E/2}$ is done starting with zero order terms and then 2, 3,...

The orbit's descriptive elements (the semimajor axis a, the excentricity e, the inclination i, the node argument Ω, and the perihelion argument ϖ are those of Newcomb. These elements are expressed relative to the 1850.0 ecliptic averaged over short periods, which will serve as our point of departure (t = 0 for 1850.0) in determining the integration constants of the sought after solutions. Also, we used more recent values for the masses of Venus, Earth, Mars, and Saturn than the ones Newcomb used.

The mean motions n_1 are the average observed values. The semi-

ORIGINAL PAGE IS
OF POOR QUALITY

major axes a_1 are related to the values of n_1 by the expression $n_1^2 a_1^3$ = constant. Now, we need for the mean motions values from which the secular perturbations have been removed. We therefore calculated the secular perturbations δn on the basis of Chapront's and Simon's work concerning the construction of planetary theory with secular terms.

In the end, we used for each planet the value n of the mean motion defined by:

$$n = n_1 - \delta n$$

and the value a of the semimajor axis obtained by

$$n^2 a^3 = \text{constant}.$$

We have assembled the elements adopted for the eight planets in Table 1.

Table 1
Planetary Elements for 1850.0

Planet	a_1 ("/yr)	a ("/yr)	n_1 (AU)	n (AU)	e	ϖ	i	Ω	m m
Mercury	5381016.3893	5381023.1732	0.3870986713	0.3870983460	0.20560396	75 07′19″37	7 00′07″00	46 33′12″24	6000000
Venus	2106641.4171	2106651.7631	0.7233322169	0.7233298487	0.00684458	129 27 34.5	3 23 35.26	75 19 47.41	408500
Earth	1295977.4496	1295975.6094	1,000000021	1.000000968	0.01677126	100 21 36,30	0		328900
Mars	689050.9354	689059.2817	1.5236914428	1.5236791387	0.09326685	333 17 52.37	1 51 02.42	48 24 03.40	3099000
Jupiter	109256.63954	109263.05033	5.202803945	5.202600424	0.04825382	11 54 26.72	1 18 41.81	98 55 58.16	1047.355
Saturn	43996.20414	43885.62112	9.538843653	9.554827367	0.05606075	90 06 39.53	2 29 39.26	112 20 51.38	3498
Uranus	15426.0928	15384.0851	19.18228185	19.21710613	0.0469055	168 15 46.9	0 46 20.54	73 14 08.0	22869
Neptune	7864.698	7843.328	30.057342	30.111791	0.0085082	43 19 43.7	1 47 01.81	130 08 00.2	19314

[Commas in tabulated material are equivalent to decimal points.]

We chose the following variables to analyze our problem:

$$h = e\sin\varpi, \quad p = \sin\tfrac{i}{2}\sin\Omega,$$
$$k = e\cos\varpi, \quad q = \sin\tfrac{i}{2}\cos\Omega. \tag{2}$$

This choice was made in order to avoid the appearance of quantities expressed in e and i in the denominators of the Lagrangian equations. Such quantities could cancel each other out. In addition, this is necessary for the resolving process because in this way the solutions /143 are expressed formally through the use of these variables and, in the algorithm of solution's construction, the second members always retain the same polynomial form.

The change in variables defined by (2) yields in the perturbation function in form (1) an expression of the form:

$$R = \Sigma S(\alpha) h_I^{r_1} h_E^{r_2} k_I^{s_1} k_E^{s_2} p_I^{t_1} p_E^{t_2} q_I^{u_1} q_E^{u_2} \cos(i_1 \lambda_I + i_2 \lambda_E) \tag{3}$$

where the summation is extended to such exponential values that $r_1 + r_2 + s_1 + s_2 + t_1 + t_2 + u_1 + u_2 \leq \omega$, where ω is the order at which it is desired to limit the calculations.

For the variables defined in (2) the Lagrange equations are written:

$$\begin{aligned}
\frac{dh}{dt} &= \frac{(1-e^2)^{1/2}}{na^2}\frac{\partial R}{\partial k} - \frac{h(1-e^2)^{1/2}}{na^2[1+(1-e^2)^{1/2}]}\frac{\partial R}{\partial \lambda} + \frac{kp}{2na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial p} + \frac{kq}{2na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial q}\\
\frac{dk}{dt} &= -\frac{(1-e^2)^{1/2}}{na^2}\frac{\partial R}{\partial h} - \frac{k(1-e^2)^{1/2}}{na^2[1+(1-e^2)^{1/2}]}\frac{\partial R}{\partial \lambda} - \frac{hp}{2na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial p} - \frac{hq}{2na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial q}\\
\frac{dp}{dt} &= \frac{1}{4na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial q} - \frac{p}{2na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial \lambda} - \frac{pk}{2na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial h} + \frac{ph}{2na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial k}\\
\frac{dq}{dt} &= -\frac{1}{4na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial p} - \frac{q}{2na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial \lambda} - \frac{qk}{2na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial h} + \frac{qh}{2na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial k}\\
\frac{1}{a}\frac{da}{dt} &= \frac{2}{na^2}\frac{\partial R}{\partial \lambda}\\
\frac{d\lambda}{dt} &= n - \frac{2}{na}\frac{\partial R}{\partial a} + \frac{(1-e^2)^{1/2}}{na^2[1+(1-e^2)^{1/2}]}\left(h\frac{\partial R}{\partial h} + k\frac{\partial R}{\partial k}\right) + \frac{1}{2na^2(1-e^2)^{1/2}}\left(p\frac{\partial R}{\partial p} + q\frac{\partial R}{\partial q}\right)
\end{aligned} \tag{4}$$

where $e^2 = h^2 + k^2$.

LAGRANGIAN METHOD

For a planet of subscript u perturbed by the seven other planets of subscript v, the perturbation function is written:

$$R_u = \sum_{v<u} \mu \frac{m_v}{a_u} \bar{R}_{uv} + \sum_{v>u} \mu \frac{m_v}{a_v} \bar{R}_{uv}$$

The first summation is extended to the planets inside the one under consideration, the second to the planets outside. We use the following notation:

$\bar{R}_{ij} = a_j/\Delta$ (Δ = distance of the two planets) and

$$\mu = \frac{n_u^2 a_u^3}{1+m_u} = \frac{n_v^2 a_v^3}{1+m_v}.$$

Limited to the second order, $\bar{R}_{uv}$ has the following expression: /144

$$\bar{R}_{uv} = C_{uv} + A_{uv}(h_u^2 + k_u^2 + h_v^2 + k_v^2) - 4A_{uv}(p_u^2 + q_u^2 + p_v^2 + q_v^2) + B_{uv}(k_u k_v + h_u h_v) + 8A_{uv}(q_u q_v + p_u p_v)$$

where C_{uv}, A_{uv}, B_{uv} are functions of $\alpha_{uv} = a_u/a_v$, which is constant here. We thus obtain, with the notation:

$$[u,v] = \frac{n_u \alpha_{uv} m_v}{1+m_u} \text{ if } v>u,$$

$$[u,v] = \frac{n_u m_v}{1+m_u} \text{ if } v<u,$$

$$\frac{dh_u}{dt} = + \sum_{v \neq u} [u,v](2A_{uv} k_u + B_{uv} k_v),$$

$$\frac{dk_u}{dt} = - \sum_{v \neq u} [u,v](2A_{uv} h_u + B_{uv} h_v),$$

$$\frac{dp_u}{dt} = - \sum_{v \neq u} [u,v](2A_{uv} q_u - 2A_{uv} q_v),$$

$$\frac{dq_u}{dt} = + \sum_{v \neq u} [u,v](2A_{uv} p_u - 2A_{uv} p_v).$$

This system is written in matrix form as:

$$\frac{dH}{dt} = E \times K, \quad \frac{dK}{dt} = -E \times H,$$

$$\frac{dP}{dt} = I \times Q, \quad \frac{dQ}{dt} = -I \times P,$$

where H is the column vector with components (h_{Me}, h_V, ..., h_N), K the column vector (k_{Me}, k_V, ..., k_N), P the column vector (p_{Me}, p_V, ..., p_N), and Q the column vector (q_{Me}, q_V, ..., q_N). The subscripts Me, V, ..., N represent Mercury, Venus, ..., Neptune, respectively. E and I are the matrices of the linear systems in excentricities and inclinations respectively.

The conventional resolution of the two Lagrangian systems gives the eigenvalues:

g_i, i = 1, 2, ..., 8 for the excentricities;

s_i, i = 1, 2, ..., 8 for the inclinations.

One of the eigenvectors in the system of inclinations in zero. We will assume $s_5 = 0$.

We determine the eigenvalues λ_{ij} associated with g_j, and μ_{ij} associated with s_j, which gives the Lagrangian solution:

$$\left.\begin{aligned} h_i &= \sum_{j=1}^{8} \lambda_{ij} M_j \sin(g_j t + \beta_j) \\ k_i &= \sum_{j=1}^{8} \lambda_{ij} M_j \cos(g_j t + \beta_j) \\ p_i &= \sum_{j=1}^{8} \mu_{ij} N_j \sin(s_j t + \delta_j) \\ q_i &= \sum_{j=1}^{8} \mu_{ij} N_j \cos(s_j t + \delta_j) . \end{aligned}\right\} \qquad (5)$$

Lastly, we calculate the 32 constants of integration M_j, β_j, N_j, δ_j from the values of h, k, p, q at t = 0.

We have assembled the eigenvalues g and s in Table 2, and in Table 3 we show the 32 constants of integration M, β, N, δ. Lastly, Table 4 gives the amplitudes of the Lagrangian solution multiplied by 10^8: $\lambda_{ij} M_j \times 10^8$ for the excentricities, and similarly in Table 5, $\mu_{ij} N_j \times 10^8$ for the inclinations. Frequencies g and s are expressed in seconds per year. λ_{ih}, μ_{ij}, M_j, N_j are dimensionless numbers.

Table 2
Frequencies in "/yr
(Lagrangian Solution)

i	g	s
1	+ 5.461369	− 5.199958
2	+ 7.346581	− 6.571387
3	+17.331295	−18.746205
4	+18.004584	−17.636111
5	+ 3.711401	0
6	+22.286552	−25.741176
7	+ 2.701787	− 2.904326
8	+ 0.633116	− 0.677520

Table 3 /145
Constants of Integration
(Lagrangian Solution)

i	M	β	N	δ
1	0.18141040	87 11′11″37	0.06274851	18 15′28″76
2	0.01909712	192 40 29.79	0.00506380	316 20 04.81
3	0.01056860	332 56 51.67	0.01222166	254 17 05.60
4	0.07300403	316 06 34.81	0.02519918	295 40 38.75
5	0.04319426	27 48 34.76	0.01383974	106 08 44.50
6	0.04837743	127 51 04.46	0.00786789	126 19 31.43
7	0.03140786	106 19 25.39	0.00880286	313 42 55.41
8	0.00923780	66 09 18.45	0.00588386	201 00 52.94

[Commas in tabulated material are equivalent to decimal points.]

ORIGINAL PAGE IS OF POOR QUALITY

Table 4

$\lambda_{ij}M_j \times 10^8$. Amplitudes of Lagrangian Solution

i \ j	1	2	3	4	5	6	7	8
Mercury	18141040	− 2318196	155897	− 170495	2414071	11364	62612	727
Vénus	631873	1909712	− 1275337	1497153	1624499	− 55447	61654	1106
Earth	405895	1490407	1056860	− 1492703	1624452	247593	65295	1284
Mars	66283	264664	3017351	7300403	1870379	1617034	86626	2053
Jupiter	− 703	− 1055	− 95	− 55	4319426	− 1563040	218379	5968
Saturn	− 627	− 1088	− 750	− 840	3404356	4837743	199229	6736
Uranus	271	265	44	46	− 4384597	− 181906	3140786	141054
Neptune	4	10	3	3	160418	− 13561	− 338902	923780

Table 5

$\mu_{ij}N_j \times 10^8$. Amplitudes of Lagrangian Solution

i \ j	1	2	3	4	5	6	7	8
Mercury	6274851	− 1781583	204668	58171	1383974	13920	− 166549	− 72367
Vénus	591896	506380	− 1341594	− 343391	1383974	6022	− 95883	− 66215
Earth	426404	408232	1222166	226117	1383974	140699	− 86614	− 64906
Mars	90534	90894	− 1794150	2519918	1383974	452917	− 62850	− 61488
Jupiter	− 1038	− 655	− 9	− 88	1383974	− 315878	− 47877	− 58499
Saturn	− 1328	− 925	− 241	− 916	1383974	786799	− 39034	− 56388
Uranus	1112	477	20	86	1383974	− 34790	880286	54715
Neptune	28	27	2	9	1383974	− 3858	− 103566	588386

INTRODUCTION OF HIGHER ORDER TERMS

We are now going to introduce the perturbation function's long period terms of fourth order h, k, p, q, as well as the perturbation function's short order terms.

Fourth Order Long Period Terms

By differentiation, these terms yield third order terms, and the Lagrangian equation for variable h_u, for example, then has the following form:

$$\frac{dh_u}{dt} = \sum_{v \neq u} [u, v]\{2A_{uu} k_u + B_{uv} k_v - P_{uv}(h_u, h_v, k_u, k_v, p_u, p_v, q_u, q_v)\} \qquad (6)$$

where P_{uv} is a homogeneous third degree polynomial.

Into polynomial P_{uv} we substitute the Lagrangian solution (5), whose numerical values are given in Tables 2, 3, 4, and 5:

$$h_u = \sum_{j=1}^{8} \lambda_{uj} M_j \sin\psi_j, \quad k_u = \sum_{j=1}^{8} \lambda_{uj} M_j \cos\psi_j,$$
$$p_u = \sum_{j=1}^{8} \mu_{uj} N_j \sin\theta_j, \quad q_u = \sum_{j=1}^{8} \mu_{uj} N_j \cos\theta_j,$$

where we have made $\psi_j = g_j t + \beta_j$ and $\theta_j = s_j t + \delta_j$. Therefore only the numerical values of amplitudes $\lambda_{uj} M_j$ and $\mu_{uj} N_j$ appear in this calculation.

Among the values of the i and j subscripts of arguments ψ_i ($i = 1, 2, \ldots, 8$) and θ_j ($j = 1, 2, \ldots, 8$), such a substitution makes combinations appear in which at most only three values of subscripts i and j are involved. For example, there will be combinations of the type $(\psi_1 + \theta_2 - \theta_4)$, $(2\psi_5 - \psi_6)$.

The expression $\sum_{v \neq u} [u, v] P_{uv}$ therefore has the form:

$$\sum_{v \neq u} [u, v] P_{uv} = \sum_{\substack{i_1, i_2, \ldots, i_8 \\ j_1, j_2, \ldots, j_8}} \xi_{u, i_1, \ldots, i_8, j_1, \ldots, j_8} \cos(i_1 \psi_1 + \cdots + i_8 \psi_8 + j_1 \theta_1 + \cdots + j_8 \theta_8) \tag{7}$$

where $\xi_{u, i_1 \ldots i_8, j_1 \ldots j_8}$ is a numerical coefficient.

The summation over integers i and j is such that:

$$\sum_{m=1}^{8} |i_m| + \sum_{m=1}^{8} |j_m| = 1 \text{ or } 3$$

We will designate that:

$$(\psi, \theta) = i_1 \psi_1 + i_2 \psi_2 + \cdots + i_8 \psi_8 + j_1 \theta_1 + j_2 \theta_2 + \cdots + j_8 \theta_8$$

and hence equation (7) takes on the form:

$$\sum_{v \neq u} [u, v] P_{uv} = \sum_{\psi, \theta} \xi_{u, \psi, \theta} \cos(\psi, \theta).$$

We also make:

$$\varepsilon = +1 \text{ if } \sum_{m=1}^{8} i_m - \sum_{m=1}^{8} j_m = +1,$$ /146

$$\varepsilon = -1 \text{ if } \sum_{m=1}^{8} i_m + \sum_{m=\cdot} i_m = -1.$$

Equation (6) is then written:

$$\frac{dh_u}{dt} = \sum_{v \neq u} [u, v]\{2A_{uv}k_u + B_{uv}k_v\} + \sum_{\psi, \theta} \xi_{u,\psi,\theta} \cos(\psi, \theta). \tag{8}$$

Substituting the Lagrangian solution into the equation in dk_u/dt similarly yields:

$$\frac{dk_u}{dt} = -\sum_{v \neq u} [u, v]\{2A_{uv}h_u + B_{uv}h_v\} - \sum_{\psi, \theta} \varepsilon \times \xi_{u,\psi,\theta} \sin(\psi, \theta). \tag{9}$$

We also calculate:

$$\frac{dp_u}{dt} = -\sum_{v \neq u} [u, v]\{2A_{uv}q_u - 2A_{uv}q_v\} + \sum_{\psi, \theta} \eta_{u,\psi,\theta} \cos(\psi, \theta).$$

$$\frac{dq_u}{dt} = \sum_{v \neq u} [u, v]\{2A_{uv}p_u - 2A_{uv}p_v\} - \sum_{\psi, \theta} \varepsilon \times \eta_{u,\psi,\theta} \sin(\psi, \theta).$$

Short Period Terms

Substituting the Lagrangian solution into the short period part of the perturbation function yields only short period terms that are first order with respect to the masses. It is only with the second mass order that we come across long period terms again.

This time we have to consider for each planet the complete system of Lagrange equations (4), which we will write for a planet of subscript u in the form: /147

$$\left.\begin{aligned}
&\frac{dh_u}{dt} = F_{h_u}, && \frac{dk_u}{dt} = F_{k_u}, \\
&\frac{dp_u}{dt} = F_{p_u}, && \frac{dq_u}{dt} = F_{q_u}, \\
&\frac{1}{a_u}\frac{da_u}{dt} = F_{a_u}, && \frac{d\lambda_u}{dt} = n_u + F_{\lambda_u}.
\end{aligned}\right\} \qquad (10)$$

We determine the short period effects argument by argument. For a short period argument $i\lambda_u + j\lambda_v$, i and j being given integers, the functions F have the form:

$$F_c \cos(i\lambda_u + j\lambda_v) + F_s \sin(i\lambda_u + j\lambda_v)$$

where F_c and F_s are polynomials in h_u, k_u, q_u, h_v, k_v, p_v, q_v, whose coefficients are functions of $\alpha_{uv} = a_u/a_v$, n_u and n_v. (In the special case in which one of the two integers i, j is zero, i.e., in the case in which the short period argument takes on the form $i\lambda_u$, the functions F_c and F_s depend on h_v, k_v, p_v, q_v, n_v for v = 1, 2, ..., 8 and on the seven quantities $\alpha_{uv} = a_u/a_v$ for $u \neq v$.)

We therefore substitute the Lagrangian solution into equations (10), which after integration yield a short period increase in the Lagrangian solution. Then by doing a Taylor expansion of the second members of equations (10), we obtain second order terms with respect to mass after substituting the first order that we have just found. We will retain only the second terms' long period parts.

Since the masses are always substituted for numerically, the terms thus found in the second members of the Lagrangian equations have the same form as those coming directly from the perturbation function's fourth order long period terms.

In contrast to the case of the perturbation function's long periods, for which we kept the fourth period terms, the criterion for choosing short period terms is numerical. What we did was to retain the beginning of the h, k, p, q expansion of all the arguments causing changes in the Lagrangian solution frequencies of more than 10^{-3} "/yr.

RESOLUTION OF THE SYSTEMS OF DIFFERENTIAL EQUATIONS

We saw that the contribution of the short period terms took on the same form as the terms coming directly from the perturbation function. We therefore have to resolve a system of differential equations having the form:

$$\frac{dh_u}{dt} = \sum_{v \neq u} [u, v]\{2A_{uv}k_u + B_{uv}k_v\} + \sum_{\psi,\theta} \xi_{u,\psi,\theta} \cos(\psi, \theta). \tag{8}$$

$$\frac{dk_u}{dt} = -\sum_{v \neq u} [u, v]\{2A_{uv}h_u + B_{uv}h_v\} - \sum_{\psi,\theta} \varepsilon \times \xi_{u,\psi,\theta} \sin(\psi, \theta) \tag{9}$$

as well as a similar system for variables p_u and q_u.

For that, we are going to use the Krylov-Bogolyubov method. This method consists of finding a solution of the form:

$$\left.\begin{aligned} h_u &= \sum_{j=1}^{8} \lambda_{uj} M_j \sin\psi_j + \sum_{\psi,\theta} M_{u,\psi,\theta} \sin(\psi,\theta) \\ k_u &= \sum_{j=1}^{8} \lambda_{uj} M_j \cos\psi_j + \sum_{\psi,\theta} M'_{u,\psi,\theta} \cos(\psi,\theta) \end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned} p_u &= \sum_{j=1}^{8} \mu_{uj} N_j \sin\theta_j + \sum_{\psi,\theta} N_{u,\psi,\theta} \sin(\psi,\theta) \\ q_u &= \sum_{j=1}^{8} \mu_{uj} N_j \cos\theta_j + \sum_{\psi,\theta} N'_{u,\psi,\theta} \cos(\psi,\theta) \end{aligned}\right\}$$

with

/148

$$\left.\begin{aligned} \frac{d\psi_j}{dt} &= g_j + B_j \\ \frac{d\theta_j}{dt} &= s_j + C_j. \end{aligned}\right\} \tag{12}$$

By differentiating system (11) and taking account of (12), we obtain:

ORIGINAL PAGE IS
OF POOR QUALITY

$$\left.\begin{aligned}\frac{dh_u}{dt} &= \sum_{j=1}^{8} \lambda_{uj} M_j \cos\psi_j \times (g_j + B_j) + \sum_{\psi,\theta} (g.s) M_{u,\psi,\theta} \cos(\psi,\theta)\\ \frac{dk_u}{dt} &= -\sum_{j=1}^{8} \lambda_{uj} M_j \sin\psi_j \times (g_j + B_j) - \sum_{\psi,\theta} (g.s) M'_{u,\psi,\theta} \sin(\psi,\theta)\end{aligned}\right\} \qquad (13)$$

where

$$(g.s) = i_1 g_1 + i_2 g_2 + \cdots + i_8 g_8 + j_1 s_1 + j_2 s_2 + \cdots + j_8 s_8 .$$

Furthermore, equations (8) and (9) yield, by plugging in (11):

$$\left.\begin{aligned}\frac{dh_u}{dt} &= \sum_{v\neq u} [u,v] \Bigg\{ 2A_{uv} \left[\sum_{j=1}^{8} \lambda_{uj} M_j \cos\psi_j + \sum_{\psi,\theta} M'_{u,\psi,\theta} \cos(\psi,\theta) \right]\\ &\quad + B_{uv} \left[\sum_{j=1}^{8} \lambda_{vj} M_j \cos\psi_j + \sum_{\psi,\theta} M'_{v,\psi,\theta} \cos(\psi,\theta) \right]\Bigg\} + \sum_{\psi,\theta} \xi_{u,\psi,\theta} \cos(\psi,\theta)\\ \frac{dk_u}{dt} &= -\sum_{v\neq u} [u,v] \Bigg\{ 2A_{uv} \left[\sum_{j=1}^{8} \lambda_{uj} M_j \sin\psi_j + \sum_{\psi,\theta} M_{u,\psi,\theta} \sin(\psi,\theta) \right]\\ &\quad + B_{uv} \left[\sum_{j=1}^{8} \lambda_{vj} M_j \sin\psi_j + \sum_{\psi,\theta} M_{v,\psi,\theta} \sin(\psi,\theta) \right]\Bigg\} - \sum_{\psi,\theta} \varepsilon \times \xi_{u,\psi,\theta} \sin(\psi,\theta) .\end{aligned}\right\} \qquad (14)$$

Hence, we have two expressions, (13) and (14), for dh_u/dk and dk_u/dt. We make equal their parts that are of third order with respect to variables h, k, p, and q:

$$\left.\begin{aligned}&\sum_{j=1}^{8} B_j \lambda_{uj} M_j \cos\psi_j + \sum_{\psi,\theta} (g.s) M_{u,\psi,\theta} \cos(\psi,\theta)\\ &\quad = \sum_{v\neq u} [u,v] \left\{ 2A_{uv} \sum_{\psi,\theta} M'_{u,\psi,\theta} \cos(\psi,\theta) + B_{uv} \sum_{\psi,\theta} M'_{v,\psi,\theta} \cos(\psi,\theta) \right\} + \sum_{\psi,\theta} \xi_{u,\psi,\theta} \cos(\psi,\theta)\\ &\sum_{j=1}^{8} B_j \lambda_{uj} M_j \sin\psi_j + \sum_{\psi,\theta} (g.s) M'_{u,\psi,\theta} \sin(\psi,\theta)\\ &\quad = \sum_{v\neq u} [u,v] \left\{ 2A_{uv} \sum_{\psi,\theta} M_{u,\psi,\theta} \sin(\psi,\theta) + B_{uv} \sum_{\psi,\theta} M_{v,\psi,\theta} \sin(\psi,\theta) \right\} + \sum_{\psi,\theta} \varepsilon \times \xi_{u,\psi,\theta} \sin(\psi,\theta) .\end{aligned}\right\} \qquad (15)$$

The method now consists of establishing argument by argument identities within each order. Two cases arise:

1) The case in which the argument (ψ,θ) is equal to ψ_j, i.e. we have:

$$\sum_{m=1}^{8} |i_m| + \sum_{m=1}^{8} |j_m| = 1 .$$

ORIGINAL PAGE IS
OF POOR QUALITY

Here, we once again come across the Lagrangian solution arguments, and establishing identities between coefficients makes it possible to determine the new frequency values. The solutions then are expressed in Fourier series of these new arguments. Establishing the identities yields:

$$\lambda_{uj} M_j B_j + g_j M'_{u,\psi_j} = \sum_{v \neq u} [u,v](2A_{uv} M'_{u,\psi_j} + B_{uv} M'_{v,\psi_j}) + \xi_{u,\psi_j}$$

and

$$\lambda_{uj} M_j B_j + g_j M_{u,\psi_j} = \sum_{v \neq u} [u,v](2A_{uv} M_{u,\psi_j} + B_{uv} M_{v,\psi_j}) + \xi_{u,\psi_j}.$$

For a given ψ_j, subtraction of these two equations furnishes: /149

$$\left(\sum_{v \neq u} [u,v] \times 2A_{uv} + g_j\right)(M'_{u,\psi_j} - M_{u,\psi_j}) + \sum_{v \neq u} [u,v] B_{uv} (M'_{v,\psi_j} - M_{v,\psi_j}) = 0.$$

The fact that and $[u,v]B_{uv}$ are not zero means that $M_{u,\psi_j} = M'_{u,\psi_j}$ whatever u and j are.

We can then write:

$$\left(\sum_{v \neq u} [u,v] \times 2A_{uv} - g_j\right) M_{u,\psi_j} + \sum_{v \neq u} [u,v] B_{uv} M_{v,\psi_j} = \lambda_{uj} M_j B_j - \xi_{u,\psi_j}.$$

In the first member of this expression, we once again come across matrix E of the Lagrangian system. Subtracting the eigenvalue g_j from the principal diagonal means that the M_{u,ψ_j} values (u = 1, 2, ..., 8) will not be independent. We then let:

$$M_{j,\psi_j} = 0$$

This is an arbitrary step in the Krylov-Bogolyubov method. It reduces to changing variables over the integration constants M_j, N_j. The choice of $M_{j,\psi_j} = 0$ does not specify the solution but imposes the choice of a certain type of expansion for the coefficients of arguments ψ_j.

Having made this choice, we then have for each ψ_j, j fixed, a system of eight equations in eight unkowns: B_j and M_{u,ψ_j}, $(u \neq j)$.

ORIGINAL PAGE IS
OF POOR QUALITY

2) Case in which the argument (ψ,θ) is random, i.e. such that:

$$\sum_{m=1}^{8} |i_m| + \sum_{m=1}^{8} |j_m| = 3.$$

Establishing argument by argument identities in equations (15) yields:

$$\sum_{r \neq u} [u,v](2A_{uv}M'_{u,\psi,\theta} + B_{uv}M'_{v,\psi,\theta}) - (g,s)M_{u,\psi,\theta} = -\xi_{u,\psi,\theta}$$

and

$$\sum_{r \neq u} [u,v](2A_{uv}M_{u,\psi,\theta} + B_{uv}M_{v,\psi,\theta}) - (g,s)M'_{u,\psi,\theta} = -\varepsilon \times \xi_{u,\psi,\theta}.$$

By subtraction, we obtain $M_{u,\psi,\theta} = \varepsilon M'_{u,\psi,\theta}$ whatever u and argument (ψ,θ) is.

We can then write:

$$\left[\sum_{v \neq u} [u,v] \times 2A_{uv} - \varepsilon(g,s)\right] M_{u,\psi,\theta} + \sum_{v \neq u} [u,v] B_{uv} M_{v,\psi,\theta} = -\varepsilon\xi_{u,\psi,\theta}.$$

This time there is no arbitrary step and the resolution of the system of eight equations in eight unknowns gives for each argument (ψ,θ) the eight values $M_{u,\psi,\theta}$ (u = 1, 2, ..., 8).

We therefore have expansions of h_u and k_u:

$$\left.\begin{aligned} h_u &= \sum_{j=1}^{8} \lambda_{uj} M_j \sin\psi_j + \sum_{\psi,\theta} M_{u,\psi,\theta} \sin(\psi,\theta) \\ k_u &= \sum_{j=1}^{8} \lambda_{uj} M_j \cos\psi_j + \sum_{\psi,\theta} \varepsilon M_{u,\psi,\theta} \cos(\psi,\theta) \end{aligned}\right\} \qquad (16)$$

And similarly we find:

$$\left.\begin{aligned} p_u &= \sum_{j=1}^{8} \mu_{uj} N_j \sin\theta_j + \sum_{\psi,\theta} N_{u,\psi,\theta} \sin(\psi,\theta) \\ q_u &= \sum_{j=1}^{8} \mu_{uj} N_j \cos\theta_j + \sum_{\psi,\theta} \varepsilon N_{u,\psi,\theta} \cos(\psi,\theta). \end{aligned}\right\} \qquad (17)$$

RESULTS AND DETERMINATION OF THE CONSTANTS OF INTEGRATION /150

In Tables 6 and 7 we give the integration constants and frequencies of the solutions obtained solely from the second order h, k, p, q long period terms. Comparison of tables 3 and 6 show how great the contribution of the perturbation function's fourth order long period terms is, especially for Mercury, Venus, Earth, and Mars. Table 7 contains the frequency modifications B_i anc C_i, as well as the frequencies' new values: $\tilde{g}_i = g_i + B_i$, $\tilde{s}_i = s_i + C_i$.

Table 6
Constants of Integration (According to the Solution Based on the Perturbation Function's Second and Fourth Order Long Periods)

i	M_i	β_i	N_i	δ_i
1	0,18454867	83 32'24".16	0,05887664	11 51'44".62
2	0,01864278	191 26 15.44	0,00323843	302 59 26.61
3	0,01204101	318 18 13.08	0,00967327	248 33 14.44
4	0,06311073	307 01 46.20	0,03227762	275 46 53.61
5	0,04297488	27 17 43.13	0,01384057	106 07 34.95
6	0,04842782	127 29 49.50	0,00786457	125 49 11.16
7	0,03210686	100 44 43.97	0,00880119	316 00 16.25
8	0,00932559	64 54 28.20	0,00592089	201 11 07.71

[Commas in tabulated material are equivalent to decimal points.]

Table 7
Modifications and New Frequencies in "/yr
(According to the Solution Based on the Perturbation Function's Second and Fourth Order Long Periods)

i	B_i	$\tilde{g}_i$	C_i	$\tilde{s}_i$
1	−0,258373	+ 5.202996	−0.443985	− 5.643943
2	−0,000721	+ 7,345860	−0.220510	− 6.791897
3	−0,130032	+17.201263	−0,152704	−18.898909
4	−0,169168	+17.835416	−0.231362	−17.867473
5	+0,018087	+ 3,729488	0	0
6	+0,322115	+22,608667	−0,606907	−26.348083
7	+0,078361	+ 2,780148	−0.083205	− 2.937531
8	+0,009180	+ 0,642296	−0,009136	− 0.686656

[Commas in tabulated material are equivalent to decimal points.]

Table 8
Constants of Integration (Complete Solution)

i	M_i	β_i	N_i	δ_i
1	0.17791613	87 03'02".09	0.05962772	12 08 17".15
2	0.02104749	193 35 04.97	0.00315338	305 13 17.19
3	0.00988829	319 43 16.64	0.01002295	249 01 59.07
4	0.06115173	307 48 39.56	0.03130279	277 56 06.16
5	0.04341616	28 30 11.60	0.01383939	106 09 11.57
6	0,04814727	127 42 54.52	0.00785328	125 38 34.23
7	0.03126134	114 46 31.58	0.00880088	316 17 35.94
8	0.00899181	72 05 25.03	0.00588806	201 17 15.59

[Commas in tabulated material are equivalent to decimal points.]

Table 9
Modifications and New Frequencies in "/yr (Complete Solution)

i	B_i	$\bar{g}_i$	C_i	$\bar{s}_i$
1	−0,262290	+ 5.199079	−0.410979	− 5.610937
2	−0,000490	+ 7.346091	−0,199640	− 6.771027
3	−0,110749	+17,220546	−0.083094	−18.529299
4	−0,147321	+17.857263	−0.182658	−17.818769
5	+0.495804	+ 4.207205	0	0
6	+3.930206	+26.216758	−0,525894	−26.267070
7	+0,363394	+ 3.065181	−0.095511	− 2.999837
8	+0,034747	+ 0.667863	−0.013911	− 0.691431

[Commas in tabulated material are equivalent to decimal points.]

Tables 8 and 9 give the constants of integration and the frequencies for the complete solutions, i.e. the solutions that take the short periods into consideration. By comparing tables 6 and 8, we can see that the integration constants are once more greatly altered. /151 Comparison of Tables 7 and 9 show that while the short period terms hardly change the frequencies related to the inside planets, the g_5 and g_6 frequencies on the contrary are changed to a much greater extent by the short period terms than by the perturbation function's fourth order long period terms.

The modification of the constants of integration originates in the magnitude of the nonlinear terms found in particular in the expressions of the elements related to the inside planets. Of course, we began by calculating these terms by numerical substitution of the Lagrangian solution in Tables 4 and 5. We then determined an analytic form of the expressions found so as to calculate the new integration /153 constants. With the help of this analytic form and by making a first order Taylor expansion about the first values of the integration constants, we obtained integration constants of sufficient accuracy after several iterations.

Table 10

$\lambda_{ij} M_j \times 10^8$. Lagrangian Solution Amplitudes

i \ j	1	2	3	4	5	6	7	8
Mercury	17791613	−2554951	145862	− 142815	2426473	11310	62320	708
Venus	619702	2104749	−1193243	1254088	1632845	− 55183	61366	1076
Earth	398077	1642622	988829	−1250360	−1632798	246415	64990	1250
Mars	65007	291694	2823123	6115173	1879988	1609341	86222	1998
Jupiter	− 689	− 1163	− 89	− 46	4341616	−1555604	217360	5809
Saturn	− 615	− 1200	− 702	− 703	3421845	4814727	198300	6557
Uranus	266	293	41	38	−4407122	− 181041	3126134	137298
Neptune	3	11	3	2	161243	− 13497	− 337321	899181

Table 11

$\mu_{ij} N_j \times 10^8$. Lagrangian Solution Amplitudes

i \ j	1	2	3	4	5	6	7	8
Mercury	5962772	−1109444	167847	72261	1383939	13894	−166511	− 72418
Venus	562458	315338	−1100237	− 426565	1383939	6010	− 95861	− 66262
Earth	405197	254218	1002295	280886	1383939	140436	− 86594	− 64951
Mars	86031	56602	−1471377	3130279	1383939	482015	− 62835	− 61532
Jupiter	− 986	− 408	− 7	− 109	1383939	−315287	− 47866	− 58541
Saturn	− 1262	− 576	− 197	− 1138	1383939	785328	− 39025	− 56429
Uranus	1057	297	16	107	1383939	− 34725	880088	54753
Neptune	27	16	1	12	1383939	− 3851	−103543	588806

Lastly, we give the totality of our solution in Tables 8 to 13. /154 Hence, Table 8 contains the 32 integration constants. Table 9 gives the B_i and C_i frequency modifications as well as the frequencies' new values: $\bar{g}_i = g_i + B_i$; $\bar{s}_i = s_i + C_i$. The amplitudes of the Lagrangian solution corresponding to the new constants are given in Table 10 for

$\lambda_{ij}M_j \times 10^8$ and in Table 11 for $\mu_{ij}N_j \times 10^8$. Finally, Tables 12 and 13 contain the amplitudes $M_{u,\psi,\theta}$ and $N_{u,\psi,\theta}$ of the higher order arguments (ψ,θ). When computing these terms, we retained only the arguments whose amplitudes are higher than 10^{-4} for the planets Mercury, Venus, Earth, and Mars, and 10^{-6} for Jupiter, Saturn, Uranus, and Neptune. The zeros found in Tables 12 and 13 are amplitudes less than the retained percisions.

Table 12

$M_{u,\psi,\theta} \times 10^6$

Argument	Mercury	Vénus	Earth	Mars
ψ_7	151	0	0	0
$\psi_6-\theta_3+\theta_6$	0	0	0	− 199
$\psi_6-\theta_3+\theta_4$	0	0	0	453
$\psi_6-\theta_4+\theta_6$	0	0	0	287
ψ_6	0	981	− 1064	− 13379
$\psi_6+\theta_3-\theta_4$	0	0	0	762
$\psi_5-\psi_6-\psi_7$	0	0	0	134
$\psi_5-2\theta_1$	254	0	0	0
$\psi_5-\theta_1-\theta_2$	− 114	0	0	0
$\psi_5-\theta_1+\theta_7$	102	0	0	0
$\psi_5-\theta_1+\theta_2$	− 275	0	0	0
$\psi_5-\theta_3+\theta_4$	150	0	0	0
ψ_5	14547	2918	2280	1092
$\psi_5+\theta_1-\theta_2$	− 2172	0	0	0
$\psi_5+\theta_1-\theta_3$	0	− 216	182	503
$\psi_5+\psi_6-\psi_7$	0	0	0	105
$2\psi_5-\psi_7$	− 118	0	0	0
$\psi_4-\psi_5+\psi_7$	0	0	0	− 180
$\psi_4-\theta_1+\theta_4$	1644	0	0	0
$\psi_4-\theta_1+\theta_3$	− 598	0	0	0
$\psi_4-\theta_3-\theta_4$	0	0	0	− 177
$\psi_4-\theta_3+\theta_6$	0	0	0	− 240
$\psi_4-\theta_3+\theta_4$	0	− 536	276	6177
$\psi_4-2\theta_4$	0	0	0	176
$\psi_4-\theta_4+\theta_6$	0	0	0	370
ψ_4	− 120	456	− 503	0
$\psi_4+\theta_4-\theta_6$	0	0	0	− 354
$\psi_4+\theta_3-\theta_4$	− 519	4126	− 3367	− 14852
$\psi_4+\theta_3-\theta_6$	0	0	0	211
$\psi_4+\psi_5-\psi_7$	0	0	0	118
$2\psi_4-\psi_6$	0	0	0	− 286
$\psi_3-2\psi_4$	0	465	− 378	− 726
$\psi_3-\psi_4-\psi_6$	0	0	0	− 133
$\psi_3-\psi_4+\psi_6$	0	0	0	184
$\psi_3-\theta_1+\theta_4$	396	0	0	0
$\psi_3-\theta_1+\theta_3$	312	0	0	0
$\psi_3-\theta_2+\theta_3$	− 237	0	0	0
$\psi_3-\theta_3+\theta_4$	− 275	2457	− 2354	8864
$\psi_3-\theta_4+\theta_6$	0	0	0	148
ψ_3	0	0	0	− 323

Table 12 (cont.)

Argument	Mercury	Vénus	Earth	Mars
$\psi_3+\theta_4-\theta_6$	0	0	0	−162
$\psi_3+\theta_3-\theta_4$	115	−901	857	−1422
$\psi_3+\theta_3-\theta_6$	0	0	0	102
$\psi_3+\psi_5-\psi_7$	0	0	0	137
$\psi_3+\psi_4-\psi_6$	0	0	0	−144
$2\psi_3-\psi_4$	0	−691	599	388
$\psi_2-2\theta_1$	−374	0	0	0
$\psi_2-\theta_1-\theta_2$	175	0	0	0
$\psi_2-\theta_1+\theta_3$	−137	0	0	0
$\psi_2-\theta_1+\theta_2$	−4711	−261	−174	0
ψ_2	2947	0	−151	0
$\psi_2+\theta_3-\theta_4$	0	111	0	0
$\psi_2+\theta_1-\theta_2$	−875	0	0	0
$\psi_2+\theta_1-\theta_3$	105	0	0	0
$\psi_2+\theta_1-\theta_7$	355	0	0	0
$\psi_2+\psi_3-\psi_4$	205	0	0	0
$\psi_1-2\psi_2$	−361	0	0	0
$\psi_1-\psi_2-\psi_5$	−3340	−131	0	0
$\psi_1-\psi_2+\psi_5$	−194	0	0	0
$\psi_1-\psi_3+\psi_4$	−591	0	0	0
$\psi_1-2\theta_1$	2231	0	0	0
$\psi_1-\theta_1-\theta_2$	−1022	0	0	0
$\psi_1-\theta_1-\theta_3$	350	0	0	0
$\psi_1-\theta_1-\theta_4$	147	0	0	0
$\psi_1-\theta_1+\theta_7$	399	0	0	0
$\psi_1-\theta_1+\theta_5$	−146	0	0	0
$\psi_1-\theta_1+\theta_4$	212	0	0	0
$\psi_1-\theta_1+\theta_3$	480	0	0	0
$\psi_1-\theta_1+\theta_2$	−8302	0	0	0
$\psi_1-2\theta_2$	119	0	0	0
$\psi_1-\theta_2+\theta_3$	−139	0	0	0
$\psi_1-\theta_3+\theta_4$	−605	0	0	0
ψ_1	0	−812	−650	0
$\psi_1+\theta_3-\theta_4$	603	0	0	0
$\psi_1+\theta_2-\theta_3$	147	0	0	190
$\psi_1+\theta_1-\theta_2$	6786	1372	1007	173
$\psi_1+\theta_1-\theta_3$	−505	164	−124	135
$\psi_1+\theta_1-\theta_4$	−209	0	0	−1087
$\psi_1+\theta_1-\theta_5$	146	0	0	0
$\psi_1+\theta_1-\theta_7$	−316	0	0	0
$\psi_1+\psi_3-\psi_4$	645	0	0	0
$\psi_1+\psi_2-\psi_5$	258	0	0	0
$2\psi_1-\psi_2$	1544	0	0	0
$2\psi_1-\psi_3$	−109	0	0	0
$2\psi_1-\psi_4$	107	0	0	0
$2\psi_1-\psi_5$	−824	−438	−337	0

Argument	Jupiter	Saturn	Uranus	Neptune
$\psi_8-\theta_7+\theta_8$	−2	−2	35	5
ψ_8	0	−3	271	0
$\psi_8+\theta_7-\theta_8$	0	0	1	−1
$\psi_7-2\psi_8$	0	0	0	−1
$\psi_7-\theta_5+\theta_7$	0	0	3	0
$\psi_7-\theta_6+\theta_7$	0	−2	2	0
$\psi_7-2\theta_7$	0	0	4	−1
$\psi_7-\theta_7-\theta_8$	0	0	−4	2
$\psi_7-\theta_7+\theta_8$	0	0	8	−4
$\psi_7-2\theta_8$	0	0	1	0

Argument	Jupiter	Saturn	Uranus	Neptune
ψ_7	−174	−198	0	−619
$\psi_7+\theta_7-\theta_8$	0	0	5	92
$\psi_7+\theta_6-\theta_7$	0	0	−2	0
$\psi_7+\theta_5-\theta_7$	0	0	−3	0
$2\psi_7-\psi_8$	0	0	−4	3
$\psi_6-2\psi_7$	0	0	−7	0
$\psi_6-\psi_7+\psi_8$	0	−1	0	0
$\psi_6-\theta_5-\theta_6$	0	5	0	0
$\psi_6-2\theta_6$	−9	15	0	0
$\psi_6-\theta_6-\theta_7$	0	0	1	0
$\psi_6-\theta_6+\theta_7$	0	0	−1	0
$\psi_6-2\theta_7$	0	0	−1	0
$\psi_6-\theta_7+\theta_8$	0	−1	0	0
ψ_6	−594	0	192	33
$\psi_6+\theta_6-\theta_7$	−1	0	22	3
$\psi_6+\theta_6-\theta_8$	0	0	1	0
$\psi_6+\theta_5-\theta_6$	0	−5	0	0
$\psi_6+\psi_7-\psi_8$	0	1	0	0
$2\psi_6-\psi_7$	−23	78	2	0
$2\psi_6-\psi_5$	0	3	0	0
$\psi_5-2\psi_6$	298	−962	27	0
$\psi_5-\psi_6-\psi_7$	−113	348	−49	0
$\psi_5-\psi_6-\psi_8$	−1	4	−1	0
$\psi_5-\psi_6+\psi_8$	0	0	−1	0
$\psi_5-\psi_6+\psi_7$	0	−6	−36	0
$\psi_5-2\psi_7$	−7	−7	−440	34
$\psi_5-\psi_7-\psi_8$	0	0	−3	−35
$\psi_5-\psi_7+\psi_8$	−1	−1	−13	−33
$\psi_5-2\psi_5$	0	0	−1	1
$\psi_5-\theta_5-\theta_7$	0	0	−5	0
$\psi_5-\theta_5+\theta_6$	−1	3	0	0
$\psi_5-2\theta_6$	7	−5	0	0
$\psi_5-\theta_6-\theta_7$	0	0	2	0
$\psi_5-\theta_6-\theta_7$	−1	6	−5	0
$\psi_5-2\theta_7$	0	1	−8	1
$\psi_5-\theta_7-\theta_8$	0	0	4	−2
$\psi_5-\theta_7+\theta_8$	0	0	−3	4
$\psi_5-2\theta_5$	0	0	−2	0
ψ_5	0	257	5257	463
$\psi_5+\theta_7-\theta_8$	1	1	15	21
$\psi_5+\theta_6-\theta_7$	0	0	5	0
$\psi_5+\theta_5-\theta_6$	1	−3	0	0
$\psi_5+\theta_5-\theta_7$	0	0	5	0
$\psi_5+\theta_3-\theta_4$	0	0	−1	0
$\psi_5+\psi_7-\psi_8$	0	0	2	−6
$\psi_5+\psi_6-\psi_7$	−111	349	15	[illegible]
$\psi_5+\psi_6-\psi_8$	−1	4	0	0
$2\psi_5-\psi_6$	−10	−50	41	0
$2\psi_5-\psi_7$	−66	−46	511	−43
[illegible]	0	0	2	[illegible]
$\psi_4-2\psi_6$	−1	3	0	[illegible]
$\psi_4-\theta_3-\theta_4$	0	−1	0	[illegible]
ψ_4	−2	8	0	[illegible]
$\psi_4+\theta_3-\theta_4$	0	3	0	[illegible]
$\psi_3-2\psi_6$	0	3	0	[illegible]
ψ_3	−2	7	0	[illegible]
ψ_2	−1	0	1	[illegible]
ψ_1	−1	−1	2	[illegible]

ORIGINAL PAGE IS OF POOR QUALITY

Table 13

$N_{u,\ell,q} \times 10^6$

Argument	Mercury	Venus	Earth	Mars
θ_4	0	0	0	−180
θ_4	306	−1138	997	0
$\theta_3 - 2\theta_4$	0	0	0	1101
θ_3	275	0	0	−1838
$2\theta_3 - \theta_4$	0	457	−378	−207
$\theta_2 - \theta_3 + \theta_4$	175	0	0	0
θ_2	−4695	0	0	0
$\theta_1 - 2\theta_2$	172	0	0	0
$\theta_1 - \theta_3 + \theta_4$	166	0	0	0
θ_1	0	2637	1988	429
$2\theta_1 - \theta_2$	−879	0	0	0
$\nu_5 - \nu_6 - \theta_6$	−105	0	0	0
$\nu_5 - \nu_7 + \theta_3$	0	0	0	−205
$\nu_4 - \nu_6 - \theta_3$	0	0	0	−416
$\nu_4 - \nu_6 - \theta_4$	0	0	0	537
$\nu_4 - \nu_6 - \theta_6$	0	0	0	−145
$\nu_4 - \nu_6 + \theta_4$	0	0	0	608
$\nu_4 - \nu_6 + \theta_3$	0	0	0	−251
$\nu_3 - \nu_4 - \theta_1$	−233	0	0	0
$\nu_3 - \nu_4 - \theta_3$	0	280	−499	7947
$\nu_3 - \nu_4 - \theta_4$	0	0	0	3757
$\nu_3 - \nu_4 + \theta_4$	320	−2029	1741	1136
$\nu_3 - \nu_4 + \theta_3$	−206	1392	−1192	0
$\nu_3 - \nu_4 + \theta_1$	−207	0	0	0
$\nu_3 - \nu_5 + \theta_3$	−221	0	0	0
$\nu_3 - \nu_6 - \theta_3$	0	0	0	−119
$\nu_3 - \nu_6 - \theta_4$	0	0	0	198
$\nu_3 - \nu_6 + \theta_4$	0	0	0	214
$\nu_2 - \nu_4 - \theta_3$	−114	0	0	0
$\nu_2 - \nu_5 - \theta_1$	170	0	0	0
$\nu_2 - \nu_5 + \theta_1$	117	0	0	0
$\nu_1 - \nu_2 - \theta_1$	−3777	0	0	0
$\nu_1 - \nu_2 - \theta_2$	3170	176	124	0
$\nu_1 - \nu_2 - \theta_3$	117	0	0	0
$\nu_1 - \nu_2 + \theta_7$	−232	0	0	0
$\nu_1 - \nu_2 + \theta_2$	787	0	0	0
$\nu_1 - \nu_2 + \theta_1$	−7950	1062	903	206
$\nu_1 - \nu_3 - \theta_1$	117	0	0	0
$\nu_1 - \nu_3 - \theta_3$	−413	0	0	0
$\nu_1 - \nu_3 - \theta_4$	−229	0	0	0
$\nu_1 - \nu_3 + \theta_1$	126	0	0	0
$\nu_1 - \nu_4 - \theta_1$	−115	0	0	0
$\nu_1 - \nu_4 - \theta_3$	277	0	0	0
$\nu_1 - \nu_4 - \theta_4$	−1021	0	0	0
$\nu_1 - \nu_4 + \theta_1$	−134	0	0	−606
$\nu_1 - \nu_5 - \theta_1$	−3353	477	401	0
$\nu_1 - \nu_5 - \theta_2$	282	0	0	0
$\nu_1 - \nu_5 + \theta_2$	1387	0	0	0
$\nu_1 - \nu_5 + \theta_1$	−1388	0	0	0
$\nu_1 + \nu_5 - \theta_1$	396	0	0	0
$\nu_1 + \nu_2 - \theta_1$	−589	0	0	0
$\nu_1 + \nu_2 - \theta_2$	138	0	0	0
$2\nu_1 - \theta_1$	1746	0	0	0
$2\nu_1 - \theta_2$	−401	0	0	0
$2\nu_1 - \theta_3$	140	0	0	0

Argument	Jupiter	Saturn	Uranus	Neptune
θ_8	−2	−2	12	0
θ_7	0	2	0	−12
$2\theta_7 - \theta_8$	0	0	1	0
θ_6	0	0	−9	0
θ_1	−3	−4	3	0
$\nu'_7 - \nu'_8 - \theta_7$	0	0	2	1
$\nu'_7 - \nu'_8 - \theta_8$	−1	−1	33	−2
$\nu'_7 - \nu'_8 + \theta_8$	0	0	−1	1
$\nu'_7 - \nu'_8 + \theta_7$	−3	−3	5	31
$2\nu'_7 - \theta_7$	0	0	3	−2
$2\nu'_7 - \theta_8$	0	0	−1	1
$\nu'_6 - \nu'_7 - \theta_6$	0	0	−1	0
$\nu'_6 - \nu'_7 - \theta_7$	−1	3	3	0
$\nu'_6 - \nu'_7 + \theta_7$	0	0	3	0
$\nu'_6 - \nu'_7 + \theta_6$	−1	0	17	−1
$\nu'_6 - \nu'_8 + \theta_6$	0	0	2	0
$\nu'_6 + \nu'_7 - \theta_6$	0	−1	1	0
$\nu'_6 + \nu'_7 - \theta_7$	0	0	−1	0
$2\nu'_6 - \theta_6$	−10	26	−1	0
$2\nu'_6 - \theta_7$	0	0	2	0
$\nu'_5 - \nu'_6 - \theta_6$	1	−22	46	0
$\nu'_5 - \nu'_6 - \theta_7$	0	−1	9	0
$\nu'_5 - \nu'_6 + \theta_8$	0	0	1	0
$\nu'_5 - \nu'_6 + \theta_7$	−3	7	9	0
$\nu'_5 - \nu'_6 + \theta_6$	8	−11	−4	0
$\nu'_5 - \nu'_7 - \theta_6$	5	−13	−3	0
$\nu'_5 - \nu'_7 - \theta_7$	−11	−8	225	−30
$\nu'_5 - \nu'_7 - \theta_8$	−2	−2	12	16
$\nu'_5 - \nu'_7 + \theta_8$	−1	−1	−2	18
$\nu'_5 - \nu'_7 + \theta_7$	−13	−11	220	−15
$\nu'_5 - \nu'_7 + \theta_6$	5	−14	5	0
$\nu'_5 - \nu'_8 - \theta_7$	0	0	−1	1
$\nu'_5 - \nu'_8 - \theta_8$	0	0	8	−1
$\nu'_5 - \nu'_8 + \theta_8$	0	0	1	0
$\nu'_5 - \nu'_8 + \theta_7$	0	0	0	8
$\nu'_5 + \nu'_7 - \theta_6$	0	1	1	0
$\nu'_5 + \nu'_7 - \theta_7$	0	0	−13	5
$\nu'_5 + \nu'_7 - \theta_8$	0	0	3	−3
$\nu'_5 + \nu'_6 - \theta_6$	10	−23	−5	0
$\nu'_5 + \nu'_6 - \theta_7$	0	−1	8	0
$2\nu'_5 - \theta_6$	−5	14	−2	0
$2\nu'_5 - \theta_7$	0	−1	21	−3
$2\nu'_5 - \theta_8$	0	0	0	3
$\nu'_3 - \nu'_4 - \theta_3$	0	−2	0	0
$\nu'_3 - \nu'_4 - \theta_4$	0	−1	0	0
$\nu'_3 - \nu'_4 + \theta_4$	0	−1	0	0
$\nu'_1 - \nu'_2 - \theta_1$	0	0	−1	0
$\nu'_1 - \nu'_2 + \theta_1$	1	−1	0	0

CONCLUSION

In this study of the long period variations of the planetary elements, we added to the Lagrangian solution the terms of third order excentricity and inclination arising from the long period portion of the perturbation function calculated for the planets as a whole. We also took into consideration the influence of short period terms of second order mass. We particularly concentrated on determining the integration constants that make the solutions agree with the mean elements when t = 1850.0 is used as time zero.

The terms calculated with these constants are grouped together in Tables 8 to 13. Notice in these results the very strong coupling that exists, for a long period problem, in the planetary system. The magnitude of the terms arising from the short periods shows that there is no point to extending a theory to the fifth order on the basis of the perturbation function's long periods if the short periods are not taken into account.

This work's essential task was therefore the comparison of the various effects according to their origin so as to have an overall view of this problem and to be able to embark on the complete construction of a long period theory. Our solution is in fact still incomplete. Even so, we should take into account the direct terms of fifth order that must have an influence, especially for Mercury, Venus, Earth, and Mars. We have yet to calculate the influence of short periods of higher orders of excentricity and inclination, and maybe even part of the third order with respect to masses in the case of the resonant argument $2\lambda_j - 5\lambda_s$ between Jupiter and Saturn. Such an investigation would be very important. However, since the largest contributions have already been considered, it would no longer present any great difficulties for the determination of integration constants.

BIBLIOGRAPHY

Hill, G.W., Astron. J., 9, 89-91 (1889).

Hill, G.W., Astron. J., 17, 81-87 (1897).

Brouwer, D., van Woerkom, A.J.J., Astron. Pap., 13, part 2 (1950).

Anolik, M.V., Krassinsky, B.S., Pius, L.J., Trudy Inst. Theor. Astron. Leningrad, 14, 3-47 (1969).

Chapront, J., Simon, J.-L., Astron. & Astrophys., 19, 231 (1972).

Kovalevsky, J., Celestial Mechanics, 4, 213 (1971).

Bogolioubov, N. Mitropolski, I., Les méthodes asymptotiques en théorie des oscillations non linéaires [Asymptotic Methods in Nonlinear Oscillations], Gauthier-Villars, Paris, 1962.